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Research article

Existence of positive solution to the boundary value problems for coupled system of nonlinear fractional differential equations

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Abstract: In this paper, we investigate the existence criteria of at least one positive solution to the three-point boundary value problems with coupled system of Riemann-Liouville type nonlinear fractional order differential equations. The analysis of this study is based on the well-known Schauder's fixed point theorem. Some new existence and multiplicity results for coupled system of Riemann-Liouville type nonlinear fractional order differential equation with three-point boundary value conditions are obtained.

Keywords: coupled system of Riemann-Liouville type fractional differential equations; three-point boundary value condition; positive solution; Schauder's fixed point theorem
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1. Introduction

Boundary value problems (for short BVPs) for nonlinear fractional order differential equation (for short NLFDE) have been addressed by several researchers during last few decades. The necessity of fractional order differential equations (for short FDEs) lies in the fact that fractional order models are more accurate than integer order models, that is, there are more degree of freedom in the fractional order models. Furthermore, fractional order derivatives provide an excellent mechanism for the description of memory and hereditary properties of various materials and processes. In applied sense, FDEs arise in various engineering and scientific disciplines for mathematical modeling in the fields of physics, chemistry, biology, mechanics, control theory of dynamical system, electrical network, statistics and economics, see for instance [1–6] and their references.

Consequently, day by day the topics in FDEs are taking an important part in various applied research. Some recent development of FDEs can be seen in [7–16] and in their references.

Now a days, many researchers devoted themselves to determine the solvability of system of nonlinear fractional order differential equations (for short SNLFDEs) with different boundary conditions, specifically to the study of existence of positive solutions to BVPs for SNLFDEs, see for instance [10,12–14,17–29] and their references.

Inspired by the above-mentioned works on existence of positive solutions to BVPs for SNLFDEs, in this paper, we establish the existence criteria of at least one positive solution to the following boundary value problem (for short BVP) for coupled system of Riemann-Liouville type nonlinear fractional order differential equations (for short NLFDEs) applying Schauder's fixed point theorem [30]:

$$\begin{cases} -D_{0^+}^{\alpha_1} u_1(t) = \lambda_1 a_1(t) f_1\big(t, u_1(t), u_2(t)\big) + g_1(t), & t \in [0,1], \alpha_1 \in (3,4], \\ -D_{0^+}^{\alpha_2} u_2(t) = \lambda_2 a_2(t) f_2\big(t, u_1(t), u_2(t)\big) + g_2(t), & t \in [0,1], \alpha_2 \in (3,4], \\ D_{0^+}^{\beta_1} u_1(0) = D_{0^+}^{\gamma_1} u_1(0) = D_{0^+}^{\delta_1} u_1(0) = 0, & u_1(1) = \eta_1 u_1(\xi_1), \\ D_{0^+}^{\beta_2} u_2(0) = D_{0^+}^{\gamma_2} u_2(0) = D_{0^+}^{\delta_2} u_2(0) = 0, & u_2(1) = \eta_2 u_2(\xi_2), \end{cases} \tag{1}$$

where, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i}$, $D_{0^+}^{\gamma_i}$ and $D_{0^+}^{\delta_i}$ are standard Riemann-Liouville fractional differential operators of order $\alpha_i \in (3,4]$, $\beta_i \in (0,1)$, $\gamma_i \in (1,2)$, $\delta_i \in (2,3)$, $(i = 1,2)$, respectively, η_i, $\xi_i \in (0,1)$ with $0 < \eta_i \xi_i^{\alpha_i - 1} < 1$, $(i = 1,2)$ and f_i, g_i, a_i and $\lambda_i, (i = 1,2)$ satisfy the following hypothesis:

(H_1) (i) $f_i \in C\big([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty)\big)$, $(i = 1,2)$

(ii) $a_i, g_i \in C\big([0,1], [0,+\infty)\big), (i = 1,2)$,

(iii) $\lambda_i, (i = 1,2)$ are positive parameters.

(H_2) $f_i\big(t, u_1(t), u_2(t)\big) > 0$, for $u_i > 0, t \in [0,1]$, $(i = 1,2)$.

To the best of our knowledge there is no any works considering the BVP for coupled system of Riemann-Liouville type NLFDEs given by (1) applying Schauder's fixed point theorem.

The rest of this work is furnished as follows. In section 2, we will provide some basic ideas of fractional calculus, certain lemmas and state Schauder's fixed point theorem. Section 3 is used to state and prove our main results, which provide some techniques to check the existence of at least one positive solutions of coupled system of Riemann-Liouville-type NLFDEs with three-point boundary conditions given by (1). In section 3 we also give some illustrative examples. Finally, we conclude this paper.

2. Preliminary notes

In this section, we introduce some necessary definitions and preliminary facts which will be used throughout this paper.

Definition 1 ([3–5])**.** Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function and $\alpha > 0$. Then the Riemann-Liouville fractional integral of order α is defined as follows:

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0,$$

where $\Gamma(\alpha)$ is the Euler Gamma function of α and provided that the integral exists.

Definition 2 ([3–5])**.** Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function and $\alpha>0$. Then the Riemann-Liouville fractional derivative of order α is defined as follows:

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of real number α and provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 1 ([10]). Suppose that $h(t)\in C[0,1]$ and (H_1) holds, then the unique solution of the BVP

$$\begin{cases} -D_{0^+}^{\alpha_1} u_1(t) = h(t), \quad t\in[0,\,1],\, 3<\alpha_1\le 4 \\ D_{0^+}^{\beta_1} u_1(0) = D_{0^+}^{\gamma_1} u_1(0) = D_{0^+}^{\delta_1} u_1(0) = 0, \quad u_1(1) = \eta_1 u_1(\xi_1), \end{cases} \tag{2}$$

is provided by

$$u_1(t) = \int_0^1 G_1(t,s) h(s) ds,$$

where the Green's function $G_1(t,s)$ is defined by

$$G_1(t,\,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} \dfrac{t^{\alpha_1-1}}{1-\eta_1\xi_1^{\alpha_1-1}}\left[(1-s)^{\alpha_1-1} - \eta_1(\xi_1-s)^{\alpha_1-1}\right] - (t-s)^{\alpha_1-1}; & 0\le s\le t\le \xi_1\le 1, \\ \dfrac{t^{\alpha_1-1}}{1-\eta_1\xi_1^{\alpha_1-1}}\left[(1-s)^{\alpha_1-1} - \eta_1(\xi_1-s)^{\alpha_1-1}\right]; & 0\le t\le s\le \xi_1\le 1, \\ \dfrac{t^{\alpha_1-1}}{1-\eta_1\xi_1^{\alpha_1-1}}(1-s)^{\alpha_1-1} - (t-s)^{\alpha_1-1}; & 0\le \xi_1\le s\le t\le 1, \\ \dfrac{t^{\alpha_1-1}}{1-\eta_1\xi_1^{\alpha_1-1}}(1-s)^{\alpha_1-1}; & 0\le \xi_1\le t\le s\le 1. \end{cases} \tag{3}$$

Remark 1. Similar as Lemma 1 the unique solution of the BVP

$$\begin{cases} -D_{0^+}^{\alpha_2} u_2(t) = h(t), \quad t\in[0,\,1],\, 3<\alpha_2\le 4 \\ D_{0^+}^{\beta_2} u_1(0) = D_{0^+}^{\gamma_2} u_1(0) = D_{0^+}^{\delta_2} u_1(0) = 0, \quad u_2(1) = \eta_2 u_2(\xi_2), \end{cases} \tag{4}$$

is provided by

$$u_2(t) = \int_0^1 G_2(t,s) h(s) ds,$$

where the Green's function $G_2(t,s)$ is defined by

$$G_2(t,\, s) = \frac{1}{\Gamma(\alpha_2)} \begin{cases} \dfrac{t^{\alpha_2-1}}{1-\eta_2\xi_2^{\alpha_2-1}}\left[(1-s)^{\alpha_2-1} - \eta_2(\xi_2 - s)^{\alpha_2-1}\right] - (t-s)^{\alpha_2-1}; & 0 \le s \le t \le \xi_2 \le 1, \\ \dfrac{t^{\alpha_2-1}}{1-\eta_2\xi_2^{\alpha_2-1}}\left[(1-s)^{\alpha_2-1} - \eta_2(\xi_2 - s)^{\alpha_2-1}\right]; & 0 \le t \le s \le \xi_2 \le 1, \\ \dfrac{t^{\alpha_2-1}}{1-\eta_2\xi_2^{\alpha_2-1}}(1-s)^{\alpha_2-1} - (t-s)^{\alpha_2-1}; & 0 \le \xi_2 \le s \le t \le 1, \\ \dfrac{t^{\alpha_2-1}}{1-\eta_2\xi_2^{\alpha_2-1}}(1-s)^{\alpha_2-1}; & 0 \le \xi_2 \le t \le s \le 1. \end{cases} \tag{5}$$

Remark 2. In view of Lemma 1 and Remark 1, the couple system of BVPs defined by (1) is equivalent to the following couple system of integral equations:

$$\begin{cases} u_1(t) = \int_0^1 G_1(t,s)\left[\lambda_1 a_1(t)\, f_1\left(s, u_1(s), u_2(s)\right) + g_1(s)\right] ds, \\ u_2(t) = \int_0^1 G_2(t,s)\left[\lambda_2 a_2(t)\, f_2\left(s, u_1(s), u_2(s)\right) + g_2(s)\right] ds, \end{cases}$$

where the Green's functions $G_i(t,s),\ (i=1,2)$ are given by (3) and (5).

Lemma 2 ([10]). The Green's functions $G_i(t,\, s),\ (i=1,2)$ defined as in (3) and (5) satisfy the following properties:

(i) $G_i(t,\, s),\ (i=1,2)$ are continuous on the unit square $[0,1]\times[0,1]$,

i.e., $G_i(t,\, s) \in C([0,1]\times[0,1])$ and $G_i(t,s) \ge 0,\ \forall\, t,s \in [0,1]$;

(ii) $\max_{t\in[0,1]} G_i(t,s) = G_i(1,s),\ (i=1,2)$;

(iii) $\min_{t\in[\lambda,1-\lambda]} G_i(t,s) \ge \theta_i(s) \max_{t\in[0,1]} G_i(t,s) = \theta_i(s) G_i(1,s),\ \lambda \in (0,1),\ (i=1,2)$.

Lemma 3. If the Green's functions $G_i(t,\, s),\ (i=1,2)$ are given as in (3) and (5), then there exist constants $\kappa_i \in (0,1),\ (i=1,2)$ such that

$$\min_{t\in[1/2,1]} G_i(t,s) \ge \kappa_i \max_{t\in[0,1]} G_i(t,s) = \kappa_i G_i(1,s),\ (i=1,2).$$

Proof. Since $t \in [1/2,\, 1]$, then from (3) we obtain that

$$\min_{t\in[1/2,1]} G_1(t,\,s)=\begin{cases}\dfrac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}-\dfrac{(1/2-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}; & 0\le s\le 1/2\le \xi_1\le 1,\\ \dfrac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}; & 0\le 1/2\le s\le \xi_1\le 1,\\ \dfrac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-(1/2-s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}; & 0\le \xi_1\le s\le 1/2\le 1,\\ \dfrac{(1/2)^{\alpha_1-1}(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}; & 0\le \xi_1\le 1/2\le s\le 1.\end{cases}$$

If we take $0\le s\le t\le \xi_1\le 1$, then

$$\begin{aligned}G_1(1,s)&=\frac{1}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\left[(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}\right]-\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\\&=\frac{(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}-(1-s)^{\alpha_1-1}\left(1-\eta_1\xi_1^{\alpha_1-1}\right)}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\\&=\frac{\eta_1\xi_1^{\alpha_1-1}(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\le\frac{(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[1/2,1]} G_1(t,\,s)&=\frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}\right]-\left(1-\eta_1\xi_1^{\alpha_1-1}\right)(1/2-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\\&=\frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1(\xi_1-s)^{\alpha_1-1}-\left(1-\eta_1\xi_1^{\alpha_1-1}\right)(1-2s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\\&=\frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1\xi_1^{\alpha_1-1}\left(1-\dfrac{s}{\xi_1}\right)^{\alpha_1-1}-\left(1-\eta_1\xi_1^{\alpha_1-1}\right)(1-2s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\\&\ge\frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-\eta_1\xi_1^{\alpha_1-1}(1-2s)^{\alpha_1-1}-\left(1-\eta_1\xi_1^{\alpha_1-1}\right)(1-2s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}\\&=\frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-(1-2s)^{\alpha_1-1}\right]}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}.\end{aligned}$$

Let σ_1 be a positive number such that $\min_{t\in[1/2,1]} G_1(t,s)\ge \sigma_1 G_1(1,s)$. Then we have

$$\sigma_1 \le \frac{(1/2)^{\alpha_1-1}\left[(1-s)^{\alpha_1-1}-(1-2s)^{\alpha_1-1}\right]}{(1-s)^{\alpha_1-1}} = \frac{(1/2)^{\alpha_1-1}(1-s)^{\alpha_1-1}-(1/2-s)^{\alpha_1-1}}{(1-s)^{\alpha_1-1}}$$

$$= (1/2)^{\alpha_1-1} - \left(\frac{1/2-s}{1-s}\right)^{\alpha_1-1} \le (1/2)^{\alpha_1-1}.$$

This means that $\sigma_1 \in (0,1)$.

If we take $0 \le t \le s \le \xi_1 \le 1$, then

$$G_1(1,s) \le \frac{(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}$$

and

$$\min_{t\in[1/2,1]} G_1(t,\ s) \ge \frac{(1/2)^{\alpha_1-1}(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}.$$

Let σ_2 be a positive number such that $\min_{t\in[1/2,1]} G_1(t,s) \ge \sigma_2 G_1(1,s)$. Then we have

$$\sigma_2 \le t^{\alpha_1-1}\left(1-\eta_1\xi_1^{\alpha_1-1}\right).$$

This means that $\sigma_2 \in (0,1)$.

If we take $0 \le \xi_1 \le s \le t \le 1$, then

$$G_1(1,s) \le \frac{(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}$$

and

$$\min_{t\in[1/2,1]} G_1(t,\ s) \ge \frac{(1/2)^{\alpha_1-1}(1-s)^{\alpha_1-1}\eta_1\xi_1^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}.$$

Let σ_3 be a positive number such that $\min_{t\in[1/2,1]} G_1(t,s) \ge \sigma_3 G_1(1,s)$. Then we have

$$\sigma_3 \le t^{\alpha_1-1}\eta_1\xi_1^{\alpha_1-1}.$$

This means that $\sigma_3 \in (0,1)$.

If we take $0 \le \xi_1 \le t \le s \le 1$, then

$$G_1(1,s) \le \frac{(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}$$

and

$$\min_{t\in[1/2,1]} G_1(t,\ s) = \frac{(1/2)^{\alpha_1-1}(1-s)^{\alpha_1-1}}{\left(1-\eta_1\xi_1^{\alpha_1-1}\right)\Gamma(\alpha_1)}.$$

Let σ_4 be a positive number such that $\min_{t\in[1/2,1]} G_1(t,s) \geq \sigma_4 G_1(1,s)$. Then we have

$$\sigma_4 \leq t^{\alpha_1-1}.$$

This means that $\sigma_3 \in (0,1)$.

Now, if we set $\kappa_1 = \min\{\sigma_1,\ \sigma_2,\ \sigma_3, \sigma_4\}$, then we obtain that

$$\min_{t\in[1/2,1]} G_1(t,s) \geq \kappa_1 G_1(1,s) = \kappa_1 \max_{t\in[0,1]} G_1(t,s).$$

Similarly, for the Green's function $G_2(t,s)$, we can prove that

$$\min_{t\in[1/2,1]} G_2(t,s) \geq \kappa_2 G_2(1,s) = \kappa_2 \max_{t\in[0,1]} G_2(t,s).$$

This completes the proof. ■

Throughout this paper let $B = \{u(t) : u \in C([0,1]),\ t \in [0,1]\}$ be a Banach space with the usual supremum norm $\|\cdot\|$. Now, if we set $X = B \times B$, where X is equipped with the norm $\|(u_1,u_2)\| = \|u_1\| + \|u_2\|$ for $(u_1,u_2) \in X$, then it is clear that X is also a Banach space. Furthermore, we define the integral operators $A_1, A_2 : X \to B$ by

$$\left(A_1(u_1,u_2)\right)(t) = \int_0^1 G_1(t,s)\left[\lambda_1 a_1(s) f_1\left(s,u_1(s),u_2(s)\right) + g_1(s)\right]ds,$$

and

$$\left(A_2(u_1,u_2)\right)(t) = \int_0^1 G_2(t,s)\left[\lambda_2 a_2(s) f_2\left(s,u_1(s),u_2(s)\right) + g_2(s)\right]ds,$$

where $G_i(t,\ s)$, $(i=1,2)$ are the Green's functions given by (3) and (5). Finally, combining the operators A_1 and A_2, we define an operator $T : X \to X$

$$\begin{aligned}\left(T(u_1,u_2)\right)(t) &= \left(\left(A_1(u_1,u_2)\right)(t), \left(A_2(u_1,u_2)\right)(t)\right)\\ &= \Big(\int_0^1 G_1(t,s)\left[\lambda_1 a_1(s) f_1\left(s,u_1(s),u_2(s)\right) + g_1(s)\right]ds, \qquad (6)\\ &\qquad \int_0^1 G_2(t,s)\left[\lambda_2 a_2(s) f_2\left(s,u_1(s),u_2(s)\right) + g_2(s)\right]ds\Big).\end{aligned}$$

Then it is easy to see that the BVP (1) has a solution $(u_1,u_2) \in X$ if and only if (u_1,u_2) is a fixed point of the operator T defined by (6) and from this context, the main objective of this study is to find the existence of fixed point of the operator T defined by (6).

For the brevity, we state only the Schauder's fixed point theorem [30], which will be used to prove the main results.

Theorem S. [30] (Schauder's Fixed Point Theorem) *Let X be a Banach space and E be a nonempty closed convex subset of X. Let T be a continuous mapping of E into a compact set $F \subset E$. Then T has a fixed point in X.*

3. Main results

This section is devoted to establishing the existence criteria of at least one positive solution to the BVP given by (1).

Let κ_1 and κ_2 be the non-negative constants given by Lemma 3 associated to the Green's functions $G_1(t,s)$ and $G_2(t,s)$ respectively. Next suppose that f_1 and f_2 are Caratheodory type functions, that is

(i) for almost all $t \in [0,1], f_1(t,\cdot): \mathbb{R}^+ \to \mathbb{R}$ and $f_2(t,\cdot): \mathbb{R}^+ \to \mathbb{R}$ are continuous.

(ii) for every $r \in \mathbb{R}^+, f_1(\cdot,\mathrm{r}): [0,1] \to \mathbb{R}$ and $f_1(\cdot,\mathrm{r}): [0,1] \to \mathbb{R}$ are measurable.

Throughout this paper, we use the following notations:

if for almost all $t \in [0,1]$, $m \geq 0, m \in L^1(0,1)$, then we denote $m \succ 0$,

$$M^* = \max\left\{\sup_{t\in[0,1]}\int_0^1 \frac{G_1(t,s)}{t^{\alpha_1-1}} g_1(s)ds,\ \sup_{t\in[0,1]}\int_0^1 \frac{G_2(t,s)}{t^{\alpha_2-1}} g_2(s)ds\right\},$$

and

$$M_* = \min\left\{\inf_{t\in[0,1]}\int_0^1 \frac{G_1(t,s)}{t^{\alpha_1-1}} g_1(s)ds,\ \inf_{t\in[0,1]}\int_0^1 \frac{G_2(t,s)}{t^{\alpha_2-1}} g_2(s)ds\right\}.$$

Finally, we define a set S as follows

$$S = \{(u_1,u_2) \in X : u_1(t),\, u_2(t) \geq 0,\, t \in [0,1]\}.$$

We are now in position to present and prove the main results.

Theorem 1. *Consider the BVP for coupled system of Riemann-Liouville-type NLFDEs given by* (1), *along with Caratheodory functions* f_1 and f_2. *Suppose that there exist* $m \succ 0$ *and* $\mu > 0$ *such that the following conditions are satisfied:*

(H_3) $0 \leq f_1(t,u_1,u_2),\ f_2(t,u_1,u_2) \leq \dfrac{m(t)}{u_1^{\mu}},\ \forall (u_1,u_2) \in S,\ t \in [0,1]$ *and* $u_1 \neq 0$;

(H_4) $\lambda_1 \displaystyle\int_0^1 \frac{G_1(1,s)a_1(s)m(s)}{s^{\mu(\alpha^*-1)}} ds < +\infty,\ \lambda_2 \int_0^1 \frac{G_2(1,s)a_2(s)m(s)}{s^{\mu(\alpha^*-1)}} ds < +\infty$, *where* $\alpha^* = \max\{\alpha_1,\alpha_2\}$. *If* $M_* > 0$, *then the BVP given by* (1) *has at least one positive solution.*

Poof. Since, the solution of the BVP given by (1) is equivalent to the fixed point of the integral operator T defined by (6), so we have to prove that the integral operator T defined by (6) exist a fixed point.

Let $\Psi=\left\{(u_1,u_2)\in S: t^{\alpha^*-1}p\le u_1(t),\, u_2(t)\le t^{\alpha_*-1}P,\ \forall t\in[0,1]\right\}$, where $\alpha_*=\min\{\alpha_1,\alpha_2\}$ and $P>p>0$ are undetermined positive constants. Then it is clear that Ψ is a bounded closed convex subset of X .

It is obvious that operator $T:\Psi\to\Psi$ is continuous. To prove $T(\Psi)\subset\Psi$, let us fix $p=M_*$ and from assumption, we have $p>0$. Now for all $t\in[0,1]$ and $(u_1,u_2)\in\Psi$, we yield that

$$A_1(u_1,u_2)(t)\ge\int_0^1 G_1(t,s)g_1(s)ds\ge t^{\alpha_1-1}M_*=t^{\alpha_1-1}p\ge t^{\alpha^*-1}p.$$

On the other hand, if we put

$$N^*=\max\left\{\lambda_1\frac{1}{t^{\alpha_1-1}}\int_0^1\frac{G_1(1,s)a_1(s)m(s)}{s^{\mu(\alpha^*-1)}}ds,\ \lambda_2\frac{1}{t^{\alpha_2-1}}\int_0^1\frac{G_2(1,s)a_2(s)m(s)}{s^{\mu(\alpha^*-1)}}ds\right\}. \quad (7)$$

then using (H_3), we get

$$\begin{aligned}A_1(u_1,u_2)(t)&\le\lambda_1\int_0^1 G_1(1,s)a_1(s)f_1(s,u_1(s),u_2(s))ds+\int_0^1 G_1(1,s)g_1(s)ds\\&\le\lambda_1\int_0^1\frac{G_1(1,s)a_1(s)m(s)}{u_1^{\mu}(s)}ds+t^{\alpha_1-1}M^*\\&\le t^{\alpha_1-1}\left(\frac{N^*}{p^{\mu}}+M^*\right)\le t^{\alpha_*-1}\left(\frac{N^*}{p^{\mu}}+M^*\right).\end{aligned}$$

Now, if we set $P=\left(\frac{N^*}{p^{\mu}}+M^*\right)$, then we obtain that

$$t^{\alpha^*-1}p\le A_1(u_1,u_2)(t)\le t^{\alpha_*-1}P. \quad (8)$$

Similarly, for the operator $A_1(u_1,u_2)$, we obtain that

$$t^{\alpha^*-1}p\le A_2(u_1,u_2)(t)\le t^{\alpha_*-1}P. \quad (9)$$

Hence, from (6)–(8) we have

$$t^{\alpha^*-1}p\le T(u_1,u_2)(t)\le t^{\alpha_*-1}P.$$

This means that $T(\Psi)\subset\Psi$.

Now for all $t_1,\, t_2\in[0,1],\, t_1>t_2$ and using (8) and (9), we get

$$
\begin{aligned}
&\|T(u_1,u_2)(t_1)-T(u_1,u_2)(t_2)\| \\
&\quad \le \|T(u_1,u_2)(t_1)\|-\|T(u_1,u_2)(t_2)\| \\
&\quad = \|A_1(u_1,u_2)(t_1),A_2(u_1,u_2)(t_1)\|-\|A_1(u_1,u_2)(t_2),A_2(u_1,u_2)(t_2)\| \\
&\quad = \|A_1(u_1,u_2)(t_1)\|+\|A_2(u_1,u_2)(t_1)\|-\left(\|A_1(u_1,u_2)(t_2)\|+\|A_2(u_1,u_2)(t_2)\|\right) \\
&\quad = \left(\|A_1(u_1,u_2)(t_1)\|-\|A_1(u_1,u_2)(t_2)\|\right)+\left(\|A_2(u_1,u_2)(t_1)\|-\|A_2(u_1,u_2)(t_2)\|\right) \\
&\quad \le P\left\|t_1^{\alpha_*-1}-t_2^{\alpha_*-1}\right\|+P\left\|t_1^{\alpha_*-1}-t_2^{\alpha_*-1}\right\| \\
&\quad = 2P\left\|t_1^{\alpha_*-1}-t_2^{\alpha_*-1}\right\| \le 2P\|t_1-t_2\|.
\end{aligned}
$$

This tells us that $T(\Psi)$ is equicontinuous. Hence from the Arzela-Ascoli theorem [31], we conclude that $T:\Psi\to\Psi$ is completely continuous operator and it ensure that T is a continuous operator from a bounded closed convex subset of X to the compact subset of that bounded closed convex subset.

Thus, in view of Theorem **S** (Schauder's Fixed Point Theorem) the integral operator T given by (6) has at least one fixed point which is positive and this means that the BVP given (1) has at least one positive solution.

This completes the proof. ■

Theorem 2. *Consider the BVP as like Theorem 1 and assume that* (H_4) *holds. Suppose that there exist* $m \succ 0$, $\widehat{m} \succ 0$ *and* $0<\mu<1$ *such that the following condition is satisfied:*

$$
(H_5)\ \frac{\widehat{m}(t)}{u_1^{\mu}} \le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2) \le \frac{m(t)}{u_1^{\mu}},\ \forall (u_1,u_2)\in S,\ t\in[0,1]\, and\ u_1\neq 0.
$$

If $M_*=0$, *then the BVP given* (1) *has at least one positive solution.*

Proof. To prove this theorem, we follow the proof the Theorem 1 and just search the positive constants $P>p>0$ such that $T(\Psi)\subset\Psi$. So, from Theorem 1 we can obtain that

$$
A_1(u_1,u_2)(t) \le t^{\alpha_*-1}\left(\frac{N^*}{p^{\mu}}+M^*\right),
$$

where N^* is given by (7).

Now, if we set

$$
\widehat{N}^* = \min\left\{\lambda_1\kappa_1 \frac{1}{t^{\alpha_1-1}}\int_{\frac{1}{2}}^{1}\frac{G_1(1,s)a_1(s)\widehat{m}(s)}{s^{\mu(\alpha^*-1)}}ds,\ \lambda_2\kappa_2 \frac{1}{t^{\alpha_2-1}}\int_{\frac{1}{2}}^{1}\frac{G_2(1,s)a_2(s)\widehat{m}(s)}{s^{\mu(\alpha^*-1)}}ds\right\}, \tag{10}
$$

where κ_1 and κ_2 be the non-negative constants given by Lemma 3, then using (H_5), we get

$$A_1(u_1,u_2)(t) \ge \lambda_1 \int_0^1 G_1(t,s)a_1(s)f_1(s,u_1(s),u_2(s))ds$$
$$\ge \lambda_1 \int_{\frac{1}{2}}^1 G_1(t,s)a_1(s)f_1(s,u_1(s),u_2(s))ds$$
$$\ge \lambda_1 \kappa_1 \int_0^1 \frac{G_1(1,s)a_1(s)\widehat{m}(s)}{u_1^{\mu}(s)}ds$$
$$\ge t^{\alpha_1 - 1}\frac{\widehat{N}^*}{P^{\mu}} \ge t^{\alpha^*-1}\frac{\widehat{N}^*}{P^{\mu}}.$$

Hence, if we consider p and P satisfying $\left(\frac{N^*}{p^{\mu}} + M^*\right) \le P$ and $\frac{\widehat{N}^*}{P^{\mu}} \ge p$ then using the same process as like the Theorem 1, we conclude that T is a continuous operator from a bounded closed convex subset of X to the compact subset.

Thus, in view of Theorem **S** the integral operator T given by (6) has at least one fixed point which is positive and this means that the BVP given (1) has at least one positive solution.

This completes the proof. ■

Theorem 3. *Consider the BVP as like Theorem* **1.** *and assume that there exist* $m \succ 0$, $\widehat{m} > 0$ *and* $0 < \mu < 1$ *such that* (H_4) *and* (H_5) *hold. If* $M^* < 0$ *with the following condition*

$$(H_6)\ M_* \ge \left[\frac{\widehat{N}^*}{(N^*)^{\mu}}\mu^2\right]^{\frac{1}{1-\mu^2}}\left(1 - \frac{1}{\mu^2}\right),$$

then the BVP given (1) *has at least one positive solution.*

Proof. To prove this theorem, we follow the proof the Theorem 2 and just search the positive constants $P > p > 0$ such that

$$\frac{N^*}{p^{\mu}} \le P \text{ and } \left(\frac{\widehat{N}^*}{P^{\mu}} + M_*\right) \ge p. \tag{11}$$

Now, if we fix $P = \frac{N^*}{p^{\mu}}$, then $\frac{\widehat{N}^*}{(N^*)^{\mu}}p^{\mu^2} + M_* \ge p$ implies that either $\frac{\widehat{N}^*}{P^{\mu}} + M_* \ge p$ or, $M_* \ge p - \frac{\widehat{N}^*}{(N^*)^{\mu}}p^{\mu^2} = \varphi(p)$. It is clear that the minimum value of $\varphi(p)$ occur at $p = p_0 = \left[\frac{\widehat{N}^*\mu^2}{(N^*)^{\mu}}\right]^{\frac{1}{1-\mu^2}}$. Hence, if we put $p = p_0$, then we obtain that

$$M_* \ge \varphi(p_0) = \left[\frac{\widehat{N}^*\mu^2}{(N^*)^{\mu}}\right]^{\frac{1}{1-\mu^2}} - \frac{\widehat{N}^*}{(N^*)^{\mu}}\left[\frac{\widehat{N}^*\mu^2}{(N^*)^{\mu}}\right]^{\frac{\mu^2}{1-\mu^2}} \ge \left[\frac{\widehat{N}^*\mu^2}{(N^*)^{\mu}}\right]^{\frac{1}{1-\mu^2}}\left(1 - \frac{1}{\mu^2}\right).$$

Therefore, for $M_* \ge \varphi(p_0)$ (11) is satisfied. Consequently, (H_6) is satisfied. Thus, in view of Theorem 2 and Theorem **S** the integral operator T given by (6) has at least one fixed point which is positive and this means that the BVP given (1) has at least one positive solution.

This completes the proof. ■

Now, we give some illustrative examples.

Example 1. Consider the BVP for coupled system of Riemann-Liouville-type NLFDEs provided by

$$\begin{cases} -D_{0^+}^{10/3}u_1(t) = t\left[u_1(t)+u_2(t)\right]^4 + t^2, \quad t\in[0,1], \\ -D_{0^+}^{13/3}u_2(t) = 2t^2\left[u_1(t)+u_2(t)\right]^6 + t^3, \quad t\in[0,1], \\ D_{0^+}^{1/2}u_1(0) = D_{0^+}^{4/3}u_1(0) = D_{0^+}^{9/4}u_1(0) = 0, \quad u_1(1) = \frac{1}{2}u_1\left(\frac{1}{2}\right), \\ D_{0^+}^{2/3}u_2(0) = D_{0^+}^{3/2}u_2(0) = D_{0^+}^{3/2}u_2(0) = 0, \quad u_2(1) = \frac{1}{3}u_2\left(\frac{1}{3}\right). \end{cases} \tag{12}$$

where for all $u_1, u_2 > 0$, $f_1(t,u_1,u_2) = \left[u_1(t)+u_2(t)\right]^4 > 0$, $f_2(t,u_1,u_2) = \left[u_1(t)+u_2(t)\right]^6 > 0$, $\alpha_1 = \frac{10}{3}, \alpha_2 = \frac{13}{4} \in (3,4], \beta_1 = \frac{1}{2}, \beta_2 = \frac{2}{3} \in (0,1), \gamma_1 = \frac{4}{3}, \gamma_2 = \frac{3}{2} \in (1,2), \delta_1 = \frac{9}{4}, \delta_2 = \frac{5}{2} \in (2,3)$, $\eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}, \xi_1 = \frac{1}{2} \in (0,1), \xi_2 = \frac{1}{3} \in (0,1), 0 < \eta_1\xi_1^{\alpha_1-1} < 1, 0 < \eta_2\xi_2^{\alpha_2-1} < 1$, $\lambda_1 = 1 > 0, \lambda_2 = 2 > 0$, for all $t\in[0,1]$ $a_1(t) = t > 0, a_2(t) = t^2 > 0$, and $g_1(t) = t^2, g_2(t) = t^3$. For the above values it is clear that (H_1) and (H_2) are satisfied.

Now if we consider $m(t) = u_1(t)\left[u_1(t)+u_2(t)\right]^7$ and $\mu = 1$, then by direct calculation we obtain that $0 \le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2) \le \frac{m(t)}{u_1^{\mu}},\ \forall\ t\in[0,1], \lambda_1\int_0^1 \frac{G_1(1,s)a_1(s)m(s)}{s^{\mu(\alpha^*-1)}}ds < +\infty,$ and $\lambda_2\int_0^1 \frac{G_2(1,s)a_2(s)m(s)}{s^{\mu(\alpha^*-1)}}ds < +\infty$, for $\alpha^* = \max\{\alpha_1,\alpha_2\}$, that is the conditions (H_3) and (H_4) are satisfied. Furthermore, since $G_1(t,s), G_2(t,s) \ge 0$, then we get

$$\begin{aligned} M_* &= \min\left\{ \inf_{t\in[0,1]}\int_0^1 \frac{G_1(t,s)}{t^{\frac{10}{3}-1}}s^2 ds,\ \inf_{t\in[0,1]}\int_0^1 \frac{G_2(t,s)}{t^{\frac{13}{4}-1}}s^3 ds \right\} \\ &= \min\left\{ \inf_{t\in[0,1]}\int_0^1 \frac{s^2G_1(t,s)}{t^{\frac{7}{3}}}ds,\ \inf_{t\in[0,1]}\int_0^1 \frac{s^3G_2(t,s)}{t^{\frac{9}{4}}}ds \right\} \\ &> 0. \end{aligned}$$

Therefore, all the conditions of Theorem 1 are satisfied by BVP (12). Hence by an application of Theorem 1, we can say that the BVP (12) has at least one positive solution.

Example 2. Consider the BVP for coupled system of Riemann-Liouville-type NLFDEs provided by

$$\begin{cases} -D_{0^+}^{7/2}u_1(t) = t\left[u_1(t)+u_2(t)\right]^{\frac{1}{4}} + \left(t^3 - \frac{1}{4}\right), \quad t\in[0,\,1], \\ -D_{0^+}^{10/3}u_2(t) = t^2\left[u_1(t)+u_2(t)\right]^{\frac{1}{3}} + \left(t^2 - \frac{1}{3}\right), \quad t\in[0,\,1], \\ D_{0^+}^{1/3}u_1(0) = D_{0^+}^{3/2}u_1(0) = D_{0^+}^{9/4}u_1(0) = 0, \quad u_1(1) = u_1\left(\frac{1}{2}\right), \\ D_{0^+}^{1/3}u_2(0) = D_{0^+}^{3/2}u_2(0) = D_{0^+}^{9/4}u_2(0) = 0, \quad u_2(1) = u_2\left(\frac{1}{2}\right). \end{cases} \tag{13}$$

where for all $u_1,u_2>0,\ f_1(t,u_1,u_2)=\left[u_1(t)+u_2(t)\right]^{\frac{1}{4}}>0,\ f_2(t,u_1,u_2)=\left[u_1(t)+u_2(t)\right]^{\frac{1}{3}}>0,$ $\alpha_1=\frac{7}{2},\alpha_2=\frac{10}{3}\in(3,4],\beta_1=\beta_2=\frac{1}{3}\in(0,1),\ \gamma_1=\gamma_2=\frac{3}{2}\in(1,2),\delta_1=\delta_2=\frac{9}{4}\in(2,3),$ $\eta_1=\eta_2=1,\ \xi_1=\xi_2=\frac{1}{2}\in(0,1),\ 0<\eta_1\xi_1^{\alpha_1-1}<1,\ 0<\eta_2\xi_2^{\alpha_2-1}<1$, $\lambda_1=\lambda_2=1>0,$ for all $t\in[0,1]$ $a_1(t)=t>0,\ a_2(t)=t^2>0,$ and $g_1(t)=\left(t^3-\frac{1}{4}\right),\ g_2(t)=\left(t^2-\frac{1}{3}\right)$. For the above values it is clear that (H_1) and (H_2) are satisfied.

Now if we consider $m(t)=u_1(t)[u_1(t)+u_2(t)]^{\frac{1}{2}},\ \widehat{m}(t)=u_1(t)[u_1(t)+u_2(t)]^{\frac{1}{5}}\ and\ \mu=\frac{1}{2}$, then by direct calculation we obtain that $0\le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2)\le\frac{m(t)}{u_1^{\mu}},\ \forall\ t\in[0,1],$ $\frac{\widehat{m(t)}}{u_1^{\mu}}\le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2)\le\frac{m(t)}{u_1^{\mu}},\ \forall\ t\in[0,1],\lambda_1\int_0^1\frac{G_1(1,s)a_1(s)m(s)}{s^{\mu(\alpha^*-1)}}ds<+\infty,$ and $\lambda_2\int_0^1\frac{G_2(1,s)a_2(s)m(s)}{s^{\mu(\alpha^*-1)}}ds<+\infty,$ for $\alpha^*=\max\{\alpha_1,\alpha_2\}$, that is the conditions $(H_3),(H_4)$ and (H_5) are satisfied. Furthermore, since $\max_{t\in[0,1]}G_i(t,s)=G_i(1,s),\ (i=1,2)$, then we have

$$\begin{aligned} M_* &= \min\left\{\inf_{t\in[0,1]}\int_0^1\frac{G_1(t,s)}{t^{\frac{7}{2}-1}}\left(s^3-\frac{1}{4}\right)ds,\ \inf_{t\in[0,1]}\int_0^1\frac{G_2(t,s)}{t^{\frac{10}{4}-1}}\left(s^2-\frac{1}{3}\right)ds\right\} \\ &= \min\left\{\inf_{t\in[0,1]}\int_0^1\frac{G_1(1,s)}{t^{\frac{5}{2}}}\left(s^3-\frac{1}{4}\right)ds,\ \inf_{t\in[0,1]}\int_0^1\frac{G_2(1,s)}{t^{\frac{6}{4}}}\left(s^2-\frac{1}{3}\right)ds\right\} \\ &= 0. \end{aligned}$$

Therefore, all the conditions of Theorem 2 are satisfied by BVP (13). Hence by an application of Theorem 2, we can say that the BVP (13) has at least one positive solution.

Example 3. Consider the BVP for coupled system of Riemann-Liouville-type NLFDEs provided by

$$\begin{cases} -D_{0^+}^{7/2}u_1(t) = t\left[u_1(t)+u_2(t)\right]^{\frac{1}{4}} + \left(t^2-1\right), & t\in[0,1], \\ -D_{0^+}^{10/3}u_2(t) = t^2\left[u_1(t)+u_2(t)\right]^{\frac{1}{3}} + \left(t^3-1\right), & t\in[0,1], \\ D_{0^+}^{1/3}u_1(0) = D_{0^+}^{3/2}u_1(0) = D_{0^+}^{9/4}u_1(0) = 0, & u_1(1) = u_1\left(\frac{1}{2}\right), \\ D_{0^+}^{1/3}u_2(0) = D_{0^+}^{3/2}u_2(0) = D_{0^+}^{9/4}u_2(0) = 0, & u_2(1) = u_2\left(\frac{1}{2}\right). \end{cases} \tag{14}$$

where for all $u_1,u_2>0$, $f_1(t,u_1,u_2)=\left[u_1(t)+u_2(t)\right]^{\frac{1}{4}}>0$, $f_2(t,u_1,u_2)=\left[u_1(t)+u_2(t)\right]^{\frac{1}{3}}>0$, $\alpha_1=\frac{7}{2},\alpha_2=\frac{10}{3}\in(3,4],\beta_1=\beta_2=\frac{1}{3}\in(0,1),\ \gamma_1=\gamma_2=\frac{3}{2}\in(1,2),\delta_1=\delta_2=\frac{9}{4}\in(2,3)$, $\eta_1=\eta_2=1,\ \xi_1=\xi_2=\frac{1}{2}\in(0,1),\ 0<\eta_1\xi_1^{\alpha_1-1}<1,\ 0<\eta_2\xi_2^{\alpha_2-1}<1$, $\lambda_1=\lambda_2=1>0$, for all $t\in[0,1]$ $a_1(t)=t>0,\ a_2(t)=t^2>0$, and $g_1(t)=\left(t^2-1\right),\ g_2(t)=\left(t^3-1\right)$. For the above values it is clear that (H_1) and (H_2) are satisfied.

Now if we consider $m(t)=u_1(t)[u_1(t)+u_2(t)]^{\frac{1}{2}},\ \widehat{m}(t)=u_1(t)[u_1(t)+u_2(t)]^{\frac{1}{5}}\ and\ \mu=\frac{1}{2}$, then by direct calculation we obtain that $0\le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2)\le\frac{m(t)}{u_1^{\mu}},\ \forall\ t\in[0,1]$,

$\frac{\widehat{m}(t)}{u_1^{\mu}}\le f_1(t,u_1,u_2),\ f_2(t,u_1,u_2)\le\frac{m(t)}{u_1^{\mu}},\ \forall\ t\in[0,1],\lambda_1\int_0^1\frac{G_1(1,s)a_1(s)m(s)}{s^{\mu(\alpha^*-1)}}ds<+\infty$,

$\lambda_2\int_0^1\frac{G_2(1,s)a_2(s)m(s)}{s^{\mu(\alpha^*-1)}}ds<+\infty$, for $\alpha^*=\max\{\alpha_1,\alpha_2\}$, and $M_*\ge\left[\frac{\hat{N}^*}{(N^*)^{\mu}}\mu^2\right]^{\frac{1}{1-\mu^2}}\left(1-\frac{1}{\mu^2}\right)$, that is the conditions $(H_3),(H_4),(H_5)$ and (H_6) are satisfied.

Furthermore, since $G_1(t,s),G_2(t,s)\ge 0$, then we have

$$M^*=\max\left\{\sup_{t\in[0,1]}\int_0^1\frac{G_1(t,s)}{t^{\frac{7}{2}-1}}\left(t^2-1\right)ds,\ \sup_{t\in[0,1]}\int_0^1\frac{G_2(t,s)}{t^{\frac{10}{3}-1}}\left(t^3-1\right)ds\right\}<0$$

Therefore, all the conditions of Theorem 3 are satisfied by BVP (14). Hence by an application of Theorem 3, we can say that the BVP (14) has at least one positive solution.

4. Conclusion

In this paper, some new existence criteria of at least one positive solution to the three-point BVP for coupled system of Riemann-Liouville-type NLFDEs given by (1) have been studied by applying Schauder's fixed point theorem. Proven theorems (Theorem 1-3) of this paper have been used as the efficient method to checked the existence of at least one positive solution to the coupled system of BVP for NLFDEs given by (1). The established results provide an easy and straightforward technique to cheek the existence of positive solutions to the considered BVP given by (1). Moreover, the results of this paper extend the corresponding results of Han and Yang [10] and Hao and Zhai [27].

Conflict of interest

The authors declare that they have no conflict of interests.

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