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Research article

Some (p, q)-Hardy type inequalities for (p, q)-integrable functions

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Abstract: In this paper, we study some (p, q)-Hardy type inequalities for (p, q)-integrable functions. Moreover, we also study (p, q)-Hölder integral inequality and (p, q)-Minkowski integral inequality for two variables. By taking $p = 1$ and $q \to 1$, our results reduce to classical results on Hardy type inequalities, Hölder integral inequality and Minkowski integral inequality for two variables.

Keywords: Hardy type inequalities; Minkowski integral inequality; (p, q)-calculus; (p, q)-integrable function
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1. Introduction

In mathematics, quantum calculus, also known as q-calculus, is the study of calculus without limit. In q-calculus, we obtain q-analogues of mathematical formulas that can be recaptured as q tends to one. The history of q-calculus can be traced back to Euler, who first introduced q-calculus in the track of Newton's work on infinite series. Then, in 1910, F. H. Jackson [1] presented a systematic study of q-calculus and defined the q-definite integral, which is known as the q-Jackson integral. In recent years, the interest in q-calculus been arising due to high demand of mathematics in this field. The q-calculus numerous applications in various fields of mathematics and other areas such as combinatorics, dynamical systems, fractals, number theory, orthogonal polynomials, special functions, mechanics and also for scientific problems in some applied areas, see [2–14] for more details.

Along with the development of the theory and application of q-calculus, the theory of q-calculus based on two parameters (p, q)-integers has also presented and recieved more attention during the last few dacades. In 1991, R. Chakrabati and R. Jagannathan [15] introduced the (p, q)-calculus. Next, P. N. Sadjang [16] studied the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas.

Recently, M. Tunç and E. Göv [17] defined the (p,q)-derivative and (p,q)-integral on finite intervals. Moreover, they studied some properties of (p,q)-calculus and (p,q)-analogue of some important integral inequalities. The (p,q)-integral inequalities have been studied and rapidly developed during this period by many authors, see [18–26] and the references therein.

Mathematical inequalities was applied in various branches of mathematics as analysis, differential equations, geometry, etc. One typical such example is Hardy inequality. Let us just mention that in 1920, G. H. Hardy [27] presented the following famous inequality for f is a non-negative integrable function and $s > 1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^s dx \leq \left(\frac{s}{s-1}\right)^s \int_0^\infty f^s(x)dx, \tag{1.1}$$

which is now known as Hardy inequality.

Hardy inequality has been studied by a large number of authors during the twentieth century. Over the last twenty years a large number of papers have appeared in the literature which deal with the simple proofs, various generalizations and discrete analogue of Hardy inequality, see [28–35] for more details.

In 2014, L. Maligranda et al. [36] studied a q-analogue of Hardy inequality (1.1) and some related inequalities. It seems to be a huge new research area to study these so called q-Hardy type inequalities. They obtained more general results on q-Hardy type inequalities. By taking $q \to 1$, we obtain classical results on Hardy inequality (1.1). Next, L. E. Persson and S. Shaimardan [37] studied some q-analogue of Hardy type inequalities for the Riemann-Liouville fractional integral operator, see [38, 39] for more details.

The purpose of this paper is to study some (p,q)-Hardy type inequalities for (p,q)-integrable functions by using (p,q)-derivative and (p,q)-integral. Moreover, we also study (p,q)-Hölder integral inequality and (p,q)-Minkowski integral inequality for two variables. By taking $q \to 1$ and $p = 1$, our results reduce to classical results on Hardy type inequalities, Hölder integral inequality and Minkowski integral inequality for two variables.

2. Preliminaries

In this section, we recall some known concepts and basic results of (p,q)-calculus. Throughout this paper, we let p, q be constants with $0 < q < p \leq 1$ and $[a,b] \subseteq \mathbb{R}$. We give some definitions and theorems for (p,q)-calculus, which will be used in these papers [16–23].

First, we give some (p,q)-notation, which would appear in this study quite frequently. For any real number n, the (p,q)-analogue of n is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \tag{2.1}$$

and

$$[-n]_{p,q} = \frac{1}{(pq)^n}[n]_{p,q}. \tag{2.2}$$

If $p = 1$, then (2.1) reduces to

$$[n]_q = \frac{1-q^n}{1-q}, \tag{2.3}$$

which is q-analogue of n.

Definition 2.1. [18] If $f : [a, b] \to \mathbb{R}$ is a continuous function, then (p, q)-derivative of the function f on $[a, b]$ at x is defined by

$$ {}_aD_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}, \quad x \neq a. \tag{2.4}$$

The function f is said to be a (p, q)-differentiable function on $[a, b]$ if ${}_aD_{p,q}f(x)$ exists for all $x \in [a, b]$.

Since $f : [a, b] \to \mathbb{R}$ is a continuous function, we have ${}_aD_{p,q}f(a) = \lim_{x\to a} D_{p,q}f(x)$. In Definition 2.1, if $a = 0$, then ${}_0D_{p,q}f = D_{p,q}f$ is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0. \tag{2.5}$$

And, if $p = 1$, then $D_{p,q}f(x) = D_qf(x)$, which is the q-derivative of the function f, and also if $q \to 1$ in (2.5), then it reduces to a classical derivative.

Example 1. Define function $f : [a, b] \to \mathbb{R}$ by $f(x) = x^2 + x + c$, where c is a constant. Then for $x \neq a$, we have

$$\begin{aligned} {}_aD_{p,q}(x^2 + x + c) &= \frac{\left[(px + (1-p)a)^2 + px + (1-p)a + c\right] - \left[(qx + (1-q)a)^2 + qx + (1-q)a + c\right]}{(p-q)(x-a)} \\ &= \frac{(p+q)x^2 + 2ax[1 - (p+q)] + a^2[(p+q) - 2] + (x-a)}{(x-a)} \\ &= \frac{x(p+q)(x-a) - a(p+q)(x-a) + 2a(x-a) + (x-a)}{(x-a)} \\ &= (p+q)(x-a) + 2a + 1. \end{aligned} \tag{2.6}$$

Theorem 2.2. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions and c, d are constants, then the following formulas hold:*

(i) ${}_aD_{p,q}[cf(x) \pm dg(x)] = c\,{}_aD_{p,q}f(x) \pm d\,{}_aD_{p,q}g(x)$;
(ii) ${}_aD_{p,q}[f(x)g(x)] = f(px + (1-p)a)\,{}_aD_{p,q}g(x) + g(qx + (1-q)a)\,{}_aD_{p,q}f(x)$;
(iii) ${}_aD_{p,q}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(px + (1-p)a)\,{}_aD_{p,q}f(x) - f(px + (1-p)a)\,{}_aD_{p,q}g(x)}{g(px + (1-p)a)g(qx + (1-q)a)}$.

The proof of this theorem is given in [17].

Definition 2.3. [18] If $f : [a, b] \to \mathbb{R}$ is a continuous function and $0 < a < b$, then the (p, q)-integral is defined by

$$\int_a^b f(x)\, {}_ad_{p,q}x = (p-q)(b-a) \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}b + \left(1 - \frac{q^k}{p^{k+1}}\right)a\right). \tag{2.7}$$

And, f is said to be a (p, q)-integrable function on $[a, b]$ if $\int_a^b f(x)\, {}_ad_{p,q}x$ exists for all $x \in [a, b]$. If $a = 0$ in (2.7), then one can get the classical (p, q)-integral defined by

$$\int_0^b f(x) d_{p,q}x = (p-q)b \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}b\right), \tag{2.8}$$

and

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x. \tag{2.9}$$

If $p = 1$ in (2.8), then we have the classical q-integral [11].

Example 2. Define function $f : [a, b] \to \mathbb{R}$ by $f(x) = 2x + c$, where c is a constant. Then we have

$$\begin{aligned}\int_a^b f(x)\ {}_ad_{p,q}x &= \int_a^b (2x+c)\ {}_ad_{p,q}x \\ &= 2(p-q)(b-a)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(\frac{q^k}{p^{k+1}}b + \left(1-\frac{q^k}{p^{k+1}}\right)a\right) + (p-q)(b-a)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}(c) \\ &= \frac{2(b-a)(b-a(1-p-q))}{p+q} + (b-a)c. \end{aligned}\tag{2.10}$$

If $p = 1$, we have (2.10) reduces to the q-integral of $f(x) = 2x + c$. Furthermore, if $a = 0$, $p = 1$ and $q \to 1$, (2.10) reduces to the classical integral.

The proofs of the following theorems are given in [17].

Theorem 2.4. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions, $t \in [a, b]$ and α is a constant, then the following formulas hold:*

(i) ${}_aD_{p,q}\int_a^t f(x)\ {}_ad_{p,q}x = f(t)$;

(ii) $\int_c^t {}_aD_{p,q}f(x)\ {}_ad_{p,q}x = f(t) - f(c)$ *for* $c \in (a, t)$;

(iii) $\int_a^t [f(x) + g(x)]\ {}_ad_{p,q}x = \int_a^b f(x)\ {}_ad_{p,q}x + \int_a^b g(x)\ {}_ad_{p,q}x$;

(iv) $\int_a^t \alpha f(x)\ {}_ad_{p,q}x = \alpha\int_a^b f(x)\ {}_ad_{p,q}x$;

(v) $\int_0^t x^\alpha d_{p,q}x = \frac{b^{\alpha+1}}{[\alpha+1]_{p,q}}$;

(vi) $\int_c^t f(px + (1-p)a)\ {}_aD_{p,q}g(x)\ {}_ad_{p,q}x = (fg)(x)\big|_c^t - \int_c^t g(qx + (1-q)a)\ {}_aD_{p,q}f(x)\ {}_ad_{p,q}x.$

Theorem 2.5. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions and $r > 1$ with $1/r + 1/s = 1$, then*

$$\int_a^b |f(t)g(t)|\ {}_ad_{p,q}t \le \left(\int_a^b |f(t)|^r\ {}_ad_{p,q}t\right)^{1/r}\left(\int_a^b |g(t)|^s\ {}_ad_{p,q}t\right)^{1/s}. \tag{2.11}$$

Theorem 2.6. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions and $0 < r < 1$ with $1/r + 1/s = 1$, then*

$$\int_a^b |f(t)g(t)|\ {}_ad_{p,q}t \ge \left(\int_a^b |f(t)|^r\ {}_ad_{p,q}t\right)^{1/r}\left(\int_a^b |g(t)|^s\ {}_ad_{p,q}t\right)^{1/s}. \tag{2.12}$$

The proof is similar to the proof of Lemma 2.2 in [32].

Definition 2.7. [16] The improper (p, q)-integral of $f(x)$ on $[0, \infty]$ is defined by

$$\int_0^{\infty} f(x)\ {}_0d_{p,q}x = (p-q) \sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}\right). \tag{2.13}$$

And, f is said to be a (p, q)-integrable function on $[0, \infty]$ if $\int_0^{\infty} f(x) d_{p,q}x$ exists for all $x \in [0, \infty]$.

And, if $p = 1$ then (2.13) reduces to

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{k=-\infty}^{\infty} q^k f\left(q^k\right), \tag{2.14}$$

which is an improper q-integral of $f(x)$ on $[0, \infty]$ and appeared in [13].

Definition 2.8. [23] Let $h : [0, b] \times [0, d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function in each variable; let $0 < q_i < p_i \leq 1$, where $i = 1, 2$. The definite $(p_1 p_2, q_1 q_2)$-integral on $[0, b] \times [0, d]$ is defined by

$$\int_0^t \int_0^s h(z, w) d_{p_1,q_1} z\, d_{p_2,q_2} w = (p_1 - q_1)(p_2 - q_2) st \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{q_1^n}{p_1^{n+1}}\right)\left(\frac{q_2^m}{p_2^{m+1}}\right) h\left(\frac{q_1^n}{p_1^{n+1}} s, \frac{q_2^m}{p_2^{m+1}} t\right) \tag{2.15}$$

for $(s, t) \subset [0, b] \times [0, d]$. And, h is said to be a $(p_1 p_2, q_1 q_2)$-integrable function on $[0, b] \times [0, d]$ if $\int_0^t \int_0^s h(z, w) d_{p_1,q_1} z\, d_{p_2,q_2} w$ exists for all $(s, t) \subset [0, b] \times [0, d]$.

Example 3. Define function $h : [0, b] \times [0, d] \subset \mathbb{R}^2 \to \mathbb{R}$ by $h(z, w) = z^n w^m$, where m, n are constants. Then we have

$$\begin{aligned} \int_0^t \int_0^s h(z, w) d_{p_1,q_1} z d_{p_2,q_2} w &= \int_0^t \int_0^s z^n w^m d_{p_1,q_1} z d_{p_2,q_2} w \\ &= \left((p_1 - q_1) s^{n+1} p_1^{-n-1} \sum_{i=0}^{\infty} \left(\frac{q_1^n}{p_1^{n+1}}\right)^i\right) \\ &\quad \times \left((p_2 - q_2) t^{m+1} p_2^{-m-1} \sum_{j=0}^{\infty} \left(\frac{q_2^m}{p_2^{m+1}}\right)^j\right) \\ &= \left(\frac{s^{n+1}}{[n+1]_{p_1,q_1}}\right)\left(\frac{t^{m+1}}{[m+1]_{p_2,q_2}}\right). \end{aligned} \tag{2.16}$$

Theorem 2.9. *If $f : [0, b] \to \mathbb{R}$ is a non-negative function, $r > 1$ and f^r is a q-integrable function on $[0, \infty]$, then*

$$\left[\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t) d_q t\right)^r d_q x\right]^{1/r} \leq \frac{1}{\left[\frac{r-1}{r}\right]_q} \left[\int_0^{\infty} f^r(t) d_q t\right]^{1/r}. \tag{2.17}$$

The proof of this theorem is given in [36].

3. Main results

In this section, we first present the (p, q)-Hölder inequality for two variables.

Theorem 3.1. *If h and g are functions defined on $[0,b]\times[0,d]$ and $m_1, m_2 > 1$ with $1/m_1 + 1/m_2 = 1$, then*

$$\int_0^t\int_0^s |h(z,w)g(z,w)|\, d_{p_1,q_1}z d_{p_2,q_2}w \le \left(\int_0^t\int_0^s |h(z,w)|^{m_1}\, d_{p_1,q_1}z d_{p_2,q_2}w\right)^{1/m_1} \times\left(\int_0^t\int_0^s |g(z,w)|^{m_2}\, d_{p_1,q_1}z d_{p_2,q_2}w\right)^{1/m_2}. \tag{3.1}$$

Proof. From Definition 2.8 and the discrete Hölder inequality, we get

$$\begin{aligned}
&\int_0^t\int_0^s |h(z,w)g(z,w)|\, d_{p_1,q_1}z d_{p_2,q_2}w \\
&\quad= (p_1-q_1)(p_2-q_2)st\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{q_1^n}{p_1^{n+1}}\right)\left(\frac{q_2^m}{p_2^{m+1}}\right)\left|h\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)g\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)\right| \\
&\quad= (p_1-q_1)(p_2-q_2)st\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|h\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)\left(\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right)^{1/m_1}\right| \\
&\qquad\times\left|g\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)\left(\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right)^{1/m_2}\right| \\
&\quad\le\left((p_1-q_1)(p_2-q_2)st\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|h\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)\right|^{m_1}\left(\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right)\right)^{1/m_1} \\
&\qquad\times\left((p_1-q_1)(p_2-q_2)st\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|g\left(\frac{q_1^n}{p_1^{n+1}}s,\frac{q_2^m}{p_2^{m+1}}t\right)\right|^{m_2}\left(\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right)\right)^{1/m_2} \\
&\quad=\left(\int_0^t\int_0^s |h(z,w)|^{m_1} d_{p_1,q_1}z d_{p_2,q_2}w\right)^{1/m_1}\left(\int_0^t\int_0^s |g(z,w)|^{m_2} d_{p_1,q_1}z d_{p_2,q_2}w\right)^{1/m_2}.
\end{aligned}$$

The proof is completed. □

Next, we present the (p,q)-Minkowski integral inequality for two variables.

Theorem 3.2. *If h is (p_1p_2, q_1q_2)-integrable function on $[0,b]\times[0,d]$, $\alpha\in(0,1]$ and $1\le r<\infty$, then*

$$\left(\int_0^b\left|\int_0^d h(x,y)d_{p,q}y\right|^r d_{p,q}x\right)^{1/r}\le\int_0^d\left(\int_0^b |h(x,y)|^r\, d_{p,q}x\right)^{1/r} d_{p,q}y. \tag{3.2}$$

Proof. For $r=1$, (3.2) holds by Fubini's Theorem. Next, for $1<r<\infty$ and $1/r+1/z=1$, by using Fubini's Theorem and Theorem 2.5, we have

$$\begin{aligned}
\left(\int_0^b\left|\int_0^d h(x,y)d_{p,q}y\right|^r d_{p,q}x\right) &\le \int_0^b\left|\int_0^d h(x,t)d_{p,q}t\right|^{r-1}\left(\int_0^d |h(x,y)|\, d_{p,q}y\right)d_{p,q}x \\
&= \int_0^d\left(\int_0^b\left|\int_0^d h(x,t)d_{p,q}t\right|^{r-1}|h(x,y)|\, d_{p,q}x\right)d_{p,q}y
\end{aligned}$$

$$\leq \int_0^d \left[\int_0^b \left(\left| \int_0^d h(x,t) d_{p,q} t \right|^r d_{p,q} x \right)^{1/z} \left(\int_0^b |h(x,y)|^r d_{p,q} x \right)^{1/r} \right] d_{p,q} y$$

$$\leq \left(\int_0^b \left| \int_0^d h(x,t) d_{p,q} t \right|^r d_{p,q} x \right)^{1/z} \int_0^d \left(\int_0^b |h(x,y)|^r d_{p,q} x \right)^{1/r} d_{p,q} y. \quad (3.3)$$

Consequently, we obtain

$$\left(\int_0^b \left| \int_0^d h(x,t) d_{p,q} y \right|^r d_{p,q} x \right)^{1-1/z} \leq \int_0^d \left(\int_0^b |h(x,y)|^r d_{p,q} x \right)^{1/r} d_{p,q} y.$$

Therefore,

$$\left(\int_0^b \left| \int_0^d h(x,y) d_{p,q} y \right|^r d_{p,q} x \right)^{1/r} \leq \int_0^d \left(\int_0^b |h(x,y)|^r d_{p,q} x \right)^{1/r} d_{p,q} y.$$

The proof is completed. □

Remark 3.3. If $p = 1$, then (3.2) reduces to

$$\left(\int_0^b \left| \int_0^d h(x,y) d_q y \right|^r d_q x \right)^{1/r} \leq \int_0^d \left(\int_0^b |h(x,y)|^r d_q x \right)^{1/r} d_q y. \quad (3.4)$$

One easily see that when $q \to 1$ in (3.4), the inequality turns into a Minkowski integral inequality for two variables.

Theorem 3.4. *If $f : [0,b] \to \mathbb{R}$ is a non-negative function, $r > 1$ and f^r is (p,q)-integrable function on $[0,\infty)$, then*

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) d_{p,q} t \right)^r d_{p,q} x \right)^{1/r} \leq \frac{1}{[\frac{r-1}{r}]_{p,q}} \left(\int_0^\infty f^r(t) d_{p,q} t \right)^{1/r}. \quad (3.5)$$

Proof. By Definition 2.3 and Theorem 3.2, it is easy to see that

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) d_{p,q} t \right)^r d_{p,q} x \right)^{1/r} = \left(\int_0^\infty \left(\int_0^1 f(xs) d_{p,q} s \right)^r d_{p,q} x \right)^{1/r}$$
$$\leq \int_0^1 \left(\int_0^\infty f^r(xs) d_{p,q} x \right)^{1/r} d_{p,q} s.$$

Thus, by Theorem 2.4(vi), we get

$$\int_0^1 \left(\int_0^\infty f^r(xs) d_{p,q} x \right)^{1/r} d_{p,q} s = \int_0^1 \left(\int_0^\infty \frac{1}{s} f^r(t) d_{p,q} x \right)^{1/r} d_{p,q} s$$
$$= \frac{1}{\left[\frac{r-1}{r}\right]_{p,q}} \left(\int_0^\infty f^r(t) d_{p,q} t \right)^{1/r}. \quad (3.6)$$

This completes the proof. □

Remark 3.5. If $p = 1$, then (3.5) reduces to

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^1 f(xs)d_qs\right)^r d_qx\right)^{1/r} \leq \frac{1}{[\frac{r-1}{r}]_q}\left(\int_0^\infty f^r(t)d_qt\right)^{1/r}, \tag{3.7}$$

which is q-Hardy inequality in [36] and when $q \to 1$ in (3.7), the inequality reduces to (1.1).

Theorem 3.6. *If $f : [0, b] \to \mathbb{R}$ is a non-negative function, $r \geq 1, z < r - 1$ and $t^z f^r$ is (p, q)-integrable function on $[0, \infty)$, then*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)d_{p,q}t\right)^r x^z d_{p,q}x\right)^{1/r} \leq \frac{1}{[\frac{r-z-1}{r}]_{p,q}^r}\left(\int_0^\infty t^z f^r(t)d_{p,q}t\right)^{1/r}. \tag{3.8}$$

Proof. The proof is similar to the proof of Theorem 3.4. □

Remark 3.7. If $p = 1$, then (3.8) reduces to

$$\left(\int_0^\infty \left(\int_0^1 f(xs)x^{z/r}d_qs\right)^r d_qx\right)^{1/r} \leq \frac{1}{[\frac{r-z-1}{r}]_q}\left(\int_0^\infty u^z f^r(u)d_qu\right)^{1/r}, \tag{3.9}$$

when $z = 0$, we get the inequality (3.7).

Theorem 3.8. *If $f_1, f_2, \ldots, f_n$ are non-negative (p, q)-integrable functions on $[0, b]$, then*

$$\int_0^\infty \left(\frac{\prod_{i=1}^n F_i(x)}{x^n}\right)^{r/n} d_{p,q}x \leq \left(\frac{1}{n[\frac{r-1}{r}]_{p,q}}\right)^r \int_0^\infty \left(\sum_{i=1}^n f_i(x)\right)^r d_{p,q}x, \tag{3.10}$$

where

$$F_i(x) = \int_0^x f_i(t)d_{p,q}t,$$

for $i = 1, 2, \ldots, n$.

Proof. By Jensen's inequality, we have

$$\left(\prod_{i=1}^n F_i(x)\right)^{r/n} \leq \frac{1}{n^r}\left(\sum_{i=1}^n F_i(x)\right)^r.$$

Thus, we obtain

$$\int_0^\infty \left(\frac{\prod_{i=1}^n F_i(x)}{x^n}\right)^{r/n} d_{p,q}x \leq \frac{1}{n^r}\int_0^\infty \left(\frac{\sum_{i=1}^n F_i(x)}{x}\right)^r d_{p,q}x. \tag{3.11}$$

Applying Theorem 3.4 to (3.11), we have

$$\int_0^\infty \left(\frac{\prod_{i=1}^n F_i(x)}{x^n}\right)^{r/n} d_{p,q}x \leq \left(\frac{1}{n[\frac{r-1}{r}]_{p,q}}\right)^r \int_0^\infty \left(\sum_{i=1}^n f_i(t)\right)^r d_{p,q}t. \tag{3.12}$$

This completes the proof. □

Remark 3.9. If $p = 1$, then (3.10) reduces to

$$\int_0^\infty \left(\frac{\prod_{i=1}^n F_i(x)}{x^n}\right)^{r/n} d_q x \le \left(\frac{1}{n[\frac{r-1}{r}]_{p,q}}\right)^r \int_0^\infty \left(\sum_{i=1}^n f_i(t)\right)^r d_{p,q}t, \tag{3.13}$$

which is q-Hardy inequality with many functions, and when $q \to 1$ in (3.13), the inequality above appeared in [33].

Theorem 3.10. *If $f : [0, b] \to \mathbb{R}$ is a non-negative function, $r > 1, 0 < a < b < \infty$ and f^r is (p, q)-integrable function on $[a, b]$, then*

$$\left(\int_a^b \left(\frac{1}{x}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x\right)^{1/r} \le \frac{1}{[\frac{r-1}{r}]_{p,q}} \left(\int_a^b f^r(t)d_{p,q}t\right)^{1/r}. \tag{3.14}$$

Proof. By Definition 2.3, it is easy to see that

$$\left(\int_a^b \left(\frac{1}{x}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x\right)^{1/r} = \left(\int_a^b \left(\int_0^1 f(xs)d_{p,q}s\right)^r d_{p,q}x\right)^{1/r}.$$

From Theorem 3.2, we have

$$\left(\int_a^b \left(\int_0^1 f(xs)d_{p,q}s\right)^r d_{p,q}x\right)^{1/r} \le \left(\int_0^1 \left(\int_a^b f^r(xs)d_{p,q}x\right) d_{p,q}s\right)^{1/r}.$$

Thus, by Theorem 2.4(vi), we get

$$\begin{aligned}
\left(\int_0^1 \left(\int_a^b f^r(xs)d_{p,q}x\right) d_{p,q}s\right)^{1/r} &= \left(\int_0^1 \left(\int_0^b f^r(xs)d_{p,q}x - \int_0^a f^r(xs)d_{p,q}x\right) d_{p,q}s\right)^{1/r} \\
&= \left(\int_0^1 \frac{1}{s}\left(\int_a^b f^r(t)d_{p,q}t\right) d_{p,q}s\right)^{1/r} \\
&= \frac{1}{[\frac{r-1}{r}]_{p,q}} \left(\int_a^b f^r(t)d_{p,q}t\right)^{1/r}.
\end{aligned}$$

This completes the proof. $\square$

Remark 3.11. If $p = 1$, then (3.14) reduces to

$$\left(\int_a^b \left(\frac{1}{x}\int_0^x f(t)d_q t\right)^r d_q x\right)^{1/r} \le \frac{1}{[\frac{r-1}{r}]_q} \left(\int_a^b f^r(t)d_q t\right)^{1/r}, \tag{3.15}$$

which is q-Hardy inequality about integration from a to b, and when $q \to 1$ in (3.15), the inequality reduces to [29].

Theorem 3.12. *Let $f \ge 0, g > 0, \frac{x}{g(x)}$ be non-increasing, $r > 1$ and let f be (p, q)-integrable function on $[0, x]$. Then*

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x \le \frac{1}{[1 - 1/r]_{p,q}^r} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r d_{p,q}x. \tag{3.16}$$

Proof. From Theorem 2.5, we get

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x \leq \int_0^\infty g^{-r}(x)\int_0^x t^{1-1/r}f^r(t)d_{p,q}t\left(\int_0^x t^{-1/r}d_{p,q}t\right)^{r-1} d_{p,q}x.$$

And by Theorem 2.4(vi), we obtain

$$\left(\int_0^x t^{-1/r}d_{p,q}t\right)^{r-1} = \frac{x^{(1-1/r)(r-1)}}{[1-1/r]_{p,q}^{r-1}}.$$

Since the assumption of the function $\frac{x}{g(x)}$, we have

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x &\leq \frac{1}{[1-1/r]_{p,q}^{r-1}}\int_0^\infty g^{-r}(x)\int_0^x x^{(1-1/r)(r-1)}t^{1-1/r}f^r(t)d_{p,q}t\,d_{p,q}x\\
&= \frac{1}{[1-1/r]_{p,q}^{r-1}}\int_0^\infty t^{1-1/r}f^r(t)\int_t^\infty x^{(1-1/r)(r-1)}g^{-r}(x)d_{p,q}x\,d_{p,q}t\\
&\leq \frac{1}{[1-1/r]_{p,q}^{r-1}}\int_0^\infty t^{1-1/r}f^r(t)\left(\frac{t}{g(t)}\right)^r\int_t^\infty x^{1/r-2}d_{p,q}x\,d_{p,q}t\\
&= \frac{1}{[1-1/r]_{p,q}^{r}}\int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^r d_{p,q}t.
\end{aligned}$$

This proof is completed. □

Remark 3.13. If $p = 1$, then (3.16) reduces to a generalization of q-Hardy inequality as

$$\int_0^\infty \left(\frac{1}{g^r(x)}\int_0^x f(t)d_qt\right)^r d_qx \leq \frac{1}{[1-1/r]_q^r}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r d_qx. \tag{3.17}$$

Also if $q \to 1$, then (3.17) reduces to the well known generalization of Hardy inequality as

$$\int_0^\infty \left(\frac{1}{g^r(x)}\int_0^x f(t)dt\right)^r dx \leq \frac{1}{(1-1/r)^r}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r dx, \tag{3.18}$$

which appeared in [34] .

The following results concern the converse inequalities.

Theorem 3.14. *Let $f \geq 0, g > 0, \frac{x}{g(x)}$ be non-decreasing, $0 < r < 1$ and let f be (p,q)-integrable function on $[0,x]$. Then*

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x \geq \frac{1}{[1+1/r]_{p,q}^{r-1}[1/r-1]_{p,q}}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r d_{p,q}x. \tag{3.19}$$

Proof. From Theorem 2.6, we get

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x \geq \int_0^\infty g^{-r}(x)\int_0^x t^{1/r-1}f^r(t)d_{p,q}t\left(\int_0^x t^{1/r}d_{p,q}t\right)^{r-1} d_{p,q}x.$$

And by Theorem 2.4(vi), we obtain

$$\left(\int_0^x t^{1/r} d_{p,q}t\right)^{r-1} = \frac{x^{(1+1/r)(r-1)}}{[1+1/r]_{p,q}^{r-1}}.$$

Since the assumption of the function $\frac{x}{g(x)}$, we have

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x &\geq \frac{1}{[1+1/r]_{p,q}^{r-1}}\int_0^\infty g^{-r}(x)\int_0^x x^{(1+1/r)(r-1)}t^{1/r-1}f^r(t)d_{p,q}t\, d_{p,q}x \\
&= \frac{1}{[1+1/r]_{p,q}^{r-1}}\int_0^\infty t^{1/r-1}f^r(t)\int_t^\infty x^{(1+1/r)(r-1)}g^{-r}(x)d_{p,q}x\, d_{p,q}t \\
&\geq \frac{1}{[1+1/r]_{p,q}^{r-1}}\int_0^\infty t^{1/r-1}f^r(t)\left(\frac{t}{g(t)}\right)^r\int_t^\infty x^{-1/r}d_{p,q}x\, d_{p,q}t \\
&= \frac{1}{[1+1/r]_{p,q}^{r-1}[1/r-1]_{p,q}}\int_0^\infty \left(\frac{tf(t)}{g(t)}\right)^r d_{p,q}t.
\end{aligned}$$

This proof is completed. □

Remark 3.15. If $p = 1$, then (3.19) reduces to a converse generalization of q-Hardy inequality as

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_qx \geq \frac{1}{[1+1/r]_q^{r-1}[1/r-1]_q}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r d_qx. \tag{3.20}$$

Also if $q \to 1$, then (3.20) reduces to the well known converse generalization of Hardy inequality as

$$\int_0^\infty \left(\frac{1}{g(x)}\int_0^x f(t)d_{p,q}t\right)^r d_{p,q}x \geq \frac{1}{(1+1/r)^{r-1}(1/r-1)}\int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^r dx, \tag{3.21}$$

which appeared in [34] .

4. Conclusion

In this paper, we establish (p, q)-Hardy type inequalities for (p, q)-integral functions. We also obtain (p, q)-Hölder integral inequality and (p, q)-Minkowski integral inequality for two variables. Our work improves the results of Hardy type inequalities and the generalization of Hardy type inequalities. By taking $q \to 1$ and $p = 1$, our results gives classical inequality formulas.

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Conflict of interest

The authors declare that there are no conflict of interest in this paper.

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