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Research article

The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order

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Abstract: In this paper, we discuss the existence of solutions to a boundary value problem of differential equations of variable order, which is a piecewise constant function. Our results are based on the Schauder fixed point theorem. Then, under some assumptions on the nonlinear term, we obtain a generalized Lyapunov-type inequality to the two-point boundary value problem considered. To the best of our knowledge, there is no paper dealing with Lyapunov-type inequalities for boundary value problems in term of variable order. In addition, some examples of the obtained inequalities are given.

Keywords: derivatives and integrals of variable order; differential equations of variable order; piecewise constant functions; existence; generalized Lyapunov-type inequality
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1. Introduction

In this paper, we consider the existence of solutions and a generalized Lyapunov-type inequality to the following boundary value problem for differential equation of variable order

$$\begin{cases} D_{0+}^{q(t)}x(t) + f(t,x) = 0, \quad 0 < t < T, \\ x(0) = 0, \quad x(T) = 0, \end{cases} \tag{1.1}$$

where $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable order([1–4]) defined by

$$D_{0+}^{q(t)}x(t) = \frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))}x(s)ds, \quad t > 0, \tag{1.2}$$

and

$$I_{0+}^{2-q(t)}x(t)=\int_0^t\frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))}x(s)ds,\quad t>0, \tag{1.3}$$

denotes integral of variable order $2-q(t)$, $1<q(t)\leq 2$, $0\leq t\leq T$. $f:(0,T]\times\mathbb{R}\to\mathbb{R}$ is given continuous function satisfying some assumption conditions.

The operators of variable order, which fall into a more complex operator category, are the derivatives and integrals whose order is the function of certain variables. The variable order fractional derivative is an extension of constant order fractional derivative. In recent years, the operator and differential equations of variable order have been applied in engineering more and more frequently, for the examples and details, see [1–17].

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in different research areas and engineering, such as physics, chemistry, control of dynamical systems etc. Recently, many people paid attention to the existence and uniqueness of solutions to boundary value problems for fractional differential equations. Although the existing literature on solutions of boundary value problems of fractional order (constant order) is quite wide, few papers deal with the existence of solutions to boundary value problems of variable order. According to (1.2) and (1.3), it is obviously that when $q(t)$ is a constant function, i.e. $q(t)\equiv q$ (q is a finite positive constant), then $I_{0+}^{q(t)}, D_{0+}^{q(t)}$ are the usual Riemann-Liouville fractional integral and derivative [18].

The following properties of fractional calculus operators D_{0+}^{q}, I_{0+}^{q} play an important part in discussing the existence of solutions of fractional differential equations.

Proposition 1.1. [18] The equality $I_{0+}^{\gamma}I_{0+}^{\delta}f(t)=I_{0+}^{\gamma+\delta}f(t)$, $\gamma>0,\delta>0$ holds for $f\in L(0,b), 0<b<+\infty$.

Proposition 1.2. [18] The equality $D_{0+}^{\gamma}I_{0+}^{\gamma}f(t)=f(t)$, $\gamma>0$ holds for $f\in L(0,b), 0<b<+\infty$.

Proposition 1.3. [18] Let $1<\alpha\leq 2$. Then the differential equation

$$D_{0+}^{\alpha}f=0$$

has solutions

$$f(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}, c_1, c_2\in R.$$

Proposition 1.4. [18] Let $1<\alpha\leq 2$, $f(t)\in L(0,b)$, $D_{0+}^{\alpha}f\in L(0,b)$. Then the following equality holds

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t)=f(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}, c_1, c_2\in R.$$

These properties play a very important role in considering the existence of the solutions of differential equations for the Riemnn-Liouville fractional derivative, for details, please refer to [18]. However, from [1, 2, 16], for general functions $h(t), g(t)$, we notice that the semigroup property doesn't hold, i.e., $I_{a+}^{h(t)}I_{a+}^{g(t)}\neq I_{a+}^{h(t)+g(t)}$. Thus, it brings us extreme difficulties, we can't get these properties like Propositions 1.1-1.4 for the variable order fractional operators (integral and derivative). Without these properties for variable order fractional derivative and integral, we can hardly consider the existence of solutions of differential equations for variable order derivative by means of nonlinear functional analysis (for instance, some fixed point theorems).

Let's take Proposition 1.1 for example. To begin with the simplest case, *Example* 1.5. Let $p(t) = t$, $q(t) = 1$, $f(t) = 1, 0 \le t \le 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=1}$ defined in (1.3).

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{s-1}}{\Gamma(s)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds = \int_0^1 \frac{(1-s)^{s-1} s}{\Gamma(s)} ds \\ \approx 0.472.$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^s}{\Gamma(s+1)} ds = \int_0^1 \frac{(1-s)^s}{s\Gamma(s)} ds \approx 0.686.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=1}.$$

As a result, the Propositions 1.2 and Propositions 1.4 do not hold for $D_{0^+}^{p(t)}$ and $I_{0^+}^{p(t)}$, such as, for function $f \in L(0, T), 0 < p(t) < 1, 0 \le t \le T$, we get

$$D_{0^+}^{p(t)} I_{0^+}^{p(t)} f(t) = D^1 (I_{0^+}^{1-p(t)} I_{0^+}^{p(t)} f(t)) \neq D^1 I_{0^+}^{1-p(t)+p(t)} f(t) = f(t),\ t \in (0, T],$$

since we know that $I_{0^+}^{1-p(t)} I_{0^+}^{p(t)} f(t) \neq I_{0^+}^{1-p(t)+p(t)} f(t)$ for general function f.

Now, we can conclude that Propositions 1.1–1.4 do not hold for $D_{0+}^{q(t)}$ and $I_{0+}^{q(t)}$.

So, one can not transform a differential equation of variable order into an equivalent interval equation without the Propositions 1.1–1.4. It is a difficulty for us in dealing with the boundary value problems of differential equations of variable order. Since the equations described by the variable order derivatives are highly complex, difficult to handle analytically, it is necessary and significant to investigate their solutions.

In [16], by means of Banach Contraction Principle, Zhang considered the uniqueness result of solutions to initial value problem of differential equation of variable order

$$\begin{cases} D_{0+}^{p(t)} x(t) = f(t, x), 0 < t \le T, \\ x(0) = 0, \end{cases} \tag{1.4}$$

where $0 < T < +\infty$, $D_{0+}^{p(t)}$ denotes derivative of variable order $p(t)$ ([1–4]) defined by

$$D_{0+}^{p(t)} x(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} x(s) ds, t > 0. \tag{1.5}$$

and $\frac{1}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(t)} x(s) ds$ is integral of variable order $1 - p(t)$ for function $x(t)$. And $p : [0, T] \to (0, 1]$ is a piecewise constant function with partition $P = \{[0, T_1], (T_1, T_2], (T_2, T_3], \cdots, (T_{N^*-1}, T]\}$ (N^* is a given natural number) of the finite interval $[0, T]$, i.e.

$$p(t) = \sum_{k=1}^{N^*} q_k I_k(t), t \in [0, T],$$

where $0 < q_k \leq 1, k = 1, 2, \cdots, N^*$ are constants, and I_k is the indicator of the interval $[T_{k-1}, T_k], k = 1, 2, \cdots, N^*$(here $T_0 = 0, T_{N^*} = T$), that is $I_k = 1$ for $t \in [T_{k-1}, T_k]$, $I_k = 0$ for elsewhere.

In [17], the authors studied the Cauchy problem for variable order differential equations with a piecewise constant order function [19]. Inspired by these works, we will study the boundary value problem (1.1) for variable order differential equation with a piecewise constant order function $q(t)$ in this paper.

Lyapunov's inequality is an outstanding result in mathematics with many different applications, see [20–25] and references therein. The result, as proved by Lyapunov [20] in 1907, asserts that if $h : [a, b] \to R$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} y''(t) + h(t)y(t) = 0, a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.6}$$

to have a nontrivial solution is given by

$$\int_a^b |h(s)|ds > \frac{4}{b-a}, \tag{1.7}$$

where $-\infty < a < b < +\infty$.

Lyapunov's inequality has taken many forms, including versions in the context of fractional (noninteger order) calculus, where the second-order derivative in (1.6) is substituted by a fractional operator of order α,

$$\begin{cases} D^\alpha_{a+} y(t) + h(t)y(t) = 0, a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.8}$$

where D^α_{a+} is the Riemann-Liouville derivative of order $\alpha \in (1, 2]$ and $h : [a, b] \to \mathbb{R}$ is a continuous function. If (1.8) has a nontrivial solution, then

$$\int_a^b |h(s)|ds > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}.$$

A Lyapunov fractional inequality can also be obtained by considering the fractional derivative in in the sense of Caputo instead of Riemann-Liouville [22]. More recently, there are some results of Lyapunov type inequalities for fractional boundary value problems. see [23, 24]. In [25], authors obtained a generalization of inequality to boundary value problem as following

$$\begin{cases} D^\alpha_{a+} y(t) + h(t)f(y) = 0, a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.9}$$

where D^α_{a+} is the Riemann-Liouville derivative, $1 < \alpha \leq 2$, and $h : [a, b] \to \mathbb{R}$ is a Lebesgue integrable function. Under some assumptions on the nonlinear term f, authors got a generalization of inequality to the boundary value problem (1.9).

$$\int_a^b |h(s)|ds > \frac{4^{\alpha-1}\Gamma(\alpha)\eta}{(b-a)^{\alpha-1} f(\eta)}, \tag{1.10}$$

where η is maximum value of nontrivial solution to the boundary value problem (1.9).

Motivated by [21–25] and the above results, we focus on a generalized Lyapunov-type inequality to the boundary value problem (1.1) under certain assumptions of nonlinear term.

The paper is organized as following. In Section 2, we provide some necessary definitions associated with the boundary value problem (1.1). In Section 3, we establish the existence of solutions for the boundary value problem (1.1) by using the Schauder fixed point theorem. In Section 4, we investigative the generalized Lyapunov-type inequalities to the boundary value problem (1.1). In section 5, we give some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions that will be used to prove our main results.

Definition 2.1. A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e. a set of the form $[a,b],(a,b),[a,b)$ or $(a,b]$); a point $\{a\}$; or the empty set $\emptyset$.

Definition 2.2. If I is a generalized interval. A partition of I is a finite set P of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in P.

Example 2.3. The set $P = \{\{1\},(1,6),[6,7),\{7\},(7,8]\}$ of generalized intervals is a partition of $[1,8]$.

Definition 2.4. Let I be a generalized interval, let $f : I \to R$ be a function, and let P a partition of I. f is said to be piecewise constant with respect to P if for every $J \in P$, f is constant on J.

Example 2.5. The function $f : [1,6] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3, & 1 \le x < 3, \\ 4, & x = 3, \\ 5, & 3 < x < 6, \\ 2, & x = 6, \end{cases}$$

is piecewise constant with respect to the partition $\{[1,3],\{3\},(3,6),\{6\}\}$ of $[1,6]$.

The following example illustrates that the semigroup property of the variable order fractional integral doesn't holds for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition of finite interval $[a,b]$.

Example 2.6. Let $p(t) = \begin{cases} 3, & 0 \le t \le 1, \\ 2, & 1 < t \le 4, \end{cases}$ $q(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 3, & 1 < t \le 4, \end{cases}$ and $f(t) = 1, 0 \le t \le 4$. We'll verify $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=2}$, here, the variable order fractional integral is defined in (1.3). For $1 \le t \le 4$, we have

$$\begin{aligned} & I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \\ = \quad & \int_0^1 \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} d\tau ds + \int_1^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} d\tau ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{(t-s)^2}{\Gamma(3)} \int_0^s \frac{(s-\tau)}{\Gamma(2)} d\tau ds + \int_1^t \frac{(t-s)}{\Gamma(2)} [\int_0^1 \frac{(s-\tau)}{\Gamma(2)} d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} d\tau] ds \\
&= \int_0^1 \frac{(t-s)^2 s^2}{2\Gamma(3)} ds + \int_1^t (t-s)[\frac{s^2}{2} - \frac{(s-1)^2}{2} + \frac{(s-1)^3}{6}] ds \\
&= \int_0^1 \frac{(t-s)^2 s^2}{2\Gamma(3)} ds + \frac{1}{6}\int_1^t (t-s)(s^3 - 3s^2 + 9s - 4) ds,
\end{aligned}
$$

thus, we have

$$
\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2} &= \int_0^1 \frac{(2-s)^2 s^2}{2\Gamma(3)} ds + \frac{1}{6}\int_1^2 (2-s)(s^3 - 3s^2 + 9s - 4) ds \\
&= \frac{2}{15} + \frac{17}{60} = \frac{5}{12}
\end{aligned}
$$

$$
\begin{aligned}
I_{0+}^{p(t)+q(t)} f(t)|_{t=2} &= \int_0^2 \frac{(2-s)^{p(s)+q(s)-1}}{\Gamma(p(s)+q(s))} ds \\
&= \int_0^1 \frac{(2-s)^{3+2-1}}{\Gamma(3+2)} ds + \int_1^2 \frac{(2-s)^{2+3-1}}{\Gamma(2+3)} ds \\
&= \frac{31}{120} + \frac{1}{120} = \frac{4}{15}.
\end{aligned}
$$

Therefore, we obtain

$$
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=2},
$$

which implies that the semigroup property of the variable order fractional integral doesn't hold for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition $[0,1], (1,4]$ of finite interval $[0,4]$.

3. Existence result of solutions

We need the following assumptions:

(H_1) Let $n^* \in N$ be an integer, $P = \{[0, T_1], (T_1, T_2], (T_2, T_3], \cdots, (T_{n^*-1}, T]\}$ be a partition of the interval $[0, T]$, and let $q(t) : [0, T] \to (1, 2]$ be a piecewise constant function with respect to P, i.e.,

$$
q(t) = \sum_{k=1}^{N} q_k I_k(t) = \begin{cases} q_1, & 0 \le t \le T_1, \\ q_2, & T_1 < t \le T_2, \\ \cdots, & \cdots, \\ q_{n^*}, & T_{n^*-1} < t \le T_{n^*} = T, \end{cases} \tag{3.1}
$$

where $1 < q_k \le 2$ $(k = 1, 2, \cdots, n^*)$ are constants, and I_k is the indicator of the interval $[T_{k-1}, T_k]$, $k = 1, 2, \cdots, n^*$ (here $T_0 = 0, T_{n^*} = T$), that is, $I_k(t) = 1$ for $t \in [T_{k-1}, T_k]$ and $I_k(t) = 0$ for elsewhere.

(H_2) Let $t^r f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 \le r < 1)$, there exist constants $c_1 > 0, c_2 > 0, 0 < \gamma < 1$ such that

$$t^r |f(t, x(t))| \le c_1 + c_2 |x(t)|^\gamma, 0 \le t \le T, x(t) \in \mathbb{R}.$$

In order to obtain our main results, we firstly carry on essential analysis to the boundary value problem (1.1).

By (1.2) , the equation of the boundary value problem (1.1) can be written as

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))} x(s)ds + f(t, x) = 0, \quad 0 < t < T, \tag{3.2}$$

According to (H_1), Eq (3.2) in the interval $(0, T_1]$ can be written as

$$D_{0+}^{q_1} x(t) + f(t, x) = 0, \quad 0 < t \le T_1. \tag{3.3}$$

Equation (3.2) in the interval $(T_1, T_2]$ can be written by

$$\frac{d^2}{dt^2} \Big(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x(s)ds + \int_{T_1}^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s)ds \Big) + f(t, x) = 0, \tag{3.4}$$

and Eq (3.2) in the interval $(T_2, T_3]$ can be written by

$$\begin{aligned} &\frac{d^2}{dt^2} \left(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x(s)ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s)ds + \int_{T_2}^t \frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)} x(s)ds \right) \\ &+ f(t, x) = 0. \end{aligned} \tag{3.5}$$

In the same way, Eq (3.2) in the interval $(T_{i-1}, T_i]$, $i = 4, 5, \cdots, n^* - 1$ can be written by

$$\frac{d^2}{dt^2} \left(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x(s)ds + \cdots + \int_{T_{i-1}}^t \frac{(t-s)^{1-q_i}}{\Gamma(2-q_i)} x(s)ds \right) + f(t, x) = 0. \tag{3.6}$$

As for the last interval (T_{n^*-1}, T), similar to above argument, Eq (3.2) can be written by

$$\frac{d^2}{dt^2} \left(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x(s)ds + \cdots + \int_{T_{n^*-1}}^t \frac{(t-s)^{1-q_{n^*}}}{\Gamma(2-q_{n^*})} x(s)ds \right) + f(t, x) = 0. \tag{3.7}$$

Remark 3.1. From the arguments above, we find that, according to condition (H_1), in the different interval, the equation of the boundary value problem (1.1) must be represented by different expression. For instance, in the interval $(0, T_1]$, the equation of the boundary value problem (1.1) is represented by (3.3); in the interval $(T_1, T_2]$, the equation of the boundary value problem (1.1) is represented by (3.4); in the interval $(T_2, T_3]$, the equation of the boundary value problem (1.1) is represented by (3.5), etc. But, as far as we know, in the different intervals, the equation of integer order or constant fractional order problems may be represented by the same expression. Based these facts, different than integer order or constant fractional order problems, in order to consider the existence results of solution to the boundary value problem (1.1), we need consider the relevant problem defined in the different interval, respectively.

Now, based on arguments previous, we present definition of solution to the boundary value problem (1.1), which is fundamental in our work.

Definition 3.2. We say the boundary value problem (1.1) has a solution, if there exist functions $x_i(t), i = 1, 2, \cdots, n^*$ such that $x_1 \in C[0, T_1]$ satisfying equation (3.3) and $x_1(0) = 0 = x_1(T_1)$; $x_2 \in C[0, T_2]$ satisfying equation (3.4) and $x_2(0) = 0 = x_2(T_2)$; $x_3 \in C[0, T_3]$ satisfying equation (3.5) and $x_3(0) = 0 = x_3(T_3)$; $x_i \in C[0, T_i]$ satisfying equation (3.6) and $x_i(0) = 0 = x_i(T_i)(i = 4, 5, \cdots, n^* - 1)$; $x_{n^*} \in C[0, T]$ satisfying equation (3.7) and $x_{n^*}(0) = x_{n^*}(T) = 0$.

Theorem 3.3. Assume that conditions (H_1) and (H_2) hold, then the boundary value problem (1.1) has one solution.

Proof. According the above analysis, the equation of the boundary value problem (1.1) can be written as Eq (3.2) . Equation (3.2) in the interval $(0, T_1]$ can be written as

$$D_{0+}^{q_1} x(t) + f(t, x) = 0, \quad 0 < t \le T_1.$$

Now, we consider the following two-point boundary value problem

$$\begin{cases} D_{0+}^{q_1} x(t) + f(t, x) = 0, \quad 0 < t < T_1, \\ x(0) = 0, \quad x(T_1) = 0. \end{cases} \tag{3.8}$$

Let $x \in C[0, T_1]$ be solution of the boundary value problem (3.8). Now, applying the operator $I_{0+}^{q_1}$ to both sides of the above equation. By Propositions 1.4, we have

$$x(t) = d_1 t^{q_1 - 1} + d_2 t^{q_1 - 2} - \frac{1}{\Gamma(q_1)} \int_0^t (t - s)^{q_1 - 1} f(s, x(s)) ds, \quad 0 < t \le T_1.$$

By $x(0) = 0$ and the assumption of function f, we could get $d_2 = 0$. Let $x(t)$ satisfying $x(T_1) = 0$, thus we can get $d_1 = I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1}$. Then, we have

$$x(t) = I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1 - 1} - I_{0+}^{q_1} f(t, x), 0 \le t \le T_1 \tag{3.9}$$

Conversely, let $x \in C[0, T_1]$ be solution of integral Eq (3.9), then, by the continuity of function $t^r f$ and Proposition 1.2, we can easily get that x is the solution of boundary value problem (3.8).

Define operator $T : C[0, T_1] \to C[0, T_1]$ by

$$Tx(t) = I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1 - 1} - I_{0+}^{q_1} f(t, x(t)), \quad 0 \le t \le T_1.$$

It follows from the properties of fractional integrals and assumptions on function f that the operator $T : C[0, T_1] \to C[0, T_1]$ defined above is well defined. By the standard arguments, we could verify that $T : C[0, T_1] \to C[0, T_1]$ is a completely continuous operator.

In the next analysis, we take

$$M(r, q) = \max \left\{ \frac{2}{(1 - r)\Gamma(q_1)}, \frac{2}{(1 - r)\Gamma(q_2)}, \cdots, \frac{2}{(1 - r)\Gamma(q_{n^*})} \right\}.$$

Let $\Omega = \{x \in C[0, T_1] : \|x\| \le R\}$ be a bounded closed convex subset of $C[0, T_1]$, where

$$R = \max \left\{ 2c_1 M(r, q)(T + 1)^2, \left(2c_2 M(r, q)(T + 1)^2\right)^{\frac{1}{1-\gamma}} \right\}.$$

For $x \in \Omega$ and by (H_2), we have

$$\begin{aligned}
|Tx(t)| \leq\ & \frac{T_1^{1-q_1} t^{q_1-1}}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} |f(s, x(s))| ds + \frac{1}{\Gamma(q_1)} \int_0^{t} (t-s)^{q_1-1} |f(s, x(s))| ds \\
\leq\ & \frac{2}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} |f(s, x(s))| ds \\
\leq\ & \frac{2}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} s^{-r} (c_1 + c_2 |x(s)|^{\gamma}) ds \\
\leq\ & \frac{2T_1^{q_1-1}}{\Gamma(q_1)} \int_0^{T_1} s^{-r} (c_1 + c_2 R^{\gamma}) ds \\
\leq\ & \frac{2T_1^{q_1-1} T_1^{1-r}}{(1-r)\Gamma(q_1)} (c_1 + c_2 R^{\gamma}) \\
\leq\ & M(r, q) T_1^{q_1-r} (c_1 + c_2 R^{\gamma}) \\
\leq\ & M(r, q)(T + 1)^2 (c_1 + c_2 R R^{\gamma-1}) \\
\leq\ & \frac{R}{2} + \frac{R}{2} = R,
\end{aligned}$$

which means that $T(\Omega) \subseteq \Omega$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_1 \in \Omega$, which is a solution of the boundary value problem (3.8).

Also, we have obtained that Eq (3.2) in the interval $(T_1, T_2]$ can be written by (3.4). In order to consider the existence result of solution to (3.4), we rewrite (3.4) as following

$$\frac{d^2}{dt^2} \int_0^{T_1} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s) ds + \frac{d^2}{dt^2} \int_{T_1}^{t} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s) ds = f(t, x). \ \ T_1 < t \leq T_2,$$

For $0 \leq s \leq T_1$, we take $x(s) \equiv 0$, then, by the above equation, we get

$$D_{T_1}^{q_2} x(t) + f(t, x) = 0, T_1 < t < T_2.$$

Now, we consider the following boundary value problem

$$\begin{cases} D_{T_1+}^{q_2} x(t) + f(t, x) = 0, \ \ T_1 < t < T_2, \\ x(T_1) = 0, \ \ x(T_2) = 0, \end{cases} \tag{3.10}$$

Let $x \in C[T_1, T_2]$ be solution of the boundary value problem (3.10). Now, applying operator $I_{T_1+}^{q_2}$ on both sides of equation to boundary value problem (3.10) and by Propositions 1.4, we have

$$x(t) = d_1 (t - T_1)^{q_2-1} + d_2 (t - T_1)^{q_2-2} - \frac{1}{\Gamma(q_2)} \int_{T_1}^{t} (t-s)^{q_2-1} f(s, x(s)) ds, \ \ T_1 < t \leq T_2.$$

By $x(T_1)=0, x(T_2)=0$, we have $d_2=0$ and $d_1=I_{T_1+}^{q_2}f(T_2,x)(T_2-T_1)^{1-q_2}$. Then, we have

$$x(t)=I_{0+}^{q_2}f(T_2,x)(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}-\frac{1}{\Gamma(q_2)}\int_{T_1}^{t}(t-s)^{q_2-1}f(s,x(s))ds,\ \ T_1\le t\le T_2.$$

Conversely, let $x\in C[T_1,T_2]$ be solution of integral equation above, then, by the continuity assumption of function $t^r f$ and Proposition (1.2), we can get that x is solution solution of the boundary value problem (3.10).

Define operator $T: C[T_1,T_2]\to C[T_1,T_2]$ by

$$Tx(t)=I_{0+}^{q_2}f(T_2,x)(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}-\frac{1}{\Gamma(q_2)}\int_{T_1}^{t}(t-s)^{q_2-1}f(s,x(s))ds.$$

It follows from the continuity of function $t^r f$ that operator $T: C[T_1,T_2]\to C[T_1,T_2]$ is well defined. By the standard arguments, we know that $T: C[T_1,T_2]\to C[T_1,T_2]$ is a completely continuous operator.

For $x\in\Omega$ and by (H_2), we get

$$\begin{aligned}
|Tx(t)| \le\ & \frac{(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}}{\Gamma(q_2)}\int_{T_1}^{T_2}(T_2-s)^{q_2-1}|f(s,x(s))|ds+\frac{1}{\Gamma(q_2)}\int_{T_1}^{t}(t-s)^{q_2-1}|f(s,x(s))|ds\\
\le\ & \frac{2}{\Gamma(q_2)}\int_{T_1}^{T_2}(T_2-s)^{q_2-1}|f(s,x(s))|ds\\
\le\ & \frac{2}{\Gamma(q_2)}\int_{T_1}^{T_2}(T_2-s)^{q_2-1}s^{-r}(c_1+c_2|x(s)|^{\gamma})ds\\
\le\ & \frac{2T_2^{q_2-1}}{\Gamma(q_2)}\int_{T_1}^{T_2}s^{-r}(c_1+c_2R^{\gamma})ds\\
=\ & \frac{2T_2^{q_2-1}(T_2^{1-r}-T_1^{1-r})}{(1-r)\Gamma(q_2)}(c_1+c_2R^{\gamma})\\
\le\ & \frac{2T_2^{q_2-r}}{(1-r)\Gamma(q_2)}(c_1+c_2R^{\gamma})\\
\le\ & M(r,q)(T+1)^2(c_1+c_2RR^{\gamma-1})\\
\le\ & \frac{R}{2}+\frac{R}{2}=R,
\end{aligned}$$

which means that $T(\Omega)\subseteq\Omega$. Then the Schauder fixed point theorem assures that operator T has one fixed point $\widetilde{x}_2\in\Omega$, which is one solution of the following integral equation, that is,

$$\begin{aligned}\widetilde{x}_2(t)=\ & I_{0+}^{q_2}f(T_2,\widetilde{x}_2)(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}\\ & -\frac{1}{\Gamma(q_2)}\int_{T_1}^{t}(t-s)^{q_2-1}f(s,\widetilde{x}_2(s))ds,\quad T_1\le t\le T_2.\end{aligned}\tag{3.11}$$

Applying operator $D_{T_1+}^{q_2}$ on both sides of (3.11), by Proposition 1.2, we can obtain that

$$D_{T_1+}^{q_2}\widetilde{x}_2(t) + f(t, \widetilde{x}_2) = 0, \quad T_1 < t \le T_2,$$

that is, $\widetilde{x}_2(t)$ satisfies the following equation

$$\frac{d^2}{dt^2}\frac{1}{\Gamma(2-q_2)}\int_{T_1}^{t}(t-s)^{1-q_2}\widetilde{x}_2(s)ds + f(t, \widetilde{x}_2) = 0, \quad T_1 < t \le T_2. \tag{3.12}$$

We let

$$x_2(t) = \begin{cases} 0, & 0 \le t \le T_1, \\ \widetilde{x}_2(t), & T_1 < t \le T_2 \end{cases} \tag{3.13}$$

hence, from (3.12), we know that $x_2 \in C[0, T_2]$ defined by (3.13) satisfies equation

$$\frac{d^2}{dt^2}\left(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x_2(s)ds + \int_{T_1}^{t}\frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x_2(s)ds\right) + f(t, x_2) = 0,$$

which means that $x_2 \in C[0, T_2]$ is one solution of (3.4) with $x_2(0) = 0, x_2(T_2) = \widetilde{x}_2(T_2) = 0$.

Again, we have known that Eq (3.2) in the interval $(T_2, T_3]$ can be written by (3.5). In order to consider the existence result of solution to Eq (3.5), for $0 \le s \le T_2$, we take $x(s) \equiv 0$, then, by (3.5), we get

$$D_{T_2}^{q_3}x(t) + f(t, x) = 0, \;\; T_2 < t < T_3.$$

Now, we consider the following boundary value problem

$$\begin{cases} D_{T_2+}^{q_3}x(t) + f(t, x) = 0, \;\; T_2 < t < T_3, \\ x(T_2) = 0, \;\; x(T_3) = 0. \end{cases} \tag{3.14}$$

By the standard way, we know that the boundary value problem (3.14) exists one solution $\widetilde{x}_3 \in \Omega$. Since $\widetilde{x}_3$ satisfies equation

$$D_{T_2+}^{q_3}\widetilde{x}_3(t) + f(t, \widetilde{x}_3) = 0, \quad T_2 < t \le T_3,$$

that is, $\widetilde{x}_3(t)$ satisfies the following equation

$$\frac{d^2}{dt^2}\frac{1}{\Gamma(2-q_3)}\int_{T_2}^{t}(t-s)^{1-q_3}\widetilde{x}_3(s)ds + f(t, \widetilde{x}_3) = 0, \quad T_2 < t \le T_3. \tag{3.15}$$

We let

$$x_3(t) = \begin{cases} 0, & 0 \le t \le T_2, \\ \widetilde{x}_3(t), & T_2 < t \le T_3, \end{cases} \tag{3.16}$$

hence, from (3.15), we know that $x_3 \in C[0, T_3]$ defined by (3.16) satisfies equation

$$\begin{aligned}\frac{d^2}{dt^2}\Bigg(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x_3(s)ds &+ \int_{T_1}^{T_2}\frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x_3(s)ds \\ &+ \int_{T_2}^{t}\frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)}x_3(s)ds\Bigg) + f(t, x_3) = 0,\end{aligned}$$

which means that $x_3 \in C[0, T_3]$ is one solution of (3.5) with $x_3(0) = 0, x(T_3) = \widetilde{x}_3(T_3) = 0$.

By the similar way, in order to consider the existence of solution to Eq (3.6) defined on $[T_{i-1}, T_i]$ of (3.2), we can investigate the following two-point boundary value problem

$$\begin{cases} D^{q_i}_{T_{i-1}+}x(t) + f(t,x) = 0, & T_{i-1} < t < T_i, \\ x(T_{i-1}) = 0, \quad x(T_i) = 0. \end{cases} \tag{3.17}$$

By the same arguments previous, we obtain that the Eq (3.6) defined on $[T_{i-1}, T_i]$ of (3.2) has solution

$$x_i(t) = \begin{cases} 0, & 0 \le t \le T_{i-1}, \\ \widetilde{x}_i(t), & T_{i-1} < t \le T_i, \end{cases} \tag{3.18}$$

where $\widetilde{x}_i \in \Omega$ with $\widetilde{x}_i(T_{i-1}) = 0 = \widetilde{x}_i(T_i)$, $i = 4, 5, \cdots, n^* - 1$.

Similar to the above argument, in order to consider the existence result of solution to Eq (3.7), we may consider the following boundary value problem

$$\begin{cases} D^{q_{n^*}}_{T_{n^*-1}+}x(t) + f(t,x) = 0, & T_{n^*-1} < t < T_{n^*} = T, \\ x(T_{n^*-1}) = 0, \quad x(T) = 0. \end{cases} \tag{3.19}$$

So by the same considering, for $T_{n^*-1} \le t \le T$ we get

$$x(t) = (T - T_{n^*-1})^{1-q_{n^*}}(t - T_{n^*-1})^{q_{n^*}-1} I^{q_{n^*}}_{T_{n^*-1}+} f(T, x) - I^{q_{n^*}}_{T_{n^*-1}+} f(t, x).$$

Define operator $T : C[T_{n^*-1}, T] \to C[T_{n^*-1}, T]$ by

$$Tx(t) = (T - T_{n^*-1})^{1-q_{n^*}}(t - T_{n^*-1})^{q_{n^*}-1} I^{q_{n^*}}_{T_{n^*-1}+} f(T, x) - \frac{1}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{t} (t-s)^{q_{n^*}-1} f(s, x(s))ds,$$

$T_{n^*-1} \le t \le T$. It follows from the continuity assumption of function $t^r f$ that operator $T : C[T_{n^*-1}, T] \to C[T_{n^*-1}, T]$ is well defined. By the standard arguments, we note that $T : C[T_{n^*-1}, T] \to C[T_{n^*-1}, T]$ is a completely continuous operator.

For $x \in \Omega$ and by (H_2), we get

$$\begin{aligned} |Tx(t)| \le\ & \frac{(T - T_{n^*-1})^{1-q_{n^*}}(t - T_{n^*-1})^{q_{n^*}-1}}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T} (T-s)^{q_{n^*}-1}|f(s, x(s))|ds \\ & + \frac{1}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{t} (t-s)^{q_{n^*}-1}|f(s, x(s))|ds \\ \le\ & \frac{2}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T} (T-s)^{q_{n^*}-1}|f(s, x(s))|ds \\ \le\ & \frac{2}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T} (T-s)^{q_{n^*}-1}s^{-r}(c_1 + c_2|x(s)|^{\gamma})ds \\ \le\ & \frac{2T^{q_{n^*}-1}}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T} s^{-r}(c_1 + c_2R^{\gamma})ds \end{aligned}$$

$$
\begin{aligned}
&\le \frac{2T^{q_{n^*}-1}(T^{1-r}-T_{n^*-1}^{1-r})}{(1-r)\Gamma(q_{n^*})}(c_1+c_2R^\gamma)\\
&\le \frac{2(T+1)^2}{(1-r)\Gamma(q_{n^*})}(c_1+c_2R^\gamma)\\
&\le M(r,q)(T+1)^2(c_1+c_2RR^{\gamma-1})\\
&\le \frac{R}{2}+\frac{R}{2}=R,
\end{aligned}
$$

which means that $T(\Omega)\subseteq\Omega$. Then the Schauder fixed point theorem assures that operator T has one fixed point $\widetilde{x}_{n^*}\in\Omega$, which is one solution of the following integral equation, that is,

$$
\begin{aligned}
\widetilde{x}_{n^*}(t) &= (T-T_{n^*-1})^{1-q_{n^*}}(t-T_{n^*-1})^{q_{n^*}-1}I_{T_{n^*-1}+}^{q_{n^*}}f(T,\widetilde{x}_{n^*})\\
&\quad -\frac{1}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{t}(t-s)^{q_{n^*}-1}f(s,\widetilde{x}_{n^*}(s))ds,\, T_{n^*-1}\le t\le T. \qquad (3.20)
\end{aligned}
$$

Applying operator $D_{T_{n^*-1}+}^{q_{n^*}}$ on both sides of (3.20), by Proposition 1.2, we can obtain that

$$
D_{T_{n^*-1}+}^{q_{n^*}}\widetilde{x}_{n^*}(t)+f(t,\widetilde{x}_{n^*})=0,\quad T_{n^*-1}<t\le T,
$$

that is, $\widetilde{x}_T(t)$ satisfies the following equation

$$
\frac{d^2}{dt^2}\frac{1}{\Gamma(2-q_{n^*})}\int_{T_{n^*-1}}^{t}(t-s)^{1-q_{n^*}}\widetilde{x}_{n^*}(s)ds+f(t,\widetilde{x}_{n^*})=0,\quad T_{n^*-1}<t\le T. \qquad (3.21)
$$

We let

$$
x_{n^*}(t)=\begin{cases} 0, & 0\le t\le T_{n^*-1},\\ \widetilde{x}_{n^*}(t), & T_{n^*-1}<t\le T,\end{cases} \qquad (3.22)
$$

hence, from (3.21), we know that $x_{n^*}\in C[0,T]$ defined by (3.22) satisfies equation

$$
\begin{aligned}
&\frac{d^2}{dt^2}\Big(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x_{n^*}(s)ds+\cdots\\
&\qquad +\int_{T_{n^*-1}}^{t}\frac{(t-s)^{1-q_{n^*}}}{\Gamma(2-q_{n^*})}x_{n^*}(s)ds\Big)+f(t,x_{n^*})=0.
\end{aligned}
$$

for $T_{n^*-1}<t<T$, which means that $x_{n^*}\in C[0,T]$ is one solution of (3.7) with $x_{n^*}(0)=0, x_{n^*}(T)=\widetilde{x}_{n^*}(T)=0$.

As a result, we know that the boundary value problem (1.1) has a solution. Thus we complete the proof. □

Remark 3.4. For condition (H_2), if $\gamma\ge 1$, then using similar way, we can obtain the existence result of solution to the boundary value problem (1.1) provided that we impose some additional conditions on c_1, c_2.

4. The generalized Lyapunov-type inequalities

In this section, we investigate the generalized Lyapunov-type inequalities for the boundary value problem (1.1).

Now, we explore characters of Green functions to the boundary value problems (3.8), (3.10), (3.14), $\cdots$, (3.17) and (3.19).

Proposition 4.1. *Assume that $t^r f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $(0 \le r < 1)$ is continuous function, $q(t) : [0, T] \to (1, 2]$ satisfies (H_1), then the Green functions*

$$G_i(t,s) = \begin{cases} \frac{1}{\Gamma(q_i)}[(T_i - T_{i-1})^{1-q_i}(t - T_{i-1})^{q_i-1}(T_i - s)^{q_i-1} - (t-s)^{q_i-1}], \\ \qquad\qquad T_{i-1} \le s \le t \le T_i, \\ \frac{1}{\Gamma(q_i)}(T_i - T_{i-1})^{1-q_i}(t - T_{i-1})^{q_i-1}(T_i - s)^{q_i-1}, \\ \qquad\qquad T_{i-1} \le t \le s \le T_i, \end{cases} \tag{4.1}$$

of the boundary value problems (3.8), (3.10), (3.14), $\cdots$, (3.17) *and* (3.19) *satisfy the following properties:*

(1) $G_i(t, s) \ge 0$ *for all* $T_{i-1} \le t, s \le T_i$*;*

(2) $\max_{t\in[T_{i-1},T_i]} G_i(t, s) = G_i(s, s)$*,* $s \in [T_{i-1}, T_i]$*;*

(3) $G_i(s, s)$ *has one unique maximum given by*

$$\max_{s\in[T_{i-1},T_i]} G_i(s,s) = \frac{1}{\Gamma(q_i)}(\frac{T_i - T_{i-1}}{4})^{q_i-1},$$

where $i = 1, 2, \cdots, n^*$*,* $T_0 = 0, T_{n^*} = T$*.*

Proof. From the proof of Theorem 3.1, we know that Green functions of the boundary value problems (3.8), (3.10), (3.14), $\cdots$, (3.17) and (3.19) are given by (4.1).

Using a similar way, we can verify these three results. In fact, let

$$g(t,s) = (T_i - T_{i-1})^{1-q_i}(t - T_{i-1})^{q_i-1}(T_i - s)^{q_i-1} - (t-s)^{q_i-1}, T_{i-1} \le s \le t \le T_i.$$

We see that

$$\begin{aligned} g_t(t,s) \quad &= (q_i - 1)[(T_i - T_{i-1})^{1-q_i}(t - T_{i-1})^{q_i-2}(T_i - s)^{q_i-1} - (t-s)^{q_i-2}] \\ &\le (q_i - 1)[(T_i - T_{i-1})^{1-q_i}(t - s)^{q_i-2}(T_i - T_{i-1})^{q_i-1} - (t-s)^{q_i-2}] \\ &= 0, \end{aligned}$$

which means that $g(t, s)$ is nonincreasing with respect to t, so $g(t, s) \ge g(T_i, s) = 0$ for $T_{i-1} \le s \le t \le T_i$. Thus, together this with the expression of $G_i(t, s)$, we get that $G_i(t, s) \ge 0$ for all $T_{i-1} \le t, s \le T_i$, $i = 1, 2, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$.

Since $g(t, s)$ is nonincreasing with respect to t, it holds that $g(t, s) \le g(s, s)$ for $T_{i-1} \le s \le t \le T_i$. On the other hand, for $T_{i-1} \le t \le s \le T_i$, we have

$$(T_i - T_{i-1})^{1-q_i}(t - T_{i-1})^{q_i-1}(T_i - s)^{q_i-1} \le (T_i - T_{i-1})^{1-q_i}(s - T_{i-1})^{q_i-1}(T_i - s)^{q_i-1}.$$

These assure that $\max_{t\in[T_{i-1},T_i]} G_i(t,s) = G_i(s,s)$, $s \in [T_{i-1}, T_i]$, $i = 1, 2, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$.

Next, we verify (3) of Proposition 4.1. Obviously, the maximum points of $G_i(s,s)$ are not T_{i-1} and T_i, $i = 1, 2, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$. For $s \in (T_{i-1}, T_i)$, $i = 1, 2, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$, we have that

$$\begin{aligned}\frac{dG_i(s,s)}{ds} &= \frac{1}{\Gamma(q_i)}(T_i - T_{i-1})^{1-q_i}(q_i - 1)[(s - T_{i-1})^{q_i-2}(T_i - s)^{q_i-1} \\ &\quad - (s - T_{i-1})^{q_i-1}(T_i - s)^{q_i-2}] \\ &= \frac{1}{\Gamma(q_i)}(T_i - T_{i-1})^{1-q_i}(q_i - 1)(s - T_{i-1})^{q_i-2}(T_i - s)^{q_i-2}[T_i + T_{i-1} - 2s],\end{aligned}$$

which implies that the maximum points of $G_i(s,s)$ is $s = \frac{T_{i-1}+T_i}{2}$, $i = 1, 2, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$. Hence, for $i = 1, 2, \cdots, n^*, T_0 = 0, T_{n^*} = T$,

$$\max_{s\in[T_{i-1},T_i]} G_i(s,s) = G\Big(\frac{T_{i-1} + T_i}{2}, \frac{T_{i-1} + T_i}{2}\Big) = \frac{1}{\Gamma(q_i)}(\frac{T_i - T_{i-1}}{4})^{q_i-1}.$$

Thus, we complete this proof. □

Theorem 4.2. Let (H_1) holds and $t^r f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $(0 \le r < 1)$ be a continuous function. Assume that there exists nonnegative continuous function $h(t)$ defined on $[0,T]$ such that

$$t^r|f(t,x)| \le h(t)|x(t)|, 0 \le t \le T, x(t) \in R$$

If the boundary value problem (1.1) has a nontrivial solution x, then

$$\int_0^T s^{-r}h(s)ds > \sum_{i=1}^{n^*} \Gamma(q_i)\Big(\frac{4}{T_i - T_{i-1}}\Big)^{q_i-1}. \tag{4.2}$$

Proof. Let x be a nontrivial solution of the boundary value problem (1.1). Using Definition 3.2 and the proof of Theorem 3.3, we know that

$$x(t) = \begin{cases} x_1(t), \quad 0 \le t \le T_1 \\ x_2(t) = \begin{cases} 0, \quad 0 \le t \le T_1, \\ \widetilde{x}_1(t), \quad T_1 < t \le T_2, \end{cases} \\ \\ x_3(t) = \begin{cases} 0, \quad 0 \le t \le T_2, \\ \widetilde{x}_2(t), \quad T_2 < t \le T_3, \end{cases} \\ \vdots \\ x_i(t) = \begin{cases} 0, \quad 0 \le t \le T_{i-1}, \\ \widetilde{x}_i(t), \quad T_{i-1} < t \le T_i, \end{cases} \\ \vdots \\ x_{n^*}(t) = \begin{cases} 0, \quad 0 \le t \le T_{n^*-1}, \\ \widetilde{x}_{n^*}(t), \quad T_{n^*-1} < t \le T, \end{cases} \end{cases} \tag{4.3}$$

where $x_1 \in C[0,T_1]$ is nontrivial solution of the boundary value problem (3.8) with $a_1 = 0$, $\widetilde{x}_2 \in C[T_1,T_2]$ is nontrivial solution of the boundary value problem (3.10) with $a_2 = 0$, $\widetilde{x}_3 \in C[T_2,T_3]$ is nontrivial solution of the boundary value problem (3.14) with $a_3 = 0$, $\widetilde{x}_i \in C[T_{i-1},T_i]$ is nontrivial solution of the boundary value problem (3.17) with $a_i = 0$, $\widetilde{x}_{n^*} \in C[T_{n^*-1},T]$ is nontrivial solution of the boundary value problem (3.19). From (4.3) and Proposition 4.1 , we have

$$\begin{aligned}
\|x_1\|_{C[0,T_1]} = \max_{0\le t\le T_1}|x_1(t)| \le & \max_{0\le t\le T_1}\int_0^{T_1} G_1(t,s)|f(s,x_1(s))|ds \\
\le & \int_0^{T_1} G_1(s,s)s^{-r}h(s)|x_1(s)|ds \\
< & \frac{\|x_1\|_{C[0,T_1]}}{\Gamma(q_1)}(\frac{T_1}{4})^{q_1-1}\int_0^{T_1} s^{-r}h(s)ds,
\end{aligned}$$

which implies that

$$\int_0^{T_1} s^{-r}h(s)ds > \Gamma(q_1)(\frac{4}{T_1})^{q_1-1}. \tag{4.4}$$

$$\begin{aligned}
\|x_2\|_{C[0,T_2]} = \max_{T_1\le t\le T_2}|\widetilde{x}_2(t)| = & \max_{T_1\le t\le T_2}\left|\int_{T_1}^{T_2} G_2(t,s)f(s,\widetilde{x}_2(s))ds\right| \\
\le & \int_{T_1}^{T_2} G_2(s,s)s^{-r}h(s)|\widetilde{x}_2(s)|ds \\
< & \frac{\|\widetilde{x}_2\|_{C[T_1,T_2]}}{\Gamma(q_2)}(\frac{T_2-T_1}{4})^{q_2-1}\int_{T_1}^{T_2} s^{-r}h(s)ds, \\
= & \frac{\|x_2\|_{C[0,T_2]}}{\Gamma(q_2)}(\frac{T_2-T_1}{4})^{q_2-1}\int_{T_1}^{T_2} s^{-r}h(s)ds,
\end{aligned}$$

which implies that

$$\int_{T_1}^{T_2} s^{-r}h(s)ds > \Gamma(q_2)(\frac{4}{T_2-T_1})^{q_2-1}. \tag{4.5}$$

Similar, for $i = 3,4,\cdots,n^*$ $(T_{n^*} = T)$, we have

$$\begin{aligned}
\|x_i\|_{C[0,T_i]} & = \max_{T_{i-1}\le t\le T_i}|\widetilde{x}_i(t)| = \max_{T_{i-1}\le t\le T_i}\left|\int_{T_{i-1}}^{T_i} G_i(t,s)f(s,\widetilde{x}_i(s))ds\right| \\
& \le \int_{T_{i-1}}^{T_i} G_i(s,s)s^{-r}h(s)|\widetilde{x}_i(s)|ds \\
& < \frac{\|\widetilde{x}_i\|_{C[T_{i-1},T_i]}}{\Gamma(q_i)}(\frac{T_i-T_{i-1}}{4})^{q_i-1}\int_{T_{i-1}}^{T_i} s^{-r}h(s)ds,
\end{aligned}$$

$$= \frac{\|x_i\|_{C[0,T_i]}}{\Gamma(q_i)}(\frac{T_i - T_{i-1}}{4})^{q_i-1} \int_{T_{i-1}}^{T_i} s^{-r}h(s)ds,$$

which implies that

$$\int_{T_{i-1}}^{T_i} s^{-r}h(s)ds > \Gamma(q_i)\Big(\frac{4}{T_i - T_{i-1}}\Big)^{q_i-1}.$$

So, we get

$$\int_0^T s^{-r}h(s)ds > \sum_{i=1}^{n^*} \Gamma(q_i)\Big(\frac{4}{T_i - T_{i-1}}\Big)^{q_i-1}.$$

We complete the proof. □

Remark 4.3. We notice that if $r = 0$ and $q(t) = q$, q is a constant, i.e., BVP (1.1) is a fractional differential equation with constant order, then by similar arguments as done in [22], we get

$$\int_0^T h(s)ds > \Gamma(q)\Big(\frac{4}{T}\Big)^{q-1}.$$

So, the inequalities (4.2) is a generalized Lyapunov-type inequality for the boundary value problem (1.1).

5. Examples

Example 5.1. Let us consider the following nonlinear boundary value problem

$$\begin{cases} D_{0^+}^{q(t)} x(t) + t^{-0.5} \frac{|x|^{\frac{1}{2}}}{1+x^2} = 0, \ \ 0 < t < 2, \\ u(0) = u(2) = 0, \end{cases} \tag{5.1}$$

where

$$q(t) = \begin{cases} 1.2, \ \ 0 \le t \le 1, \\ 1.6, \ \ 1 < t \le 2. \end{cases}$$

We see that $q(t)$ satisfies condition (H_1); $t^{0.5}f(t, x(t)) = \frac{|x(t)|^{\frac{1}{2}}}{1+x(t)^2} : [0, 2] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, we have

$$t^{0.5}|f(t, x(t))| = \frac{|x(t)|^{\frac{1}{2}}}{1 + x(t)^2} \le |x(t)|^{\frac{1}{2}}.$$

Let $r = 0.5$, $c_1 = c_2 = 1$ and $\gamma = \frac{1}{2}$. We could verify that $f(t, x) = t^{-0.5} \frac{|x|^{\frac{1}{2}}}{1+x^2}$ satisfies condition (H_2). This suggests that the boundary value problem (5.1) has a solution by the conclusion of Theorem 3.3.

Example 5.2. Let us consider the following linear boundary value problem

$$\begin{cases} D_{0^+}^{q(t)} x(t) + t^{0.4} = 0, \ \ 0 < t < 3, \\ u(0) = 0, \ u(3) = 0, \end{cases} \tag{5.2}$$

where

$$q(t) = \begin{cases} 1.2, & 0 \le t \le 1, \\ 1.5, & 1 < t \le 2, \\ 1.8, & 2 < t \le 3. \end{cases}$$

We see that $q(t)$ satisfies condition (H_1); $f(t, x(t)) = t^{0.4} : [0, 3] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, $|f(t, x(t))| = t^{0.4} \le 3^{0.4}$, thus we could take suitable constants to verify $f(t, x) = t^{0.4}$ satisfies condition (H_2). Then Theorem 3.3 assures the boundary value problem (5.2) has a solution.

In fact, we know that equation of (5.1) can been divided into three expressions as following

$$D_{0+}^{1.2} x(t) + t^{0.4} = 0, \quad 0 < t \le 1. \tag{5.3}$$

For $1 < t \le 2$,

$$\frac{d^2}{dt^2}\Big(\int_0^1 \frac{(t-s)^{-0.2}}{\Gamma(0.8)} x(s)ds + \int_1^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)} x(s)ds \Big) + t^{0.4} = 0. \tag{5.4}$$

For $2 < t \le 3$,

$$\frac{d^2}{dt^2}\Big(\int_0^1 \frac{(t-s)^{-0.2}}{\Gamma(0.8)} x(s)ds + \int_1^2 \frac{(t-s)^{-0.5}}{\Gamma(0.5)} x(s)ds + \int_2^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)} x(s)ds \Big) + t^{0.4} = 0. \tag{5.5}$$

By [18], we can easily obtain that the following boundary value problems

$$\begin{cases} D_{0+}^{1.2} x(t) + t^{0.4} = 0, \quad 0 < t \le 1, \\ x(0) = 0, x(1) = 0 \end{cases}$$

$$\begin{cases} D_{1+}^{1.5} x(t) = \frac{d^2}{dt^2} \int_1^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)} x(s)ds + t^{0.4} = 0, \quad 1 < t < 2, \\ x(1) = 0, x(2) = 0 \end{cases}$$

$$\begin{cases} D_{2+}^{1.8} x(t) = \frac{d^2}{dt^2} \int_2^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)} x(s)ds + t^{0.4} = 0, \quad 2 < t < 3, \\ x(2) = 0, x(3) = 0 \end{cases}$$

respectively have solutions

$$\begin{aligned} x_1(t) &= \frac{\Gamma(1.4)}{\Gamma(2.6)}(t^{0.2} - t^{1.6}) \in C[0, 1]; \\ \widetilde{x_2}(t) &= \frac{\Gamma(1.4)}{\Gamma(2.9)}((t-1)^{0.5} - (t-1)^{1.9}) \in C[1, 2]; \\ \widetilde{x_3}(t) &= \frac{\Gamma(1.4)}{\Gamma(3.2)}((t-2)^{0.8} - (t-2)^{2.2}) \in C[2, 3]. \end{aligned}$$

It is known by calculation that

$$x_1(t), 0 \le t \le 1, \qquad x_2(t) = \begin{cases} 0, & 0 \le t \le 1, \\ \widetilde{x_2}(t), & 1 < t \le 2, \end{cases} \quad x_3(t) = \begin{cases} 0, & 0 \le t \le 2, \\ \widetilde{x_3}(t), & 2 < t \le 3, \end{cases} \tag{5.6}$$

are the solutions of (5.3)–(5.5), respectively. By Definition 3.2 and (5.6), we know that

$$x(t)=\begin{cases} x_1(t)=\frac{\Gamma(1.4)}{\Gamma(2.6)}(t^{0.2}-t^{1.6}), & 0\le t\le 1,\\ x_2(t)=\begin{cases}0, & 0\le t\le 1,\\ \frac{\Gamma(1.4)}{\Gamma(2.9)}((t-1)^{0.5}-(t-1)^{1.9}), & 1<t\le 2,\end{cases}\\ x_3(t)=\begin{cases}0, & 0\le t\le 2,\\ \frac{\Gamma(1.4)}{\Gamma(3.2)}((t-2)^{0.8}-(t-2)^{2.2}), & 2<t\le 3\end{cases}\end{cases}$$

is one solution of the boundary value problem (5.2).

6. Conclusions

In this paper, we consider a two-points boundary value problem of differential equations of variable order, which is a piecewise constant function. Based the essential difference about the variable order fractional calculus (derivative and integral) and the integer order and the constant fractional order calculus (derivative and integral), we carry on essential analysis to the boundary value problem (1.1). According to our analysis, we give the definition of solution to the boundary value problem (1.1). The existence result of solution to the boundary value problem (1.1) is derived. We present a Lyapunov-type inequality for the boundary value problems (1.1). Since the variable order fractional calculus (derivative and integral) and the integer order and the constant fractional order calculus (derivative and integral) has the essential difference, it is interesting and challenging about the existence, uniqueness of solutions, Lyapunov-type inequality, etc, to the boundary value problems of differential equations of variable order.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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