Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(6): 1582–1595.
DOI:10.3934/math.2019.6.1582
Received: 07 July 2019
Accepted: 25 September 2019
Published: 12 October 2019

Research article

Conformable differential operator generalizes the Briot-Bouquet differential equation in a complex domain

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Abstract: Very recently, a new local and limit-based extension of derivatives, called conformable derivative, has been formulated. We define a new conformable derivative in the complex domain, derive its differential calculus properties as well as its geometric properties in the field of geometric function theory. In addition, we employ the new conformable operator to generalize the Briot-Bouquet differential equation. We establish analytic solutions for the generalized Briot-Bouquet differential equation by using the concept of subordination and superordination. Examples of special normalized functions are illustrated in the sequel.

Keywords: conformable derivative; subordination and superordination; univalent function; analytic function; Briot-Bouquet differential equation
Mathematics Subject Classification: 30C45, 30C55

1. Introduction

The conformable derivative of differential functions $f(t)$ is given by the formula

$$\mathcal{D}^{\alpha} f(t) = t^{1-\alpha}\frac{df(t)}{dt},$$

which has applications in physical sciences and engineering (see [1]and [2]). We know that a critique of the conformable derivative is that, also conformable at the limit

$$\lim_{\alpha\to 1}\mathcal{D}^{\alpha} f(t) = \frac{df(t)}{dt}$$

and it is not conformable at the limit

$$\lim_{\alpha\to 0}\mathcal{D}^{\alpha} f(t) \neq f(t).$$

Definition 1.1. Conformable differential operator. Let $\alpha\in[0,1]$. A differential operator $\mathcal{D}^{\alpha}$ is conformable if and only if $\mathcal{D}^{0}$ is the identity operator and $\mathcal{D}^{1}$ is the classical differential operator. Specifically, $\mathcal{D}^{\alpha}$ is conformable if and only if for differentable function $f = f(t)$,

$$\mathcal{D}^{0} f(t) = f(t) \;\; and \;\; \mathcal{D}^{1} f(t) = \frac{d}{dt} f(t) = f'(t).$$

Anderson et al. [2] also noted that in control theory, a proportional-derivative controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and E is the error between the state variable and the process variable (also see [6]).

It is the aim of the present paper to define a "*complex conformable derivative of order* α; $\alpha \in [0,\infty)$" that satisfies the conformable differential operator criteria and then derive its differential calculus properties as well as its geometric properties in the field of geometric function theory. We note that all the previous definitions of conformable differential operators dealt with $\alpha \in (0,1)$ while our definition (given in the next section) is for $\alpha \in [0,\infty)$.

2. Complex conformable derivative (CCD)

We shall start with the definition of complex conformable derivative. Let Λ^* be the class of complex valued functions that are analytic in the complex open unit disk $\mathbb{U} := \{z = re^{i\theta} : |z| < 1,\ 0 \le \theta < 2\pi\}$ and Λ be the subclass of Λ^* consisting of analytic functions f that are normalized by $f(0) = f'(0) - 1 = 0$.

Definition 2.1. For non-negative real numbers α let $[[\alpha]]$ be the integer part of α. For $f \in \Lambda^*$, we define a new complex conformable derivative $\mathcal{D}^{\alpha} f$ of order α in $\mathbb{U}$ by

$$\begin{aligned} \mathcal{D}^{\alpha} f(z) &= \mathcal{D}^{\alpha-[[\alpha]]}(\mathcal{D}^{[[\alpha]]} f(z)) \\ &= \frac{\kappa_1(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\,(\mathcal{D}^{[[\alpha]]} f(z)) \\ &+ \frac{\kappa_0(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\,(z(\mathcal{D}^{[[\alpha]]} f(z))'), \end{aligned} \tag{2.1}$$

where for $\nu = \alpha - [[\alpha]] \in [0,1)$,

$$\begin{aligned} &\mathcal{D}^{0} f(z) = f(z) \\ &\mathcal{D}^{\nu} f(z) = \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\, f(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,(zf'(z)) \\ &\mathcal{D}^{1} f(z) = zf'(z),\ \ldots, \\ &\mathcal{D}^{[[\alpha]]} f(z) = \mathcal{D}\left(\mathcal{D}^{[[\alpha]]-1} f(z)\right), \end{aligned}$$

the functions $\kappa_1, \kappa_0 : [0,1] \times \mathbb{U} \to \mathbb{U}$ are analytic in $\mathbb{U}$ so that $\kappa_1(\nu,z) \neq -\kappa_0(\nu,z)$,

$$\lim_{\nu\to 0} \kappa_1(\nu,z) = 1, \quad \lim_{\nu\to 1} \kappa_1(\nu,z) = 0, \quad \kappa_1(\nu,z) \neq 0,\ \forall z \in \mathbb{U},\ \nu \in (0,1),$$

and

$$\lim_{\nu\to 0}\kappa_0(\nu,z)=0,\quad \lim_{\nu\to 1}\kappa_0(\nu,z)=1,\quad \kappa_0(\nu,z)\neq 0,\ \forall z\in\mathbb{U}\ \nu\in(0,1).$$

We note that if α assumes only non-negative integer values, that is if $\alpha-[[\alpha]]=0$, then we have the Sàlàgean differential operator [8]. In the following lemma we present the differential calculus (linearity rule, product rule, quotient rule and constant function rule) properties for our newly defined complex conformable derivative.

Lemma 2.2. *Let $\alpha\in[0,\infty)$, $\nu=\alpha-[[\alpha]]$, $\kappa_1,\kappa_0:[0,1]\times\mathbb{U}\to\mathbb{U}$ be given by Definition 2.1 and assume that the functions f and g are differentiable as needed. Then the complex conformable differential $\mathcal{D}^\alpha$ given by (2.1) satisfies the following calculus of operation rules.*
(i) Linearity Rule:

$$\mathcal{D}^\alpha[af(z)+bg(z)]=a\mathcal{D}^\alpha f(z)+b\mathcal{D}^\alpha g(z),\quad a,\ b\ \in\mathbb{R};$$

(ii) Product Rule:

$$\begin{aligned}\mathcal{D}^\alpha[f(z)g(z)]=&\sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(f^{([[\alpha]]-k)}(z)\mathcal{D}^{\alpha-[[\alpha]]}g^{(k)}(z)\Big)\\&+\sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(g^{(k)}(z)\mathcal{D}^{\alpha-[[\alpha]]}f^{([[\alpha]]-k)}(z)\Big)\\&-\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(f^{([[\alpha]]-k)}(z)g^{(k)}(z)\Big).\end{aligned}$$

(iii) Quotient Rule:

$$\begin{aligned}\mathcal{D}^\alpha\left(\frac{f(z)}{g(z)}\right)=&\frac{g(z)\mathcal{D}^\nu\left(\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\big[f(z)g^k(z)\big]^{([[\alpha]])}}{g^k(z)}\right)}{(g(z))^2}\\&-\frac{\left(\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\big[f(z)g^k(z)\big]^{([[\alpha]])}}{g^k(z)}\right)\mathcal{D}^\nu g(z)}{(g(z))^2}\\&+\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\big[f(z)g^k(z)\big]^{([[\alpha]])}}{g^k(z)}}{g(z)}\right).\end{aligned}$$

(iv) Constant Function Rule: If K is a constant function, then

$$\mathcal{D}^\alpha K=\begin{cases}\dfrac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}K & if\quad \alpha\in[0,1);\\ 0 & if\quad \alpha\in[1,\infty).\end{cases}$$

Proof. (i) First consider the case $\alpha \in [0,1)$. Then $\alpha - [[\alpha]] = \alpha - 0 = \alpha \in [0,1)$ and so by Definition 2.1 it follows that

$$\begin{aligned}
\mathcal{D}^{\alpha}[af(z)+bg(z)] &= \frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\Big(af(z)+bg(z)\Big) + \frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\Big(z[af(z)+bg(z)]'\Big) \\
&= a\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}f(z)\Big) + b\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}g(z)\Big) \\
&\quad + \frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\left[a\left(zf'(z)\right)+b\left(zg'(z)\right)\right] \\
&= a\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}f(z)\Big) + b\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}g(z)\Big) \\
&\quad + a\frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\left(zf'(z)\right) + b\frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\left(zg'(z)\right) \\
&= a\left[\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}f(z)\Big) + \frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\left(zf'(z)\right)\right] \\
&\quad + b\left[\Big(\frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}g(z)\Big) + \frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}\left(zg'(z)\right)\right] \\
&= a\mathcal{D}^{\alpha}f(z) + b\mathcal{D}^{\alpha}g(z).
\end{aligned}$$

Next, consider the case $\alpha \in [1,\infty)$. Since $[[\alpha]]$ is the integer part of α, it follows from the Definition 2.1 that

$$\mathcal{D}^{[[\alpha]]}\left[af(z)+bg(z)\right] = a\mathcal{D}^{[[\alpha]]}f(z) + b\mathcal{D}^{[[\alpha]]}g(z).$$

Once again, according to Definition 2.1, we obtain

$$\begin{aligned}
\mathcal{D}^{\alpha}[af(z)+bg(z)] &= \mathcal{D}^{\alpha-[[\alpha]]}\left\{\mathcal{D}^{[[\alpha]]}[af(z)+bg(z)]\right\} \\
&= \mathcal{D}^{\alpha-[[\alpha]]}\left\{a\mathcal{D}^{[[\alpha]]}f(z)+b\mathcal{D}^{[[\alpha]]}g(z)\right\} \\
&= a\mathcal{D}^{\alpha-[[\alpha]]}\left\{\mathcal{D}^{[[\alpha]]}f(z)\right\} + b\mathcal{D}^{\alpha-[[\alpha]]}\left\{\mathcal{D}^{[[\alpha]]}g(z)\right\} \\
&= a\mathcal{D}^{\alpha}f(z) + b\mathcal{D}^{\alpha}g(z).
\end{aligned}$$

(ii) For $\alpha \in [0,\infty)$ set $\nu = \alpha - [[\alpha]] \in [0,1)$. First we will show that

$$\mathcal{D}^{\nu}[f(z)g(z)] = f(z)\mathcal{D}^{\nu}g(z) + g(z)\mathcal{D}^{\nu}f(z) - \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f(z)g(z).$$

The product rule for the differentiable functions f and g in conjunction with the definition of the operator $\mathcal{D}^{\nu}$ yield

$$\mathcal{D}^{\nu}[f(z)g(z)] = \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,[f(z)g(z)] + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,(z[f(z)g(z)]')$$

$$= f(z)\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}g(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(zf(z)g'(z)+zg(z)f'(z)\right)$$

$$= f(z)\left[\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}g(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,(zg'(z))\right]$$

$$+ g(z)\left[\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\,(zf'(z))\right]$$

$$- \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f(z)g(z)$$

$$= f(z)\mathcal{D}^{\nu}g(z) + g(z)\mathcal{D}^{\nu}f(z) - \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f(z)g(z).$$

Since $[[\alpha]]$ is the integer part of α,, from Definition 2.1, where $\mathcal{D}^{[[\alpha]]}f(z) = \mathcal{D}\left(\mathcal{D}^{[[\alpha]]-1}f(z)\right)$, it follows that

$$\mathcal{D}^{[[\alpha]]}\,[f(z)g(z)] = \sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}f^{([[\alpha]]-k)}(z)g^{(k)}(z).$$

Now, from what we proved for $\mathcal{D}^{\nu}[f(z)g(z)]$ and $\mathcal{D}^{[[\alpha]]}[f(z)g(z)]$, we obtain

$$\begin{aligned}
\mathcal{D}^{\alpha}[f(z)g(z)] &= \mathcal{D}^{\alpha-[[\alpha]]}\left\{\mathcal{D}^{[[\alpha]]}[f(z)g(z)]\right\}\\
&= \mathcal{D}^{\nu}\left\{\sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}f^{([[\alpha]]-k)}(z)g^{(k)}(z)\right\}\\
&= \sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\mathcal{D}^{\nu}\left\{f^{([[\alpha]]-k)}(z)g^{(k)}(z)\right\}\\
&= \sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(f^{([[\alpha]]-k)}(z)\mathcal{D}^{\nu}g^{(k)}(z) + g^{(k)}(z)\mathcal{D}^{\nu}f^{([[\alpha]]-k)}(z)\\
&- \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f^{([[\alpha]]-k)}(z)g^{(k)}(z)\Big)\\
&= \sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(f^{([[\alpha]]-k)}(z)\mathcal{D}^{\nu}g^{(k)}(z)\Big) + \sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(g^{(k)}(z)\mathcal{D}^{\nu}f^{([[\alpha]]-k)}(z)\Big)\\
&- \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\sum_{k=0}^{[[\alpha]]}\binom{[[\alpha]]}{k}\Big(f^{([[\alpha]]-k)}(z)g^{(k)}(z)\Big).
\end{aligned}$$

(iii) First we will show that

$$\mathcal{D}^{\nu}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\mathcal{D}^{\nu}f(z) - f(z)\mathcal{D}^{\nu}g(z)}{(g(z))^2} + \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\frac{f(z)}{g(z)} \tag{2.2}$$

where $\alpha \in [0,\infty)$ and $\nu = \alpha - [[\alpha]] \in [0,1)$. Now by the quotient rule for the differentiable functions f

and g and the definition for the operator $\mathcal{D}^\nu$ it follows that

$$\mathcal{D}^\nu\left(\frac{f(z)}{g(z)}\right) = \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{f(z)}{g(z)}\right) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left[z\left(\frac{f(z)}{g(z)}\right)'\right]$$

$$= \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{f(z)}{g(z)}\right)$$

$$+ \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(z\frac{g(z)f'(z)-f(z)g'(z)}{(g(z))^2}\right)$$

$$= \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{f(z)}{g(z)}\right)$$

$$+ \frac{g(z)\left(z\frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f'(z)\right) - f(z)\left(z\frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}g'(z)\right)}{(g(z))^2}$$

$$+ \frac{\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(g(z)f(z)\right) - \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(f(z)g(z)\right)}{(g(z))^2}$$

$$= \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{f(z)}{g(z)}\right)$$

$$+ \frac{g(z)\left(\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}f(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}(zf'(z))\right)}{(g(z))^2}$$

$$- \frac{f(z)\left(\frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}g(z) + \frac{\kappa_0(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}(zg'(z))\right)}{(g(z))^2}$$

$$= \frac{g(z)\mathcal{D}^\nu f(z)}{(g(z))^2} - \frac{f(z)\mathcal{D}^\nu g(z)}{(g(z))^2} + \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{f(z)}{g(z)}\right).$$

Next, from the Definition 2.1, where $\mathcal{D}^{[[\alpha]]}f(z) = \mathcal{D}\left(\mathcal{D}^{[[\alpha]]-1}f(z)\right)$, it follows that

$$\begin{aligned}
\mathcal{D}^{\alpha}\left(\frac{f(z)}{g(z)}\right) &= \mathcal{D}^{\alpha-[[\alpha]]}\left(\mathcal{D}^{[[\alpha]]}\left[\frac{f(z)}{g(z)}\right]\right) = \mathcal{D}^{\nu}\left(\mathcal{D}^{[[\alpha]]}\left[\frac{f(z)}{g(z)}\right]\right) \\
&= \mathcal{D}^{\nu}\left(\frac{\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\left[f(z)g^k(z)\right]^{([[\alpha]])}}{g^k(z)}}{g(z)}\right) \\
&= \frac{g(z)\mathcal{D}^{\nu}\left(\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\left[f(z)g^k(z)\right]^{([[\alpha]])}}{g^k(z)}\right)}{(g(z))^2} \\
&- \frac{\left(\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\left[f(z)g^k(z)\right]^{([[\alpha]])}}{g^k(z)}\right)\mathcal{D}^{\nu}g(z)}{(g(z))^2} \\
&+ \frac{\kappa_1(\nu,z)}{\kappa_1(\nu,z)+\kappa_0(\nu,z)}\left(\frac{\sum_{k=0}^{[[\alpha]]}(-1)^k\binom{[[\alpha]]+1}{k+1}\frac{\left[f(z)g^k(z)\right]^{([[\alpha]])}}{g^k(z)}}{g(z)}\right).
\end{aligned}$$

(iv) For $\alpha\in[0,1)$ and for the constant function K we get

$$\mathcal{D}^{\alpha}K = \frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}K + \frac{\kappa_0(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}[z(K)'] = \frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)}K.$$

For $\alpha\in[1,\infty)$, Definition 2.1 yields

$$\mathcal{D}^{\alpha} = \mathcal{D}^{\alpha-[[\alpha]]}\left(\mathcal{D}^{[[\alpha]]}K\right) = \mathcal{D}^{\alpha-[[\alpha]]}(0) = 0.$$

□

3. Geometric properties of CCD

We shall need the following basic definitions throughout this paper. A function $g \in \Lambda^*$ is said to be univalent in $\mathbb{U}$ if it never takes the same value twice; that is, if $z_1\neq z_2$ in $\mathbb{U}$ then $g(z_1)\neq g(z_2)$ or equivalently, if $g(z_1) = g(z_2)$ then $z_1 = z_2$. Without loss of generality, we can use Λ in place of Λ^* for our univalent functions. This is because if $g(z) = b_0 + b_1 z + b_2 z^2 + ... \in \Lambda^*$ and is univalent in $\mathbb{U}$ then the transformation function $f(z) = \frac{g(z)-b_0}{b_1} \in \Lambda$ is also univalent in $\mathbb{U}$. Note that the univalency of the function g guarantees that $b_1 \neq 0$ (e.g., see Goodman [4], Chapter 2). We let $\mathcal{S}$ denote the class of such functions $f \in \Lambda$ that are univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be starlike with respect to origin in $\mathbb{U}$ if the linear segment joining the origin to every other point of $f(z : |z| = r < 1)$ lies entirely in $f(z : |z| = r < 1)$. In more picturesque language, the requirement is that every point of $f(z : |z| = r < 1)$ be visible from the origin. A function $f \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if the linear segment joining any two points of $f(z : |z| = r < 1)$ lies entirely in $f(z : |z| = r < 1)$. In other words, a function $f \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if it is starlike with respect to each and every of its points. We denote the class of function $f \in \mathcal{S}$ that are starlike with respect to origin by $\mathcal{S}^*$ and convex in $\mathbb{U}$ by C. Closely related to the classes $\mathcal{S}^*$ and C is the class $\mathcal{P}$ of all functions ϕ analytic in $\mathbb{U}$ and having positive real part in $\mathbb{U}$ with $\phi(0) = 1$. In fact $f \in \mathcal{S}^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$ and $f \in C$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$. In general, for $\epsilon \in [0, 1)$ we let $\mathcal{P}(\epsilon)$ consist of functions ϕ analytic in $\mathbb{U}$ with $\phi(0) = 1$ so that $\Re(\phi(z)) > \epsilon$ for all $z \in \mathbb{U}$. Note that $\mathcal{P}(\epsilon_2) \subset \mathcal{P}(\epsilon_1) \subset \mathcal{P}(0) \equiv \mathcal{P}$ for $0 < \epsilon_1 < \epsilon_2$ (e.g., see Duren [3] or Goodman [4]). For functions f and g in Λ we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ so that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$ (see [3], [4] or [7]). Evidently $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Finally, for $\mathcal{P}(A, B)$ being the class of functions $p(z) = (1 + A\omega(z))/(1 + B\omega(z)) \prec (1 + Az)/(1 + Bz)$ where ω is the Schwarz function and $-1 \le B < A \le 1$ we have $\mathcal{P}(A, B) \subset \mathcal{P}(\frac{1-A}{1-B})$ (e.g., see Janowski [5]).

In this section we explore the conditions on the complex conformable derivative $\mathcal{D}^\alpha$ yielding interesting geometric properties in relation to functions with positive real part.

Theorem 3.1. *For fixed $\epsilon \in (0, 1)$ and $\alpha \in [0, \infty)$ set $\kappa_0(\alpha - [[\alpha]], z) = \frac{\epsilon}{1-\epsilon}\, \kappa_1(\alpha - [[\alpha]], z)$. Then*

$$\frac{\mathcal{D}^{\alpha+2} f(z)}{\mathcal{D}^{\alpha+1} f(z)} \in \mathcal{P} \implies \frac{\mathcal{D}^{\alpha+1} f(z)}{\mathcal{D}^{\alpha} f(z)} \in \mathcal{P}.$$

Proof. For $\kappa_0(\alpha - [[\alpha]], z) = \frac{\epsilon}{1-\epsilon}\, \kappa_1(\alpha - [[\alpha]], z)$, by Definition 2.1 we have

$$\mathcal{D}^\alpha f(z) = (1-\epsilon)\left(\mathcal{D}^{[[\alpha]]} f(z)\right) + \epsilon z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)' \;,$$

$$\mathcal{D}^{\alpha+1} f(z) = z\left(\mathcal{D}^{[[\alpha]]} f(z)\right)' + \epsilon z^2 \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'' \;,$$

and

$$\mathcal{D}^{\alpha+2} f(z) = z\left(\mathcal{D}^{[[\alpha]]} f(z)\right)' + (1+2\epsilon) z^2 \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'' + \epsilon z^3 \left(\mathcal{D}^{[[\alpha]]} f(z)\right)''' \;.$$

Observe that $\Re\left(\frac{\mathcal{D}^{\alpha+2} f(z)}{\mathcal{D}^{\alpha+1} f(z)}\right) > 0$ if and only if

$$\Re\left\{1 + \frac{(1+\epsilon) z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'' + \epsilon z^2 \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'''}{(\mathcal{D}^{[[\alpha]]} f(z))' + \epsilon z\, (\mathcal{D}^{[[\alpha]]} f(z))''}\right\} > 0.$$

Or equivalently, if and only if

$$\Re\left\{1 + \frac{z\left[(1-\epsilon)\left(\mathcal{D}^{[[\alpha]]} f(z)\right) + \epsilon z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'\right]''}{\left[(1-\epsilon)\,(\mathcal{D}^{[[\alpha]]} f(z)) + \epsilon z\, (\mathcal{D}^{[[\alpha]]} f(z))'\right]'}\right\} > 0. \tag{3.1}$$

The inequality 3.1 in conjunction with the definition of convex functions yield that $(1-\epsilon)\left(\mathcal{D}^{[[\alpha]]} f(z)\right) + \epsilon z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'$ is convex. Since every convex function is also starlike, it follows that

$$\Re\left\{\frac{z\left[(1-\epsilon)\left(\mathcal{D}^{[[\alpha]]} f(z)\right) + \epsilon z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'\right]'}{(1-\epsilon)\,(\mathcal{D}^{[[\alpha]]} f(z)) + \epsilon z\, (\mathcal{D}^{[[\alpha]]} f(z))'}\right\} > 0. \tag{3.2}$$

The inequality 3.2 holds if and only if $\Re\left(\frac{\mathcal{D}^{\alpha+1} f(z)}{\mathcal{D}^{\alpha} f(z)}\right) > 0$ and this completes the proof. □

Our second theorem determines the best possible necessary condition to be imposed on the operator $\mathcal{D}^{[[\alpha]]}$ for $[\mathcal{D}^{\alpha+1}f(z)]/[z(\mathcal{D}^{[[\alpha]]}f(z))']$ to be of positive real part.

Theorem 3.2. *For fixed* $\epsilon \in (0,1)$ *and* $\alpha \in [0,\infty)$ *set* $\kappa_1(\alpha-[[\alpha]],z) = \frac{\epsilon}{1-\epsilon}\,\kappa_0(\alpha-[[\alpha]],z)$*. If* $\mathcal{D}^{[[\alpha]]}f(z) \in C$ *then*

$$\frac{\mathcal{D}^{\alpha+1}f(z)}{z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'} \in \mathcal{P}(\epsilon).$$

Proof. Applying the differential operator rule (2.1) to $\mathcal{D}^{\alpha+1}f(z) = \mathcal{D}\left(\mathcal{D}^{\alpha}f(z)\right)$ yields

$$\begin{aligned}
\mathcal{D}^{\alpha+1}f(z) &= \mathcal{D}^{\alpha-[[\alpha]]}\left(\mathcal{D}^{[[\alpha]]+1}f(z)\right)\\
&= \mathcal{D}^{\alpha-[[\alpha]]}\left\{\mathcal{D}\left[\mathcal{D}^{[[\alpha]]}f(z)\right]\right\} = \mathcal{D}^{\alpha-[[\alpha]]}\left\{z\left[\mathcal{D}^{[[\alpha]]}f(z)\right]'\right\}\\
&= \frac{\kappa_1(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z\left[\mathcal{D}^{[[\alpha]]}f(z)\right]'\right\}\\
&+ \frac{\kappa_0(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z\left[\left(\mathcal{D}^{[[\alpha]]}f(z)\right)' + z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)''\right]\right\}\\
&= \frac{\kappa_1(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z\left[\mathcal{D}^{[[\alpha]]}f(z)\right]'\right\}\\
&+ \frac{\kappa_0(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z\left[\mathcal{D}^{[[\alpha]]}f(z)\right]'\right\}\\
&+ \frac{\kappa_0(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z^2\left[\mathcal{D}^{[[\alpha]]}f(z)\right]''\right\}\\
&= z\left[\mathcal{D}^{[[\alpha]]}f(z)\right]' + \frac{\kappa_0(\alpha-[[\alpha]],z)}{\kappa_1(\alpha-[[\alpha]],z)+\kappa_0(\alpha-[[\alpha]],z)}\left\{z^2\left[\mathcal{D}^{[[\alpha]]}f(z)\right]''\right\}.
\end{aligned} \tag{3.3}$$

Dividing both sides of the Eq. 3.3 by $z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'$ and using the relation

$$\kappa_1(\alpha-[[\alpha]],z) = \frac{\epsilon}{1-\epsilon}\,\kappa_0(\alpha-[[\alpha]],z)$$

we obtain

$$\frac{\mathcal{D}^{\alpha+1}f(z)}{z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'} = 1 + (1-\epsilon)\frac{z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)''}{\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'}.$$

On the other hand, from the convexity of $\mathcal{D}^{[[\alpha]]}f(z)$ it follows that

$$\Re\left\{1 + \frac{z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)''}{\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'}\right\} > 0.$$

Therefore

$$\Re\left\{\frac{\mathcal{D}^{\alpha+1}f(z)}{z\left(\mathcal{D}^{[[\alpha]]}f(z)\right)'}\right\} > \epsilon.$$

□

3.1. The Briot-Bouquet differential equation

A class of complex differential equations is a collection of differential equations whose results are expressions of a complex variable. Assembling integrals contains special paths to income, which yields singularities and branch points of the equation must study. Existence and uniqueness theorems include the utility of majorants and minorants (or subordination and superordination concepts) (see [7]). Study of rational first ODEs in the complex domain implies the discovery of new transcendental special functions which are now known as Briot-Bouquet differential transcendent

$$\lambda f(z)+(1-\lambda)\frac{z(f(z))'}{f(z)}=h(z),\quad h(0)=f(0),\ \lambda\in[0,1].$$

Many applications of these equations in the geometric function theory have newly performed in [7]. Our aim is to generalize this type of equation by using the conformable operator and establish its solutions by using the subordination relations. By using the conformable differential operator (3.3), we have the conformable Briot-Bouquet differential equation

$$\lambda f(z)+(1-\lambda)\Big(\frac{z(\mathcal{D}^{\alpha}f(z))'}{\mathcal{D}^{\alpha}f(z)}\Big)=h(z),\quad h(0)=f(0),\ z\in\mathbb{U}. \tag{3.4}$$

The subordination conditions and distortion bounds for a class of complex conformable derivative are given in the next theorem. A trivial solution of (3.4) is given when $\lambda=1$. Therefore, our study is delivered for the case, $f\in\Lambda$ and $\lambda=0$.
Next result shows the behavior of the solution of (3.4).

Theorem 3.3. *For $f\in\Lambda$, $\alpha\in[0,\infty)$ and h is univalent convex in $\mathbb{U}$ if*

$$\Big(\frac{z(\mathcal{D}^{\alpha}f(z))'}{\mathcal{D}^{\alpha}f(z)}\Big)\prec h(z),\quad z\in\mathbb{U} \tag{3.5}$$

then

$$\mathcal{D}^{\alpha}f(z)\prec z\exp\Big(\int_0^z\frac{h(\omega(\xi))-1}{\xi}d\xi\Big),$$

where ω is a Schwarz function in $\mathbb{U}$. Furthermore,

$$|z|\exp\left(\int_0^1\frac{h(\omega(-\eta))-1}{\eta}d\eta\right)\le\left|\mathcal{D}^{\alpha}f(z)\right|\le|z|\exp\left(\int_0^1\frac{h(\omega(\eta))-1}{\eta}d\eta\right).$$

Proof. The subordination condition 3.5 means that there exists a Schwarz function ω so that

$$\Big(\frac{z(\mathcal{D}^{\alpha}f(z))'}{\mathcal{D}^{\alpha}f(z)}\Big)=h(\omega(z)),\quad z\in\mathbb{U}.$$

This implies that

$$\Big(\frac{(\mathcal{D}^{\alpha}f(z))'}{\mathcal{D}^{\alpha}f(z)}\Big)-\frac{1}{z}=\frac{h(\omega(z))-1}{z}.$$

Integrating both sides of the above equations yields

$$\log\frac{\mathcal{D}^{\alpha}f(z)}{z}=\int_0^z\frac{h(\omega(\xi))-1}{\xi}d\xi. \tag{3.6}$$

Again, by subordination, we get

$$\mathcal{D}^{\alpha} f(z) \prec z \exp\Big(\int_0^z \frac{h(\omega(\xi)) - 1}{\xi} d\xi \Big).$$

In addition, we note that the function $h(z)$ maps the disk $0 < |z| < \eta \le 1$ onto a region which is convex and symmetric with respect to the real axis, that is

$$h(-\eta|z|) \le \Re(h(\omega(\eta z))) \le h(\eta|z|), \quad \eta \in (0, 1], |z| \neq \eta,$$

which yields the following inequalities:

$$h(-\eta) \le h(-\eta|z|), \quad h(\eta|z|) \le h(\eta)$$

and

$$\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta \le \Re\Big(\int_0^1 \frac{h(\omega(\eta)) - 1}{\eta} d\eta \Big) \le \int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta.$$

By using the above relations and Eq. (3.6), we conclude that

$$\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta \le \log \Big| \frac{\mathcal{D}^{\alpha} f(z)}{z} \Big| \le \int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta.$$

This equivalence to the inequality

$$\exp\Big(\int_0^1 \frac{h(\omega(-\eta|z|)) - 1}{\eta} d\eta \Big) \le \Big| \frac{\mathcal{D}^{\alpha} f(z)}{z} \Big| \le \exp\Big(\int_0^1 \frac{h(\omega(\eta|z|)) - 1}{\eta} d\eta \Big).$$

This completes the proof. □

4. Examples

Our first example uses the quotient rule given by Lemma 2.2.

Example 4.1. Let $\alpha \in [0, 1)$. Now for $\mathcal{D}^{\alpha}\left(\frac{z}{1-z}\right)$ let $f(z) = z$ and $g(z) = 1 - z$ and apply the quotient rule (2.2) to get

$$\mathcal{D}^{\alpha}\left(\frac{z}{1-z}\right) = \frac{z\left(1 - \frac{\kappa_1(\alpha,z)}{\kappa_1(\alpha,z)+\kappa_0(\alpha,z)} z\right)}{(1-z)^2} = \frac{z\,(1 - \epsilon z)}{(1-z)^2}. \tag{4.1}$$

If $\epsilon \to 1$ then $\mathcal{D}^{\alpha}\left(\frac{z}{1-z}\right)$ is convex and $\epsilon \to 0$ then $\mathcal{D}^{\alpha}\left(\frac{z}{1-z}\right)$ is starlike.

In the following example we demonstrate Definition 2.1 for the given analytic functions $\kappa_1(\alpha - [[\alpha]], z)$ and $\kappa_0(\alpha - [[\alpha]], z)$ operating on the convex function $f(z) = log\Big(\frac{1}{1-z}\Big)$.

Example 4.2. For $\alpha \in [0, \infty)$ and $\nu = \alpha - [[\alpha]]$ let $\kappa_1(\nu, z) = (1 - \nu)\left(\frac{1+z}{1-z}\right)^{\nu}$ and $\kappa_0(\nu, z) = \nu\left(\frac{1+z}{1-z}\right)^{1-\nu}$. Then, according to Definition 2.1, the complex conformable derivative $\mathcal{D}^{\alpha} f$ is given by

$$\mathcal{D}^{\alpha} f(z) = \frac{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu}}{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu} + \nu\left(\frac{1+z}{1-z}\right)^{1-\nu}} \left(\mathcal{D}^{[[\alpha]]} f(z)\right) + \frac{\nu\left(\frac{1+z}{1-z}\right)^{1-\nu}}{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu} + \nu\left(\frac{1+z}{1-z}\right)^{1-\nu}} z \left(\mathcal{D}^{[[\alpha]]} f(z)\right)'.$$

If we choose $\alpha \in [1,2)$ then for the function $f(z) = log\left(\frac{1}{1-z}\right)$ that is convex in $\mathbb{U}$ the complex conformable derivative $\mathcal{D}^{\alpha} f$ is be given by

$$\mathcal{D}^{\alpha}\left(f(z)\right) = \frac{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu}}{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu} + \nu\left(\frac{1+z}{1-z}\right)^{1-\nu}}\left(\frac{1}{1-z}\right) + \frac{\nu\left(\frac{1+z}{1-z}\right)^{1-\nu}}{(1-\nu)\left(\frac{1+z}{1-z}\right)^{\nu} + \nu\left(\frac{1+z}{1-z}\right)^{1-\nu}}\left(\frac{z}{(1-z)^2}\right).$$

If $\nu \to 0$ then $\mathcal{D}^{\alpha}\left(log\left(\frac{1}{1-z}\right)\right)$ is convex and if $\nu \to 1$ then $\mathcal{D}^{\alpha}\left(log\left(\frac{1}{1-z}\right)\right)$ is starlike. For $\nu \in (0,1)$ let us, for example, choose $\alpha = 1.3$ then the complex conformable derivative $\mathcal{D}^{1.3}\left(log\left(\frac{1}{1-z}\right)\right)$ becomes

$$\begin{aligned}\mathcal{D}^{1.3}\left(log\left(\frac{1}{1-z}\right)\right) &= \frac{0.7\left(\frac{1+z}{1-z}\right)^{0.3}}{0.7\left(\frac{1+z}{1-z}\right)^{0.3} + 0.3\left(\frac{1+z}{1-z}\right)^{0.7}}\left(\frac{1}{1-z}\right) + \frac{0.3\left(\frac{1+z}{1-z}\right)^{0.7}}{0.7\left(\frac{1+z}{1-z}\right)^{0.3} + (0.3\left(\frac{1+z}{1-z}\right)^{0.7}}\left(\frac{z}{(1-z)^2}\right)\\ &= \frac{7}{10} + \frac{104}{125}z + \frac{7957}{6250}z^2 + \frac{134114}{78125}z^3 + \dots .\end{aligned}$$

In the next example we show that the required convexity condition for $\mathcal{D}^{[[\alpha]]} f(z)$ is the best possible condition for Theorem 3.2 to hold and this convexity condition cannot be replaced by the larger class of starlike functions.

Example 4.3. (i) For $\alpha \in [0,\infty)$ and $\epsilon \in [0,1)$ we let

$$\kappa_1(\alpha - [[\alpha]], z) = \frac{\epsilon}{1-\epsilon}\,\kappa_0(\alpha - [[\alpha]], z).$$

Then the convex function $\mathcal{D}^{[[\alpha]]} f(z) = \frac{z}{1-z}$ yields

$$\frac{\mathcal{D}^{\alpha+1} f(z)}{z\left(\mathcal{D}^{[[\alpha]]} f(z)\right)'} = \frac{1+(1-2\epsilon)z}{1-z} \in \mathcal{P}(1-2\epsilon, -1) \subset \mathcal{P}(\epsilon),$$

(ii) For $\alpha \in [0,\infty)$ and $\epsilon \in [0,1)$ we let $\kappa_1(\alpha - [[\alpha]], z) = \frac{\epsilon}{1-\epsilon}\,\kappa_0(\alpha - [[\alpha]], z)$.

Then for $n = 2, 3, \dots$, the convex binomial $\mathcal{D}^{[[\alpha]]} f(z) = z + \frac{1}{n^2} z^n$ yields

$$\frac{\mathcal{D}^{\alpha+1} f(z)}{z\left(\mathcal{D}^{[[\alpha]]} f(z)\right)'} = \frac{1 + \frac{n(1-\epsilon)+\epsilon}{n} z^{n-1}}{1 + \frac{1}{n} z^{n-1}} \in \mathcal{P}\left(\frac{n(1-\epsilon)+\epsilon}{n}, \frac{1}{n}\right) \subset \mathcal{P}(\epsilon),$$

But for the starlike binomial

$$\mathcal{D}^{[[\alpha]]} f(z) = z + \frac{1}{3} z^3,$$

we get

$$\frac{\mathcal{D}^{\alpha+1} f(z)}{z\left(\mathcal{D}^{[[\alpha]]} f(z)\right)'} = \frac{1+(3-2\epsilon)z^2}{1+z^2} \notin \mathcal{P}.$$

5. Conclusion

In conclusion, it is worth it to note that the nature of the complex conformable derivatives and their applications is a yet to be fully explored territory and it is expected that the present paper triggers the future research on this topic.

Acknowledgments

The author wishes to express his profound gratitude to the anonymous referee for his/her careful reading of the manuscript and the very useful comments that have been implemented in the final version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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