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Research article

A novel entropy measure of Pythagorean fuzzy soft sets

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Abstract: Pythagorean fuzzy soft set (PFSS) is one of the useful extension of the Pythagorean fuzzy set (PFS) to deal with the vagueness and uncertainties in the data. The major advantages of PFSS over the other existing sets are to consider the parameterized tool of the family of PFS. Keeping this advantage, in this paper we define some new entropy measures for PFSS to compute the degree of fuzziness of the set. The axiomatic definition and their validity are stated. The larger the entropy, the lesser the vagueness and so, the decision making based on entropy is a useful one. Further, a decision-making algorithm is explored to solve the decision-making problem under the PFSS environment. A numerical example is given to validate the method and compare their performance with the existing intuitionistic fuzzy soft set entropy measures.

Keywords: soft sets; Pythagorean fuzzy soft sets; decision-making; entropy measures
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1. Introduction

In the modern decision-making process, the system becomes more and more complex which results in the difficulty for the decision-makers to pick their decision smoothly. This occurs mainly due to the presence of a large number of uncertainties in the collected information. Traditionally, the information collected either from the logbooks or respective centers is in the form of crisp or interval numbers. However, with the increasing complexities of the systems day-by-day, it is difficult to express the information in a precise way. In other words, the major difficulty faced by crisp sets is that they are not capable of dealing with uncertainty and vagueness. Therefore, if we are utilized the collected information as such in the analysis then the computed results may diverge from the original choice. To handle such ambiguity in the data, a theory of fuzzy set (FS) was given by Zadeh [1] in which each element is characterized by its membership degree (MD) lying between 0 and 1. Later on, Atanassov [2] prolongs the FSs to intuitionistic FSs (IFSs) by adding non-membership degrees (NMDs) along with MDs such that their sum can't pass one. Yager [3] developed the Pythagorean FS (PFS) in which the square sum of the MDs and NMDs are less than one. The major benefits of such extended FSs are that they represent uncertain information using MD, NMD and the degree of hesitancy.

In the literature, many scholars have utilized the conditions of IFS & PFS as well as the concept of information measures such as distance, entropy, similarity to developing the algorithms for solving the decision-making problems (DMPs). For instance, Hung and Yang [4] introduced the similarity measures (SM) between the two distinctive IFSs based on Hausdorff distance. Garg and Kumar [5] presented the SM between IFSs using the idea of connection numbers. Peng and Garg [6] presented multiparametric SM on PFS to solve pattern recognition problems. The distance or similarity measures the degree of discrimination between the given sets. However, to measure the degree of fuzziness of the set, a concept of entropy measure play a vital role in the information measure theory. Burillo and Bustince [7] initiated the idea of entropy for IFS. The axiomatic definition of entropy is given by Hung and Yang [8] for IFSs. Vlachos and Sergiadis [9] formulated a mathematical model for entropy between FS and IFS. Zhang et al. [10] designed the entropy for the vague set. Garg et al. [11] presented the generalized entropy of order α and degree β for IFS. Garg [12] discussed an algorithm for solving DMP with entropy weight and aggregation operators. Garg [13] presented the different and generalized form of the entropy measures for IFSs. Selvachandran et al. [14] presented the vague entropy measure for a complex soft set. A comprehensive overview of various measures on PFS is reviewed by Peng and Selvachandran [15].

All the above-stated approaches are widely applicable in many fields and areas. However, these theories have restrictions in view of their inadequacy over the parameterizations tool. To deal it with completely, a concept of soft set [16], introduced by Molodtsov, is a good parametrization tool, in which object is evaluated over some specific parameters. Ali et al. [17] defined some operations for SS. Later on, Maji et al. [18,19] respectively combined the SS with FS and IFS to present the concept of fuzzy SS (FSS) and intuitionistic FSS (IFSS). Feng et al. [20] presented an approach by combining the SS with fuzzy and rough sets. In IFSS, the information is collected by taking more than one parameter as compared to IFSs. Taking the advantages of FSS and IFSS, several researchers put forward diverse approaches to solve the DMPs. For example, Bora et al. [21] discussed some basic properties of it. However, in terms of information measures, researchers made an effort to present an algorithm for various kinds of DMPs by using similarity measures [22–25], aggregation operators [26, 27], distance measures [28, 29]. Apart from it, extensions of IFSS namely generalized IFSS [30, 31] and group generalized IFSS [32] was presented to illuminate decision-making issues. Hayat et al. [33] presented the application of generalized IFSS to the selection of design process. On the other hand, Majumdar and Samanta [34] presented the entropy measures for the soft sets. Jiang et al. [35] proposed the entropy measures for IFSSs. Liu et al. [36] defined the similarity and entropy measures for FSS. Selvachandran et al. [37] defined the distance and entropy measures of generalized IFSS. A concise critique of the soft set theory is given in Zhan and Alcantud [38].

Currently, research on the hypothetical and application perspectives of SSs and its different extensions are advancing quickly. To address it completely, Peng et al. [39] combined SS with PFS and presented the notion of Pythagorean fuzzy SS (PFSS). From their structure, it is noted that FSS and IFSS are the special cases of PFSS. Recently, Athira et al. [40] presented the entropy and distance measures for PFSS. To the best of authors' knowledge, very little work has been investigated on the theory of the PFSS. Thus, keeping the flexibility of PFSSs to examine the considered information in a better manner than IFSS or FSS, this paper focused on enriching the theory of PFSS by defining new entropy measures. Since entropy of a set represent the degree of fuzziness of the information and hence the primary objective of the work is carried out as (i) to present a new entropy measures under the PFSS environment, (ii) to develop an algorithm based on proposed measure to solve the DMPs, (iii) a numerical example is exhibited to demonstrate the approach. The major advantage of the proposed measures is that under some special restrictions, the suggested measures reduce entropy measures for IFSS as well as FSS.

The rest of the paper is organized as follows. In Section 2, some basic concepts about SS are reviewed. Section 3 presents entropy measures for PFSS and examine their basic features. In Section 4, an approach is presented based on proposed entropy measures and demonstrate it with a numerical example. Finally, some conclusion is drawn in Section 5.

2. Preliminaries

In this section, some basic review of FSS, IFSS, PFSS is presented over the set $\mathbf{U}$ and parameter $\mathbf{E}$.

Definition 2.1. [16] A map $\mathbf{F} : \mathbf{E} \to \mathbf{K}^{\mathbf{U}}$ is called as soft set, where $\mathbf{K}^{\mathbf{U}}$ is a set of all subsets of $\mathbf{U}$.

Definition 2.2. [20] Let $A, B \subset \mathbf{E}$ and $(\mathbf{F}, A)$, $(\mathbf{G}, B)$ be two SSs over $\mathbf{U}$. Then, the basic operations over them are stated as

1) $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ if $A \subseteq B$ and $\mathbf{F}(e) \subseteq \mathbf{G}(e)$, $\forall e \in A$.

2) $(\mathbf{F}, A) = (\mathbf{G}, B)$ if $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ and $(\mathbf{G}, B) \subseteq (\mathbf{F}, A)$.

3) Complement: $(\mathbf{F}, A)^c = (\mathbf{F}^c, A)$, where $\mathbf{F}^c : A \to \mathbf{K}^{\mathbf{U}}$ defined as $\mathbf{F}^c(e) = \mathbf{U} - \mathbf{F}(e)$, $\forall e \in A$.

Definition 2.3. [18] A map $\mathbf{F} : \mathbf{E} \to \mathbf{F}^{\mathbf{U}}$ is called FSS defined as

$$\mathbf{F}_{u_i}(e_j) = \left\{(u_i, \zeta_j(u_i)) \mid u_i \in \mathbf{U}\right\}, \tag{2.1}$$

where $\mathbf{F}^{\mathbf{U}}$ be a set of all fuzzy subsets of $\mathbf{U}$ and $\zeta_j(u_i)$ is MD of an expert u_i over parameter $e_j \in \mathbf{E}$.

Definition 2.4. [18] For $A, B \subset \mathbf{E}$ and $(\mathbf{F}, A)$, $(\mathbf{G}, B)$ be any two FSSs over $\mathbf{U}$, then

1) $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ if, $A \subset B$ and $\mathbf{F}(e) \le \mathbf{G}(e)$ for each $e \in A$.

2) $(\mathbf{F}, A) = (\mathbf{G}, B)$ are equal if $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ and $(\mathbf{G}, B) \subseteq (\mathbf{F}, A)$.

3) Complement: $(\mathbf{F}^c, A)$ where for each $a \in A$, $\mathbf{F}^c(e) = 1 - \mathbf{F}(e)$ for all $e \in U$.

Definition 2.5. [19] A mapping $\mathbf{F} : \mathbf{E} \to \mathbf{IF}^{\mathbf{U}}$ is called IFSS defined as

$$\mathbf{F}_{u_i}(e_j) = \left\{(u_i, \zeta_j(u_i), \vartheta_j(u_i)) \mid u_i \in \mathbf{U}\right\}, \tag{2.2}$$

where $\mathbf{IF}^{\mathbf{U}}$ is the intuitionistic fuzzy subsets of $\mathbf{U}$ and ζ_j and ϑ_j are MD and NMD respectively, with $0 \le \zeta_j, \vartheta_j, \zeta_j + \vartheta_j \le 1$ for all $u_i \in \mathbf{U}$.

Definition 2.6. [3] A PFS $\mathcal{P}$ on $\mathbf{U}$ is stated as

$$\mathcal{P} = \{(u, \zeta_{\mathcal{P}}(u), \vartheta_{\mathcal{P}}(u)) \mid u \in \mathbf{U}\}, \tag{2.3}$$

where $\zeta_{\mathcal{P}}, \vartheta_{\mathcal{P}} : \mathbf{U} \to [0,1]$ denote the "MD and the NMD", respectively, with $(\zeta_{\mathcal{P}}(u))^2 + (\vartheta_{\mathcal{P}}(u))^2 \leq 1$ $\forall u \in \mathbf{U}$. A pair (ζ, ϑ) is called Pythagorean fuzzy numbers (PFNs).

Definition 2.7. Let $\mathcal{P}_1 = (\zeta_1, \vartheta_1)$ and $\mathcal{P}_2 = (\zeta_2, \vartheta_2)$ be two PFNs, then

1) $\mathcal{P}_1 \cup \mathcal{P}_2 = (\max(\zeta_1, \zeta_2), \min(\vartheta_1, \vartheta_2))$.

2) $\mathcal{P}_1 \cap \mathcal{P}_2 = (\min(\zeta_1, \zeta_2), \max(\vartheta_1, \vartheta_2))$.

3) Complement: $\mathcal{P}_1^c = (\vartheta_1, \zeta_1)$.

Definition 2.8. [39] A pair $(\mathbf{F}, \mathbf{E})$ is called PFSS if a map $\mathbf{F} : \mathbf{E} \to \mathbf{P}^{\mathbf{U}}$ defined as

$$\mathbf{F}_{u_i}(e_j) = \left\{(u_i, \zeta_j(u_i), \vartheta_j(u_i)) \mid u_i \in \mathbf{U}\right\}, \tag{2.4}$$

where $\mathbf{P}^{\mathbf{U}}$ is the Pythagorean fuzzy subset of $\mathbf{U}$ and ζ, ϑ satisfies $\zeta^2 + \vartheta^2 \leq 1$ for all $u_i \in \mathbf{U}$.

A pair $(\mathbf{F}, \mathbf{E})$ is termed as PFSS and denote $\mathbf{F}_{u_i}(e_j) = (\zeta_{\mathbf{F}(e_j)}(u_i), \vartheta_{\mathbf{F}(e_j)}(u_i))$ called as Pythagorean fuzzy soft number (PFSN) with $\zeta^2_{\mathbf{F}(e_j)} + \vartheta^2_{\mathbf{F}(e_j)} \leq 1$ for $\zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)} \in [0,1]$.

Remark 2.1. For a given set, if $\zeta^2 + \vartheta^2 \leq 1$ and $\zeta + \vartheta \leq 1$ holds then PFSS reduces to IFSS.

Remark 2.2. We denote PFSS($\mathbf{U}$) be the collections of all PFSSs.

Definition 2.9. [39] For $A, B \subset \mathbf{E}$ and $(\mathbf{F}, A), (\mathbf{G}, B)$ are two PFSSs over $\mathbf{U}$. Then, the basic operations over them are listed as

1) $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ if $A \subseteq B$ and $\zeta_{\mathbf{F}(e)} \leq \zeta_{\mathbf{G}(e)}, \vartheta_{\mathbf{F}(e)} \geq \vartheta_{\mathbf{G}(e)}$ for all $e \in A$.

2) $(\mathbf{F}, A) = (\mathbf{G}, B)$ if $(\mathbf{F}, A) \subseteq (\mathbf{G}, B)$ and $(\mathbf{G}, B) \subseteq (\mathbf{F}, A)$.

3) Complement: $(\mathbf{F}^c, A)$ where $\mathbf{F}^c = (\vartheta_A, \zeta_A)$.

3. Entropy measure on PFSSs

In this section, we present the axiomatic definition for entropy on PFSS p. For it, let $p = (\mathbf{F}, \mathbf{E})$, $\tilde{p} = (\mathbf{G}, \mathbf{E})$, $\mathbf{U} = \{u_1, u_2, \ldots, u_n\}$ and $\mathbf{E} = \{e_1, e_2, \ldots, e_m\}$ be universe and parameter sets.

Definition 3.1. A real function E : PFSS($\mathbf{U}$) $\to [0, mn]$ is said to be an entropy on PFSS($\mathbf{U}$) if it satisfies the following properties:

P1) $E(p) = 0$ iff p is a soft set.

P2) $E(p) = mn$ if $\zeta_{\mathbf{F}(e_j)}(u_i) = \vartheta_{\mathbf{F}(e_j)}(u_i)$, for all $e_j \in \mathbf{E}$ and $u_i \in \mathbf{U}$.

P3) $E(p) = E(p^c)$.

P4) $E(p) \geq E(\tilde{p})$ if $\zeta_{\mathbf{F}(e_j)}(u_i) \leq \zeta_{\mathbf{G}(e_j)}(u_i)$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) \geq \vartheta_{\mathbf{G}(e_j)}(u_i)$ for, $\zeta_{\mathbf{G}(e_j)}(u_i) \leq \vartheta_{\mathbf{G}(e_j)}(u_i)$ or $\zeta_{\mathbf{F}(e_j)}(u_i) \geq \zeta_{\mathbf{G}(e_j)}(u_i)$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) \leq \vartheta_{\mathbf{G}(e_j)}(u_i)$ for $\zeta_{\mathbf{G}(e_j)}(u_i) \geq \vartheta_{\mathbf{G}(e_j)}(u_i)$.

Next, we present two kinds of entropy measures from PFSSs($\mathbf{U}$) to $\mathbb{R}^+ \cup \{0\}$.

Definition 3.2. For PFSS $p = \left(\zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)}\right)$, two new proposed entropy measures E_1, E_2 : PFSS($\mathbf{U}$) $\to$ $\mathbb{R}^+ \cup \{0\}$ are stated as

$$E_1(p) = \frac{1}{\sqrt{2}-1} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\sin\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \sin\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) - 1 \right) \quad (3.1)$$

and

$$E_2(p) = \frac{1}{\sqrt{2}-1} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\cos\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \cos\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) - 1 \right) \quad (3.2)$$

Theorem 3.1. The measures defined in Definition 3.2 are the valid entropy measures.

Proof. For PFSS $p = \left(\zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)}\right)$, to prove the proposed measures are valid, for it, we have to show that it satisfy the properties as given in Definition 3.1.

P1) When p is a soft sets, $\zeta_{\mathbf{F}(e_j)}(u_i) = 0$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) = 1$ or $\zeta_{\mathbf{F}(e_j)}(u_i) = 1$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) = 0$ where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Thus in both case,

$$\sin\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \sin\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) - 1 = 0$$

and

$$\cos\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \cos\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) - 1 = 0.$$

Hence, by Eqs. (3.1), (3.2), we get $E_1(p) = E_2(p) = 0$.

On the other hand, assume $E_1(p) = E_2(p) = 0$ for PFSS p, then by Definition 3.2, we have

$$\sin\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \sin\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) = 1$$

and

$$\cos\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i) - \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) + \cos\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i) + \vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4} \right) = 1$$

which implies that $\vartheta^2_{\mathbf{F}(e_j)} - \zeta^2_{\mathbf{F}(e_j)} = \pm 1$.

But for PFSS p, the values of $(\zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)})$ satisfying the condition $\vartheta^2_{\mathbf{F}(e_j)} - \zeta^2_{\mathbf{F}(e_j)} = \pm 1$ in the domain $0 \le \zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)} \le 1$ and $0 \le \zeta^2_{\mathbf{F}(e_j)} + \vartheta^2_{\mathbf{F}(e_j)} \le 1$ is $(1, 0)$ or $(0, 1)$. Hence, p is a soft set.

P2) If $\zeta_{\mathbf{F}(e_j)}(u_i) = \vartheta_{\mathbf{F}(e_j)}(u_i)$ then,

$$\sin\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i)-\vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4}\right)+\sin\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i)+\vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4}\right)-1=\sqrt{2}-1$$

and

$$\cos\left(\frac{\pi\left(1+\zeta^2_{\mathbf{F}(e_j)}(u_i)-\vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4}\right)+\cos\left(\frac{\pi\left(1-\zeta^2_{\mathbf{F}(e_j)}(u_i)+\vartheta^2_{\mathbf{F}(e_j)}(u_i)\right)}{4}\right)-1=\sqrt{2}-1$$

Hence, by Eqs. (3.1), (3.2), we have $E_1(p) = E_2(p) = mn$.

P3) For PFSS $p = \left(\zeta_{\mathbf{F}(e_j)}, \vartheta_{\mathbf{F}(e_j)}\right)$, we have $p^c = \left(\vartheta_{\mathbf{F}(e_j)}, \zeta_{\mathbf{F}(e_j)}\right)$. Thus, by Eqs. (3.1), (3.2), we get the required result.

P4) In order to prove the fourth property, consider the functions $F_1(x,y)$ and $F_2(x,y)$ such that,

$$F_1(x,y)=\sin\left(\frac{\pi(1+x^2-y^2)}{4}\right)+\sin\left(\frac{\pi(1-x^2+y^2)}{4}\right)-1$$

and

$$F_2(x,y)=\cos\left(\frac{\pi(1+x^2-y^2)}{4}\right)+\cos\left(\frac{\pi(1-x^2+y^2)}{4}\right)-1$$

where $x, y \in [0,1]$ and $0 \le x^2 +^2 \le 1$.

The partial derivatives with respect to x and y are obtained as,

$$\begin{aligned}
\frac{\partial F_1(x,y)}{\partial x} &= \frac{\pi x}{2}\left(\cos\left(\frac{\pi(1+x^2-y^2)}{4}\right)-\cos\left(\frac{\pi(1-x^2+y^2)}{4}\right)\right);\\
\frac{\partial F_1(x,y)}{\partial y} &= \frac{\pi y}{2}\left(\cos\left(\frac{\pi(1-x^2+y^2)}{4}\right)-\cos\left(\frac{\pi(1+x^2-y^2)}{4}\right)\right);\\
\frac{\partial F_2(x,y)}{\partial x} &= \frac{\pi x}{2}\left(\sin\left(\frac{\pi(1-x^2+y^2)}{4}\right)-\sin\left(\frac{\pi(1+x^2-y^2)}{4}\right)\right);\\
\text{and}\quad \frac{\partial F_2(x,y)}{\partial y} &= \frac{\pi y}{2}\left(\sin\left(\frac{\pi(1+x^2-y^2)}{4}\right)-\sin\left(\frac{\pi(1-x^2+y^2)}{4}\right)\right).
\end{aligned}$$

Critical points are obtained as $x = y$ by solving the equations

$$\frac{\partial F_1(x,y)}{\partial x}=0,\ \frac{\partial F_1(x,y)}{\partial y}=0,\ \frac{\partial F_2(x,y)}{\partial x}=0\quad \textit{and}\quad \frac{\partial F_2(x,y)}{\partial y}=0.$$

Also obtained as $\frac{\partial F_1(x,y)}{\partial x} \ge 0$ when $x \le y$ and $\frac{\partial F_1(x,y)}{\partial x} \le 0$ when $x \ge y$, $\frac{\partial F_1(x,y)}{\partial y} \le 0$ when $x \le y$ and $\frac{\partial F_1(x,y)}{\partial y} \ge 0$ when $x \ge y$. Similar results hold for F_2 also. Thus, F_1 and F_2 increasing with respect to x when $x \leq y$ and decreasing when $x \geq y$. Also F_1 and F_2 decreasing with respect to y when $x \leq y$ and increasing when $x \geq y$.

Now, for PFSSs $p = (\mathbf{F}, \mathbf{E})$, $\tilde{p} = (\mathbf{G}, \mathbf{E})$, and by using the property of the functions F_1 and F_2 in Eqs. (3.1), (3.2) we can be concluded that $E_1(p) \geq E_1(\tilde{p})$ and $E_2(p) \geq E_2(\tilde{p})$ if $\zeta_{\mathbf{F}(e_j)}(u_i) \leq \zeta_{\mathbf{G}(e_j)}(u_i)$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) \geq \vartheta_{\mathbf{G}(e_j)}(u_i)$ for, $\zeta_{\mathbf{G}(e_j)}(u_i) \leq \vartheta_{\mathbf{G}(e_j)}(u_i)$ or $\zeta_{\mathbf{F}(e_j)}(u_i) \geq \zeta_{\mathbf{G}(e_j)}(u_i)$ and $\vartheta_{\mathbf{F}(e_j)}(u_i) \leq \vartheta_{\mathbf{G}(e_j)}(u_i)$ for, $\zeta_{\mathbf{G}(e_j)}(u_i) \geq \vartheta_{\mathbf{G}(e_j)}(u_i)$.

Hence E_1 and E_2 are the valid entropies for PFSSs. □

Remark 3.1. From the definition of proposed entropy, if the membership value and non-membership value corresponding to each parameter sets come nearer then entropy increases and attain its maximum when both are equal. Also, the same values are obtained from E_1 and E_2.

4. Proposed algorithm based on entropy measure

This section presents an algorithm to solve the decision-making problems under the PFSS environment. Also, the numerical example is given to demonstrate it.

4.1. Proposed algorithm

The PFSS is an extension of the existing sets such as FSS, IFSS and hence it is a valuable tool to represent the information during any decision-making process. To address it completely, consider a set of t alternatives $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \ldots, \mathcal{V}^{(t)}$ evaluated by n experts $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$. Each expert $\mathcal{U}_j$ judge the alternatives over the parameters $\mathbf{E} = \{e_1, e_2, \ldots, e_m\}$ and gives their rating values in terms of PFSNs $\mathfrak{B}_{ij}^{(d)} = \left(\zeta_{ij}^{(d)}, \vartheta_{ij}^{(d)}\right)$ such that $0 \leq \zeta_{ij}^{(d)}, \vartheta_{ij}^{(d)} \leq 1$ and $\zeta_{ij}^{(d)} + \vartheta_{ij}^{(d)} \leq 1$, where $1 \leq i \leq n$; $1 \leq j \leq m$; $1 \leq d \leq t$. Then, the problem aims to pick the finest alternative among them. For it, the presented algorithm explain a method to solve the above stated problem using the concept of the entropy measures. The steps of the suggested algorithm is given as below.

Step 1: Arrange the collection information of each alternative $\mathcal{V}^{(d)}; d = 1, 2, \ldots, t$ in a matrix format as:

$$\left(\mathcal{V}^{(d)}, \mathbf{E}\right) = \begin{matrix} \\ \mathcal{U}_1 \\ \mathcal{U}_2 \\ \vdots \\ \mathcal{U}_n \end{matrix} \begin{matrix} \begin{matrix} e_1 & e_2 & \ldots & e_m \end{matrix} \\ \begin{pmatrix} \mathfrak{B}_{11}^{(d)} & \mathfrak{B}_{12}^{(d)} & \ldots & \mathfrak{B}_{1m}^{(d)} \\ \mathfrak{B}_{21}^{(d)} & \mathfrak{B}_{22}^{(d)} & \ldots & \mathfrak{B}_{2m}^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{B}_{n1}^{(d)} & \mathfrak{B}_{n2}^{(d)} & \ldots & \mathfrak{B}_{nm}^{(d)} \end{pmatrix} \end{matrix} \tag{4.1}$$

Step 2: Compute the measurement values p_d ($d = 1, 2, \ldots, t$) of entropies by using either Eq. (3.1) or Eq. (3.2) for each alternative. For instance, using the proposed E_1 entropy, the values of $p_d = E_1\left(\mathcal{V}^{(d)}\right)$ are computed as

$$p_d = \frac{1}{\sqrt{2}-1} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\sin\left(\frac{\pi\left(1+\left(\zeta_{ij}^{(d)}\right)^2-\left(\vartheta_{ij}^{(d)}\right)^2\right)}{4} \right) + \sin\left(\frac{\pi\left(1-\left(\zeta_{ij}^{(d)}\right)^2+\left(\vartheta_{ij}^{(d)}\right)^2\right)}{4} \right) - 1 \right) \tag{4.2}$$

Step 3: As larger the entropy implies that lesser is the vagueness and hence provide more better decision. Thus, compute $p_s = \max_{d=1,2,\ldots,t}\{p_d\}$ and choose the index value.

Step 4: Choose the optimal alternative based on the index terms obtained from Step 3.

4.2. Numerical example

To demonstrate the approach, we provide a numerical example for DMPs under PFSS environment.

Example 4.1. Consider a DMP regarding the selection of the car from a particular company. For it, a person want to select a car from their three different alternatives $\mathcal{V}^{(1)}$ (Hyundi Pvt. Ltd.), $\mathcal{V}^{(2)}$ (Toyota Pvt. Ltd.), and $\mathcal{V}^{(3)}$ (Tata Motors Pvt. Ltd.). To address it completely and remove the hesitation between them, they hire a five experts $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{U}_4$ and $\mathcal{U}_5$ to evaluate the each alternative under the four major set of parameters $\mathbf{E} = \{e_1, e_2, e_3, e_4\}$ where e_1="expensive", e_2= "better mileage", e_3="good engine capacity" and e_4="warranty". Then, the steps of the presented algorithm are executed here to find the best alternatives.

Step 1: The evaluation of each expert $\mathcal{U}_i$ over e_j for the alternative is summarized as below.

$$\left(\mathcal{V}^{(1)}, \mathbf{E}\right) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \quad\mathcal{U}_1 & \quad\quad\mathcal{U}_2 & \quad\quad\mathcal{U}_3 & \quad\quad\mathcal{U}_4 & \quad\quad\mathcal{U}_5 \end{array} \\ \begin{pmatrix} (0.4, 0.5) & (0.2, 0.4) & (0.4, 0.4) & (0.5, 0, 3) & (0.6, 0.2) \\ (0.6, 0.2) & (0.5, 0.6) & (0.7, 0.6) & (0.6, 0.5) & (0.4, 0.4) \\ (0.3, 0.7) & (0.4, 0.8) & (0.3, 0.6) & (0.4, 0.6) & (0.6, 0.3) \\ (0.5, 0.2) & (0.5, 0.2) & (0.5, 0.2) & (0.6, 0.2) & (0.6, 0.2) \end{pmatrix} \end{array};$$

$$\left(\mathcal{V}^{(2)}, \mathbf{E}\right) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \quad\mathcal{U}_1 & \quad\quad\mathcal{U}_2 & \quad\quad\mathcal{U}_3 & \quad\quad\mathcal{U}_4 & \quad\quad\mathcal{U}_5 \end{array} \\ \begin{pmatrix} (0.8, 0.2) & (0.7.0.2) & (0.7, 0.1) & (0.8, 0.1) & (0.9, 0.2) \\ (0.5, 0.5) & (0.6, 0.4) & (0.7, 0.6) & (0.6, 0.2) & (0.7, 0.4) \\ (0.7, 0.3) & (0.6, 0.3) & (0.6, 0.4) & (0.7, 0.4) & (0.6, 0.3) \\ (0.5, 0.2) & (0.6, 0.1) & (0.5, 0.2) & (0.6, 0.2) & (0.6, 0.2) \end{pmatrix} \end{array};$$

and

$$\left(\mathcal{V}^{(3)}, \mathbf{E}\right) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \quad\mathcal{U}_1 & \quad\quad\mathcal{U}_2 & \quad\quad\mathcal{U}_3 & \quad\quad\mathcal{U}_4 & \quad\quad\mathcal{U}_5 \end{array} \\ \begin{pmatrix} (0.6, 0.4) & (0.8, 0.3) & (0.7, 0.4) & (0.8, 0.3) & (0.7, 0.4) \\ (0.9, 0.5) & (0.6, 0.6) & (0.8, 0.5) & (0.6, 0.2) & (0.7, 0.1) \\ (0.7, 0.1) & (0.8, 0.2) & (0.6, 0.3) & (0.7, 0.1) & (0.6, 0.2) \\ (0.3, 0.2) & (0.4, 0.2) & (0.4, 0.3) & (0.6, 0.5) & (0.6, 0.3) \end{pmatrix} \end{array};$$

Step 2: Without loss of generality, we taken the entropy measure E_1 and compute their measurement values as

$$E_1\left(\mathcal{V}^{(1)}\right) = \frac{1}{\sqrt{2}-1}\left(\sin\left(\frac{\pi\left(1+(0.4)^2-(0.5)^2\right)}{4}\right) + \sin\left(\frac{\pi\left(1-(0.4)^2+(0.5)^2\right)}{4}\right) - 1\right)$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{2}-1}\left(\sin\left(\frac{\pi\left(1+(0.2)^2-(0.4)^2\right)}{4}\right)+\sin\left(\frac{\pi\left(1-(0.2)^2+(0.4)^2\right)}{4}\right)-1\right)\\
&+ \ldots\\
&+ \frac{1}{\sqrt{2}-1}\left(\sin\left(\frac{\pi\left(1+(0.6)^2-(0.2)^2\right)}{4}\right)+\sin\left(\frac{\pi\left(1-(0.6)^2+(0.2)^2\right)}{4}\right)-1\right)\\
&= 18.7362
\end{aligned}
$$

Similarity, we get $E_1\left(\mathcal{V}^{(2)}\right) = 16.9694$ and $E_1\left(\mathcal{V}^{(3)}\right) = 17.1242$.

Step 3: From their values, we get the ordering as $E_1\left(\mathcal{V}^{(2)}\right) < E_1\left(\mathcal{V}^{(3)}\right) < E_1\left(\mathcal{V}^{(1)}\right)$.

Step 4: The best alternative is $\mathcal{V}^{(1)}$) model, i.e., Hyundi Pvt. Ltd. car.

4.3. Comparative analysis

The efficiency of the presented approach over the existing approaches under the IFSS or PFSS are examined with the following numerical example.

Example 4.2. Consider three PFSSs $\mathcal{V}^{(1)}$, $\mathcal{V}^{(2)}$ and $\mathcal{V}^{(3)}$ which are evaluated over the three experts and three parameters whose rating values are summarized in terms of IFSSs as follows:

$$
\left(\mathcal{V}^{(1)},\mathbf{E}\right)=\begin{pmatrix}(0.3,0.2) & (0.6,0) & (0.5,0.4)\\(0.6,0.3) & (0.7,0.2) & (0.4,0.3)\\(0.8,0.1) & (0.8,0.1) & (0.6,0.1)\end{pmatrix};
$$

$$
\left(\mathcal{V}^{(2)},\mathbf{E}\right)=\begin{pmatrix}(0.6,0.2) & (0.8,0.1) & (0.8,0.1)\\(0.5,0.5) & (0.7,0.2) & (0.5,0.4)\\(0.7,0.1) & (0.6,0.3) & (0.6,0.3)\end{pmatrix};
$$

and

$$
\left(\mathcal{V}^{(3)},\mathbf{E}\right)=\begin{pmatrix}(0.5,0.4) & (0.4,0.1) & (0.6,0.2)\\(0.6,0.2) & (0.7,0.1) & (0.8,0.1)\\(0.9,0) & (0.5,0.1) & (0.6,0.3)\end{pmatrix}
$$

As IFSS is a special case of PFSS, so we have implemented the proposed entropy on these given information and compared their performance over the existing measures under IFSS [35] as well as the PFSS [40]. The expressions for these existing entropy measures are summarized as follows.

$$
\begin{aligned}
En_1(\mathcal{V}) &= \sum_{j=1}^{m}\sum_{i=1}^{n}\left(1-\zeta_{\mathbf{F}(e_j)}(u_i)-\vartheta_{\mathbf{F}(e_j)}(u_i)\right),\\
En_2(\mathcal{V}) &= \sum_{j=1}^{m}\sum_{i=1}^{n}\left(1-\zeta_{\mathbf{F}(e_j)}(u_i)-\vartheta_{\mathbf{F}(e_j)}(u_i)\right)e^{1-\zeta_{\mathbf{F}(e_j)}(u_i)-\vartheta_{\mathbf{F}(e_j)}(u_i)},\\
En_3(\mathcal{V}) &= \sum_{j=1}^{m}\sum_{i=1}^{n}\left(1-\zeta^3_{\mathbf{F}(e_j)}(u_i)-\vartheta^3_{\mathbf{F}(e_j)}(u_i)\right),
\end{aligned}
$$

$$En_4(\mathcal{V}) = \sum_{j=1}^{m}\sum_{i=1}^{n}\left(1 - \zeta^4_{\mathbf{F}(e_j)}(u_i) - \vartheta^4_{\mathbf{F}(e_j)}(u_i)\right).$$

The results corresponding to these four existing entropy measures along with the proposed measures are listed in Table 1. It is seen from these computed results that the best alternative for the given problem is $\mathcal{V}^{(1)}$ while the worst one is either $\mathcal{V}^{(2)}$ or $\mathcal{V}^{(3)}$. Also, it is noted that the best alternative by the proposed approach results coincides with the results of the existing measures which itself sates the consistency of the proposed measure.

Table 1. Comparative analysis with existing measures.

	With IFSS entropy [35]		With PFSS entropy [40]		Proposed entropy measure
	En_1	En_2	En_3	En_4	
$\mathcal{V}^{(1)}$	2.0000	2.7836	6.6320	7.4104	7.6134
$\mathcal{V}^{(2)}$	1.0000	1.1517	6.1300	7.0790	7.4615
$\mathcal{V}^{(3)}$	1.9000	2.5960	6.3430	7.1175	7.2992

5. Conclusions

In this paper, we developed the entropy measures for the PFSSs. The PFSS is a valuable tool to represent the uncertainties in the data in a more fruitful manner as compared to the other existing sets. Also, the existing sets such as FSS, IFSS, etc., are considered as a special case of the PFSS. Keeping these advantages, the presented paper developed the new entropy measures to compute the degree of the fuzziness for the PFSS. The axiomatic representation and their properties are studied. Further, based on the proposed method, a decision-making algorithm is stated to solve the DMPs. The validity of the proposed measure is demonstrated through a numerical example and compares their performance with some of the existing entropy measures under the IFSS and PFSS environment. In the future, we shall investigate more properties of PFSS and solve some DMPs using diverse fuzzy conditions [41, 42].

Conflict of interest

All authors declare no conflicts of interest.

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