Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6030–6042.
DOI: 10.3934/math.2020386
Received: 20 May 2020
Accepted: 17 July 2020
Published: 23 July 2020

Research article

Fractional integral versions of Hermite-Hadamard type inequality for generalized exponentially convexity

Hengxiao Qi[1], **Muhammad Yussouf**[2], **Sajid Mehmood**[3], **Yu-Ming Chu**[4,5,*] **and Ghulam Farid**[6,*]

[1] Party School of Shandong Provincial Committee of the Communist Party of China (Shandong Administration College), Jinan 250014, China
[2] Department of Mathematics, University of Sargodha, Sargodha, Pakistan
[3] Govt Boys Primary school Sherani, Hazro, Attock, Pakistan
[4] Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China
[5] Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, P. R. China
[6] Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan

* **Correspondence:** Email: chuyuming2005@126.com; faridphdsms@hotmail.com.

Abstract: In this paper, we establish generalized fractional versions of Hermite-Hadamard inequalities for exponentially $(\alpha, h-m)$-convex functions, exponentially $(h-m)$-convex functions and exponentially (α, m)-convex functions. These inequalities arise when using the generalized fractional integral operators containing Mittag-Leffler function via a monotonically increasing function. The presented results hold at the same time for various kinds of convexities and well-known fractional integral operators. Moreover, the established inequalities reproduce several known results which are part of the existing literature.

Keywords: convex function; Hermite-Hadamard inequality; generalized fractional integral operators; Mittag-Leffler function
Mathematics Subject Classification: 26B25, 26A33, 26A51, 33E12

1. Introduction and preliminaries

Convexity is very important in the field of mathematical analysis and optimization theory. It is a basic concept in mathematics which has been extended and generalized in different ways by using various techniques. For example one of the generalizations is exponentially $(\alpha, h-m)$-convexity, that contains $(\alpha, h-m)$-convexity, exponentially $(h-m)$-convexity, $(h-m)$-convexity, exponentially (α, m)-convexity, (α, m)-convexity and several related convexities.

Definition 1. [1] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Then a function $\eta : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) is said to be exponentially $(\alpha, h - m)$-convex, if inequality (1.1) must holds for all $\alpha, m \in [0, 1]$, $a_1, a_2 \in I$, $\tau \in (0, 1)$ and $\varsigma \in \mathbb{R}$:

$$\eta(\tau a_1 + m(1 - \tau)a_2) \leq h(\tau^\alpha)\frac{\eta(a_1)}{e^{\varsigma a_1}} + mh(1 - \tau^\alpha)\frac{\eta(a_2)}{e^{\varsigma a_2}}. \tag{1.1}$$

If we put $\alpha = 1$ in (1.1), then we get the following definition of exponentially $(h - m)$-convex functions:

Definition 2. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. Then a function $\eta : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) is said to be exponentially $(h - m)$-convex, if inequality (1.2) must holds for all $m \in [0, 1]$, $a_1, a_2 \in I$, $\tau \in (0, 1)$ and $\varsigma \in \mathbb{R}$:

$$\eta(\tau a_1 + m(1 - \tau)a_2) \leq h(\tau)\frac{\eta(a_1)}{e^{\varsigma a_1}} + mh(1 - \tau)\frac{\eta(a_2)}{e^{\varsigma a_2}}. \tag{1.2}$$

If we put $h(\tau) = \tau$ in (1.1), then we get the following definition of exponentially (α, m)-convex functions:

Definition 3. A function $\eta : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) is said to be exponentially (α, m)-convex, if inequality (1.3) must holds for all $\alpha, m \in [0, 1]$, $a_1, a_2 \in I$, $\tau \in (0, 1)$ and $\varsigma \in \mathbb{R}$:

$$\eta(\tau a_1 + m(1 - \tau)a_2) \leq \tau^\alpha\frac{\eta(a_1)}{e^{\varsigma a_1}} + m(1 - \tau^\alpha)\frac{\eta(a_2)}{e^{\varsigma a_2}}. \tag{1.3}$$

Remark 1.

1. If we fix $\alpha = 1$ and $h(\tau) = \tau^s$ in (1.1), we recover the definition of exponentially (s, m)-convexity defined by Qiang et al. in [2].
2. If we fix $\alpha = m = 1$ and $h(\tau) = \tau^s$ in (1.1), we recover the definition of exponentially s-convexity defined by Mehreen et al. in [3].
3. If we fix $\alpha = m = 1$ and $h(\tau) = \tau$ in (1.1), we recover the definition of exponentially convexity defined by Awan et al. in [4].
4. If we fix $\varsigma = 0$ in (1.1), we recover the definition of $(\alpha, h - m)$-convexity defined by Farid et al. in [5].
5. If we fix $\varsigma = \alpha = 0$ and $\alpha = 1$ in (1.1), we recover the definition of $(h - m)$-convexity defined by Özdemir et al. in [6].
6. If we fix $\varsigma = 0$ and $h(\tau) = \tau$ in (1.1), we recover the definition of (α, m)-convexity defined by Mihesan in [7].
7. If we fix $\varsigma = 0$, $\alpha = 1$ and $h(\tau) = \tau^s$ in (1.1), we recover the definition of (s, m)-convexity defined by Efthekhari in [8].
8. If we fix $\varsigma = 0$, $\alpha = m = 1$ and $h(\tau) = \tau^s$ in (1.1), we recover the definition of s-convexity defined by Hudzik and Maligranda in [9].
9. If we fix $\varsigma = 0$, $\alpha = 1$ and $h(\tau) = \tau$ in (1.1), we recover the definition of m-convexity defined by Toader in [10].
10. If we fix $\varsigma = 0$ and $\alpha = m = 1$in (1.1), we recover the definition of h-convexity defined by Varosanec in [11].
11. If we fix $\varsigma = 0$, $\alpha = m = 1$ and $h(\tau) = \tau$ in (1.1), we recover the definition of convexity.

A convex function is elegantly interpreted in the coordinate plane by the well known Hermite-Hadamard inequality [12], stated as follows:

Theorem 1.1. *Let $\eta : [a_1, a_2] \to \mathbb{R}$ be a convex function such that $a_1 < a_2$. Then following inequality holds:*

$$\eta\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \eta(\tau) d\tau \leq \frac{\eta(a_1) + \eta(a_2)}{2}.$$

The Hermite-Hadamard inequality is generalized in various ways by using different fractional integral operators (see, for example [13, 4, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 3, 29, 30]). In this paper we will further generalize this inequality by using a new generalized convexity and fractional integral operators containing an extended generalized Mittag-Leffler function. The results of this paper also generalize results of [17, 18, 19, 20, 22, 24, 25, 29, 30].

In [31], Andrić et al. defined the generalized fractional integral operators containing generalized Mittag-Leffler function as follows:

Definition 4. Let $\kappa, \theta, \delta, l, \omega, c \in \mathbb{C}$, $\Re(\theta), \Re(\delta), \Re(l) > 0$, $\Re(c) > \Re(\omega) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\theta)$. Let $\eta \in L_1[a_1, a_2]$ and $\psi \in [a_1, a_2]$. Then the generalized fractional integral operators $\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_1^+}\eta$ and $\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_2^-}\eta$ are defined by:

$$\left(\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_1^+}\eta\right)(\psi; p) = \int_{a_1}^{\psi} (\psi - \tau)^{\delta-1} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\psi - \tau)^{\theta}; p)\eta(\tau) d\tau, \tag{1.4}$$

$$\left(\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_2^-}\eta\right)(\psi; p) = \int_{\psi}^{a_2} (\tau - \psi)^{\delta-1} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\tau - \psi)^{\theta}; p)\eta(\tau) d\tau, \tag{1.5}$$

where $E^{\omega,r,q,c}_{\theta,\delta,l}(\tau; p)$ is the generalized Mittag-Leffler function defined as follows:

$$E^{\omega,r,q,c}_{\theta,\delta,l}(\tau; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\varsigma + nq, c - \varsigma)}{\beta(\varsigma, c - \varsigma)} \frac{(c)_{nq}}{\Gamma(\theta n + \delta)} \frac{\tau^n}{(l)_{nr}}.$$

In [32], Farid defined the following unified integral operators:

Definition 5. Let $\eta, \mu : [a_1, a_2] \to \mathbb{R}$, (with $0 < a_1 < a_2$) be two functions such that η is positive and integrable on $[a_1, a_2]$ and μ is differentiable and strictly increasing on $[a_1, a_2]$. Also, let $\frac{\gamma}{\psi}$ be an increasing function on $[a_1, \infty)$ and $\kappa, \delta, l, \omega, c \in \mathbb{C}$, $\Re(\delta), \Re(l) > 0$, $\Re(c) > \Re(\omega) > 0$ with $p \geq 0$, $\theta, r > 0$ and $0 < q \leq r + \theta$. Then for $\psi \in [a_1, a_2]$ the integral operators ${}_{\mu}\Upsilon^{\gamma,\omega,r,q,c}_{\theta,\delta,l,a_1^+}\eta$ and ${}_{\mu}\Upsilon^{\gamma,\omega,r,q,c}_{\theta,\delta,l,a_2^-}\eta$ are defined by:

$$\left({}_{\mu}\Upsilon^{\gamma,\omega,r,q,c}_{\theta,\delta,l,a_1^+}\eta\right)(\psi; p) = \int_{a_1}^{\psi} \frac{\gamma(\mu(\psi) - \mu(\tau))}{\mu(\psi) - \mu(\tau)} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\mu(\psi) - \mu(\tau))^{\theta}; p)\eta(\tau) d(\mu(\tau)), \tag{1.6}$$

$$\left({}_{\mu}\Upsilon^{\gamma,\omega,r,q,c}_{\theta,\delta,l,a_2^-}\eta\right)(\psi; p) = \int_{\psi}^{a_2} \frac{\gamma(\mu(\tau) - \mu(\psi))}{\mu(\tau) - \mu(\psi)} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\mu(\tau) - \mu(\psi))^{\theta}; p)\eta(\tau) d(\mu(\tau)). \tag{1.7}$$

If we put $\gamma(\psi) = \psi^{\delta}$ in (1.6) and (1.7), then we get the following generalized fractional integral operators containing Mittag-Leffler function:

Definition 6. Let $\eta, \mu : [a_1, a_2] \to \mathbb{R}$, (with $0 < a_1 < a_2$) be two functions such that η is positive and integrable on $[a_1, a_2]$ and μ is differentiable and strictly increasing on $[a_1, a_2]$. Also, let $\kappa, \delta, l, \omega, c \in \mathbb{C}$, $\Re(\delta), \Re(l) > 0$, $\Re(c) > \Re(\omega) > 0$ with $p \geq 0$, $\theta, r > 0$ and $0 < q \leq r + \theta$. Then for $\psi \in [a_1, a_2]$ the integral operators ${}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_1^+}\eta$ and ${}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_2^-}\eta$ are defined by:

$$\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_1^+}\eta\right)(\psi;p) = \int_{a_1}^{\psi} (\mu(\psi) - \mu(\tau))^{\delta-1} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\mu(\psi) - \mu(\tau))^{\theta}; p)\eta(\tau)d(\mu(\tau)), \tag{1.8}$$

$$\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_2^-}\eta\right)(\psi;p) = \int_{\psi}^{a_2} (\mu(\tau) - \mu(\psi))^{\delta-1} E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa(\mu(\tau) - \mu(\psi))^{\theta}; p)\eta(\tau)d(\mu(\tau)). \tag{1.9}$$

Remark 2. Operators (1.8) and (1.9) are the generalizations of the following fractional integral operators:

1. Choosing $\mu(\psi) = \psi$, we recover the fractional integral operators defined in (1.4) and (1.5).
2. Choosing $\mu(\psi) = \psi$ and $p = 0$, we recover the fractional integral operators defined by Salim-Faraj in [33].
3. Choosing $\mu(\psi) = \psi$ and $l = r = 1$, we recover the fractional integral operators defined by Rahman et al. in [34].
4. Choosing $\mu(\psi) = \psi$, $p = 0$ and $l = r = 1$, we recover the fractional integral operators defined by Srivastava-Tomovski in [35].
5. Choosing $\mu(\psi) = \psi$, $p = 0$ and $l = r = q = 1$, we recover the fractional integral operators defined by Prabhakar in [36].
6. Choosing $\mu(\psi) = \psi$ and $\kappa = p = 0$, we recover the Riemann-Liouville fractional integral operators.

In [26], Mehmood et al. given the following formulas which we will use frequently:

$$\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_1^+}1\right)(\psi;p) = (\mu(\psi) - \mu(a_1))^{\delta} E^{\omega,r,q,c}_{\theta,\delta+1,l}(\kappa(\mu(\psi) - \mu(a_1))^{\theta}; p) := {}_{\mu}\chi^{\delta}_{\kappa,a_1^+}(\psi;p), \tag{1.10}$$

$$\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\kappa,a_2^-}1\right)(\psi;p) = (\mu(a_2) - \mu(\psi))^{\delta} E^{\omega,r,q,c}_{\theta,\delta+1,l}(\kappa(\mu(a_2) - \mu(\psi))^{\theta}; p) := {}_{\mu}\chi^{\delta}_{\kappa,a_2^-}(\psi;p). \tag{1.11}$$

The aim of this paper is to establish the generalized Hermite-Hadamard inequalities for exponentially $(\alpha, h - m)$-convex functions, exponentially $(h - m)$-convex functions and exponentially (α, m)-convex functions. These inequalities are produced by using the generalized fractional integral operators (1.8) and (1.9) containing Mittag-Leffler function via a monotone increasing function. These inequalities lead to produce the Hermite-Hadamard inequalities for various kinds of convexities (see Remark 1) and well-known fractional integral operators (see Remark 2).

In the upcoming section we prove the Hermite-Hadamard inequalities for generalized fractional integral operators (1.8) and (1.9) via exponentially $(\alpha, h - m)$-convex functions. Further we present them for generalized fractional integral operators (1.8) and (1.9) via exponentially $(h - m)$-convex functions. Also we give these inequalities for exponentially (α, m)-convex functions.

2. Fractional Hermite-Hadamard inequalities for exponentially $(\alpha, h - m)$-convex functions

First we give the following Hermite-Hadamard inequality for exponentially $(\alpha, h - m)$-convex functions via further generalized fractional integral operators.

Theorem 2.1. *Let $\eta : [a_1, ma_2] \subset [0,\infty) \to \mathbb{R}$, $0 < a_1 < ma_2$ be a positive, integrable and exponentially $(\alpha, h-m)$-convex function. Let $\mu : [a_1, ma_2] \to \mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)D(\varsigma)_{\mu}\chi^{\delta}_{\bar{\kappa},a_1^+}(\mu^{-1}(m\mu(a_2));p) \tag{2.1}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa},a_1^+}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)+m^{\delta+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}m^\theta,a_2^-}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)$$
$$\leq (m\mu(a_2)-\mu(a_1))^{\delta}\left[\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\right.$$
$$\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(\tau^\alpha)d\tau+m\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)$$
$$\left.\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(1-\tau^\alpha)d\tau\right],\qquad \bar{\kappa}=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^\theta},$$

where $D(\varsigma)=e^{\varsigma\mu(a_2)}$ for $\varsigma<0$, $D(\varsigma)=e^{\varsigma\mu(a_1)}$ for $\varsigma\geq 0$.

Proof. From exponentially $(\alpha, h-m)$-convexity of η, we have

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\leq h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\tau\mu(a_1)+m(1-\tau)\mu(a_2))}{e^{\varsigma(\tau\mu(a_1)+m(1-\tau)\mu(a_2))}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)}{e^{\varsigma\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)}}. \tag{2.2}$$

Multiplying (2.2) by $\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)$ and integrating over $[0,1]$, we have

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)d\tau \tag{2.3}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\frac{\eta(\tau\mu(a_1)+m(1-\tau)\mu(a_2))}{e^{\varsigma(\tau\mu(a_1)+m(1-\tau)\mu(a_2))}}d\tau$$
$$+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\frac{\eta\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)}{e^{\varsigma\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)}}d\tau.$$

Putting $\mu(\psi)=\tau\mu(a_1)+m(1-\tau)\mu(a_2)$ and $\mu(\phi)=(1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)$ in (2.3), then by using (1.8), (1.9) and (1.10), the first inequality of (2.1) can be achieved.

Again from exponentially $(\alpha, h-m)$-convexity of η, we have the following inequalities:

$$\eta(\tau\mu(a_1)+m(1-\tau)\mu(a_2))\leq h(\tau^\alpha)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh(1-\tau^\alpha)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}, \tag{2.4}$$

$$\eta\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)\leq mh(1-\tau^\alpha)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}+h(\tau^\alpha)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}. \tag{2.5}$$

Multiplying (2.4) by $h\left(\frac{1}{2^\alpha}\right)$ and (2.5) by $mh\left(\frac{2^\alpha-1}{2^\alpha}\right)$, then adding resulting inequalities, we have

$$h\left(\frac{1}{2^\alpha}\right)\eta(\tau\mu(a_1)+m(1-\tau)\mu(a_2))+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\eta\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right) \tag{2.6}$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)h(\tau^\alpha)$$
$$+m\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)h(1-\tau^\alpha).$$

Now multiplying (2.6) by $\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2^\alpha}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\eta(\tau\mu(a_1)+m(1-\tau)\mu(a_2))d\tau \tag{2.7}$$
$$+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\eta\left((1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)\right)d\tau$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(\tau^\alpha)d\tau$$
$$+m\left(h\left(\frac{1}{2^\alpha}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(1-\tau^\alpha)d\tau.$$

Putting $\mu(\psi)=\tau\mu(a_1)+m(1-\tau)\mu(a_2)$ and $\mu(\phi)=(1-\tau)\frac{\mu(a_1)}{m}+\tau\mu(a_2)$ in (2.7), then by using (1.8) and (1.9), the second inequality of (2.1) can be achieved. □

If we choose $\alpha=1$ in (2.1), then we get following Hermite-Hadamard inequality for exponentially $(h-m)$-convex functions.

Corollary 2.2. *Let $\eta:[a_1,ma_2]\subset[0,\infty)\to\mathbb{R}$, $0<a_1<ma_2$ be a positive, integrable and exponentially $(h-m)$-convex functions. Let $\mu:[a_1,ma_2]\to\mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)D(\varsigma)_\mu\chi^\delta_{\bar\kappa,a_1^+}(\mu^{-1}(m\mu(a_2));p) \tag{2.8}$$
$$\leq h\left(\frac{1}{2}\right)\left[\left({}_\mu\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar\kappa,a_1^+}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)+m^{\delta+1}\left({}_\mu\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar\kappa m^\theta,a_2^-}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)\right]$$
$$\leq(m\mu(a_2)-\mu(a_1))^\delta h\left(\frac{1}{2}\right)\left[\left(\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+m\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(\tau)d\tau\right.$$
$$\left.+m\left(\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+m\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)h(1-\tau)d\tau\right],\qquad\bar\kappa=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^\theta},$$

where $D(\varsigma)=e^{\varsigma\mu(a_2)}$ for $\varsigma<0$, $D(\varsigma)=e^{\varsigma\mu(a_1)}$ for $\varsigma\geq0$.

If we choose $h(\tau)=\tau$ in (2.1), then we get following Hermite-Hadamard inequality for exponentially (α,m)-convex functions.

Corollary 2.3. *Let $\eta : [a_1, ma_2] \subset [0,\infty) \to \mathbb{R}$, $0 < a_1 < ma_2$, be a positive, integrable and exponentially (α, m)-convex function. Let $\mu : [a_1, ma_2] \to \mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned}
&\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right) D(\varsigma)_{\mu}\chi^{\delta}_{\bar{\kappa},a_1^+}(\mu^{-1}(m\mu(a_2));p) \qquad (2.9)\\
&\le \frac{1}{2^{\alpha}}\Big[\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa},a_1^+}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)\\
&+m^{\delta+1}\left(2^{\alpha}-1\right)\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}m^{\theta},a_2^-}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)\Big]\\
&\le \frac{(m\mu(a_2)-\mu(a_1))^{\delta}}{2^{\alpha}}\Bigg[\left(\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+m\left(2^{\alpha}-1\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\\
&\times\int_0^1 \tau^{\delta+\alpha-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)d\tau + m\left(\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+m\left(2^{\alpha}-1\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)\\
&\times\int_0^1 \tau^{\delta-1}(1-\tau^{\alpha})E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)d\tau\Bigg],\qquad \bar{\kappa}=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^{\theta}},
\end{aligned}$$

where $D(\varsigma) = e^{\varsigma\mu(a_2)}$ for $\varsigma < 0$, $D(\varsigma) = e^{\varsigma\mu(a_1)}$ for $\varsigma \ge 0$.

Remark 3.
1. If we choose $\varsigma = p = 0$, $\alpha = m = 1$, $\mu(\psi) = \psi$ and $h(\tau) = \tau$ in (2.1), we recover the result in [17, Theorem 2.1].
2. If we choose $\varsigma = p = 0$, $\alpha = 1$, $\mu(\psi) = \psi$ and $h(\tau) = \tau$ in (2.1), we recover the result in [18, Theorem 3].
3. If we choose $\varsigma = 0$, $\alpha = 1$ and $\mu(\psi) = \psi$ in (2.1), we recover the result in [24, Theorem 2.1].
4. If we choose $\varsigma = 0$, $\alpha = m = 1$, $\mu(\psi) = \psi$ and $h(\tau) = \tau$ in (2.1), we recover the result in [25, Theorem 2.1].
5. If we choose $\varsigma = 0$, $\alpha = 1$, $\mu(\psi) = \psi$ and $h(\tau) = \tau$ in (2.1), we recover the result in [25, Theorem 3.1].
6. If we choose $\varsigma = p = \kappa = 0$, $\alpha = m = 1$, $\mu(\psi) = \psi$ and $h(\tau) = \tau$ in (2.1), we recover the result in [29, Theorem 2].

In the following we give another version of the Hermite-Hadamard inequality for exponentially $(\alpha, h-m)$-convex functions via further generalized fractional integral operators.

Theorem 2.4. *Let $\eta : [a_1, ma_2] \subset [0,\infty) \to \mathbb{R}$, $0 < a_1 < ma_2$ be a positive, integrable and exponentially $(\alpha, h-m)$-convex functions. Let $\mu : [a_1, ma_2] \to \mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)D(\varsigma)_{\mu}\chi^{\delta}_{\bar{\kappa}2^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^{+}}(\mu^{-1}(m\mu(a_2));p) \tag{2.10}$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}2^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^{+}}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)$$
$$+m^{\delta+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}(2m)^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2m}\right)\right)^{-}}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)$$
$$\leq\frac{(m\mu(a_2)-\mu(a_1))^{\delta}}{2^{\delta}}\left[\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\right.$$
$$\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{\tau^{\alpha}}{2^{\alpha}}\right)d\tau+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)$$
$$\left.\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{(2-\tau)^{\alpha}}{2^{\alpha}}\right)d\tau\right],\qquad\bar{\kappa}=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^{\theta}},$$

where $D(\varsigma)=e^{\varsigma\mu(a_2)}$ *for* $\varsigma<0$, $D(\varsigma)=e^{\varsigma\mu(a_1)}$ *for* $\varsigma\geq 0$.

Proof. From exponentially $(\alpha,h-m)$-convexity of η, we have

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)}{e^{\varsigma\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right)}{e^{\varsigma\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right)}}. \tag{2.11}$$

Multiplying (2.11) by $\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)$ and integrating over $[0,1]$, we have

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)d\tau \tag{2.12}$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)\frac{\eta\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)}{e^{\varsigma\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)}}d\tau$$
$$+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)\frac{\eta\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right)}{e^{\varsigma\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right)}}d\tau.$$

Putting $\mu(\psi)=\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)$ and $\mu(\phi)=\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)$ in (2.12), then by using (1.8), (1.9) and (1.10), the first inequality of (2.10) can be achieved.

Again from exponentially $(\alpha,h-m)$-convexity of η, we have the following inequalities:

$$\eta\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)\leq h\left(\frac{\tau^{\alpha}}{2^{\alpha}}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{(2-\tau)^{\alpha}}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}, \tag{2.13}$$

$$\eta\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right)\leq mh\left(\frac{(2-\tau)^{\alpha}}{2^{\alpha}}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}+h\left(\frac{\tau^{\alpha}}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}. \tag{2.14}$$

Multiplying (2.13) by $h\left(\frac{1}{2^{\alpha}}\right)$ and (2.14) by $mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)$, then adding resulting inequalities, we have

$$h\left(\frac{1}{2^{\alpha}}\right)\eta\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\eta\left(\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}+\frac{\tau}{2}\mu(a_2)\right) \tag{2.15}$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)h\left(\frac{\tau^{\alpha}}{2^{\alpha}}\right)$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)h\left(\frac{(2-\tau)^{\alpha}}{2^{\alpha}}\right).$$

Now multiplying (2.15) by $\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)\eta\left(\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)\right)d\tau \tag{2.16}$$
$$+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)\eta\left(\frac{\tau}{2}\mu(a_2)+\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}\right)d\tau$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{\tau}{2}\right)d\tau$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{2-\tau}{2}\right)d\tau.$$

Putting $\mu(\psi)=\frac{\tau}{2}\mu(a_1)+m\frac{(2-\tau)}{2}\mu(a_2)$ and $\mu(\phi)=\frac{\tau}{2}\mu(a_2)+\frac{(2-\tau)}{2}\frac{\mu(a_1)}{m}$ in (2.16), then by using (1.8) and (1.9), the second inequality of (2.10) can be achieved. □

If we choose $\alpha=1$ in (2.10), then we get following Hermite-Hadamard inequality for exponentially $(h-m)$-convex functions.

Corollary 2.5. *Let $\eta:[a_1,ma_2]\subset[0,\infty)\to\mathbb{R}$, $0<a_1<ma_2$ be a positive, integrable and exponentially $(h-m)$-convex functions. Let $\mu:[a_1,ma_2]\to\mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)D(\varsigma)_{\mu}\chi^{\delta}_{\bar{\kappa}2^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^{+}}(\mu^{-1}(m\mu(a_2));p) \tag{2.17}$$
$$\leq h\left(\frac{1}{2}\right)\left[\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}2^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^{+}}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)\right.$$
$$\left.+m^{\delta+1}\left({}_{\mu}\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}(2m)^{\theta},\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2m}\right)\right)^{-}}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)\right]$$
$$\leq\frac{(m\mu(a_2)-\mu(a_1))^{\delta}}{2^{\delta}}h\left(\frac{1}{2}\right)\left[\left(\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+m\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\right.$$
$$\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{\tau}{2}\right)d\tau+m\left(\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+m\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)$$
$$\left.\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^{\theta};p)h\left(\frac{2-\tau}{2}\right)d\tau\right],\qquad\bar{\kappa}=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^{\theta}},$$

where $D(\varsigma)=e^{\varsigma\mu(a_2)}$ for $\varsigma<0$, $D(\varsigma)=e^{\varsigma\mu(a_1)}$ for $\varsigma\geq 0$.

If we choose $h(\tau) = \tau$ in 2.10, then we get following Hermite-Hadamard inequality for exponentially (α, m)-convex functions.

Corollary 2.6. *Let $\eta : [a_1, ma_2] \subset [0,\infty) \to \mathbb{R}$, $0 < a_1 < ma_2$ be a positive, integrable and exponentially (α, m)-convex functions. Let $\mu : [a_1, ma_2] \to \mathbb{R}$ be differentiable and strictly increasing. Then for generalized fractional integral operators, the following inequalities hold:*

$$\eta\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)D(\varsigma)_\mu\chi^{\delta}_{\bar{\kappa}2^\theta,\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^+}(\mu^{-1}(m\mu(a_2));p) \tag{2.18}$$
$$\leq \frac{1}{2^\alpha}\left[\left({}_\mu\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}2^\theta,\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2}\right)\right)^+}\eta\circ\mu\right)(\mu^{-1}(m\mu(a_2));p)\right.$$
$$\left.+m^{\delta+1}\left(2^\alpha-1\right)\left({}_\mu\Upsilon^{\omega,r,q,c}_{\theta,\delta,l,\bar{\kappa}(2m)^\theta,\left(\mu^{-1}\left(\frac{\mu(a_1)+m\mu(a_2)}{2m}\right)\right)^-}\eta\circ\mu\right)\left(\mu^{-1}\left(\frac{\mu(a_1)}{m}\right);p\right)\right]$$
$$\leq \frac{(m\mu(a_2)-\mu(a_1))^\delta}{2^{\delta+\alpha}}\left[\left(\frac{\eta(\mu(a_1))}{e^{\varsigma\mu(a_1)}}+m\left(2^\alpha-1\right)\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}\right)\right.$$
$$\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\left(\frac{\tau}{2}\right)^\alpha d\tau+m\left(\frac{\eta(\mu(a_2))}{e^{\varsigma\mu(a_2)}}+m\left(2^\alpha-1\right)\frac{\eta\left(\frac{\mu(a_1)}{m^2}\right)}{e^{\varsigma\frac{\mu(a_1)}{m^2}}}\right)$$
$$\left.\times\int_0^1\tau^{\delta-1}E^{\omega,r,q,c}_{\theta,\delta,l}(\kappa\tau^\theta;p)\left(\frac{(2-\tau)^\alpha}{2^\alpha}\right)d\tau\right],\qquad \bar{\kappa}=\frac{\kappa}{(m\mu(a_2)-\mu(a_1))^\theta},$$

where $D(\varsigma) = e^{\varsigma\mu(a_2)}$ for $\varsigma < 0$, $D(\varsigma) = e^{\varsigma\mu(a_1)}$ for $\varsigma \geq 0$.

Remark 4.
1. If we choose $\varsigma = p = 0$, $\alpha = 1$ and $\mu(\psi) = \psi$ in (2.10), we recover the result in [19, Theorem 3.10].
2. If we choose $\varsigma = p = \kappa = 0$, $\alpha = 1$ and $\mu(\psi) = \psi$ in (2.10), we recover the result in [20, Theorem 2.1].
3. If we choose $\varsigma = 0$, $\alpha = 1$ and $\mu(\psi) = \psi$ in (2.10), we recover the result in [22, Theorem 2.11].
4. If we choose $\varsigma = p = \kappa = 0$, $\alpha = m = 1$ and $\mu(\psi) = \psi$ in (2.10), we recover the result in [30, Theorem 4].

3. Conclusions

In this article, we have proposed the generalized fractional Hermite-Hadamard inequalities for a generalized convexity. The results are applicable for fractional integral operators containing Mittag-Leffler functions in their kernels. Also they hold for exponentially $(\alpha, h-m)$-convex functions, exponentially $(h-m)$-convex functions and exponentially (α, m)-convex functions which are further linked with several known classes of convex functions. The readers can deduce a plenty of fractional integral inequalities of their choice of fractional integral operators from Remark 2 and convex function of any kind from Remark 1.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (Grant Nos. 11971142, 11871202, 61673169, 11701176, 11626101, 11601485).

Conflict of interest

The authors do not have any competing interest

References

1. X. Yang, G. Farid, K. Mahreen, et al. *On an integral and consequent fractional integral operators via generalized convexity*, AIMS Math., to appear.

2. X. Qiang, G. Farid, J. Pečarić, et al. *Generalized fractional integral inequalities for exponentially (s, m)-convex functions*, J. Inequal. Appl., **2020** (2020), 70.

3. N. Mehreen, M. Anwar, *Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications*, J. Inequal. Appl., **2019** (2019), 92.

4. M. U. Awan, M. A. Noor, K. I. Noor, *Hermite-Hadamard inequalities for exponentially convex functions*, Appl. Math. Inf. Sci., **12** (2018), 405–409.

5. G. Farid, A. U. Rehman, Q. U. Ain, *k-fractional integral inequalities of Hadamard type for $(h-m)$-convex functions*, Comput. Methods Differ. Equ., **8** (2020), 119–140.

6. M. E. Özdemir, A. O. Akdemri, E. Set, *On $(h - m)$-convexity and Hadamard-type inequalities*, Transylv. J. Math. Mech., **8** (2016), 51–58.

7. V. G. Mihesan, *A generalization of the convexity*, Seminar on Functional Equations, Approx. and Convex, Cluj-Napoca, Romania, 1993.

8. N. Efthekhari, *Some remarks on (s, m)-convexity in the second sense*, J. Math. Ineq., **8** (2014), 489–495.

9. H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequ. Math. **48** (1994), 100–111.

10. G. H. Toader, *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim. Cluj-Naploca (Romania), (1984), 329–338.

11. S. Varosanec, *On h-convexity*, J. Math. Anal. Appl., **326** (2007), 303–311.

12. J. Hadamard, *Eťude sur les proprieťeś des fonctions entiŕes et en particulier deúne fonction consideré per riemann*, J. Math. Pures Appl., (1893), 171–216.

13. G. Abbas, G. Farid, *Hadamard and Fejér-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals*, J. Anal., **25** (2017), 107–119.

14. F. Chen, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Chin. J. Math., **2014** (2014), 1–7.

15. H. Chen, U. N. Katugampola, *Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl., **446** (2017), 1274–1291.

16. S. S. Dragomir, I. Gomm, *Some Hermite-Hadamard type inequalities for functions whose exponentials are convex*, Stud. Univ. Babes-Bolyai Math., **60** (2015), 527–534.

17. G. Farid, *Hadamard and Fejér-Hadamard inequalities for generalized fractional integral involving special functions*, Konuralp J. Math., **4** (2016), 108–113.

18. G. Farid, *A Treatment of the Hadamard inequality due to m-convexity via generalized fractional integral*, J. Fract. Calc. Appl., **9** (2018), 8–14.

19. G. Farid, G. Abbas, *Generalizations of some fractional integral inequalities for m-convex functions via generalized Mittag-Leffler function*, Studia Univ. Babes-Bolyai, Math., **63** (2018), 23–35.

20. G. Farid, A. U. Rehman, B. Tariq, *On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals*, Studia Univ. Babes-Bolyai, Math., **62** (2017), 141–150.

21. G. Farid, A. U. Rehman, S. Mehmood, *Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function*, J. Math. Comput. Sci., **8** (2018), 630–643.

22. G. Farid, K. A. Khan, N. Latif, et al. *General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2018** (2018), 243.

23. İ. İşcan, *Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals*, Stud. Univ. Babeş-Bolyai Math., **60** (2015), 355–366.

24. S. M. Kang, G. Farid, W. Nazeer, et al. *$(h-m)$-convex functions and associated fractional Hadamard and Fejér-Hadamard inequalities via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2019** (2019), 255.

25. S. M. Kang, G. Farid, W. Nazeer, et al. *Hadamard and Fejér-Hadamard inequalities for extended generalized fractional integrals involving special functions*, J. Inequal. Appl., **2018** (2018), 119.

26. S. Mehmood, G. Farid, K. A. Khan, et al. *New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m-convex function*, Eng. Appl. Sci. Lett., **3** (2020), 45–55.

27. S. Mehmood, G. Farid, K. A. Khan, et al. *New fractional Hadamard and Fejér-Hadamard inequalities associated with exponentially $(h-m)$-convex function*, Eng. Appl. Sci. Lett., **3** (2020), 9–18.

28. S. Mehmood, G. Farid, *Fractional Hadamard and Fejér-Hadamard inequalities for exponentially m-convex function*, Stud. Univ. Babeş-Bolyai Math., to appear.

29. M. Z. Sarikaya, E. Set, H. Yaldiz, et al. *Hermite- Hadamard inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model., **57** (2013), 2403–2407.

30. M. Z. Sarikaya, H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes, **17** (2016), 1049–1059.

31. M. Andrić, G. Farid, J. Pečarić, *A further extension of Mittag-Leffler function*, Fract. Calc. Appl. Anal., **21** (2018), 1377–1395.

32. G. Farid, *A unified integral operator and further its consequences*, Open J. Math. Anal., **4** (2020), 1–7.

33. T. O. Salim, A. W. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with integral calculus*, J. Frac. Calc. Appl., **3** (2012), 1–13.

34. G. Rahman, D. Baleanu, M. A. Qurashi, et al. *The extended Mittag-Leffler function via fractional calculus*, J. Nonlinear Sci. Appl., **10** (2017), 4244–4253.

35. H. M. Srivastava, Z. Tomovski, *Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernal*, Appl. Math. Comput., **211** (2009), 198–210.

36. T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19** (1971), 7–15.

AIMS Press