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Research article

Integral inequalities of Hermite-Hadamard type for quasi-convex functions with applications

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Abstract: There is a strong connection between convexity and inequalities. So, techniques from each concept applies to the other due to the strong correlation between them; specifically, in the past few years. In this attempt, we consider the Hermite–Hadamard inequality and related inequalities for the class of functions whose absolute value of the third derivative are quasi-convex functions. Finally, the applications of our findings for special functions and particular functions are pointed out.

Keywords: Hermite-Hadamard inequality; quasi-convex function; Hölder inequality
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1. Introduction

Many research papers have studied the properties of convex functions that make this concept interesting in mathematical analysis [1–4]. In recent years, important generalizations have been made in the context of convexity: quasi-convex [5], pseudo-convex [6], invex and preinvex [7], strongly convex [8], approximately convex [9], MT-convex [10], (α, m)-convex [11], and strongly (s, m)-convex [12–15]. Here, we recall the notion of convexity: A function $g : [\beta_1, \beta_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$g(tx+(1-t)y) \leq tg(x)+(1-t)g(y), \quad x,y \in [\beta_1,\beta_2],\ t \in [0,1]. \tag{1.1}$$

Now, we recall our basic definition, the so-called quasi-convex function.

Definition 1.1 ([16]). A function $g:[\beta_1,\beta_2] \to \mathbb{R}$ is said quasi-convex on $[\beta_1,\beta_2]$ if

$$g(tx+(1-t)y) \leq \max\{g(x),g(y)\}, \tag{1.2}$$

for any $x,y \in [\beta_1,\beta_2]$ and $t \in [0,1]$.

It is important to note that, any convex function is a quasi-convex but the reverse is not true. In the following example we explain that fact.

Example 1.1 ([5]). The function $h:[-2,2] \to \mathbb{R}$, defined by

$$h(s) = \begin{cases} 1, & \text{for } s \in [-2,-1], \\ s^2, & \text{for } s \in (-1,2], \end{cases}$$

is not convex on $[-2,2]$ but it is easy to see that the function is quasi-convex on $[-2,2]$.

Notice that h is quasi-convex if and only if all the level sets of h are intervals (convex sets of the line).

The use of the convex function to study the integral inequalities have been deeply investigated, especially for the well-known inequality of Hermite-Hadamard type (HH-type inequality). The HH-type inequalities are one of the most important type inequalities and have a strong relationship to convex functions. In 1893 Hermite and Hadamard [17] found independently that for any convex function $g:[\beta_1,\beta_2] \to \mathbb{R}$, the inequality

$$g\left(\frac{\beta_1+\beta_2}{2}\right) \leq \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx \leq \frac{g(\beta_1)+g(\beta_2)}{2}, \tag{1.3}$$

holds.

In the field of mathematical analysis, many scholars have focused on defining new convexity and implementing of the problems based on their features. The features that make the results different from each other include lower and higher order derivative of the function. The differential equations with impulse perturbations lie in a special significant position in the theory of differential equations. Among them, integral inequality methods are the important tools to investigate the qualitative characteristics of solutions of different kinds of equations such as differential equations, difference equations, partial differential equations, and impulsive differential equations; see [18–23] for more details.

The HH-type inequality (1.3) has been applied to various convex functions like s-geometrically convex functions [24], GA-convex functions [25], MT-convex function [10], (α,m)-convex functions [26] and many other types can be found in [27]. Besides, the HH-type inequality (1.3) has been applied to a numerous type of convex functions in the sense of fractional calculus like F-convex functions [28], λ_ψ-convex functions [29], MT-convex functions [30], (α,m)-convex functions [11], new class of convex functions [31] and many other types can be found in the literature. Meanwhile, it has been applied to other models of fractional calculus like standard Riemann-Liouville fractional operators [32, 33], conformable fractional operators [34–36], generalized fractional operators [37], ψ-RL-fractional operators [38, 39], Tempered fractional operators [40], and AB- and Prabhakar fractional operators [41].

In view of the above indices, we extend the work done in [42] to establish some modified HH-type inequalities for the 3-times differentiable quasi-convex functions.

2. Main results

This section deals with our main results. Throughout this paper, we mean $g \in L[\beta_1, \beta_2]$ that the function g is differential and continuous on $[\beta_1, \beta_2]$.

Lemma 2.1. *Suppose that $g : J \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $\beta_1, \beta_2 \in J$ with $\beta_1 < \beta_2$. If $g''' \in L[\beta_1, \beta_2]$, then we have*

$$g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24} g''\left(\frac{\beta_1+\beta_2}{2}\right)$$
$$= \frac{(\beta_2-\beta_1)^3}{96}\left[\int_0^1 t^3 g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)dt + \int_0^1 (t-1)^3 g'''\left(t\beta_2 + (1-t)\frac{\beta_1+\beta_2}{2}\right)dt\right]. \quad (2.1)$$

Proof. By applying integration by parts three times to get

$$\begin{aligned} J_1 &:= \int_0^1 t^3 g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)dt \\ &= \frac{2}{\beta_2-\beta_1} g''\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{6}{\beta_2-\beta_1}\int_0^1 t^2 g''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)dt \\ &= \frac{2}{\beta_2-\beta_1} g''\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{12}{(\beta_2-\beta_1)^2} g'\left(\frac{\beta_1+\beta_2}{2}\right) \\ &+ \frac{24}{(\beta_2-\beta_1)^2}\int_0^1 t g'\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)dt \\ &= \frac{2}{\beta_2-\beta_1} g''\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{12}{(\beta_2-\beta_1)^2} g'\left(\frac{\beta_1+\beta_2}{2}\right) + \frac{48}{(\beta_2-\beta_1)^3} g\left(\frac{\beta_1+\beta_2}{2}\right) \\ &- \frac{48}{(\beta_2-\beta_1)^3}\int_0^1 g\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)dt. \end{aligned} \quad (2.2)$$

Making use of change of the variable $x = t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1$ for $t \in [0, 1]$ and multiplying by $\frac{(\beta_2-\beta_1)^3}{96}$ on both sides, we obtain

$$\frac{(\beta_2-\beta_1)^3}{96} J_1 = \frac{(\beta_2-\beta_1)^2}{48} g''\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{\beta_2-\beta_1}{8} g'\left(\frac{\beta_1+\beta_2}{2}\right) + \frac{1}{2} g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\frac{\beta_1+\beta_2}{2}} g(x)dx. \quad (2.3)$$

Analogously, we can deduce

$$\begin{aligned}\frac{(\beta_2-\beta_1)^3}{96}J_2 &:= \frac{(\beta_2-\beta_1)^3}{96}\int_0^1 (t-1)^3 g'''\Big(t\beta_2+(1-t)\frac{\beta_1+\beta_2}{2}\Big)dt\\ &= \frac{(\beta_2-\beta_1)^2}{48}g''\left(\frac{\beta_1+\beta_2}{2}\right)+\frac{\beta_2-\beta_1}{8}g'\left(\frac{\beta_1+\beta_2}{2}\right)\\ &\quad+\frac{1}{2}g\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{1}{\beta_2-\beta_1}\int_{\frac{\beta_1+\beta_2}{2}}^{\beta_2} g(x)dx. \end{aligned} \tag{2.4}$$

Finally, by adding (2.3) and (2.4), we get the required identity (2.1). □

Remark 2.1. *Notice that f being quasi-convex is not equivalent to $|f|$ being quasi-convex. For instance, $g(x) = x^2 - 1$ is only quasi-convex (but not $|g(x)|$), whereas $g(x) = 1$ if $x \in [-1, 1]$ and $g(x) = -1$ otherwise, is not quasi-convex, but $|g(x)| = 1$ is quasi-convex.*

Theorem 2.1. *Suppose that $g : J \subseteq [0, +\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[\beta_1, \beta_2]$, where $\beta_1, \beta_2 \in J$ with $\beta_1 < \beta_2$. If $|g'''|$ is quasi-convex function on $[\beta_1, \beta_2]$, then we have*

$$\left|g\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx+\frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \le \frac{(\beta_2-\beta_1)^3}{384}K, \tag{2.5}$$

where $K = \max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_1)|\right\} + \max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_2)|\right\}$.

Proof. Making use of Lemma 2.1 and the quasi-convexity $|g'''|$, we have that

$$\begin{aligned}&\left|g\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx+\frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right|\\ &\le \frac{(\beta_2-\beta_1)^3}{96}\Bigg[\int_0^1 t^3\left|g'''\left(t\frac{\beta_1+\beta_2}{2}+(1-t)\beta_1\right)\right|dt\\ &+\int_0^1 (t-1)^3\left|g'''\left(t\beta_2+(1-t)\frac{\beta_1+\beta_2}{2}\right)\right|dt\Bigg]\\ &\le \frac{(\beta_2-\beta_1)^3}{96}\int_0^1 t^3\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_1)|\right\}dt\\ &+\frac{(\beta_2-\beta_1)^3}{96}\int_0^1 (1-t)^3\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_2)|\right\}dt\\ &\le \frac{(\beta_2-\beta_1)^3}{384}\left[\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_1)|\right\}+\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|, |g'''(\beta_2)|\right\}\right].\end{aligned}$$

This rearranges to the desired result. □

Example 2.1. To clarify the following expression occurs in Theorem 2.1

$$p := \frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right), \tag{2.6}$$

we consider the function $g(x) = \frac{x}{x^2+2}$ on the interval $[\beta_1, \beta_2] = [0, 1]$. Then, we have

$$y(x) := g''(x) = \frac{2x\left(4x^2 - 3\right)}{\left(x^2 + 2\right)^2};$$
$$p = \frac{1}{24} g''\left(\frac{1}{2}\right) = -\frac{46}{2187}.$$

Figure 1 demonstrates the intersections and relationships between the functions $g(x), y(x)$ and the point p geometrically.

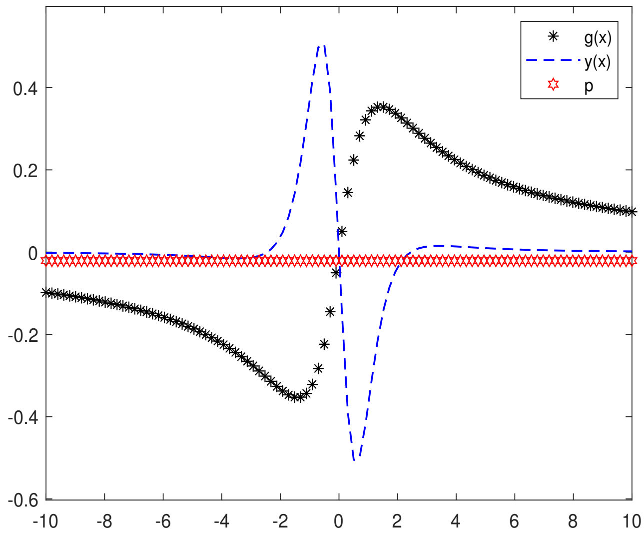

Figure 1. Plot illustration for the expression (2.6).

Corollary 2.1. *Let the assumptions of Theorem 2.1 be valid and let*

$$H := \left| g\left(\frac{\beta_1 + \beta_2}{2}\right) - \frac{1}{\beta_2 - \beta_1} \int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2 - \beta_1)^2}{24} g''\left(\frac{\beta_1 + \beta_2}{2}\right) \right|.$$

Then,

(i) if $|g'''|$ is increasing, then we have

$$H \le \frac{(\beta_2 - \beta_1)^3}{384} \left[|g'''(\beta_2)| + \left| g'''\left(\frac{\beta_1 + \beta_2}{2}\right) \right| \right], \tag{2.7}$$

(ii) if $|g'''|$ is decreasing, then we have

$$H \le \frac{(\beta_2 - \beta_1)^3}{384} \left[|g'''(\beta_1)| + \left| g'''\left(\frac{\beta_1 + \beta_2}{2}\right) \right| \right], \tag{2.8}$$

(iii) if $g'''\left(\frac{\beta_1 + \beta_2}{2}\right) = 0$, then we have

$$H \le \frac{(\beta_2 - \beta_1)^3}{384} \left[|g'''(\beta_1)| + |g'''(\beta_2)| \right], \tag{2.9}$$

(iv) if $g'''(\beta_1) = g'''(\beta_2) = 0$, then we have

$$H \leq \frac{(\beta_2-\beta_1)^3}{384}\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|. \tag{2.10}$$

Theorem 2.2. *Suppose that $g : J \subseteq [0,+\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[\beta_1,\beta_2]$, where $\beta_1,\beta_2 \in J$ with $\beta_1 < \beta_2$. If $|g'''|^q$ is quasi-convex function on $[\beta_1,\beta_2]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then we have*

$$\left|g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}K_q, \tag{2.11}$$

where $K_q = \left(\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_1)|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_2)|^q\right\}\right)^{\frac{1}{q}}$.

Proof. Let $p > 1$. Then from Lemma 2.1 and using the Hölder inequality, we can deduce

$$\begin{aligned}
&\left|g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \\
&\qquad\leq \frac{(\beta_2-\beta_1)^3}{96}\left[\int_0^1 t^3\left|g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)\right|dt\right. \\
&\qquad\quad \left. + \int_0^1 (1-t)^3\left|g'''\left(t\beta_2 + (1-t)\frac{\beta_1+\beta_2}{2}\right)\right|dt\right] \\
&\leq \frac{(\beta_2-\beta_1)^3}{96}\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad + \frac{(\beta_2-\beta_1)^3}{96}\left(\int_0^1 (1-t)^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|g'''\left(tb + (1-t)\frac{\beta_1+\beta_2}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

The quasi-convexity of $|g'''|^q$ on $[\beta_1,\beta_2]$ implies that

$$\int_0^1\left|g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)\right|^q dt \leq \max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_1)|^q\right\},$$

and

$$\int_0^1\left|g'''\left(tb + (1-t)\frac{\beta_1+\beta_2}{2}\right)\right|^q dt \leq \max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_2)|^q\right\}.$$

Therefore, we obtain

$$\left|g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}K_q,$$

where we used the identities

$$\int_0^1 t^{3p}dt = \int_0^1 (1-t)^{3p}dt = \frac{1}{3p+1}.$$

Thus, our proof is completely done. □

Corollary 2.2. *Let the assumptions of Theorem 2.2 be valid and let*

$$H = \left| g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24} g''\left(\frac{\beta_1+\beta_2}{2}\right)\right|.$$

Then,

(i) if $|g'''|$ is increasing, then we have

$$H \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}\left[|g'''(\beta_2)| + \left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|\right], \tag{2.12}$$

(ii) if $|g'''|$ is decreasing, then we have

$$H \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}\left[|g'''(\beta_1)| + \left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|\right], \tag{2.13}$$

(iii) if $g'''\left(\frac{\beta_1+\beta_2}{2}\right) = 0$, then we have

$$H \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}\Big[|g'''(\beta_1)| + |g'''(\beta_2)|\Big], \tag{2.14}$$

(iv) if $g'''(\beta_1) = g'''(\beta_2) = 0$, then we have

$$H \leq \frac{(\beta_2-\beta_1)^3}{96(3p+1)^{\frac{1}{p}}}\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|. \tag{2.15}$$

Theorem 2.3. *Suppose that $g : J \subseteq [0,+\infty) \to \mathbb{R}$ is a differentiable function such that $g''' \in L[\beta_1,\beta_2]$, where $\beta_1,\beta_2 \in J$ with $\beta_1 < \beta_2$. If $|g'''|^q$ is quasi-convex function on $[\beta_1,\beta_2]$ and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then we have*

$$\left| g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24} g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \leq \frac{(\beta_2-\beta_1)^3}{384}K_q, \tag{2.16}$$

where K_q is as before.

Proof. From Lemma 2.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left| g\left(\frac{\beta_1+\beta_2}{2}\right) - \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx + \frac{(\beta_2-\beta_1)^2}{24} g''\left(\frac{\beta_1+\beta_2}{2}\right)\right| \\
&\qquad \leq \frac{(\beta_2-\beta_1)^3}{96}\left[\int_0^1 t^3\left|g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)\right| dt\right. \\
&\qquad\quad \left. + \int_0^1 (1-t)^3\left|g'''\left(t\beta_2 + (1-t)\frac{\beta_1+\beta_2}{2}\right)\right| dt\right] \\
&\leq \frac{(\beta_2-\beta_1)^3}{96}\left(\int_0^1 t^3 dt\right)^{\frac{1}{p}}\left(\int_0^1 t^3\left|g'''\left(t\frac{\beta_1+\beta_2}{2} + (1-t)\beta_1\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(\beta_2-\beta_1)^3}{96}\left(\int_0^1(1-t)^3dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^3\left|g'''\left(tb+(1-t)\frac{\beta_1+\beta_2}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Then, by using the quasi-convexity of $|g'''|^q$ on $[\beta_1,\beta_2]$, we have

$$\int_0^1 t^3\left|g'''\left(t\frac{\beta_1+\beta_2}{2}+(1-t)\beta_1\right)\right|^q dt \le \frac{1}{4}\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_1)|^q\right\},$$

and

$$\int_0^1 (1-t)^3\left|g'''\left(tb+(1-t)\frac{\beta_1+\beta_2}{2}\right)\right|^q dt \le \frac{1}{4}\max\left\{\left|g'''\left(\frac{\beta_1+\beta_2}{2}\right)\right|^q, |g'''(\beta_2)|^q\right\}.$$

Therefore, we obtain

$$\left|g\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}g(x)dx+\frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right|\le\frac{(\beta_2-\beta_1)^3}{384}K_q,$$

Hence, our proof is completely done. □

Remark 2.2. *If* $q=\frac{p}{p-1}$ $(p>1)$*, the constants of Theorem 2.2 are improved, since* $\frac{1}{(3p+1)^{\frac{1}{p}}}<1$.

The following corollary improves the inequalities (2.7)–(2.10).

Corollary 2.3. *Let the assumptions of Theorem 2.3 be valid and let*

$$H=\left|g\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}g(x)dx+\frac{(\beta_2-\beta_1)^2}{24}g''\left(\frac{\beta_1+\beta_2}{2}\right)\right|.$$

Then,

(i) If $|g'''|$ *is increasing, we obtain* (2.7).
(ii) If $|g'''|$ *is decreasing, we obtain* (2.8).
(iii) If $g'''\left(\frac{\beta_1+\beta_2}{2}\right)=0$*, we obtain* (2.9)*,*
(iv) if $g'''(\beta_1)=g'''(\beta_2)=0$*, we obtain* (2.10).

3. Applications

3.1. Applications for special means

Consider the special means of positive real numbers $\beta_1>0$ and $\beta_2>0$, define by:

- Arithmetic Mean:

$$A(\beta_1,\beta_2)=\frac{\beta_1+\beta_2}{2}.$$

- Logarithmic mean:

$$L(\beta_1,\beta_2)=\frac{\beta_2-\beta_1}{\ln|\beta_2|-\ln|\beta_1|},\ |\beta_1|\ne|\beta_2|,\ \beta_1,\beta_2\ne 0.$$

- Generalized log-mean:

$$L_p(\beta_1,\beta_2) = \Big[\frac{{\beta_2}^{p+1} - {\beta_1}^{p+1}}{(p+1)(\beta_2-\beta_1)}\Big]^{\frac{1}{p}},\ p \in \mathbb{Z} \setminus \{-1,0\},\ \beta_1 \neq \beta_2.$$

Remark 3.1. *Let* $0 < \alpha \leq 1$ *and* $x > 0$*. Then, we consider*

$$g(x) = \frac{x^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}, \quad g'''(x) = x^{\alpha}.$$

For each $x, y > 0$ *and* $t \in [0,1]$*, we see that* $(tx + (1-t)y)^{\alpha} \leq t^{\alpha}x^{\alpha} + (1-t)^{\alpha}y^{\alpha}$*, then we see that* $g'''(x)$ *is* α*-convex function on* $(0,+\infty)$ *and* $g\left(\frac{\beta_1+\beta_2}{2}\right) = \frac{A^{\alpha+3}(\beta_1,\beta_2)}{(\alpha+1)(\alpha+2)(\alpha+3)}$*. Furthermore, we have*

$$\begin{aligned}\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} g(x)dx &= \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\left[\frac{\beta_2^{\alpha+4} - \beta_1^{\alpha+4}}{(\alpha+4)(\beta_2-\beta_1)}\right] \\ &= \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)} L_{\alpha+3}^{\alpha+3}(\beta_1,\beta_2).\end{aligned}$$

Above we used the definition of α*-convexity [11]: A function* $g : [0,r] \to \mathbb{R}, r > 0$ *is said to be* α*-convex, if the following holds:*

$$g(tx + (1-t)y) \leq t^{\alpha}g(x) + (1-t^{\alpha})g(y), \quad x, y \in [0,r],\ t, \alpha \in [0,1].$$

Proposition 3.1. *Let* $0 < \alpha \leq 1$ *and*$\beta_1, \beta_2 \in \mathbb{R}^{+}$ *with* $\beta_1 < \beta_2$*, then we have*

$$\begin{aligned}&\frac{384}{(\alpha+1)(\alpha+2)(\alpha+3)}\left|A^{\alpha+3}(\beta_1,\beta_2) - L_{\alpha+3}^{\alpha+3}(\beta_1,\beta_2) + \frac{(\beta_2-\beta_1)^2(\alpha+2)(\alpha+3)}{24}A^{\alpha+1}(\beta_1,\beta_2)\right| \\ &\qquad \leq (\beta_2-\beta_1)^3\Big[\max\left\{g\left(\frac{\beta_1+\beta_2}{2}\right)^{\alpha}, \beta_1^{\alpha}\right\} + \max\left\{g\left(\frac{\beta_1+\beta_2}{2}\right)^{\alpha}, \beta_2^{\alpha}\right\}\Big].\end{aligned}$$

Proof. Since x^{α} is quasi-convex for each $x > 0$ and $\alpha \in (0,1)$ because every non-decreasing continuous function is also quasi-convex, so the assertion follows from inequality (2.5) with $g(x) = \frac{x^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}$. □

Proposition 3.2. *Let* $\beta_1, \beta_2 \in \mathbb{R}$ *such that* $\beta_1 < \beta_2$ *and* $[\beta_1,\beta_2] \subset (0,+\infty)$*, then we have*

$$\left|A^{-1}(\beta_1,\beta_2) - L^{-1}(\beta_1,\beta_2) + \frac{(\beta_2-\beta_1)^2}{12}A^{-3}(\beta_1,\beta_2)\right| \leq \frac{(\beta_2-\beta_1)^3}{64}\left[\beta_1^{-4} + g\left(\frac{\beta_1+\beta_2}{2}\right)^{-4}\right].$$

Proof. The assertion follows from inequality (2.11) with $g(x) = \frac{1}{x},\ x \in [\beta_1,\beta_2]$. □

Proposition 3.3. *Let* $\beta_1, \beta_2 \in \mathbb{R}$ *with* $0 < \beta_1 < \beta_2$ *and* $n \in \mathbb{N}$*,* $k \geq 5$*, then for all* $q \geq 1$*, we have*

$$\begin{aligned}&\left|A^{k}(\beta_1,\beta_2) - L_k^k(\beta_1,\beta_2) + \frac{(\beta_2-\beta_1)^2 k!}{24(k-2)!}A^{k-2}(\beta_1,\beta_2)\right| \\ &\qquad \leq \frac{k!(\beta_2-\beta_1)^3}{2^{k+2}3(3p+1)^{\frac{1}{p}}(k-3)!}\left[(\beta_1+\beta_2)^{k-3} + 2^{k-3}\beta_2^{k-3}\right].\end{aligned}$$

Proof. Let $g(x) = x^k,\ x \in [\beta_1, \beta_2],\ k \in \mathbb{N}$ with $k \geq 5$, then we have

$$g'''(x) = \frac{k!}{(k-3)!} x^{k-3},$$

and it is easy to see that g''' is an increasing and quasi-convex function. Then, by applying Corollary 2.2(i) above, we have

$$\left| A^k(\beta_1, \beta_2) - L_k^k(\beta_1, \beta_2) + \frac{(\beta_2 - \beta_1)^2 k!}{24(k-2)!} A^{k-2}(\beta_1, \beta_2) \right| \leq \frac{(\beta_2 - \beta_1)^3}{96(3p+1)^{\frac{1}{p}}} K,$$

where

$$K = \frac{k!}{(k-3)!} \left[\max \left\{ \left| \frac{\beta_1 + \beta_2}{2} \right|^{k-3}, |\beta_1|^{k-3} \right\} + \max \left\{ \left| \frac{\beta_1 + \beta_2}{2} \right|^{k-3}, |\beta_2|^{k-3} \right\} \right].$$

Then, by applying Corollary 2.1(9) to g above, we get

$$\left| A^k(\beta_1, \beta_2) - L_k^k(\beta_1, \beta_2) + \frac{(\beta_2 - \beta_1)^2 k!}{24(k-2)!} A^{k-2}(\beta_1, \beta_2) \right|$$
$$\leq \frac{k!(\beta_2 - \beta_1)^3}{2^{k+2} 3(3p+1)^{\frac{1}{p}} (k-3)!} \left[(\beta_1 + \beta_2)^{k-3} + 2^{k-3} \beta_2^{k-3} \right],$$

and this completes the proof. □

3.2. Application for particular functions

Here, we consider two particular functions.

- First, we define $g : \mathbb{R} \to \mathbb{R}$, by $g(x) = e^x$.

Then, we have $g'''(x) = e^x$ and

$$\|g'''\|_\infty = \sup_{t \in [\beta_1, \beta_2]} |g'''(t)| = e^{\beta_2},$$

By applying inequality (2.7) for above $\|g'''\|_\infty$, we can deduce

$$\left| \left(1 + \frac{(\beta_2 - \beta_1)^2}{24} \right) e^{\frac{\beta_1 + \beta_2}{2}} - \frac{1}{\beta_2 - \beta_1} \left(e^{\beta_2} - e^{\beta_1} \right) \right| \leq \frac{(\beta_2 - \beta_1)^3}{384} \|g'''\|_\infty = \frac{(\beta_2 - \beta_1)^3}{384} e^{\beta_2}.$$

Particularly for $\beta_1 = 0$, it follows that

$$\left| \left(1 + \frac{{\beta_2}^2}{24} \right) e^{\frac{\beta_2}{2}} - \frac{1}{\beta_2} (e^{\beta_2} - 1) \right| \leq \frac{{\beta_2}^3}{384} e^{\beta_2}, \tag{3.1}$$

and for $\beta_2 = 1$, it follows that

$$\left| \frac{25}{24} \sqrt{e} - e + 1 \right| \leq \frac{e}{384}.$$

- Now, we define $g : \mathbb{R}^{+} \to \mathbb{R}$, by $g(x) = \frac{1}{x}$.

Then, since $|g'''(x)| = \frac{6}{x^4}$ is quasi-convex in $[\beta_1, \beta_2] \subset \mathbb{R}^{+}$ and

$$\|g'''\|_{\infty} = \sup_{t\in[\beta_1,y]} |g'''(t)| = \frac{6}{{\beta_1}^4},\ 0 < \beta_1 < y.$$

By applying inequality (2.8) to the function g above, we get

$$\left|\frac{2}{\beta_1+\beta_2} - \frac{\ln\beta_2 - \ln\beta_1}{\beta_2-\beta_1} + \frac{(\beta_2-\beta_1)^2}{12}\left(\frac{2}{\beta_1+\beta_2}\right)^3\right| \le \frac{(\beta_2-\beta_1)^3}{384} 2\|g'''\|_{\infty} = \frac{(\beta_2-\beta_1)^3}{32{\beta_1}^4}. \tag{3.2}$$

In view of (3.2) and Proposition 3.2, we can deduce

$$\begin{aligned}\left|A^{-1}(\beta_1,\beta_2) - L^{-1}(\beta_1,\beta_2) + \frac{(\beta_2-\beta_1)^2}{12}A^{-3}(\beta_1,\beta_2)\right| &\le \frac{(\beta_2-\beta_1)^3}{64}\left[\beta_1^{-4} + g\left(\frac{\beta_1+\beta_2}{2}\right)^{-4}\right] \\ &\le \frac{(\beta_2-\beta_1)^3}{32{\beta_1}^4}.\end{aligned}$$

For further illustration on the inequalities (3.1) and (3.2), we present some plot examples. Figures 2 and 3 illustrate the inequalities (3.1) and (3.2), respectively.

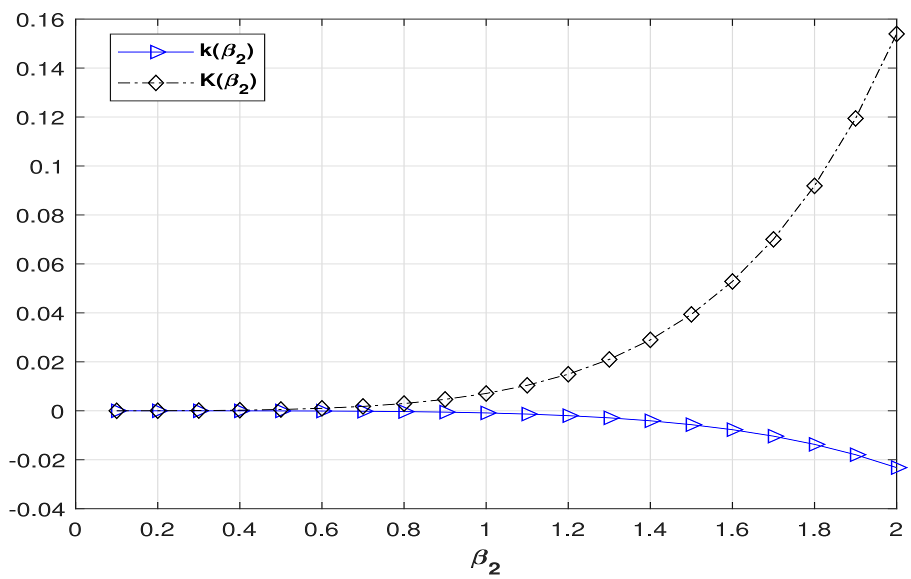

Figure 2. Plot illustration for inequality (3.1).

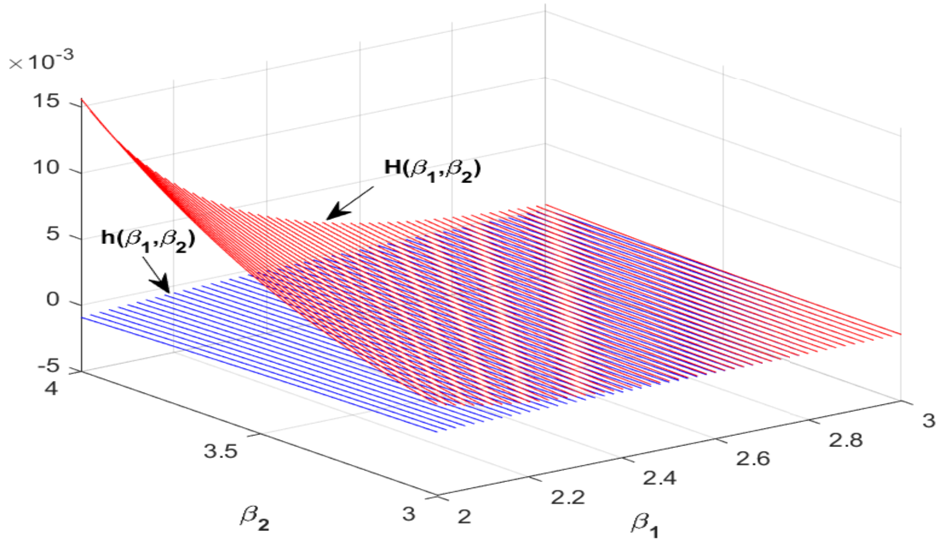

Figure 3. Plot illustration for inequality (3.2).

Let

$$k(\beta_2) = \left(1 + \frac{\beta_2^2}{24}\right) e^{\frac{\beta_2}{2}} - \frac{1}{\beta_2}(e^{\beta_2} - 1),$$
$$K(\beta_2) = \frac{\beta_2^3}{384} e^{\beta_2},$$

and

$$h(\beta_1, \beta_2) = \frac{2}{\beta_1 + \beta_2} - \frac{\ln\beta_2 - \ln\beta_1}{\beta_2 - \beta_1} + \frac{(\beta_2 - \beta_1)^2}{12}\left(\frac{2}{\beta_1 + \beta_2}\right)^3,$$
$$H(\beta_1, \beta_2) = \frac{(\beta_2 - \beta_1)^3}{32\beta_1^4}.$$

Furthermore, Figures 4 and 5 show $K(\beta_2) - k(\beta_2)$ and $D(\beta_1, \beta_2) := H(\beta_1, \beta_2) - h(\beta_1, \beta_2)$, receptively. From Figure 5, we can see that all values of $D(\beta_1, \beta_2)$ are positive which confirms the validity of (3.2).

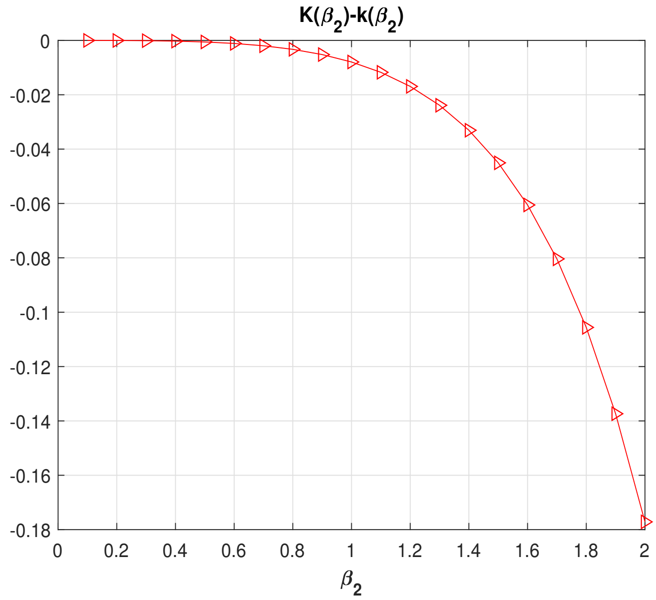

Figure 4. Plot illustration for $K(\beta_2) - k(\beta_2)$.

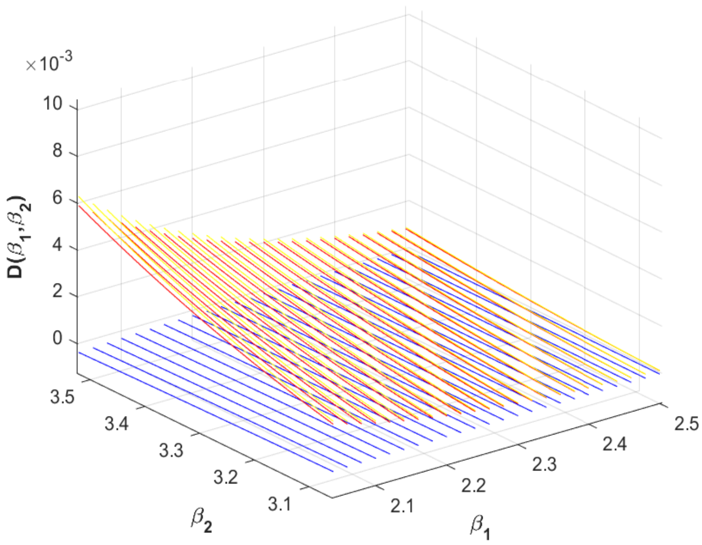

Figure 5. Plot illustration for $D(\beta_1, \beta_2)$.

4. Conclusions

In this paper we have established new Hermite–Hadamard inequality mainly motivated by Alomari et al in [42] for quasi-convex functions with $g \in C^3([\beta_1, \beta_2])$ such that $g''' \in L([\beta_1, \beta_2])$ and we give some applications to some special means and for some particular functions. We hope that the ideas used in this paper may inspire interested readers to explore some new applications.

We believe that our results, this new understanding of Hermite-Hadamard integral inequalities for quasi-convex functions, will be vital information for the future studies of these models of integral inequality. One can obtain the similar results for other kind of convex functions

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Conflict of interest

The authors declare no conflict of interest.

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