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Research article

On the property of bases of multiple systems in Sobolev-Liouville classes

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Abstract: In the present work we consider the question of preservation of the baseness property for the system of vectors $\varphi = \{\varphi_n\}_{n\in Z^N}$ in the Sobolev-Liouville and Besov classes at small perturbations with the purpose of the further application of obtained results to study decomposition on root vectors of differential operators.

Keywords: Banach Space; Hilbert Space; Sobolev Space; Normed Vector Space

1. Introduction

We say that a series $\sum_{n\in Z^N} c_n\varphi_n$ converges on rectangulars if there exists the limit of the partial sums $S_m = \sum_{|n_1|\le m_1}\sum_{|n_2|\le m_2}\cdots\sum_{|n_N|\le m_N} c_n\varphi_n$ as $\min\limits_{1\le j\le N} m_j \to \infty$.

Let us remind that a system of elements $\varphi = \{\varphi_n\}_{n\in Z^N}$ is called a basis of the Banach space E at summation on rectangulars if any vector $x \in E$ decomposes uniquely in the series

$$x = \sum_{n\in Z^N} c_n\varphi_n \tag{1.1}$$

which is convergent with respect to the norm of the space E at summation by rectangulars. Hence we exclude from consideration Banach spaces which do not possess the property of approximation (see [1] and [2]).

Factors c_n in (1.1) are linear functionals:

$$c_n = f_n(x)\ , \quad n \in Z^N$$

and, according to the well known Banach theorem (see, for example, [3], [4]), there is a constant C_φ such that

$$\left\|\varphi_n\right\|^{-1} \le \|f_n\| \le C_\varphi \left\|\varphi_n\right\|^{-1} .$$

A system of elements $\psi = \{\psi_n\}_{n\in Z^N}$ from the Banach space is said to be ω-linear independent at summation by rectangulars if the equality $\sum_{n\in Z^N} c_n\psi_n = 0$ at summation on rectangulars is impossible at

$$\sum_{n=1}^{\infty} |c_n|^2 \cdot \|\, \psi_n \,\|^2 > 0.$$

2. Main Results

Theorem 2.1. *Let $\{\varphi_n\}_{n\in Z^N}$ be a normed basis in the Banach space E at summation by rectangulars. Further, let the system $\{\psi_n\}_{n\in Z^N}$ be ω–linear independent at summation by rectangulars and $\sum_{n\in Z^N} \|\varphi_n - \psi_n\| < \infty$. Then $\{\psi_n\}_{n\in Z^N}$ is also a basis in E at summation by rectangulars.*

Proof. Fix an N-dimensional vector $\beta = (\beta_1, \beta_2, \dots, \beta_N)$ with nonnegative integer components $\beta_1, \beta_2, \dots, \beta_N$ and define as,

$$\tilde{\psi}_n = \begin{cases} \varphi_n & as \quad |n_1| \le \beta_1, \quad |n_2| \le \beta_2, \dots, |n_N| \le \beta_N, \\ \psi_n & as \quad or\ |n_1| > \beta_1,\ or\ |n_2| > \beta_2, \dots, or\ |n_N| > \beta_N, \quad here\ n \in Z^N. \end{cases}$$

Let us introduce the operator $S : E \to E$ which compares to each element

$$x = \sum_{n\in Z^N} f_n(x)\,\varphi_n = \lim_{\substack{\min m_j \to\infty \\ 1\le j\le N}} \sum_{|n_1|\le m_1} \sum_{|n_2|\le m_2} \dots \sum_{|n_N|\le m_N} f_n(x)\,\varphi_n$$

to the element

$$S\,x = \sum_{n\in Z^N} f_n(x)\left(\varphi_n - \widetilde{\psi}_n\right).$$

Obviously, for sufficiently large $\mu = \min\limits_{1\le j\le N} \beta$, we have

$$\|S\,x\| \le C_\varphi \,\|x\| \sum_{or\ |n_1|>\beta_1} \ \sum_{or\ |n_2|>\beta_2} \dots \sum_{or\ |n_N|>\beta_N} \|\varphi_n - \psi_n\| < \varepsilon\,\|x\|\,.$$

Hence, for the operator U defined by equality

$$U\,x = x - S\,x = \sum_{n\in Z^N} f_n(x)\,\widetilde{\psi}_n,$$

there is an inverse linear operator U^{-1}. Acting on both parts of the equality

$$U^{-1}x = \sum_{n\in Z^N} f_n\left(U^{-1}x\right)\varphi_n$$

with the operator U, we obtain

$$x = \sum_{n\in Z^N} f_n\left(U^{-1}x\right)\widetilde{\psi}_n,$$

which implies that the system $\left\{\widetilde{\psi}_n\right\}_{n\in Z^N}$ forms a basis in E at summation on rectangulars, i.e. each vector $x \in E$ is decomposed uniquely in the series

$$x = \sum_{n\in Z^N} f_n\left(U^{-1}x\right)\widetilde{\psi}_n = \lim_{\substack{\min m_j\to\infty \\ 1\le j\le N}} \sum_{|n_1|\le m_1}\sum_{|n_2|\le m_2}\cdots\sum_{|n_N|\le m_N} f_n\left(U^{-1}x\right)\widetilde{\psi}_n$$

which is convergent with respect to the norm of the space E at summation on rectangulars.

Since the system $\left\{\widetilde{\psi}_n\right\}_{n\in Z^N}$forms a basis in E at summation on rectangulars, then

$$\psi_k = \sum_{|n_1|\le\beta_1}\sum_{|n_2|\le\beta_2}\cdots\sum_{|n_N|\le\beta_N} f_n\left(U^{-1}\psi_k\right)\varphi_n + \sum_{or\ |n_1|>\beta_1}\sum_{or\ |n_2|>\beta_2}\cdots\sum_{or\ |n_N|>\beta_N} f_n\left(U^{-1}\psi_k\right)\psi_n = x_k^1 + x_k^2,$$

here $k = (k_1, k_2, \ldots, k_N)$ is a multi-index with components $|k_1| \le \beta_1, |k_2| \le \beta_2, \ldots, |k_N| \le \beta_N$, and

$$\begin{aligned} x_k^1 &= \sum_{|n_1|\le\beta_1}\sum_{|n_2|\le\beta_2}\cdots\sum_{|n_N|\le\beta_N} f_n\left(U^{-1}\psi_k\right)\varphi_n, \\ x_k^2 &= \sum_{or\ |n_1|>\beta_1}\sum_{or\ |n_2|>\beta_2}\cdots\sum_{or\ |n_N|>\beta_N} f_n\left(U^{-1}\psi_k\right)\psi_n \end{aligned}$$

ω-linear independence of $\{\psi_n\}_{n\in Z^N}$ at summation on rectangulars implies linear independence of $\left\{x_k^1\right\}$. As concepts of linear independence and baseness are equivalent in finite dimensional space,

$$\varphi_n = \sum_{|k_1|\le\beta_1}\sum_{|k_2|\le\beta_2}\cdots\sum_{|k_N|\le\beta_N} \alpha_{nk} x_k^1$$

is a multi-index with components $|n_1| \le \beta_1, \ |n_2| \le \beta_2, \ \ldots, \ |n_N| \le \beta_N$ for $n = (n_1, n_2, \ldots, n_N) \in Z^N$.

Hence we have

$$\begin{aligned} x = &\sum_{|n_1|\le\beta_1}\sum_{|n_2|\le\beta_2}\cdots\sum_{|n_N|\le\beta_N} f_n\left(U^{-1}x\right)\varphi_n + \sum_{or\ |n_1|>\beta_1}\sum_{or\ |n_2|>\beta_2}\cdots\sum_{or\ |n_N|>\beta_N} f_n\left(U^{-1}x\right)\psi_n = \\ = &\sum_{|k_1|\le\beta_1}\sum_{|k_2|\le\beta_2}\cdots\sum_{|k_N|\le\beta_N} (T)\,\psi_k + \sum_{or\ |n_1|>\beta_1}\sum_{or\ |n_2|>\beta_2}\cdots\sum_{or\ |n_N|>\beta_N} f_n\left(U^{-1}x\right)\psi_n = \\ = &\sum_{|k_1|\le\beta_1}\sum_{|k_2|\le\beta_2}\cdots\sum_{|k_N|\le\beta_N} (T)\,\psi_k + \sum_{or\ |s_1|>\beta_1}\sum_{or\ |s_2|>\beta_2}\cdots\sum_{or\ |s_N|>\beta_N} \\ &\left(\sum_{|k_1|\le\beta_1}\sum_{|k_2|\le\beta_2}\cdots\sum_{|k_N|\le\beta_N} (T)\, f_s\left(U^{-1}\psi_k\right)\right)\psi_s + \sum_{or\ |n_1|>\beta_1}\sum_{or\ |n_2|>\beta_2}\cdots\sum_{or\ |n_N|>\beta_N} f_n\left(U^{-1}x\right)\psi_n \end{aligned}$$

Here ,

$$T = \sum_{|n_1|\le\beta_1}\sum_{|n_2|\le\beta_2}\cdots\sum_{|n_N|\le\beta_N} f_n\left(U^{-1}x\right)a_{nk}$$

It means that the system $\{\psi_n\}_{n\in Z^N}$ is a basis in the Banach space E at summation on rectangulars. Hence, Theorem (2.1) is proved. □

When $N = 1$ Theorem (2.1) was proved in [6]

Remark 1. At absence of ω-linear independence of the system $\{\psi_n\}_{n\in Z^N}$ at summation on rectangulars, one states that the system $\{\psi_n\}_{n\in Z^N}$ is a basis (probably, overfilling) with the finite defect in the Banch space E.

A function $f(x) \in L_p\left(T^N\right)$ belongs to the space $W_p^s\left(T^N\right)$, if all its partial derivatives $D^\alpha f$ (in the sense of the theory of distributions) of the order $|\alpha| = s$ belong to $L_p\left(T^N\right)$, i.e. the norm

$$\|f\|_{W_p^s(T^N)} = \|f\|_{L_p(T^N)} + \sum_{|\alpha|=s} \|D^\alpha f\|_{L_p(T^N)} ,$$

where $1 \le p < \infty$, $s = 0,\ 1,\ 2,\ ...$, is finite.

In the case of $N = 1$ belonging of a function $f(x)$ to the class $W_p^s\left(T^N\right)$ it means that $f(x)$ has $s-1$ continuous derivatives, $f^{(s-1)}(x)$ is absolutely continuous, and $f^{(s)}(x)$ belongs to $L_p(T)$.

Corollary 1. *Let $\psi_n(x) = (2\pi)^{-\frac{N}{p}} \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|\right)^{-1} \cdot \exp(i\lambda_n x) + \alpha_n(x)$, where $\lambda_n \ne \lambda_m$ as $n \ne m$, be an ω-linear independent system of functions satisfying the following conditions:*

1. $\sum_{n\in Z^N} |\lambda_n - n| < \infty$;
2. $\sum_{n\in Z^N} \|\alpha_n(x)\|_{W_p^s(T^N)} < \infty$.

Then the system of functions $\{\psi_n\}_{n\in Z^N}$ forms at summation on rectangulars a basis in $W_p^s\left(T^N\right)$, $1 < p < \infty$, $s = 0,\ 1,\ 2,\ \dots$.

Theorem 2.2. *Let*

$$\psi_n(x) = (2\pi)^{-\frac{N}{2}} \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \exp(i\lambda_n x) + \alpha_n(x),$$

where $\lambda_n \ne \lambda_m$ as $n \ne m$, be an ω-linear independent system of functions satisfying the following conditions:

1. $k = \sqrt{\sup_n \frac{\theta^2 + \sum_{|\alpha|=s}\left(\theta^2|\lambda_n^\alpha|^2 + |\lambda_n^\alpha - n^\alpha|^2\right)}{1+\sum_{|\alpha|=s}|n^\alpha|^2}} < 1$, *here* $\theta = \exp(MN\pi) - 1$,
2. $\sum_{n\in Z^N} \|\alpha_n(x)\|^2_{W_2^s(T^N)} < \infty$.

$$M = \sup_j \sup_{n_j} \left|\lambda_{n_j} - n_j\right| ;$$

Then the system of functions $\{\psi_n(x)\}_{n\in Z^N}$ forms the Riesz basis in the space $W_2^s\left(T^N\right)$.

Proof. It is known that the system of functions $\varphi_n(x) = (2\pi)^{-\frac{N}{2}} \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \exp(inx)$ forms an orthonormal basis in the space $W_2^s\left(T^N\right)$. The norm in this space is introduced in such a way at following :

$$\|f\|^2_{W_2^s(T^N)} = \|f\|^2_{L_2(T^N)} + \sum_{|\alpha|=s} \|D^\alpha f\|^2_{L_2(T^N)} .$$

Let

$$\tilde{\psi}_n(x) = (2\pi)^{-\frac{N}{2}} \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \exp(i\lambda_n x),$$

where $\lambda_n \neq \lambda_m$ as $n \neq m$, be an ω-linear independent system of functions satisfying the condition:

$$k = \sqrt{\sup_n \frac{\theta^2 + \sum_{|\alpha|=s}\left(\theta^2 \left|\lambda_n^\alpha\right|^2 + \left|\lambda_n^\alpha - n^\alpha\right|^2\right)}{1 + \sum_{|\alpha|=s} |n^\alpha|^2}} < 1,$$

here $\theta = \exp(MN\pi) - 1$, $M = \sup_j \sup_{n_j} \left|\lambda_{n_j} - n_j\right|$.

Further, let $\{a_n\}$ be a finite system of complex numbers. Then

$$\left\|\sum_n a_n \left(\widetilde{\psi}_n - \varphi_n\right)\right\|^2_{W_2^s(T^N)} = \left\|\sum_n a_n \left(\widetilde{\psi}_n - \varphi_n\right)\right\|^2_{L_2(T^N)} + \sum_{|\alpha|=s} \left\|D^\alpha \left(\sum_n a_n \left(\widetilde{\psi}_n - \varphi_n\right)\right)\right\|^2_{L_2(T^N)} =$$

$$= (2\pi)^{-N} \left[\left\|\sum_n a_n \cdot \left(\left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} (\exp(i\lambda_n x) - \exp(inx))\right)\right\|^2_{L_2(T^N)} + \right.$$

$$\left. + \sum_{|\alpha|=s} \left\|D^\alpha \left(\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot (\exp(i\lambda_n x) - \exp(inx))\right)\right\|^2_{L_2(T^N)}\right]$$

As we have,

$$\left\|\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot (\exp(i\lambda_n x) - \exp(inx))\right\|_{L_2(T^N)} \leq$$

$$\leq \sum_{k=1}^\infty \frac{1}{k!} \left\|\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot [i(\lambda_n - n)x]^k \cdot \exp(inx)\right\|_{L_2(T^N)}.$$

Further,

$$\left\|\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot [i(\lambda_n - n)x]^k \cdot \exp(inx)\right\|_{L_2(T^N)} =$$

$$= \Big\| \sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \left(\sum_{\beta_1+\beta_2+\ldots+\beta_N=k} \frac{k!}{\beta_1!\beta_2!\ldots\beta_N!} \cdot \right.$$

$$\left. \cdot \prod_{j=1}^N \left(\lambda_{n_j} - n_j\right)^{\beta_j} \cdot x_j^{\beta_j}\right) \cdot \exp(inx)\Big\|_{L_2(T^N)} =$$

$$= \Big\|\sum_{\beta_1+\beta_2+\ldots+\beta_N=k} \frac{k!}{\beta_1!\beta_2!\ldots\beta_N!} \cdot \prod_{j=1}^N x_j^{\beta_j} \cdot \left(\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot\right.$$

$$\cdot \prod_{j=1}^{N} \left(\lambda_{n_j} - n_j\right)^{\beta_j} \cdot \exp(inx) \Bigg\|_{L_2(T^N)} \leq \sum_{\beta_1+\beta_2+...+\beta_N=k} \frac{k!}{\beta_1!\beta_2!...\beta_N!} \cdot \pi^k \cdot$$

$$\cdot \left\| \sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \prod_{j=1}^{N} \left(\lambda_{n_j} - n_j\right)^{\beta_j} \cdot \exp(inx) \right\|_{L_2(T^N)} \leq$$

$$\sum_{\beta_1+\beta_2+...+\beta_N=k} \frac{k!}{\beta_1!\beta_2!...\beta_N!} \cdot \pi^k \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \cdot \prod_{j=1}^{N} \left|\lambda_{n_j} - n_j\right|^{2\beta_j} \cdot (2\pi)^N \right)^{\frac{1}{2}} \leq$$

$$\leq (2\pi)^{\frac{N}{2}} \sum_{\beta_1+\beta_2+...+\beta_N=k} \frac{k!}{\beta_1!\beta_2!...\beta_N!} \cdot \pi^k \cdot M^k \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \right)^{\frac{1}{2}} =$$

$$= (2\pi)^{\frac{N}{2}} \cdot \pi^k \cdot M^k \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \right)^{\frac{1}{2}} \cdot \sum_{\beta_1+\beta_2+...+\beta_N=k} \frac{k!}{\beta_1!\beta_2!...\beta_N!} =$$

$$= (2\pi)^{\frac{N}{2}} \cdot \pi^k \cdot M^k \cdot N^k \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \right)^{\frac{1}{2}},$$

where summation is carried out on all integer nonnegative $\beta_1, \beta_2, ..., \beta_N$ such that $\beta_1 + \beta_2 + ... + \beta_N = k$, $M = \sup_j \sup_{n_j} \left|\lambda_{n_j} - n_j\right|$, we get

$$\left\| \sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \left(\exp(i\lambda_n x) - \exp(inx)\right) \right\|_{L_2(T^N)} \leq$$

$$\leq (2\pi)^{\frac{N}{2}} \cdot \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \pi^k \cdot M^k \cdot N^k \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \right)^{\frac{1}{2}} =$$

$$= (2\pi)^{\frac{N}{2}} \left(\exp(MN\pi) - 1\right) \cdot \left(\sum_n |a_n|^2 \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-1} \right)^{\frac{1}{2}}.$$

Further,

$$\sum_{|\alpha|=s} \left\| D^\alpha \left(\sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot \left(\exp(i\lambda_n x) - \exp(inx)\right) \right) \right\|_{L_2(T^N)} =$$

$$= \sum_{|\alpha|=s} \left\| \sum_n a_n \cdot \left(1 + \sum_{|\alpha|=s} |n^\alpha|^2\right)^{-\frac{1}{2}} \cdot D^\alpha \left(\exp(i\lambda_n x) - \exp(inx)\right) \right\|_{L_2(T^N)}.$$

Therefore,

$$\left\|\sum_n a_n\left(\widetilde{\psi}_n-\varphi_n\right)\right\|_{W_2^s(T^N)} \le (\exp(MN\pi)-1)^2\cdot\left(\sum_n |a_n|^2\cdot\left(1+\sum_{|\alpha|=s}|n^\alpha|^2\right)^{-1}\right)+$$

$$+\sum_{|\alpha|=s}\left((\exp(MN\pi)-1)^2\cdot\left(\sum_n |a_n|^2\cdot\left|\lambda_n^\alpha\right|^2\cdot\left(1+\sum_{|\alpha|=s}|n^\alpha|^2\right)^{-1}\right)+\right.$$

$$\left.+\sum_n |a_n|^2\cdot\left|\lambda_n^\alpha-n^\alpha\right|^2\cdot\left(1+\sum_{|\alpha|=s}|n^\alpha|^2\right)^{-1}\right).$$

Hence, we have

$$\left\|\sum_n a_n\left(\widetilde{\psi}_n-\varphi_n\right)\right\|_{W_2^s(T^N)} \le k\cdot\left(\sum_n |a_n|^2\right)^{\frac{1}{2}}.$$

Since $k < 1$, then by theorem by R. Paly and N. Winner ([5], p.224) the system of functions $\left\{\widetilde{\psi}_n(x)\right\}_{n\in Z^N}$ forms a basis in the space $W_2^s\left(T^N\right)$. On the other hand, the theorem by N.K. Bary (see [3], p. 382) implies that the ω-linear system of functions $\{\psi_n(x)\}_{n\in Z^N}$, quadratically close to the Riesz basis $\left\{\widetilde{\psi}_n(x)\right\}_{n\in Z^N}$ in $W_2^s\left(T^N\right)$, is a Riesz basis in $W_2^s\left(T^N\right)$. Hence, Theorem (2.2) is proved. □

Theorem 2.3. *Let* $\psi_n(x) = (2\pi)^{-\frac{N}{p}}\cdot\left(1+|n|^2\right)^{-\frac{s}{2}}\cdot\exp(i\lambda_n x)+\alpha_n(x)$, $n\in Z^N$,*where* $\lambda_n\ne\lambda_m$, *as* $n\ne m$, *ω be an linear independent system of functions at summation on rectangles that satisfies the following conditions:*

1. $\sum_{n\in Z^N}\left\|\sum_{k\in Z^N}\left(\frac{1+|k|^2}{1+|n|^2}\right)^{\frac{s}{2}}\left(\prod_{j=1}^N\frac{\sin\left(\lambda_{n_j}-k_j\right)\pi}{\left(\lambda_{n_j}-k_j\right)\pi}-\delta_{nk}\right)\cdot\exp(ikx)\right\|_{L_p(T^N)}<\infty;$
2. $\sum_{n\in Z^N}\|\alpha_n(x)\|_{L_p^s(T^N)}<\infty.$

Then, the summation on rectangles system functions $\{\psi_n\}_{n\in Z^N}$ *forms a basis* $L_p^s\left(T^N\right)$, $1<p<\infty$.

Proof. By Theorem Sokol-Sokolowski, the system functions $\varphi_n(x) = (2\pi)^{-\frac{N}{p}}\exp(inx)$ forms a normalized basis in $L_p\left(T^N\right)$ at summation on rectangles, i.e, for every $f\in L_p\left(T^N\right)$, there is a single row $\sum_{n\in Z^N} f_n\exp(inx)$ such that

$$S_m(x) = \sum_{|n_1|\le m_1}\sum_{|n_2|\le m_2}\dots\sum_{|n_N|\le m_N} f_n\exp(inx)$$

which partial sums converges (on rectangles) to function $f(x)$ in $L_p\left(T^N\right)$ with respect to norm topology ,while $\min_{1\le j\le N} m_j\to\infty$.

Similarly, the system functions

$$\varphi_n(x) = (2\pi)^{-\frac{N}{p}}\cdot\left(1+|n|^2\right)^{-\frac{s}{2}}\cdot\exp(inx)$$

forms a normalized basis in $L_p^s\left(T^N\right)$ at summation on rectangles, ie, for every $f\in L_p^s\left(T^N\right)$, there is a single row

$$\sum_{n\in Z^N}\widetilde{f_n}\cdot\varphi_n(x)$$

such that

$$S_m(x) = \sum_{|n_1|\le m_1} \sum_{|n_2|\le m_2} ... \sum_{|n_N|\le m_N} \widetilde{f_n} \cdot \varphi_n(x)$$

which partial sums converges (on rectangles) to function $f(x)$ in $L_p^s\left(T^N\right)$ with respect to norm topology, while $\min\limits_{1\le j\le N} m_j \to \infty$.
Consequently,

$$\|f(x) - S_m(x)\|_{L_p^s(T^N)} = \left\| \sum_{n\in Z^N} (2\pi)^{-\frac{N}{p}} \cdot \widetilde{f_n} \cdot \exp(inx) \right.$$
$$\left. - \sum_{|n_1|\le m_1} \sum_{|n_2|\le m_2} ... \sum_{|n_N|\le m_N} (2\pi)^{-\frac{N}{p}} \cdot \widetilde{f_n} \cdot \exp(inx) \right\|_{L_p(T^N)} \to 0$$

while $\min\limits_{1\le j\le N} m_j \to \infty$ where $p \ge 1,\ s \ge 0$, $\widetilde{f_n} = (2\pi)^{-\frac{N}{q}} \cdot \left(1 + |n|^2\right)^{\frac{s}{2}} \cdot \int_{T^N} f(x) \cdot \exp(-inx)dx$, $\frac{1}{p} + \frac{1}{q} = 1$.
We have

$$\widetilde{\psi_n}(x) = (2\pi)^{-\frac{N}{p}} \cdot \left(1 + |n|^2\right)^{-\frac{s}{2}} \cdot \exp(i\lambda_n x)$$

where $\lambda_n \ne \lambda_m$, while $n \ne m$,be an ω - linear independent system of functions that satisfies the following conditions:

$$\sum_{n\in Z^N} \left\| \sum_{k\in Z^N} \left(\frac{1 + |k|^2}{1 + |n|^2} \right)^{\frac{s}{2}} \left(\prod_{j=1}^{N} \frac{\sin\left(\lambda_{n_j} - k_j\right)\pi}{\left(\lambda_{n_j} - k_j\right)\pi} - \delta_{nk} \right) \cdot \exp(ikx) \right\|_{L_p(T^N)} < \infty.$$

as

$$\left\|\varphi_n - \widetilde{\psi_n}\right\|_{L_p^s(T^N)} = \left\| \sum_{k\in Z^N} \left(1 + |k|^2\right)^{\frac{s}{2}} \left(\varphi_n - \widetilde{\psi_n}\right)_k \cdot \exp(ikx) \right\|_{L_p(T^N)}$$

where

$$\left(\varphi_n - \widetilde{\psi_n}\right)_k = (2\pi)^{-N} \int_{T^N} \left[\varphi_n(x) - \widetilde{\psi_n}(x)\right] \cdot \exp(-ikx)dx$$

are Fourier coefficients. Hence we get

$$\left(\varphi_n - \widetilde{\psi_n}\right)_k = (2\pi)^{-N} \int_{T^N} \left[\varphi_n(x) - \widetilde{\psi_n}(x)\right] \cdot \exp(-ikx)dx$$
$$= (2\pi)^{-N-\frac{N}{p}} \left(1 + |n|^2\right)^{-\frac{s}{2}} \int_{T^N} \left[\exp(inx) - \exp(i\lambda_n x)\right] \cdot \exp(-ikx)dx$$

$$= (2\pi)^{-N-\frac{N}{p}} \left(1 + |n|^2\right)^{-\frac{s}{2}} \left[\int_{T^N} \exp\left(i\left(n - k\right)x\right)dx - \int_{T^N} \exp\left(i\left(\lambda_n - k\right)x\right)dx \right] =$$

$$= (2\pi)^{-\frac{N}{p}} \left(1 + |n|^2\right)^{-\frac{s}{2}} \left[\delta_{nk} - (2\pi)^{-N} \int_{T^N} \exp\left(i\left(\lambda_n - k\right)x\right)dx \right] =$$

$$= (2\pi)^{-\frac{N}{p}} \left(1+|n|^2\right)^{-\frac{s}{2}} \left[\delta_{nk} - (2\pi)^{-N} \prod_{j=1}^{N} \int_{-\pi}^{\pi} \exp\left(i\left(\lambda_{n_j} - k_j\right) x_j\right) dx_j\right] =$$

$$= (2\pi)^{-\frac{N}{p}} \left(1+|n|^2\right)^{-\frac{s}{2}} \left[\delta_{nk} - (2\pi)^{-N} \prod_{j=1}^{N} \frac{1}{i(\lambda_{n_j} - k_j)} \exp\left(i\left(\lambda_{n_j} - k_j\right) x_j\right)\Big|_{-\pi}^{\pi}\right] =$$

$$= (2\pi)^{-\frac{N}{p}} \left(1+|n|^2\right)^{-\frac{s}{2}} \left[\delta_{nk} - \prod_{j=1}^{N} \frac{\sin\left(\lambda_{n_j} - k_j\right)\pi}{\left(\lambda_{n_j} - k_j\right)\pi}\right]$$

in this way,

$$\left(\varphi_n - \widetilde{\psi}_n\right)_k = (2\pi)^{-\frac{N}{p}} \left(1+|n|^2\right)^{-\frac{s}{2}} \left[\delta_{nk} - \prod_{j=1}^{N} \frac{\sin\left(\lambda_{n_j} - k_j\right)\pi}{\left(\lambda_{n_j} - k_j\right)\pi}\right]$$

hence

$$\sum_{n\in Z^N} \|\varphi_n - \psi_n\|_{L_p^s(T^N)} = \sum_{n\in Z^N} \left\|\varphi_n - \widetilde{\psi}_n - \alpha_n(x)\right\|_{L_p^s(T^N)} \le \sum_{n\in Z^N} \left\|\varphi_n - \widetilde{\psi}_n\right\|_{L_p^s(T^N)}$$

$$+ \sum_{n\in Z^N} \|\alpha_n(x)\|_{L_p^s(T^N)} =$$

$$= \sum_{n\in Z^N} \left\| \sum_{k\in Z^N} \left(\frac{1+|k|^2}{1+|n|^2}\right)^{\frac{s}{2}} \cdot (2\pi)^{-\frac{N}{p}} \left[\delta_{nk} - \prod_{j=1}^{N} \frac{\sin\left(\lambda_{n_j}-k_j\right)\pi}{\left(\lambda_{n_j}-k_j\right)\pi}\right] \cdot \exp(ikx)\right\|_{L_p(T^N)}$$

$$+ \sum_{n\in Z^N} \|\alpha_n(x)\|_{L_p^s(T^N)} < \infty.$$

□

By Theorem (2.2) we have the proof of the Theorem (2.3).

References

1. Grothendieck A., *Produits tensoriels topologiques et espaces nucleaires,* Mem. Amer. Math. Soc., **16** (1955).
2. Enflo P., *A counterexample to the approximation problem in Banach spaces,* Acta Math., **130** (1973), 309-317.
3. Gokhberg I.C., Kreyn M.G., *Introduction to the Theory of Linear non Self-adjoint Operators in the Hilbert Space,* Moscow: Nauka, 1969.

4. Gokhberg I.C., Markus A.S., *Stability for bases of Banach and Hilbert spaces,* Izvestiya AN MSSR., **5** (1962), 17-35.

5. Riesz F., Sekyofalvi-Nad B., *Lectures on Functional Analysis,* Moscow, "Mir", 1979.

6. Shakirbay G. Kasimov., *On a Property of Bases in Banach and Hilbert Spaces,* Malaysian Journal of Mathematical Sciences, **5** (2011), 229-240.

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