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Research article

A note on the Fujita exponent in fractional heat equation involving the Hardy potential†

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Abstract: In this work, we are interested on the study of the Fujita exponent and the meaning of the blow-up for the fractional Cauchy problem with the Hardy potential, namely,

$$u_t + (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p \text{ in } \boldsymbol{R}^N, u(x,0) = u_0(x) \text{ in } \boldsymbol{R}^N,$$

where $N > 2s$, $0 < s < 1$, $(-\Delta)^s$ is the fractional laplacian of order $2s$, $\lambda > 0$, $u_0 \geq 0$, and $1 < p < p_+(s, \lambda)$, where $p_+(\lambda, s)$ is the critical existence power to be given subsequently.

Keywords: Fujita exponent; fractional Cauchy heat equation with Hardy potential; blow-up; global solution

To Sandro in his 70th birthday with our friendship.

1. Introduction

In the pioneering work [10], Fujita found a critical exponent for the heat equation with a semilinear term of power type. More precisely, for the problem,

$$\begin{cases} u_t = \Delta u + u^p, \ x \in \boldsymbol{R}^N, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \ x \in \boldsymbol{R}^N, \end{cases} \tag{1.1}$$

where $1 < p < \infty$, Fujita proved that if $1 < p < 1 + \frac{2}{N}$, then there exists $T > 0$ such that the solution to problem (1.1) satisfies $\|u(\cdot, t_n)\|_\infty \to \infty$ as $t_n \to T$. However, if $p > 1 + \frac{2}{N}$, then there are both global solutions for small data as well as non-global solutions for large data. The critical value $F(0) = 1 + \frac{2}{N}$ is often called the critical Fujita blow-up exponent for the heat equation. Moreover it is proved that for $p = 1 + \frac{2}{N}$, a suitable norm of the solution goes to infinity in a finite time. We refer to [24] for a simple proof of this last fact (see also [13]).

Sugitani in [22] studies the same kind of question for *the fractional heat equation*, that is, the problem,

$$\begin{cases} u_t + (-\Delta)^s u &= u^p & \text{in } \Omega \times (0,T), \\ u(x,t) &> 0 & \text{in } \Omega \times (0,T), \\ u(x,t) &= 0 & \text{in } (\boldsymbol{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) &= u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{1.2}$$

where $N > 2s$, $0 < s < 1$, $p > 1$ and $u_0 \geq 0$ is in a suitable class of functions.

By $(-\Delta)^s$ we denote the fractional Laplacian of order $2s$ introduced by M. Riesz in [20], that is,

$$(-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \ s \in (0,1), \tag{1.3}$$

where

$$a_{N,s} = 2^{2s-1} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$$

is the normalization constant to have the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u), \ \xi \in \mathbb{R}^N, \ s \in (0,1),$$

for every $u \in \mathcal{S}(\mathbb{R}^N)$, the Schwartz class. See [8, 9, 14] and Chapter 8 of [17], for technical details and properties of the fractional Laplacian.

In [1], the authors deal with the following problem,

$$\begin{cases} u_t - \Delta u &= \lambda \dfrac{u}{|x|^2} + u^p + cf & \text{in } \Omega \times (0,T), \\ u(x,t) &> 0 & \text{in } \Omega \times (0,T), \\ u(x,t) &= 0 & \text{in } \partial\Omega \times [0,T), \\ u(x,0) &= u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{1.4}$$

where $N > 2$ and $0 \in \Omega$.

This problem is related to the classical Hardy inequality:

Hardy Inequality. *Assume $N \geq 3$. For all $\phi \in C_0^\infty(\mathbb{R}^N)$ the following inequality holds,*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx. \tag{1.5}$$

Moreover $\Lambda_N := \left(\frac{N-2}{2}\right)^2$ is optimal and is not achieved.

The blow-up in the L^∞ norm for the solution of problem (1.4) is produced in any time $t > 0$, for any nonnegative data and for all $p > 1$, according with the results by Baras-Goldstein in [3]. Therefore, the Fujita behavior in the presence of the Hardy potential must be understood in a different way.

For $\lambda > 0$, setting $\mu_1(\lambda) = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$, then it was proved that if $1 < p < 1 + \frac{2}{N-\mu_1(\lambda)}$, there exists $T^* > 0$ that is independents of the nonnegative initial datum, such that the solution u to problem (1.4) satisfies

$$\lim_{t\to T^*} \int_{B_r(0)} |x|^{-\mu_1(\lambda)} u(x,t)\, dx = \infty, \tag{1.6}$$

for any ball $B_r(0)$. Moreover for $p > 1 + \dfrac{2}{N-\mu_1(\lambda)}$, if the initial datum is small enough, there exists a global solution to (1.4). According to this behavior the corresponding *Fujita type exponent* for problem (1.4) is defined by $F(\lambda) = 1 + \dfrac{2}{N-\mu_1(\lambda)}$ and the blow-up is understood in the sense of local weighted L^1 associated to (1.6). The Hardy inequality is an expression of the *uncertainty Heisenberg principle*, hence we can say that the result in [1], explains the influence of the uncertainty principle on the diffusion problem (1.4).

The following fractional Hardy inequality appears in [9] in order to study the relativistic stability of the matter.

Theorem 1.1. *(Fractional Hardy inequality). For all $u \in C_0^\infty(\boldsymbol{R}^N)$ the following inequality holds,*

$$\int_{\boldsymbol{R}^N} |\xi|^{2s} |\hat{u}|^2\, d\xi \geq \Lambda_{N,s} \int_{\boldsymbol{R}^N} |x|^{-2s} u^2\, dx, \tag{1.7}$$

where

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

The constant $\Lambda_{N,s}$ is optimal and is not attained. Moreover, $\Lambda_{N,s} \to \Lambda_{N,1} := \left(\dfrac{N-2}{2}\right)^2$, the classical Hardy constant, when s tends to 1.

This inequality was proved in [12]. See also [5, 9, 23, 25]. The reader can find all the details of a direct proof in Section 9.2 of [17].

Recently, in [2] and related to the Hardy inequality stated in (1.7), the authors study the fractional parabolic semilinear problem,

$$\begin{cases} u_t + (-\Delta)^s u &= \lambda \dfrac{u}{|x|^{2s}} + u^p + cf & \text{in } \Omega \times (0,T), \\ u(x,t) &> 0 & \text{in } \Omega \times (0,T), \\ u(x,t) &= 0 & \text{in } (\boldsymbol{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) &= u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{1.8}$$

where $N > 2s$, $0 < s < 1$, $p > 1$, $c, \lambda > 0$, and $u_0 \geq 0$, $f \geq 0$ are in a suitable class of functions. By $(-\Delta)^s$ we denote the fractional Laplacian of order $2s$, defined in (1.3). In [2] and [4], the authors prove the existence of a critical power $p_+(s,\lambda)$ such that if $p > p_+(s,\lambda)$, the problem (1.8) has no weak positive supersolutions and a phenomenon of *complete and instantaneous blow up* happens. If $p < p_+(s, \lambda)$, there exists a positive solution for a suitable class of nonnegative data.

In this note, we deal with the corresponding *fractional Cauchy problem,*

$$u_t + (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + u^p \text{ in } \boldsymbol{R}^N \times (0, \infty), u(x, 0) = u_0(x) \text{ in } \boldsymbol{R}^N, \tag{1.9}$$

with $1 < p < p_+(s, \lambda)$ in order to find the value of the corresponding Fujita exponent.

A relevant fact in this work is that the effect of the Hardy potential produces a shift on the right of the Fujita exponent of the fractional heat equation, depending of the spectral parameter λ.

The problem (1.9) with $s \in (0, 1)$ and $\lambda = 0$ was considered in [22]. The author was able to show that $F(s) := 1 + \frac{2s}{N}$ is the associated Fujita exponent. See also [11] for some extensions.

For $\lambda > 0$, any solution to problem (1.8) is unbounded close to the origin, even for nice data (see [2]). This is the corresponding nonlocal version of the Baras-Goldstein results for the heat equation developed in [3]. Therefore, L^∞-blowup is instantaneous and free in problem (1.9) as in the local case and the blowup will be also obtained in a suitable weighted Lebesgue space.

In this work we will treat the case $s \in (0, 1)$ and $\lambda > 0$ that is more involved than the local problem for several reasons, one of them that the kernel of the fractional heat equation has not a closed form with the exception of $s = \frac{1}{2}$ and $s = 1$.

The paper is organized as follows. In Section 2 we introduce some tools about the fractional equation. The Fujita exponent $F(\lambda, s)$ for problem (1.9) in obtained in Section 3. Notice that by the Fujita exponent, we understand that, independently of the initial datum, for $1 < p < F(\lambda, s)$, any solution to (1.9) blows-up in a certain weighted norm in a finite time. The Fujita exponent verifies $F(\lambda, s) < p_+(s, \lambda)$, and the effect of the Hardy potential is reflected by the strict inequality, $F(0, s) < F(\lambda, s)$.

The critical case $p = F(\lambda, s)$ is analyzed in Subsection 3.1. In this case we are able to show a blowup of a precise norm of u that reflects the critical exponent $F(\lambda, s)$. In Section 4, for $F(\lambda, s) < p < p_+(\lambda, s)$, we prove the existence of global solutions for suitable data. This shows in some sense the optimality of our blow up results.

2. Preliminaries tools

First, we enunciate some Lemmas and notations that we will use along the paper (see [2] for a proof).

Lemma 2.1. *Let $0 < \lambda \leq \Lambda_{N,s}$. Then $v_{\pm\alpha} = |x|^{-\frac{N-2s}{2} \pm \alpha_\lambda}$ are solutions to*

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \text{ in } (\boldsymbol{R}^N \setminus \{0\}), \tag{2.1}$$

where α_λ is obtained by the identity

$$\lambda = \lambda(\alpha_\lambda) = \lambda(-\alpha_\lambda) = \frac{2^{2s}\, \Gamma(\frac{N+2s+2\alpha_\lambda}{4})\Gamma(\frac{N+2s-2\alpha_\lambda}{4})}{\Gamma(\frac{N-2s+2\alpha_\lambda}{4})\Gamma(\frac{N-2s-2\alpha_\lambda}{4})}. \tag{2.2}$$

Remark 2.2. *Notice that* $\lambda(\alpha_\lambda) = \lambda(-\alpha_\lambda) = m_{\alpha_\lambda} m_{-\alpha_\lambda}$*, with* $m_{\alpha_\lambda} = 2^s \dfrac{\Gamma(\frac{N+2s+2\alpha_\lambda}{4})}{\Gamma(\frac{N-2s-2\alpha_\lambda}{4})}$.

Lemma 2.3. *The following equivalence holds true:*

$$0 < \lambda(\alpha_\lambda) = \lambda(-\alpha_\lambda) \le \Lambda_{N,s} \text{ if and only if } 0 \le \alpha_\lambda < \frac{N-2s}{2}.$$

Remark 2.4. *Notice that we can explicitly construct two positive solutions to the homogeneous problem* (2.1)*. Henceforth, we denote*

$$\mu(\lambda) = \frac{N-2s}{2} - \alpha_\lambda \text{ and } \bar{\mu}(\lambda) = \frac{N-2s}{2} + \alpha_\lambda, \tag{2.3}$$

with $0 < \mu(\lambda) \le \dfrac{N-2s}{2} \le \bar{\mu}(\lambda) < (N-2s)$*. Since* $N - 2\mu(\lambda) - 2s = 2\alpha_\lambda > 0$ *and* $N - 2\bar{\mu}(\lambda) - 2s = -2\alpha_\lambda < 0$*, then* $|x|^{-\mu(\lambda)}$ *is the unique nonnegative solution that is locally in the energy space.*

The critical existence power $p_+(\lambda, s)$, found in [2,4], depends on s and λ, and in particular satisfies:

$$p_+(\lambda, s) := 1 + \frac{2s}{\frac{N-2s}{2} - \alpha_\lambda} = 1 + \frac{2s}{\mu(\lambda)}.$$

(See Figure 1 below).

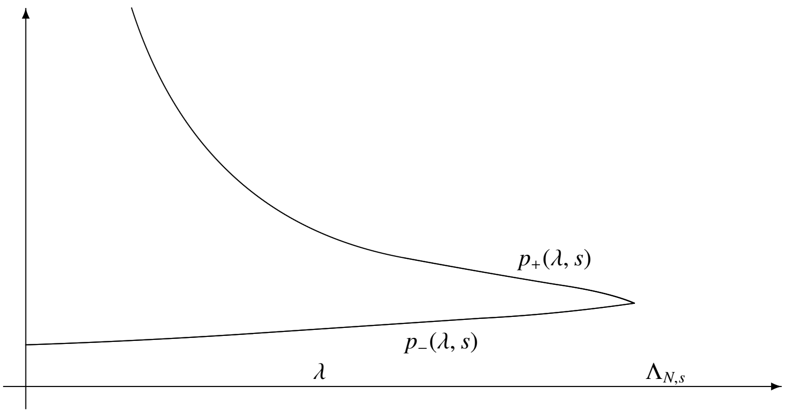

Figure 1. Fujita exponent for fractional Cauchy problem with Hardy potential.

Note that if $\lambda = \Lambda_{N,s}$, namely, $\alpha_\lambda = 0$, then $p_+(\lambda, s) = \dfrac{N+2s}{N-2s} = 2_s^* - 1$, and if $\lambda = 0$, namely, $\alpha_\lambda = \dfrac{N-2s}{2}$, then $p_+(\lambda, s) = \infty$. Noting

$$p_-(\lambda, s) = 1 + \frac{2s}{\frac{N-2s}{2} + \alpha_\lambda} = 1 + \frac{2s}{\bar{\mu}(\lambda)},$$

it follows that for $\lambda = \Lambda_{N,s}$, namely, $\alpha_\lambda = 0$, then $p_-(\lambda, s) = 2_s^* - 1$ and if $\lambda = 0$, namely, $\alpha_\lambda = \dfrac{N-2s}{2}$, then $p_-(\lambda, s) = \dfrac{N}{N-2s}$. Hence,

$$\frac{N}{N-2s} \le p_-(\lambda, s) \le 2_s^* - 1 \le p_+(\lambda, s) \le \infty.$$

3. Blow up result for the Cauchy problem

It is clear that L^∞-blowup is instantaneous and free in problem (1.9) because the solutions are unbounded at the origin.

Before starting the main blowup result we begin by precising the sense for which blow up is considered. As in the case $s = 1$, $\lambda > 0$, this phenomenon will be analyzed in a suitable weighted Lebesgue space.

Definition 3.1. *Consider $u(x,t)$ a positive solution to* (1.9)*, then we say that u blows-up in a finite time if there exists $T^* < \infty$ such that*

$$\lim_{t \to T^*} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda)} u(x,t)\, dx = \infty,$$

with $\mu(\lambda) = \dfrac{N-2s}{2} - \alpha_\lambda$.

The next proposition justifies in some sense the previous definition.

Proposition 3.2. *Let $\lambda \le \Lambda_{N,s}$ and consider u to be a nonnegative solution to problem* (1.9)*, then*

$$\int_{B_r(0)} |x|^{-\mu(\lambda)} u_0(x)\, dx < \infty, \textit{ for some } r > 0.$$

In particular, for all $t \in (0, T)$, we have

$$\int_{B_r(0)} |x|^{-\mu(\lambda)} u(x,t)\, dx < \infty.$$

The proof follows combining the approximating arguments used in [1,2].

The main blow up result of this section is the following.

Theorem 3.3. *Suppose that* $1 < p < F(\lambda, s) := 1 + \dfrac{2s}{N - \mu(\lambda)}$ *and let u be a positive solution to problem* (1.9)*. Then there exists $T^* := T^*(u_0)$ such that*

$$\lim_{t \to T^*} \int_{\mathbb{R}^N} |x|^{-\mu(\lambda)} u(x,t)\, dx = \infty.$$

Before proving Theorem 3.3, we need some analysis related to the fractional heat equation.

Let $h(x,t)$ be the *fractional Heat Kernel*, namely,

$$h_t + (-\Delta)^s h = 0 \text{ in } \mathbb{R}^N \times (0,\infty), \;\; h(x,0) = \delta_0.$$

There is no known closed form for $h(t,x)$ in real variables. However, in Fourier variables it is simply $\mathcal{F}(h)(t,\xi) = e^{-t|2\pi\xi|^{2s}}$. The properties of the kernel h were studied in [18] for $N = 1$ and in [6] for all dimensions. More precisely, since $h(x,t)$ is defined by

$$h(x,t) = \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle} e^{-(2\pi|\xi|)^{2s}t} d\xi. \tag{3.1}$$

where $0 < s < 1$ and $N \geq 2s$, then there exists a constant $C > 1$ such that

$$\frac{1}{C}\frac{1}{(1+|x|^{N+2s})} \leq h(x,1) \leq \frac{C}{(1+|x|^{N+2s})}, \qquad \text{for all } x \in \mathbb{R}^N. \tag{3.2}$$

There is a direct approach inside of the Real Analysis field and even without using Bessel functions. Such a proof is based on a celebrated result by S. N. Bernstein about the characterization of *completely monotone functions* via Laplace transform. See Section 12.5 of [17] for a detailed proof.

Notice that $h(x,t) = t^{-\frac{N}{2s}} H(\frac{|x|}{t^{\frac{1}{2s}}})$ and H is a decreasing function that satisfies

$$H(\sigma) \approx \frac{1}{(1+\sigma^2)^{\frac{N+2s}{2}}}, \quad |H'(\sigma)| \leq \frac{C}{(1+\sigma^2)^{\frac{N+2s+1}{2}}},$$

with

$$2s(-\Delta)^s H = NH + rH'.$$

See for instance [7] and [21]. We set

$$\hat{h}(x,t) = \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} h(x,t) \equiv \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} t^{-\frac{N}{2s}} H(\frac{|x|}{t^{\frac{1}{2s}}}).$$

We also have the elementary formula,

$$(-\Delta)^s(w\,v) = v(-\Delta)^s w + w(-\Delta)^s v - \int\limits_{R^N} \frac{(w(x)-w(y))(v(x)-v(y))}{|x-y|^{N+2s}} dy.$$

Hence, for $t > 0$, we have

$$\begin{aligned}(-\Delta)^s(\hat{h}(x,t)) \;\; &= \;\; \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} (-\Delta)^s h(x,t) + h(x,t)(-\Delta)^s \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} \\ &- \;\; \int_{\mathbb{R}^N} \frac{\left(\left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} - \left(\frac{|y|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)}\right)\left(h(x,t)-h(y,t)\right)}{|x'-y|^{N+2s}} dy.\end{aligned}$$

Notice that

$$\left(\left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} - \left(\frac{|y|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)}\right)\left(h(x,t)-h(y,t)\right) = t^{\frac{\mu(\lambda)}{s}-\frac{N}{2s}}(|x|^{-\mu(\lambda)} - |y|^{-\gamma})(H(\frac{|x|}{t^{\frac{1}{2s}}}) - H(\frac{|y|}{t^{\frac{1}{2s}}})) \geq 0.$$

Thus

$$(-\Delta)^s(\hat{h}(x,t)) \;\; \leq \;\; \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)} (-\Delta)^s h(x,t) + h(x,t)(-\Delta)^s \left(\frac{|x|}{t^{\frac{1}{s}}}\right)^{-\mu(\lambda)}$$

$$
\begin{aligned}
&= \Big(\frac{|x|}{t^{\frac{1}{s}}}\Big)^{-\mu(\lambda)}(-h_t(x,t)) + \frac{\lambda h(x,t)}{|x|^{2s}}\Big(\frac{|x|}{t^{\frac{1}{s}}}\Big)^{-\mu(\lambda)}\\
&= \frac{N}{2s}\frac{\hat{h}(x,t)}{t} + \frac{1}{2s}|x|t^{-\frac{N}{2s}-\frac{1}{2s}-1}H'\Big(\frac{|x|}{t^{\frac{1}{2s}}}\Big)\Big(\frac{|x|}{t^{\frac{1}{s}}}\Big)^{-\mu(\lambda)} + \frac{\lambda\hat{h}(x,t)}{|x|^{2s}}\\
&\leq \frac{N}{2s}\frac{\hat{h}(x,t)}{t} + \frac{\lambda\hat{h}(x,t)}{|x|^{2s}},
\end{aligned}
$$

where we have used the fact that $H' \leq 0$. Thus

$$
-(-\Delta)^s(\hat{h}(x,t)) + \frac{\lambda\hat{h}(x,t)}{|x|^{2s}} \geq -\frac{N}{2s}\frac{\hat{h}(x,t)}{t}.
$$

We are now in position to prove Theorem 3.3.

Proof of Theorem 3.3. We follow closely some arguments developed in [1], see also [19]. Let u be a positive solution to (1.9). Fix $\eta > 0$ to be chosen later and define the function ψ_η

$$
\psi_\eta(x) = \hat{h}(x, \frac{1}{\eta}) = \eta^{\frac{N}{2s}-\frac{\mu(\lambda)}{s}}|x|^{-\mu(\lambda)}H(\eta^{\frac{|x|}{2s}}),
$$

then by the previous computation it holds that

$$
-(-\Delta)^s\psi_\eta(x) + \frac{\lambda\psi_\eta(x)}{|x|^{2s}} \geq -\frac{N}{2s}\eta\psi_\eta(x).
$$

Notice that

$$
\int\limits_{\mathbb{R}^N} \psi_\eta(x)dx = C\eta^{-\frac{\mu(\lambda)}{2s}}.
$$

Now, using ψ_η as a test function in (1.9), we get

$$
\frac{d}{dt}\int\limits_{\mathbb{R}^N} u\psi_\eta dx = \int\limits_{\mathbb{R}^N} u^p\psi_\eta dx + \int\limits_{\mathbb{R}^N}\Big(-(-\Delta)^s\psi_\eta(x) + \frac{\lambda\psi_\eta(x)}{|x|^{2s}}\Big)udx.
$$

Thus

$$
\frac{d}{dt}\int\limits_{\mathbb{R}^N} u\psi_\eta dx \geq \int\limits_{\mathbb{R}^N} u^p\psi_\eta dx - \frac{N}{2s}\eta\int\limits_{\mathbb{R}^N}\psi_\eta(x)udx.
$$

Using Jensen inequality, there results that

$$
\int\limits_{\mathbb{R}^N} u^p\psi_\eta dx \geq C\eta^{(p-1)\frac{\mu(\lambda)}{2s}}\Big(\int\limits_{\mathbb{R}^N} u\psi_\eta dx\Big)^p.
$$

Then

$$
\frac{d}{dt}\int\limits_{\mathbb{R}^N} u\psi_\eta dx + \frac{N}{2s}\eta\int\limits_{\mathbb{R}^N}\psi_\eta(x)udx \geq C\eta^{(p-1)\frac{\mu(\lambda)}{2s}}\Big(\int\limits_{\mathbb{R}^N} u\psi_\eta dx\Big)^p.
$$

Setting

$$Y(t) = e^{\frac{N}{2s}\eta t} \int\limits_{\mathbb{R}^N} u\psi_\eta dx,$$

it follows that

$$Y'(t) \geq C\eta^{(p-1)\frac{\mu(\lambda)}{2s}} e^{-(p-1)\frac{N}{2s}\eta t} Y^p(t).$$

Integrating the previous differential inequality, we arrive to

$$\begin{aligned}\frac{1}{p-1}\Big(\frac{1}{Y^{p-1}(0)} - \frac{1}{Y^{p-1}(t)}\Big) &\geq C\eta^{(p-1)\frac{\mu(\lambda)}{2s}} \frac{1}{(p-1)\frac{N}{2s}\eta}(1 - e^{-(p-1)\frac{N}{2s}\eta t}) \\ &\geq \frac{C}{\frac{N}{2s}(p-1)}\eta^{(p-1)\frac{\mu(\lambda)}{2s}-1}(1 - e^{-(p-1)\frac{N}{2s}\eta t}).\end{aligned}$$

Therefore,

$$Y^{p-1}(t) \geq \frac{1}{\left(\frac{1}{Y^{p-1}(0)} - C\frac{2s}{N}\eta^{(p-1)\frac{\mu(\lambda)}{2s}-1}(1 - e^{-(p-1)\frac{N}{2s}\eta t})\right)}.$$

It is clear that, if for some $T < \infty$, we have

$$\frac{1}{Y^{p-1}(0)} \leq C\frac{2s}{N}\eta^{(p-1)\frac{\mu(\lambda)}{2s}-1}(1 - e^{-(p-1)\frac{N}{2s}\eta T}), \tag{3.3}$$

then $Y(T) = \infty$.

Since $(1 - e^{-(p-1)\frac{N}{2s}\eta T}) \to 1$ as $T \to \infty$, then condition (3.3) holds if

$$Y^{p-1}(0) > \frac{1}{C}\frac{N}{2s}\eta^{1-(p-1)\frac{\mu(\lambda)}{2s}}.$$

Hence

$$\eta^{(p-1)(\frac{N}{2s}-\frac{\mu(\lambda)}{s})}\Big(\int\limits_{\mathbb{R}^N} u_0(x)|x|^{-\mu(\lambda)} H(\eta^{\frac{1}{2s}}|x|)dx\Big)^{p-1} > \frac{1}{C}\frac{N}{2s}\eta^{1-(p-1)\frac{\mu(\lambda)}{2s}},$$

and then

$$\Big(\int\limits_{\mathbb{R}^N} u_0(x)|x|^{-\mu(\lambda)} H(\eta^{\frac{1}{2s}}|x|)dx\Big)^{p-1} > \frac{1}{C}\frac{N}{2s}\eta^{-(p-1)(\frac{N}{2s}-\frac{\mu(\lambda)}{2s})+1}. \tag{3.4}$$

It is clear that (3.4) holds for η small if and only if

$$-(p-1)(\frac{N}{2s} - \frac{\mu(\lambda)}{2s}) + 1 > 0$$

and then $p < F(\lambda, s)$.

Since

$$\int\limits_{\mathbb{R}^N} u_0(x)|x|^{-\mu(\lambda)} dx > C_0,$$

using the fact that H is bounded, there exists $\eta > 0$ such that

$$\Big(\int\limits_{\mathbb{R}^N} u_0(x)|x|^{-\mu(\lambda)} H(\eta^{\frac{1}{2s}}|x|)dx\Big)^{p-1} \geq 2sNC\eta^{-(p-1)(2sN\beta-\frac{\mu(\lambda)}{2s})+1}. \tag{3.5}$$

Hence the result follows. ∎

3.1. The critical case

Notice that the above argument does not hold for the critical case $p = F(\lambda, s)$. Hence in this case we will use a different argument based on a suitable apriori estimates as in [11, 16]. More precisely we have

Theorem 3.4. *Assume that* $p = F(\lambda, s) := 1 + \dfrac{2s}{N - \mu(\lambda)}$. *If u is a positive solution to problem* (1.9), *then there exists* $T^* := T^*(u_0)$ *such that*

$$\lim_{t\to T^*} \int_{R^N} |x|^{-\mu(\lambda)} u^p(x,t)\, dx = \infty.$$

Proof. We will perform the *ground state transform*, i.e., define $v(x,t) := |x|^{\mu(\lambda)} u(x,t)$, then

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = |x|^{\mu(\lambda)} Lv(x,t)$$

where

$$L(v(x,t)) := a_{N,s}\ \ p.v. \int_{\mathbb{R}^N} (v(x,t) - v(y,t))K(x,y)dy$$

and

$$K(x,y) = \frac{1}{|x|^{\mu(\lambda)}} \frac{1}{|y|^{\mu(\lambda)}} \frac{1}{|x-y|^{N+2s}}.$$

See [2, 9]. Thus v solves the parabolic equation

$$\begin{cases} |x|^{-2\mu(\lambda)} v_t + Lv &= |x|^{-\mu(\lambda)} u^p = |x|^{-\mu(\lambda)(p+1)} v^p \quad \text{in } \boldsymbol{R}^N \times (0,T), \\ |x|^{-\mu(\lambda)} v(x,0) &= u_0(x) \quad \text{in } \boldsymbol{R}^N. \end{cases} \tag{3.6}$$

It is clear that

$$\int_{R^N} |x|^{-\mu(\lambda)} u^p(x,t)\, dx = \int_{R^N} |x|^{-\mu(\lambda)(p+1)} v^p\, dx.$$

Therefore, in order to show the blowup result we will prove that

$$\lim_{t\to T^*} \int_{R^N} |x|^{-\mu(\lambda)(p+1)} v^p\, dx = \infty.$$

We argue by contradiction. Assume that $\int_{R^N} |x|^{-\mu(\lambda)(p+1)} v^p\, dx < \infty$ for all $t < \infty$. We claim that

$$\int_0^\infty \int_{R^N} |x|^{-\mu(\lambda)(p+1)} v^p dxdt \le C. \tag{3.7}$$

Let $\varphi \in C_0^\infty(\boldsymbol{R}^N)$ be such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B_1(0)$ and $\varphi = 0$ in $\boldsymbol{R}^N \backslash B_2(0)$. Define $\psi(x,t) = \varphi(\frac{t^2+|x|^{4s}}{R^2})$ with $R >> 1$. It is clear that if $t > R$, then $\psi(x,t) = 0$. Fix $T > R$, then using ψ^m as a test function in (3.6), with $1 < m < p'$, setting $Q_T = \boldsymbol{R}^N \times (0,T)$ and using Kato inequality, it holds that

$$\begin{aligned} &\iint_{Q_T} |x|^{-\mu(\lambda)(p+1)} v^p \psi^m dxdt + \int_{R^N} |x|^{-\mu(\lambda)} v(x,0)\psi^m(x,0)dx \\ &= \iint_{Q_T} |x|^{-2\mu(\lambda)} v\Big(- (\psi^m)_t dxdt + L\psi^m\Big) dxdt \\ &\le m \iint_{Q_T} |x|^{-2\mu(\lambda)} v\psi^{m-1}(-\psi_t)dxdt + m \iint_{Q_T} v\psi^{m-1} L\psi dxdt = I + J. \end{aligned} \tag{3.8}$$

We begin by estimating I. Define

$$Q_T^1 = \Big\{(x,t) \in Q_T \text{ such that } R^2 < t^2 + |x|^{4s} < 2R^2\Big\},$$

$$Q_T^2 = \Big\{(x,t) \in Q_T \text{ such that } t^2 + |x|^{4s} < 2R^2\Big\},$$

it is clear that supp$\psi_t \subset Q_T^1$ and supp$\psi \subset Q_T^1$. Then we have

$$\begin{aligned} I &\leq m \iint_{Q_T} |x|^{-2\mu(\lambda)} v\psi^{m-1}|\psi_t| dxdt \leq m \iint_{Q_T^1} |x|^{-2\mu(\lambda)} v\psi^{m-1}|\psi_t| dxdt \\ &\leq m\Big(\iint_{Q_T^1} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt\Big)^{\frac{1}{p}}\Big(\iint_{Q_T^1} |x|^{-\mu(\lambda)} \frac{|\psi_t|^{p'}}{\psi^{p'-m}} dxdt\Big)^{\frac{1}{p'}}. \end{aligned}$$

In the same way we have

$$\begin{aligned} J &\leq m \iint_{Q_T^2} v\psi^{m-1}|L\psi| dxdt \\ &\leq \Big(\iint_{Q_T^2} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt\Big)^{\frac{1}{p}}\Big(\iint_{Q_T^2} |x|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\psi|^{p'}}{\psi^{p'-m}} dxdt\Big)^{\frac{1}{p'}}. \end{aligned}$$

Now, since $p = F(\lambda, s)$ and setting $\tau = \frac{t}{R}, y = \frac{x}{R^{\frac{1}{2s}}}$, we reach that

$$\iint_{Q_T^1} |x|^{-\mu(\lambda)} \frac{|\psi_t|^{p'}}{\psi^{p'-m}} dxdt = 2^{p'} \iint_{\{1<\tau^2+|y|^{4s}<2\}} |y|^{-\mu(\lambda)}\tau^{p'} \frac{|\varphi'(\tau^2+|y|^{4s})|^{p'}}{\varphi^{p'-m}(\tau^2+|y|^{4s})} dyd\tau \equiv C_1,$$

and

$$\iint_{Q_T^2} |x|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\psi|^{p'}}{\psi^{p'-m}} dxdt = \iint_{\{\tau^2+|y|^{4s}<2\}} |y|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\theta(y,\tau)|^{p'}}{\theta^{p'-m}} dyd\tau = C_2,$$

where $\theta(y,\tau) = \varphi(\tau^2 + |y|^{4s})$. Thus

$$\begin{aligned} &\iint_{Q_T} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt \leq \\ &C_1\Big(\iint_{Q_T^2} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt\Big)^{\frac{1}{p}} + C_2\Big(\iint_{Q_T^2} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt\Big)^{\frac{1}{p}}. \end{aligned} \tag{3.9}$$

Therefore, using Young inequality, we obtain that

$$\iint_{Q_T} |x|^{-\mu(\lambda)(p+1)} v^p\psi^m dxdt \leq C,$$

where C is independent of R and T. Letting $R, T \to \infty$, we conclude that

$$\int_0^\infty \int_{\mathbf{R}^N} |x|^{-\mu(\lambda)(p+1)} v^p dxdt \leq C,$$

and the claim follows.

Recall that by (3.8) we have

$$\iint_{Q_T} |x|^{-\mu(\lambda)(p+1)} v^p \psi^m dxdt \leq I + J, \tag{3.10}$$

with

$$I \leq C\Big(\iint_{Q_T^1} |x|^{-\mu(\lambda)(p+1)} v^p \psi^m dxdt \Big)^{\frac{1}{p}}, \tag{3.11}$$

and

$$J \leq m \iint_{Q_T^2} v\psi^{m-1} |L\psi| dxdt. \tag{3.12}$$

From (3.11) and using the result of the claim we deduce that

$$I \leq C\Big(\iint_{\{R^2 < t^2 + |x|^{4s} < 2R^2\}} |x|^{-\mu(\lambda)(p+1)} v^p dxdt \Big)^{\frac{1}{p}} \to 0 \text{ as } R \to \infty. \tag{3.13}$$

Now we deal with J. For $\kappa > 0$ small enough, We have

$$\begin{aligned} J &\leq m \iint_{Q_T^2} v\psi^{m-1}(1-\psi)^{\kappa}(1-\psi)^{-\kappa} |L\psi| dxdt \\ &\leq \Big(\iint_{Q_T^2} |x|^{-\mu(\lambda)(p+1)} v^p \psi^m (1-\psi)^{\kappa} dxdt \Big)^{\frac{1}{p}} \Big(\iint_{Q_T^2} |x|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\psi|^{p'}}{\psi^{p'-m}(1-\psi)^{\kappa(p'-1)}} dxdt \Big)^{\frac{1}{p'}}. \end{aligned}$$

Using the same change of variable as above we obtain that

$$\iint_{Q_T^2} |x|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\psi|^{p'}}{\psi^{p'-m}(1-\psi)^{\kappa(p'-1)}} dxdt = \iint_{\{\tau^2 + |y|^{4s} < 2\}} |y|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\theta(y,\tau)|^{p'}}{\theta^{p'-m}(1-\theta)^{\kappa(p'-1)}} dyd\tau = C_3.$$

Thus

$$\begin{aligned} J &\leq C\Big(\iint_{Q_T^2} |x|^{-\mu(\lambda)(p+1)} v^p \psi^m (1-\psi)^{\kappa} dxdt \Big)^{\frac{1}{p}} \\ &\leq C\Big(\iint_{Q_T^1} |x|^{-\mu(\lambda)(p+1)} v^p dxdt \Big)^{\frac{1}{p}} \to 0 \text{ as } R \to \infty. \end{aligned} \tag{3.14}$$

Thus combining (3.10), (3.13) and (3.14) and letting $R \to \infty$, we conclude that

$$\int_0^{\infty} \int_{R^N} |x|^{-\mu(\lambda)(p+1)} v^p dxdt \leq 0,$$

a contradiction and then the result follows. □

Remarks 3.5. *Notice that the above blow up result holds under the hypothesis that we can choose* $\varphi \in C_0^{\infty}(B_2(0))$ *with* $0 \leq \varphi \leq 1$, $\varphi = 1$ *in* $B_1(0)$ *and*

$$\iint_{\{1 < \tau^2 + |y|^{4s} < 2\}} |y|^{-\mu(\lambda)} \tau^{p'} \frac{|\varphi'(\tau^2 + |y|^{4s})|^{p'}}{\varphi^{p'-m}(\tau^2 + |y|^{4s})} dyd\tau \equiv C_1,$$

$$\iint_{\{1 < \tau^2 + |y|^{4s} < 2\}} |y|^{\frac{\mu(\lambda)(p+1)}{p-1}} \frac{|L\theta(y,\tau)|^{p'}}{\theta^{p'-m}(1-\theta)^{\kappa(p'-1)}} dyd\tau = C_3$$

where $\theta(y,\tau) = \varphi(\tau^2 + |y|^{4s})$.

The above conditions hold choosing m *closed to* p' *and* κ *small enough.*

4. Global existence for $F(\lambda, s) < p < p_+(\lambda, s)$.

In order to show the optimality of $F(\lambda, s)$ we will prove that, under suitable condition on u_0, problem (1.9) has a global solution. To achieve this affirmation, we will show the existence of a family of global supersolutions to problem (1.9) where $F(\lambda, s) < p < p_+(\lambda, s)$.

Recall that $F(\lambda, s) = 1 + \frac{2s}{N-\mu(\lambda)}$, since $p < p_+(\lambda, s) = 1 + \frac{2s}{\mu(\lambda)}$, then $\frac{2s}{p-1} > \mu(\lambda)$. Fix $\gamma > 0$ be such that $\mu(\lambda) < \gamma < \frac{2s}{p-1}$, then for $T > 0$, we define

$$w(x, t, T) = A(T + t)^{-\theta}\Big(\frac{|x|}{(T + t)^{\beta}}\Big)^{-\gamma} H\Big(\frac{|x|}{(T + t)^{\beta}}\Big), \tag{4.1}$$

where $\theta = \frac{2s}{p-1}$ and, as above, $\beta = \frac{1}{2s}$. Notice that

$$w(x, t, T) = A(T + t)^{-\theta+\frac{\gamma}{2s}+\frac{N}{2s}}|x|^{-\gamma} h(x, t + T).$$

It is clear that

$$h_t(x, t + T) + (-\Delta)^s h(x, t + T) = 0.$$

We claim that, under suitable condition on A and T, w satisfies

$$w_t + (-\Delta)^s w - \lambda\frac{w}{r^{2s}} \geq w^p. \tag{4.2}$$

For simplicity of typing we set

$$D(x, t, T) = A(T + t)^{-\theta+\frac{\gamma}{2s}+\frac{N}{2s}}|x|^{-\gamma},$$

then

$$w(x, t, T) = D(x, t, T)h(x, t + T).$$

By a direct computations we reach that

$$\begin{aligned} &w_t + (-\Delta)^s w - \lambda\frac{w}{r^{2s}} = \\ &D(x, t, T)\Big(h_t(x, t + T) + (-\Delta)^s h(x, t + T)\Big) + h(x, t + T)\Big(D_t(x, t, T) + (-\Delta)^s D(x, t, T)\Big) \\ &- \int_{\mathbb{R}^N} \frac{(h(x, t + T) - h(y, t + T))(D(x, t, T) - D(y, t, T))}{|x - y|^{N+2s}}dy \\ &-\lambda\frac{D(x, t, T)h(x, t + T)}{|x|^{2s}}. \end{aligned}$$

Since $T > 0$, then

$$h_t(x, t + T) + (-\Delta)^s h(x, t + T) = 0.$$

On the other hand we have

$$D_t(x, t, T) + (-\Delta)^s D(x, t, T) = (-\theta + \frac{\gamma}{2s} + \frac{N}{2s})\frac{D(x, t, T)}{(T + t)} + \lambda(\gamma)\frac{D(x, t, T)}{|x|^{2s}}.$$

Since $\gamma > \mu(\lambda)$, then $\lambda(\gamma) > \lambda$.

We deal now with the mixed term

$$J(x) := -\int_{\mathbb{R}^N} \frac{(h(x,t+T)-h(y,t+T))(D(x,t,T)-D(y,t,T))}{|x-y|^{N+2s}}dy.$$

By a direct computations, it follows that

$$\begin{aligned} J(x) &= -A(T+t)^{-\theta+\frac{\gamma}{2s}}\int_{\mathbb{R}^N}\frac{(|x|^{-\gamma}-|y|^{-\gamma})(H(\frac{|x|}{(T+t)^{\beta}})-H(\frac{|y|}{(T+t)^{\beta}}))}{|x-y|^{N+2s}}dy \\ &= -AC^{N-\gamma}(T+t)^{-\theta+\frac{\gamma}{2s}-1}\int_{\mathbb{R}^N}\frac{(|x_1|^{-\gamma}-|y_1|^{-\gamma})(H(|x_1|)-H(|y_1|))}{|x_1-y_1|^{N+2s}}dy_1, \end{aligned}$$

where $x_1 = \frac{|x|}{(T+t)^{\beta}}$ and $y_1 = \frac{|y|}{(T+t)^{\beta}}$. Since H is decreasing then $J(x) \geq 0$. Therefore, combining the above estimates it holds that

$$\begin{aligned} w_t + (-\Delta)^s w - \lambda\frac{w}{r^{2s}} &= (-\theta+\frac{\gamma}{2s}+\frac{N}{2s})\frac{w(x,t,T)}{(T+t)} + (\lambda(\gamma)-\lambda)\frac{w(x,t,T)}{|x|^{2s}} + J(x) \\ &\geq (-\theta+\frac{\gamma}{2s}+\frac{N}{2s})\frac{w(x,t,T)}{(T+t)} + (\lambda(\gamma)-\lambda)\frac{w(x,t,T)}{|x|^{2s}}. \end{aligned}$$

Hence, w is a supersolution to (1.9) if we can chose $A, C > 0$ such that

$$(-\theta+\frac{\gamma}{2s}+\frac{N}{2s})\frac{w(x,t,T)}{(T+t)} + (\lambda(\gamma)-\lambda)\frac{w(x,t,T)}{|x|^{2s}} \geq w^p$$

hence

$$(-\theta+\frac{\gamma}{2s}+\frac{N}{2s})\frac{1}{(T+t)} + (\lambda(\gamma)-\lambda)\frac{1}{|x|^{2s}} \geq w^{p-1}.$$

The last inequality is equivalent to have

$$\begin{aligned} &(\frac{N+\gamma}{2s}-\theta) + (\lambda(\gamma)-\lambda)(T+t)|x|^{-\gamma-2s} \\ &\geq A^{p-1}(T+t)^{-(p-1)\theta+\frac{(p-1)\gamma}{2s}+1}|x|^{-(p-1)\gamma}H^{p-1}\Big(\frac{|x|}{(T+t)^{\beta}}\Big). \end{aligned} \tag{4.3}$$

Recall that $\theta = \frac{2}{p-1}$, since $\gamma < \frac{2s}{p-1}$, then

$$-(p-1)\theta + \frac{(p-1)\gamma}{2s} + 1 = \frac{(p-1)\gamma}{2s} - 1 < 0.$$

On the other hand we have $2s + \gamma > (p-1)\gamma$. Thus going back to (4.3), we conclude that, for any $T > 0$, we can choose A small such that w is a supersolution to (1.9) and then the claim follows.

We are now able to state the main global existence result in this section.

Theorem 4.1. *Assume that $F(\lambda, s) < p < p_+(\lambda, s)$. Let u_0 be a nonnegative function such that*

$$|x|^{\mu(\lambda)} u_0(x) \le \frac{C}{(1+|x|^2)^{\frac{N+2s}{2}}},$$

then the Cauchy problem (1.9) *has a global solution u such that $u(x,t) \le w(x,t,T)$ for all $(x,t) \in \boldsymbol{R}^N \times (0,\infty)$.*

Proof. Let u_0 be a nonnegative function such that the above condition holds, then $u_0 \in L^2(\boldsymbol{R}^N)$. According to the definition of w given in (4.1), there exist $A, T_0 > 0$ such that $u_0(x) \le w(x,0,T_0)$ for all $x \in \mathbb{R}^N$. Thus w is a supersolution to problem (1.9). Since $v(x,t) = 0$ is a strict subsolution, we can use the same classical iteration argument as in Theorem 6.2 in [1] for the heat equation and the existence result follows. For the reader convenience we give a schematic idea of the iteration argument. Let B_n be the ball in $\mathbb{R}^N$ with radius n and centered at the origin. We consider

$$v_n \in L^2((0,T), H_0^s(B_{n+1})),\ \forall T > 0,$$

the weak solutions to the following approximated problems,

$$\begin{cases} v_{nt} + (-\Delta)^s v_n &= \lambda \dfrac{1}{|x|^{2s} + \frac{1}{n}} \tilde{v}_{n-1} + \tilde{v}_{n-1}^p \text{ in } B_{n+1},\ t > 0, \\ v_n(x,0) &= u_0(x) \text{ in } B_{n+1},\ t > 0, \\ v_n(x,t) &= 0 \text{ in } \mathbb{R}^N \setminus B_{n+1},\ t > 0, \end{cases} \tag{4.4}$$

with

$$\begin{cases} v_{0t} - \Delta v_0 &= 0 \text{ in } B_1,\ t > 0, \\ v_0(x,0) &= u_0(x) \text{ in } B_1,\ t > 0, \\ v_0(x,t) &= 0 \text{ in } \mathbb{R}^N \setminus B_1,\ t > 0, \end{cases}$$

and $\tilde{v}_{n-1} = v_{n-1}$ in $B_n \times (0,T)$, $\tilde{v}_{n-1} = 0$ in $(\mathbb{R}^N \setminus B_n) \times (0,T)$. See for instance [15].

Applying the classical comparison principle for finite energy solutions, we conclude that $0 < v_0 \le v_1 \le \cdots \le v_{n-1} \le v_n \le w$ in $B_{n+1} \times (0,T_1)$ with $T_1 < T$. Hence there exists $u \in L^2(0,T_1,L^2_{loc}(\mathbb{R}^N))$ such that $v_n \uparrow u$ strongly in $L^2((0,T_1),L^2(\mathbb{R}^N))$ and $u \le w$. Using the monotonicity of v_n and the dominated convergence theorem it follows that $v_n \to u$ strongly $L^p(K \times (0,T_1))$ for all compact set $K \subset \mathbb{R}^N$. Take $\phi \in C_0^\infty(\mathbb{R}^N \times (0,T_1))$, then using ϕ as a test function in (4.4) and by letting $n \to \infty$ we easily get that u solves problem (1.9) with $u(x,0) = u_0(x)$. It is clear that $u \in L^2(0,T_1;H^s(\mathbb{R}^N))$. □

Remark 4.2. *In the general case $1 < p < p_+(\lambda)$ and under some hypotheses on u_0 it is possible to show a complete blowup in a suitable sense.*

Suppose that $u_0(x) \ge h$ where $h \ge 0$ satisfies $h \in C_0^\infty(\mathbb{R}^N)$, supp$(h) \subset B_0(R)$ and

$$\frac{1}{p+1} \int\limits_{\boldsymbol{R}^N} h^{p+1} dx > \frac{a_{N,s}}{4} \iint_{D_\Omega} \frac{(h(x)-h(y))^2}{|x-y|^{N+2s}}\, dx\, dy - \frac{\lambda}{2} \int\limits_{\boldsymbol{R}^N} \frac{h^2}{|x|^{2s}} dx. \tag{4.5}$$

Then if u is a positive solution to problem (1.9) *we have*

$$\int_{B_R(0)} u^2(x,t) dx \to \infty \text{ as } t \to T^*.$$

We argue by contradiction. Suppose that the above conditions holds and that

$$\sup_{t\in(0,T)} \int_{B_R(0)} u^2(x,t)dx \le M(T) < \infty. \tag{4.6}$$

Let w be the unique positive solution to the problem

$$\begin{cases} w_t + (-\Delta)^s w &= \lambda \dfrac{w}{|x|^{2s}+1} + w^p & \text{in } B_R(0)\times(0,T(w)),\\ w(x,t) &= 0 & \text{in } (\mathbb{R}^N\backslash B_R(0))\times(0,T(w)),\\ w(x,0) &= h(x) & \text{if } x\in B_R(0). \end{cases} \tag{4.7}$$

It is clear that $w \in L^2(0,T(h);H_0^s(B_R(0))) \cap L^\infty(B_R(0)\times(0,T(w))$. *Since u is a supersolution to* (4.7), *then* $w \le u$ *and therefore* $T(w)=\infty$. *Define the energy in time t,*

$$E(t) = \frac{a_{N,s}}{4}\iint_{D_{B_R(0)}} \frac{(w(x,t)-w(y,t))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\lambda}{2}\int_{B_R(0)} \frac{w^2}{|x|^{2s}+1}dx - \frac{1}{p+1}\int_{B_R(0)} w^{p+1}dx.$$

By a direct computations, it follows that $\frac{d}{dt}E(t) = -\langle w_t, w_t\rangle \le 0$. *Taking into consideration the hypothesis on h, we conclude that* $E(t)\le 0$ *for all t. Hence*

$$\frac{d}{dt}\int_{B_R(0)} w^2(x,t)dx \ge C\Big(\int_{B_R(0)} w^2(x,t)dx\Big)^{\frac{p+1}{2}}.$$

By integration, it holds

$$\int_{B_R(0)} w^2(x,t)dx \to \infty \text{ as } t\to T^* < \infty,$$

a contradiction with (4.6).

Remark 4.3. *Notice that*

$$p_-(\lambda,s) = 1 + \frac{2s}{\bar{\mu}(\lambda)} \ge 1 + \frac{2s}{N-\mu(\lambda)}.$$

Hence,

$$1+\frac{2s}{N} \le 1+\frac{2s}{N-\mu(\lambda)} \le p_-(\lambda,s) \le 2_s^*-1 \le p_+(\lambda,s).$$

See Figure 2.

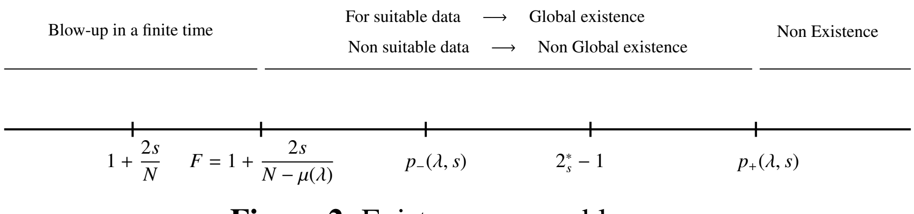

Figure 2. Existence versus blowup.

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Conflict of interest

The authors declare no conflict of interest.

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