A degree condition for fractional \((g, f, n)\)-critical covered graphs

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Abstract: A graph \(G\) is called a fractional \((g, f)\)-covered graph if for any \(e \in E(G)\), \(G\) admits a fractional \((g, f)\)-factor covering \(e\). A graph \(G\) is called a fractional \((g, f, n)\)-critical covered graph if for any \(W \subseteq V(G)\) with \(|W| = n\), \(G - W\) is a fractional \((g, f)\)-covered graph. In this paper, we demonstrate that a graph \(G\) of order \(p\) is a fractional \((g, f, n)\)-critical covered graph if \(p \geq \frac{\delta(G) + n - (b - m)(b+1) + \frac{a+b}{a+m}}{n + \frac{a+b}{a+m} + \frac{m}{a+m}}\), where \(g\) and \(f\) are integer-valued functions defined on \(V(G)\) satisfying \(a \leq g(x) \leq f(x) - m \leq b - m\) for any \(x \in V(G)\).

Keywords: graph; degree condition; fractional \((g, f)\)-factor; fractional \((g, f)\)-covered graph; fractional \((g, f, n)\)-critical covered graph

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1. Introduction

All graphs considered here are finite, undirected and simple. Let \(G\) be a graph. The vertex set and the edge set of \(G\) are denoted by \(V(G)\) and \(E(G)\), respectively. Let \(d_G(x)\) denote the degree of a vertex \(x\) in \(G\), and \(N_G(x)\) denote the neighborhood of a vertex \(x\) in \(G\). Set \(N_G[x] = N_G(x) \cup \{x\}\). Let \(X\) be a vertex subset of \(G\). We use \(G[X]\) to denote the subgraph of \(G\) induced by \(X\), and write \(G - X = G[V(G) \setminus X]\). If no two vertices in \(X\) are adjacent, then we call \(X\) an independent set of \(G\).

For two integer-valued functions \(g\) and \(f\) with \(f(x) \geq g(x) \geq 0\) for any \(x \in V(G)\), a \((g, f)\)-factor of \(G\) is defined as a spanning subgraph \(F\) of \(G\) such that \(g(x) \leq d_F(x) \leq f(x)\) for any \(x \in V(G)\). Let \(E_\varepsilon = \{e : e = xy \in E(G)\}\). A fractional \((g, f)\)-indicator function is a function \(h\) that assigns each edge of \(G\) to a number in \([0, 1]\) so that \(g(x) \leq \sum_{e \in E_\varepsilon} h(e) \leq f(x)\) for every \(x \in V(G)\). Let \(h\) be a fractional \((g, f)\)-indicator function of \(G\). Write \(E_h = \{e : e \in E(G), h(e) \geq 0\}\). If \(G_h\) is a spanning subgraph of \(G\) with \(E(G_h) = E_h\), then \(G_h\) is called a fractional \((g, f)\)-factor of \(G\). If \(h(e) \in [0, 1]\) for any \(e \in E(G)\), then \(G_h\) is just a \((g, f)\)-factor of \(G\). A graph \(G\) is said to be a fractional \((g, f)\)-covered graph if for any
In this paper, we extend Theorem 1 to fractional $(g, f, n)$-critical covered graph, and derive the following result.

**Theorem 2.** Let $a$, $b$, $m$, and $n$ be integers satisfying $m \geq 0$, $n \geq 0$, $a \geq 1$ and $b \geq a + m$, let $G$ be a graph of order $p$ with $p \geq \frac{(ab)(a+b+n+1)-(b-m)n+2}{a+m}$, and let $g$ and $f$ be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b - m$ for every $x \in V(G)$. If $\delta(G) \geq \frac{(b-m)p + (a+m)n + 2}{a+b} + n$ and for every pair of nonadjacent vertices $u$ and $v$ of $G$,

$$\max\{d_G(u), d_G(v)\} \geq \frac{(b-m)p + (a+m)n + 2}{a+b},$$

then $G$ is a fractional $(g, f, n)$-critical covered graph.

The following result holds if setting $m = 0$ in Theorem 2.

**Corollary 1.** Let $a$, $b$, and $n$ be integers satisfying $n \geq 0$ and $b \geq a + 1$, let $G$ be a graph of order $p$ with $p \geq \frac{(ab)(a+b+n+1)-(b+1)n+2}{a}$, and let $g$ and $f$ be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq \frac{b(b+1)+2}{a} + n$ and for every pair of nonadjacent vertices $u$ and $v$ of $G$,

$$\max\{d_G(u), d_G(v)\} \geq \frac{bp + an + 2}{a+b},$$
then $G$ is a fractional $(g, f, n)$-critical covered graph.

The following result holds if setting $n = 0$ in Theorem 2.

**Corollary 2.** Let $a, b$ and $m$ be integers satisfying $m \geq 0$, $a \geq 1$ and $b \geq a + m$, let $G$ be a graph of order $p$ with $p \geq \frac{(a+b)(a+b+1)+2}{a+m}$, and let $g$ and $f$ be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - m \leq b - m$ for every $x \in V(G)$. If $\delta(G) \geq \frac{(b-m)(b+1)+2}{a+m}$ and for every pair of nonadjacent vertices $u$ and $v$ of $G$,

$$\max|d_G(u), d_G(v)| \geq \frac{(b-m)p + 2}{a+b},$$

then $G$ is a fractional $(g, f)$-covered graph.

### 2. Proof of Theorem 2

The following theorem derived by Li, Yan and Zhang [23] is essential to the proof of Theorem 2.

**Theorem 3 ( [23]).** Let $G$ be a graph, and let $g$ and $f$ be integer-valued functions defined on $V(G)$ satisfying $0 \leq g(x) \leq f(x)$ for any $x \in V(G)$. Then $G$ is a fractional $(g, f)$-covered graph if and only if

$$\delta_g(S, T) = f(S) + d_{G-S}(T) - g(T) \geq \epsilon(S)$$

for each $S \subseteq V(G)$, where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$ and $\epsilon(S)$ is defined by

$$\epsilon(S) = \begin{cases} 2, & \text{if } S \text{ is not independent}, \\ 1, & \text{if } S \text{ is independent and there is an edge joining } S \text{ and } V(G) \setminus (S \cup T), \text{ or there is an edge } e = uv \\ & \text{joining } S \text{ and } T \text{ such that } d_{G-S}(v) = g(v) \text{ for } v \in T, \\ 0, & \text{otherwise}. \end{cases}$$

We now verify Theorem 2. Let $H = G - W$ for any $W \subseteq V(G)$ with $|W| = n$. In order to justify Theorem 2, it suffices to show that $H$ is a fractional $(g, f)$-covered graph. Suppose that $H$ is not a fractional $(g, f)$-covered graph. Then by Theorem 3, there exists some subset $S$ of $V(H)$ such that

$$\delta_H(S, T) = f(S) + d_{H-S}(T) - g(T) \leq \epsilon(S) - 1,$$  \hspace{1cm} (2.1)

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq g(x)\}$.

If $T = \emptyset$, then using (2.1) and $\epsilon(S) \leq |S|$ we derive $\epsilon(S) - 1 \geq \delta_H(S, T) = f(S) \geq (a+m)|S| \geq |S| \geq \epsilon(S)$, a contradiction. Therefore, we admit $T \neq \emptyset$. Next, we define

$$d_1 = \min\{d_{H-S}(x) : x \in T\}$$

and select $x_1 \in T$ with $d_{H-S}(x_1) = d_1$. Note that $d_1 \leq d_{H-S}(x) \leq g(x) \leq b - m$ holds for any $x \in T$. We shall discuss two cases.

**Case 1.** $T = N_{H[T]}[x_1]$.

It follows from $0 \leq d_1 \leq b - m$, $|S| + d_1 = |S| + d_{H-S}(x_1) \geq d_H(x_1) = d_{G-W}(x_1) \geq d_G(x_1) - |W| \geq \delta(G) - n \geq \frac{(b-m)(b+1)+2}{a+m}$, $|T| = |N_{H[T]}[x_1]| \leq d_{H-S}(x_1) + 1 = d_1 + 1 \leq b - m + 1$ and $\epsilon(S) \leq 2$ that

$$\delta_H(S, T) = f(S) + d_{H-S}(T) - g(T)$$
\[ \geq (a + m)|S| + d_{H-S}(T) - (b - m)|T| \]
\[ = (a + m)|S| + d_1|T| - (b - m)|T| \]
\[ \geq (a + m)\left(\frac{(b - m)(b + 1) + 2}{a + m} - d_1\right) - (b - m - d_1)(b - m + 1) \]
\[ = (b - m - d_1)m + 2 + (b - a - m + 1)d_1 \]
\[ \geq 2 \geq \varepsilon(S), \]

which contradicts (2.1).

**Case 2.** \( T \neq N_{H[T]}[x_1]. \)

Obviously, \( T \setminus N_{H[T]}[x_1] \neq \emptyset. \) We may define

\[ d_2 = \min\{d_{H-S}(x) : x \in T \setminus N_{H[T]}[x_1]\} \]

and select \( x_2 \in T \setminus N_{H[T]}[x_1] \) with \( d_{H-S}(x_2) = d_2. \) It is clear that \( 0 \leq d_1 \leq d_2 \leq b - m \) holds.

Note that \( x_1 x_2 \notin E(H). \) Thus, we easily see that \( x_1 x_2 \notin E(G). \) According to the hypothesis of Theorem 2 and \( H = G - W, \) the following inequalities hold:

\[
\frac{(b - m)p + (a + m)n + 2}{a + b} \leq \max\{d_G(x_1), d_G(x_2)\} = \max\{d_{H+W}(x_1), d_{H+W}(x_2)\} \leq \max\{d_H(x_1) + n, d_H(x_2) + n\} = \max\{d_H(x_1), d_H(x_2)\} + n \leq \max\{d_{H-S}(x_1) + |S|, d_{H-S}(x_2) + |S|\} + n = \max\{d_{H-S}(x_1), d_{H-S}(x_2)\} + |S| + n = \max\{d_1, d_2\} + |S| + n = d_2 + |S| + n,
\]

namely,

\[
|S| \geq \frac{(b - m)p - (b - m)n + 2}{a + b} - d_2. \tag{2.2}
\]

Note that \( p - n - |S| - |T| \geq 0 \) and \( b - m - d_2 \geq 0. \) Thus, we derive \( (p - n - |S| - |T|)(b - m - d_2) \geq 0. \) Combining this inequality with (2.1) and \( \varepsilon(S) \leq 2, \) we obtain

\[
(p - n - |S| - |T|)(b - m - d_2) \geq 0 \geq \varepsilon(S) - 2 \geq \delta_H(S, T) - 1
\]
\[
= f(S) + d_{H-S}(T) - g(T) - 1
\]
\[
\geq (a + m)|S| + d_1|N_{H[T]}[x_1]| + d_2(|T| - |N_{H[T]}[x_1]|) - (b - m)|T| - 1
\]
\[
= (a + m)|S| + (d_1 - d_2)|N_{H[T]}[x_1]| - (b - m - d_2)|T| - 1
\]
\[
\geq (a + m)|S| + (d_1 - d_2)(d_1 + 1) - (b - m - d_2)|T| - 1,
\]

where \( |T| \geq |N_{H[T]}[x_1]| + 1, \) \( d_1 - d_2 \leq 0 \) and \( |N_{H[T]}[x_1]| \leq d_1 + 1. \) Then from the above inequality we get

\[
-1 \leq (p - n)(b - m - d_2) - (a + b - d_2)|S| - (d_1 - d_2)(d_1 + 1). \tag{2.3}
\]

It follows from (2.2), (2.3), \( 0 \leq d_1 \leq d_2 \leq b - m \) and \( p \geq \frac{a + b + n + 1 - (b - m)n + 2}{a + m} \) that

\[
-1 \leq (p - n)(b - m - d_2) - (a + b - d_2)|S| - (d_1 - d_2)(d_1 + 1)
\]
\[ \leq (p - n)(b - m - d_2) - (a + b - d_2)\left(\frac{(b - m)p - (b - m)n + 2}{a + b} - d_2\right) \]
\[= -(d_1 - d_2)(d_1 + 1) \]
\[= -\frac{(a + m)p + (b - m)n - 2}{a + b}d_2 + (a + b + n + 1)d_2 - d_1(d_1 + 1) \]
\[+d_2(d_1 - d_2) - 2 \]
\[\leq -\frac{(a + m)p + (b - m)n - 2}{a + b}d_2 + (a + b + n + 1)d_2 - 2 \]
\[\leq -\frac{(a + b)(a + b + n + 1) - (b - m)n + 2 + (b - m)n - 2}{a + b}d_2 \]
\[+ (a + b + n + 1)d_2 - 2 \]
\[= -2, \]

which is a contradiction. Theorem 2 is proved. \(\square\)

3. Remark

Let us explain that \(\max\{d_G(u), d_G(v)\} \geq \frac{(b - m)p + (a + m)n + 2}{a + b} \) in Theorem 2 is best possible, namely, it can not be replaced by \(\max\{d_G(u), d_G(v)\} \geq \frac{(b - m)p + (a + m)n + 2}{a + b} - 1\). Let \(b = a + m\), \(g(x) \equiv b - m\) and \(f(x) \equiv a + m\). We construct a graph \(G = K_{(b - m)t + n} \cup ((a + m)tK_1)\) with order \(p\), where \(\cup\) means “join”. Then \(p = (a + b)t + n\) and

\[\frac{(b - m)p + (a + m)n + 2}{a + b} - 1 \leq \max\{d_G(u), d_G(v)\} \]
\[= \frac{(b - m)t + n}{a + b} \]
\[= \frac{(b - m)p + (a + m)n}{a + b} \]
\[< \frac{(b - m)p + (a + m)n + 2}{a + b} \]

for every pair of nonadjacent vertices \(u\) and \(v\) of \(G\). Let \(W = V(K_n) \subseteq V(K_{(b - m)t + n})\) and \(H = G - W = K_{(b - m)t + n} \cup ((a + m)tK_1)\). Select \(S = V(K_{(b - m)t})\) and \(T = V((a + m)tK_1)\), and \(\varepsilon(S) = 2\). Thus, we derive

\[\delta_H(S, T) = f(S) + d_{H - S}(T) - g(T) \]
\[= (a + m)|S| - (b - m)|T| \]
\[= (a + m)(b - m)t - (b - m)(a + m)t \]
\[= 0 < 2 = \varepsilon(S). \]

In light of Theorem 3, \(H\) is not a fractional \((g, f)\)-covered graph, and so \(G\) is not a fractional \((g, f)\)-critical covered graph.

4. Conclusions

In this paper, we investigate the relationship between degree conditions and the existence of fractional \((g, f, n)\)-critical covered graphs. A sufficient condition for a graph being a fractional
A $(g, f, n)$-critical covered graph is derived. Furthermore, the sharpness of the main result in this paper is illustrated by constructing a special graph class. In addition, some other graph parameter conditions for graphs being fractional $(g, f, n)$-critical covered graphs can be studied further.

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**Conflict of interest**

The author declares no conflict of interest in this paper.

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