



*Research article*

## The Riemann problem for a Two–Phase model for road traffic with fixed or moving constraints

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**Abstract:** We define two Riemann solvers for the Two–Phase traffic model proposed in [1], given by a system of two conservation laws with Lipschitz continuous flow, under fixed and moving constraints. From the traffic point of view this situation corresponds to the study of vehicular flow with fixed constraints as, for instance, a traffic light, a toll gate or a construction site. On the other hand, the presence of a slow moving large vehicle, like a bus, corresponds to the case of a moving constraint. In the latter case, we have to consider a mixed system where the conservation laws are coupled with an ordinary differential equation describing the trajectory of the large vehicle.

**Keywords:** hyperbolic systems of conservation laws; continuum traffic models; evolution problems; unilateral flux constraints; moving bottlenecks

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### 1. Introduction

In this paper we consider the Two–Phase traffic model proposed in [1] subject to fixed and moving constraints. In particular, we propose two Riemann solvers in order to describe the evolution of car traffic affected by an obstacle as, for instance, a traffic light, a toll gate, a construction site or by the presence of a slow moving large vehicle, like a bus.

The present Two–Phase traffic model consists of a system of two conservation laws with Lipschitz continuous flow and it falls in the class of macroscopic second order traffic models displaying two distinct traffic regimes or phases: the free phase  $F$  and the congested phase  $C$ . In the free phase the model reduces to a linear degenerate system, where the speed is constantly equal to  $V_{\max}$ , while in the congested phase the model is given by a  $2 \times 2$  system of conservation laws.

There is a large bibliography related to macroscopic models describing traffic dynamics. The basic one is the classical *first order* Lighthill–Whitham [2] and Richards [3] (LWR) model, consisting of a single conservation law. *Second order* models are based on two equations, as the Aw–Rascle–Zhang

(ARZ) model [4,5], the GARZ model [6] and the collapsed GARZ (CGARZ) [7]. The class here under consideration, namely that of two–phases or phase transition models as [1], is a subset of *second order* models. Thanks to the distinction in phases, models in this class are able to describe traffic evolution both with the simple structure typical of the *first order* models in the free phase and with the richest and more advanced structure of *second order* models in the congested regime. For further results related to two–phase and phase transition models, see [8–15].

We propose here two Riemann solvers for the Two–Phase traffic model introduced in [1]. The first one provides a description of traffic evolution under the presence of a fixed obstacle. From the analytical point of view, this corresponds to the study of a Riemann problem for a system of two conservation laws under a constraint on the first component of the flux. For the second Riemann solver, we have to consider a mixed PDE–ODE system describing a slowly moving large vehicle that reduces the capacity of the road. The resulting model consists of the present system of two conservation laws, coupled with an ordinary differential equation describing the trajectory of the large vehicle.

Conservation laws with pointwise unilateral constraints on the flow have been widely studied in the last decade. Their peculiarity is the possible rising of non–classical shocks at the constraint position, violating the classical Kruřkov [16] or Lax [17] entropy admissibility conditions. Scalar conservation laws with fixed flux constraints were introduced in [18], see also [19], while results in the case of second order models are provided in [20–23] and two–phase models with fixed constraints are considered in [24–26]. Problems with moving constraints have been considered in [27] in the scalar case and in [28] for the ARZ model. A Riemann problem for the GARZ model [6] with moving constraint is proposed in [29], where the distinction between free and congested traffic phases is not considered, see Remark 2.1. An extension to non local constraints is considered in [30].

The article is organized as follows. Section 2 gives a description of the model from an analytical point of view, Section 3 is dedicated to the solution of the Riemann problem for the Two–Phase traffic model with fixed constraint and, finally, Section 4 contains the definition of the Riemann problem for the mixed PDE–ODE model with moving constraint. Conclusions are gathered in the final Section 5.

## 2. Description of the Two–Phase model

The Two–Phase model introduced in [1] has been derived as an extension of the LWR model, given by a single conservation law, by assuming further assumptions on the speed. There are two key ideas in the development of this traffic model: each driver has a his/her own preferred maximal speed and there exists a maximal velocity that all the drivers have to respect. The resulting model is a system of two equations that we write in the following two equivalent forms,

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t w + v(\rho, w) \partial_x w = 0, \end{cases} \quad \begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) = 0, \end{cases} \quad (2.1)$$

$$\text{where } v(\rho, w) = \min \{V_{\max}, w \psi(\rho)\} .$$

Here  $\rho$  is the traffic density, the Lagrangian marker  $w \in [\underline{w}, \bar{w}]$  is the maximal speed of each driver,  $V_{\max}$  is the maximal velocity that all the drivers have to respect and  $\psi = \psi(\rho)$  is a decreasing function. In (2.1), left, the system is not in conservation form, while on the right, we have a  $2 \times 2$  system of conservation laws and  $(\rho, \rho w)$  are the conserved variables. Below we mostly refer to the couple  $(\rho, w)$ .

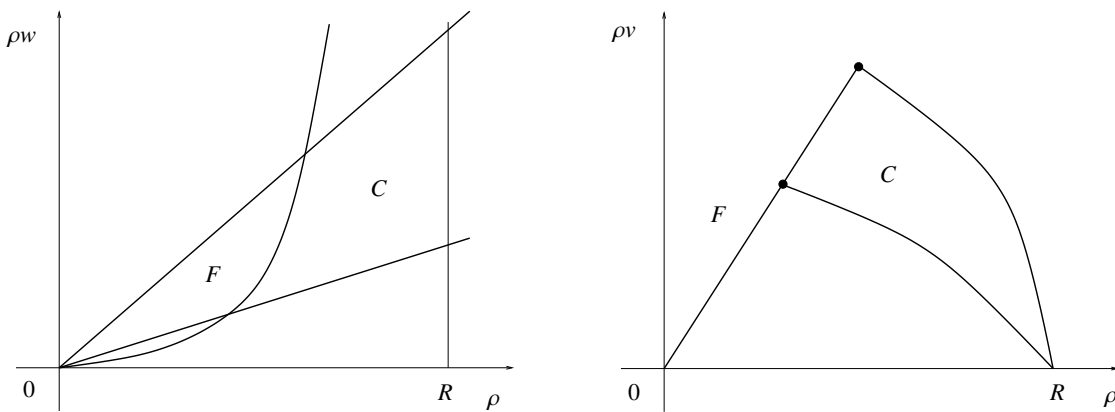
**Remark 2.1.** System (2.1), left, can be rewritten as a  $2 \times 2$  system of conservation laws in infinitely many ways, keeping the same smooth solutions. In particular, using as conserved variables quantities of the form  $(\rho, \rho f(w))$ , for a suitable smooth invertible function  $f$ , also weak solutions remain unaltered, when expressed in the  $(\rho, w)$  coordinates, see [1, Remark 5.3] for more details.

In (2.1), the presence of the minimum in the speed law naturally generates two phases: namely the free and the congested one, described by the sets

$$F = \{(\rho, w) \in [0, R] \times [\underline{w}, \bar{w}]: v(\rho, \rho w) = V_{\max}\}, \quad (2.2)$$

$$C = \{(\rho, w) \in [0, R] \times [\underline{w}, \bar{w}]: v(\rho, \rho w) = w \psi(\rho)\}, \quad (2.3)$$

represented in Figure 1. Note that  $F$  and  $C$  are closed sets and  $F \cap C \neq \emptyset$ . Note also that  $F$  is a one-dimensional manifold in the  $(\rho, \rho v)$  plane of the fundamental diagram, while it is a two-dimensional manifold in the  $(\rho, \rho w)$  coordinates.



**Figure 1.** The free phase  $F$  and the congested phase  $C$  resulting from (2.1) in the coordinates, from left to right,  $(\rho, \rho w)$  and  $(\rho, \rho v)$ .

As in [1], we recall the following assumptions:

**(H-1)**  $R, \underline{w}, \bar{w}, V_{\max}$  are positive constants, with  $V_{\max} < \underline{w} < \bar{w}$ ;  $R$  is the maximal possible density.

**(H-2)**  $\psi \in C^2([0, R]; [0, 1])$  is such that  $\psi(0) = 1$ ,  $\psi(R) = 0$ , and, for every  $\rho \in [0, R]$ ,  $\psi'(\rho) \leq 0$ ,  $\frac{d^2}{d\rho^2}(\rho \psi(\rho)) \leq 0$ .

Above, the inequality  $V_{\max} < \underline{w}$  ensures that, in principle, all drivers might exceed the speed limit or, in other words, that the speed limit does affect every driver.

Below, we list the waves that we use in the present paper and we recall the solutions to standard Riemann problems for model (2.1); see also [1].

**(FF)** *First Family Wave*: a wave connecting a left state  $(\rho_l, w_l) \in C$  with a right state  $(\rho_r, w_r) \in C$  such that  $w_l = w_r$ .

**(SW)** *Second Family Wave*: a wave connecting a left state  $(\rho_l, w_l) \in C$  with a right state  $(\rho_r, w_r) \in C$  such that  $v(\rho_l, w_l) = v(\rho_r, w_r)$ .

**(LW)** *Linear Wave*: a wave connecting two states in the free phase.

**(PT)** *Phase Transition Wave*: a wave connecting a left state  $(\rho_l, w_l) \in F$  with a right state  $(\rho_r, w_r) \in C$  satisfying  $w_l = w_r$ .

Note that in the first case **(FF)**, first family waves can be either shocks or rarefactions; in all other cases, discontinuities (linear waves, contact discontinuities or phase transitions) appear.

As in [32], we introduce also the following technical assumption:

**(H-3)** Waves of the first family in the congested phase  $C$  have negative speed.

Under the assumptions **(H-1)**, **(H-2)** and **(H-3)**, for all states  $(\rho_l, w_l)$  and  $(\rho_r, w_r) \in F \cup C$ , the Riemann problem consisting of (2.1) with initial data

$$\rho(0, x) = \begin{cases} \rho_l & \text{if } x < 0 \\ \rho_r & \text{if } x \geq 0 \end{cases} \quad w(0, x) = \begin{cases} w_l & \text{if } x < 0 \\ w_r & \text{if } x \geq 0 \end{cases} \quad (2.4)$$

admits a unique self similar weak solution constructed as follows:

**(RS1)** If  $(\rho_l, w_l), (\rho_r, w_r) \in F$ , the solution attains values in  $F$  and consists of a linear wave separating  $(\rho_l, w_l)$  from  $(\rho_r, w_r)$ .

**(RS2)** If  $(\rho_l, w_l), (\rho_r, w_r) \in C$ , the solution attains values in  $C$  and consists of a first family wave (shock or rarefaction) between  $(\rho_l, w_l)$  and a middle state  $(\rho_m, w_m)$ , followed by a second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . The middle state  $(\rho_m, w_m)$  is in  $C$  and is uniquely characterized by the two conditions  $w_m = w_l$  and  $v(\rho_m, w_m) = v(\rho_r, w_r)$ .

**(RS3)** If  $(\rho_l, w_l) \in C$  and  $(\rho_r, w_r) \in F$ , the solution attains values in  $F \cup C$  and consists of a first family wave separating  $(\rho_l, w_l)$  from a middle state  $(\rho_m, w_m)$  and by a linear wave separating  $(\rho_m, w_m)$  from  $(\rho_r, w_r)$ . The middle state  $(\rho_m, w_m)$  belongs to the intersection between  $F$  and  $C$  and is uniquely characterized by the two conditions  $w_m = w_r$  and  $v(\rho_m, w_m) = V_{\max}$ .

**(RS4)** If  $(\rho_l, w_l) \in F$  and  $(\rho_r, w_r) \in C$ , then the solution attains values in  $F \cup C$  and consists of a phase transition wave between  $(\rho_l, w_l)$  and a middle state  $(\rho_m, w_m)$ , followed by a second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . The middle state  $(\rho_m, w_m)$  is in  $C$  and is uniquely characterized by the two conditions  $w_m = w_l$  and  $v(\rho_m, w_m) = v(\rho_r, w_r)$ .

**Remark 2.2.** The *phase transition wave* as in **(PT)** connects a left state  $(\rho_l, w_l)$  in the free phase with a right state  $(\rho_r, w_r)$  in the congested phase under the condition  $w_l = w_r$ . This is the key difference between the evolution described by the present Two-Phase traffic model and that taking place in the GARZ model in [6]. Note that here phase transition waves, which do not appear in the GARZ equations, may well have both positive or negative speed. This is due to the peculiarity of model (2.1), where the introduction of the speed bound creates the distinction of the two distinct phases, free and congested, that are not present in the GARZ model. In [29] is studied the Riemann problem for the GARZ model in the case of moving constraints.

### 2.1. Other Two-Phase models

We note here some differences between the model (2.1) considered in this paper and other two-phase models found in the literature.

The first two-phase model is proposed by Colombo in [9], where the traffic dynamics is described by a first order model in the free regime and by a second order model in the congested regime. There, the basic LWR model is used in the free phase, expressing the conservation of the number of vehicles. One more equation describes the conservation of a sort of linearized momentum in the congested phase. The differences between this first two-phase model and the one considered in this paper is presented in [1, Section 3], the main one being the gap between the free and the congested phase: differently from here, in the two-phase model [9] the two phases are disconnected.

A first generalization of the model by Colombo is proposed in [12], where Goatin couples the LWR equation in the free phase with the ARZ model in the congested phase. This model is further generalized and studied under fixed point constraints in [24, 25]. More recently, [26] presents a phase transition model displaying two phases, the free one being characterized by a unique value of the velocity, similarly to (2.1). Moreover, in [26], no assumption on the intersection of the two phases is required: both cases  $F \cap C = \emptyset$  and  $F \cap C \neq \emptyset$  are treated. The presence of local point constraints on the flow is also considered.

A further two phase model, very similar to the present one, is proposed in [8]. Indeed, both these constructions are based on two intersecting phases: a one dimensional free phase and a two-dimensional congested phase. On the other hand the two models differ in how Riemann problems are solved. When the left state belongs to the free phase and the right one to the congested phase, below we use at most two waves, while in [8] also three waves can be necessary, see [31, Proposition 3.1, Proposition 3.2]. Also the derivations of the two models are completely different: according to the construction in [8], the very definition of the two phases is postulated *a priori*. Here, the two phases are obtained as a consequence of the speed limit.

### 3. The Riemann Problem with a fixed constraint

In this section we consider the Riemann Problem for the model (2.1) with a flux constraint at  $x = 0$ . From the traffic point of view this situation corresponds to the presence of a constraint on traffic flow as for instance a traffic light, a toll gate or a construction site. More precisely, we consider the following Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) = 0, \\ (\rho, w) = \begin{cases} (\rho_l, w_l), & \text{if } x < 0 \\ (\rho_r, w_r), & \text{if } x \geq 0 \end{cases} \end{cases} \quad (3.1)$$

for all states  $(\rho_l, w_l), (\rho_r, w_r) \in F \cup C$  and with a constraint on the first component of the flux at  $x = 0$ ,

$$\rho(t, 0) v(t, 0) \leq q, \quad (3.2)$$

where  $q$  is a given constant.

It is convenient to denote by  $f(\rho, w)$  the flux for system (2.1) in conservation form and by  $f_1(\rho, w)$ ,

$f_2(\rho, w)$  its components, i.e.,

$$f(\rho, w) = \begin{pmatrix} f_1(\rho, w) \\ f_2(\rho, w) \end{pmatrix} = \begin{pmatrix} \rho v(\rho, w) \\ \rho w v(\rho, w) \end{pmatrix}. \tag{3.3}$$

To define the solution to (3.1)–(3.2), the following notations will be of use. Fix  $q \in [0, \max f_1]$ , then there exist unique  $(\check{\rho}, \check{w}) \in F$  and  $(\hat{\rho}, \hat{w}) \in C$  such that:

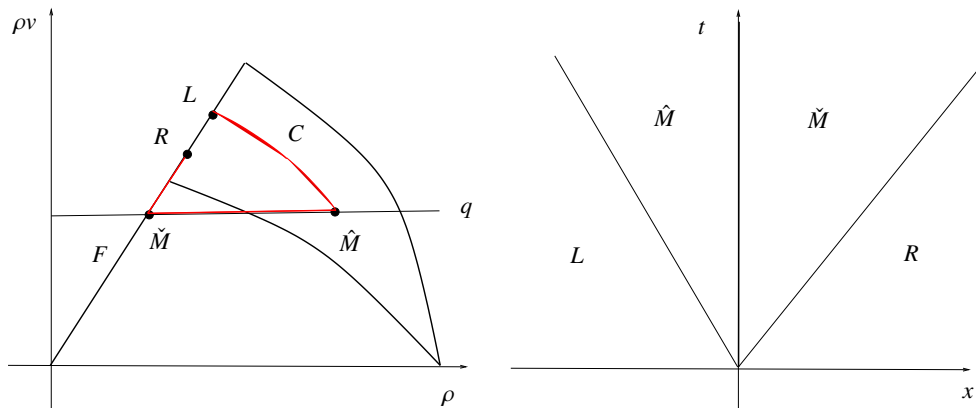
$$\begin{cases} f_1(\check{\rho}, \check{w}) = f_1(\hat{\rho}, \hat{w}) = q \\ \check{w} = \hat{w}. \end{cases}$$

With reference to the constrained Riemann problem (3.1)–(3.2), whenever the constraint is enforced,  $q$  and the Riemann data uniquely determine points  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $\check{M} \equiv (\check{\rho}, \check{w})$  such that

$$\begin{cases} f_1(\check{\rho}, \check{w}) = f_1(\hat{\rho}, \hat{w}) = q \\ \check{w} = \hat{w} = w_l \end{cases} \tag{3.4}$$

see figures 2, 3, 4, 5 and 6.

Fix  $(\rho_l, w_l), (\rho_r, w_r) \in F \cup C$ . By  $\mathcal{RS}$  we denote the unconstrained Riemann solver of conservation laws (2.1), right; see (RS1), (RS2), (RS3) and (RS4) and see also [1]. The Riemann solver  $\mathcal{RS}^q$ , i.e. the Riemann solver for (3.1) with the constraint (3.2) is defined as follows.



**Figure 2.** The constrained Riemann Problem in the first case of Definition 3.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . Here, the initial data  $(\rho_l, w_l), (\rho_r, w_r)$  are in  $F \cap C$ .

**Definition 3.1.** *The constrained Riemann solver*

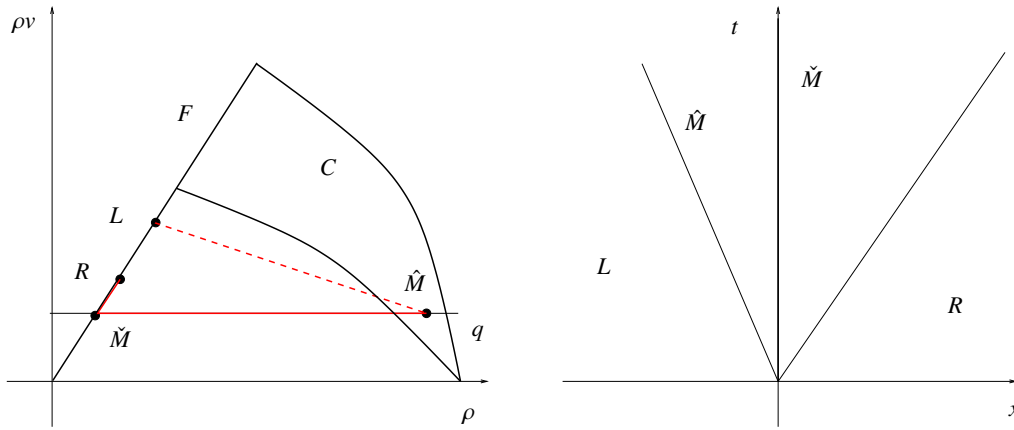
$$\mathcal{RS}^q: (F \cup C)^2 \rightarrow \mathbf{L}_{loc}^1(\mathbb{R}; F \cup C)$$

is defined as follows.

1. If  $f_1(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(0)) > q$ , then

$$\mathcal{RS}^q((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) = \begin{cases} \mathcal{RS}((\rho_l, w_l), (\hat{\rho}, \hat{w}))\left(\frac{x}{t}\right) & \text{if } x < 0 \\ \mathcal{RS}((\check{\rho}, \check{w}), (\rho_r, w_r))\left(\frac{x}{t}\right) & \text{if } x > 0. \end{cases} \tag{3.5}$$

for every  $x \in \mathbb{R}, t > 0$  and  $(\check{\rho}, \check{w}) \in F, (\hat{\rho}, \hat{w}) \in C$  defined in (3.4).



**Figure 3.** The constrained Riemann Problem in the first case of Definition 3.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . Here, the initial data  $(\rho_l, w_l), (\rho_r, w_r)$  are in  $F$ .

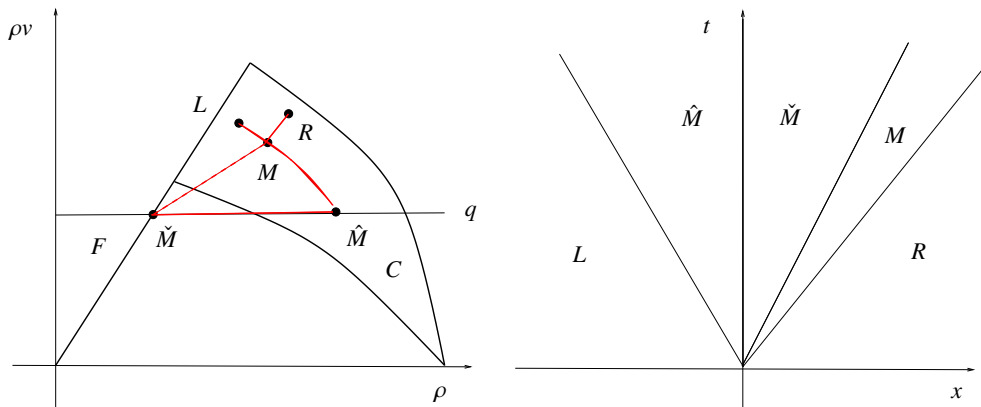
2. If  $f_1(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r)))(0) \leq q$ , then

$$\mathcal{RS}^q((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) = \mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) \tag{3.6}$$

for every  $x \in \mathbb{R}$  and  $t > 0$ .

Clearly, the first case above refers to the case when the constraint is enforced, while in the latter case the standard Lax solution to the Riemann problem satisfies the constraint.

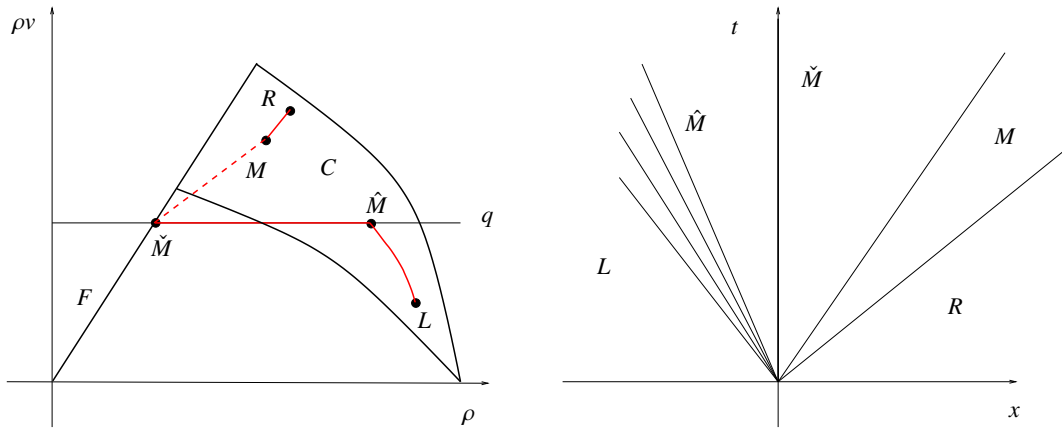
In particular, in the first case of Definition 3.1, we introduce a *non-classical shock* at the constraint location, that is a wave which satisfies the Rankine–Hugoniot condition but may violate the Lax entropy condition.



**Figure 4.** The constrained Riemann Problem in the first case of Definition 3.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . The initial data  $(\rho_l, w_l), (\rho_r, w_r)$  are in  $C$ . Here, the first family wave is a shock with negative speed.

We now present a constructive procedure showing that the Riemann solver in Definition 3.1 actually provides a unique weak, though possibly non-classical, solution to the constrained Riemann

problem (3.1)–(3.2). We limit the description to the first case, the other one being a direct consequence of (RS1)–(RS2)–(RS3)–(RS4).

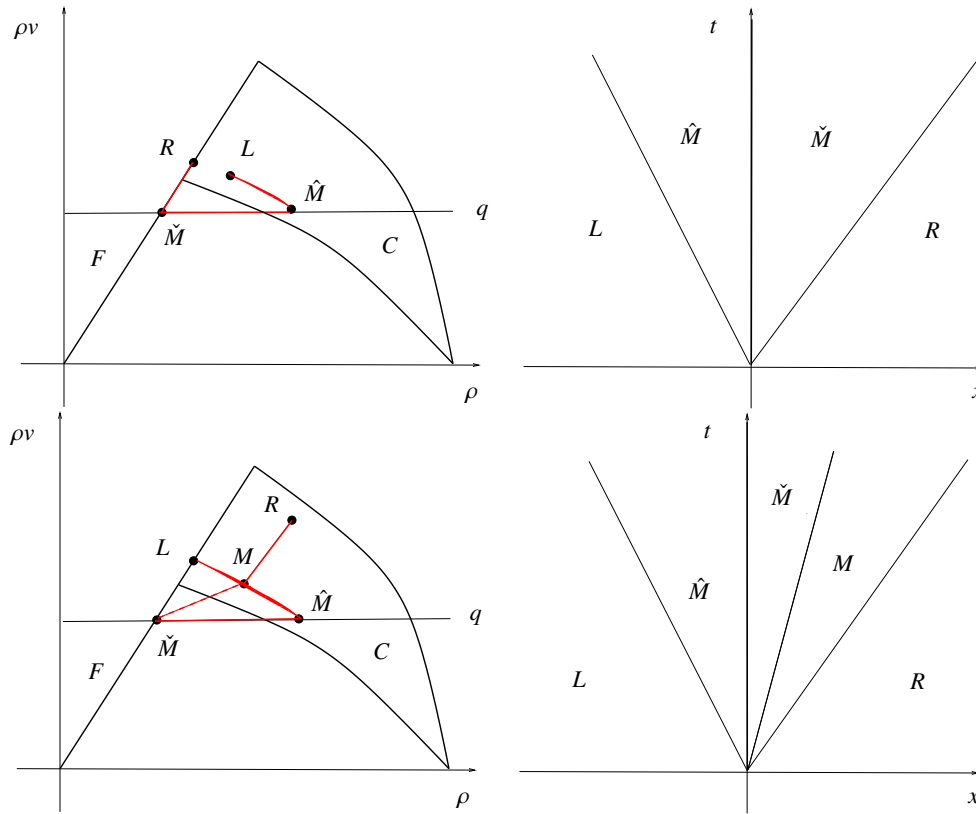


**Figure 5.** The constrained Riemann Problem in the first case of Definition 3.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . The initial data  $(\rho_l, w_l), (\rho_r, w_r) \in C$ . Here, the first family wave is a rarefaction with negative speed.

- (1) If  $(\rho_l, w_l), (\rho_r, w_r) \in F$ , then the solution consists of a phase transition wave with negative speed separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$  and a linear wave in the free phase between  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$ , see Figure 2 and Figure 3.
- (2) If  $(\rho_l, w_l), (\rho_r, w_r) \in C$ , then the solution consists of a first family wave between  $(\rho_l, w_l)$  and  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$ , a phase transition separating  $(\check{\rho}, \check{w})$  from  $(\rho_m, w_m)$  and finally a (possibly null) second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . Note that the middle state  $(\rho_m, w_m)$  is uniquely characterized by the two conditions  $v(\rho_m, w_m) = v(\rho_r, w_r)$  and  $w_m = w_l$ , see Figure 4 and Figure 5.
- (3) If  $(\rho_l, w_l) \in C$  and  $(\rho_r, w_r) \in F$ , then the solution consists of first family wave separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$  and a linear wave between  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$ , see Figure 6, above.
- (4) If  $(\rho_l, w_l) \in F$  and  $(\rho_r, w_r) \in C$ , then the solution consists of a phase transition wave with negative speed separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$ , a phase transition wave with positive speed between  $(\check{\rho}, \check{w})$  and  $(\rho_m, w_m)$  and finally a (possibly null) second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . The middle state  $(\rho_m, w_m)$  is uniquely characterized by the two conditions  $v(\rho_m, w_m) = v(\rho_r, w_r)$  and  $w_m = w_l$ , see Figure 6, below.

**Remark 3.2.** Note that the states  $(\rho_l, w_l)$  and  $(\hat{\rho}, \hat{w})$  are connected by waves of the first family or phase transition waves with negative speed. The states  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$  are connected by linear waves or by phase transition waves between  $(\check{\rho}, \check{w})$  and  $(\rho_m, w_m)$  with positive speed followed by waves of the second family between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ .





**Figure 6.** The constrained Riemann Problem in the first case of Definition 3.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . Above, if the initial data  $(\rho_l, w_l) \in C$  and  $(\rho_r, w_r) \in F$ . Below, if the initial data  $(\rho_l, w_l) \in F$  and  $(\rho_r, w_r) \in C$ .

#### 4. The Riemann Problem with a moving constraint

We now generalize the results in the previous section and consider a moving constraint that imposes on the traffic flow a  $w$  dependent bound. From the traffic point of view, we describe the dynamics of moving bottlenecks caused by a slow moving large vehicle, like a bus. The bus trajectory  $y = y(t)$  solves the following ODE

$$\dot{y}(t) = \omega(\rho(t, y(t)+), w(t, y(t)+)) , \tag{4.1}$$

where the right limit is due to the bus adjusting its speed according to the traffic situation it has in front. The discussion below ensures that  $\rho$  and  $w$  result as being of bounded variation in the space variable, so that the right traces above are well defined. The speed of the bus is given by

$$\omega(\rho, w) = \begin{cases} V_b & \text{if } V_b < v(\rho, w) \\ v(\rho, w) & \text{if } V_b \geq v(\rho, w), \end{cases} \tag{4.2}$$

with  $0 < V_b < V_{\max}$ . Thus the bus always travels with its speed  $V_b$  when traffic allows it otherwise it adapts its speed; in any case, it is not possible for the bus to overtake the cars. It is natural to assume that the slowest cars still travel faster than the bus along an empty road, that is  $V_b < \underline{w}$ .

In the case  $\dot{y} = V_b$ , so that  $v \geq V_b$  by (4.2), we introduce the flux of the main traffic  $F$  at the bus position by

$$F : [0, R] \times [\underline{w}, \bar{w}] \rightarrow \mathbb{R} \quad (4.3)$$

$$\rho, w \rightarrow \rho (v(\rho, w) - V_b).$$

As a consequence, for a driver with feature  $w \in [\underline{w}, \bar{w}]$ , the maximum available flux at the bus position is

$$F_\alpha : [\underline{w}, \bar{w}] \rightarrow \mathbb{R} \quad (4.4)$$

$$w \rightarrow \alpha(w) \max_{\rho \in [0, R]} F(\rho, w),$$

where  $\alpha(w)$ , with  $\alpha(w) \in [0, 1]$ , models the reduction of the road capacity felt by a driver with feature  $w$ . For instance, whenever  $w$  represents the individual maximal speed, an increasing  $\alpha$  corresponds to slow vehicles having more difficulties in overtaking the slow vehicle causing the constraint. Thus, we consider the following mixed ODE–PDE system consisting of the PDE model (2.1), of the ODE describing the slower vehicle motion (4.1) and with a moving constraint on the flux:

$$\begin{cases} \partial_t \rho + \partial_x (\rho v(\rho, w)) = 0 \\ \partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) = 0 \\ \dot{y}(t) = \omega(\rho(t, y(t)+), w(t, y(t)+)) \\ \rho(t, y(t)) (v(\rho(t, y(t)), w(t, y(t))) - \dot{y}(t)) \leq F_\alpha(w). \end{cases} \quad (4.5)$$

We consider the Riemann problem for system (4.5) with initial data

$$\rho(0, x) = \begin{cases} \rho_l & \text{if } x < 0 \\ \rho_r & \text{if } x \geq 0 \end{cases} \quad w(0, x) = \begin{cases} w_l & \text{if } x < 0 \\ w_r & \text{if } x \geq 0 \end{cases} \quad (4.6)$$

for all states  $(\rho_l, w_l), (\rho_r, w_r) \in F \cup C$ , and

$$y(0) = 0. \quad (4.7)$$

As in the previous section we denote by  $f_1(\rho, w) = \rho v$  and  $f_2(\rho, w) = \rho w v$  the two components of the flux for system (2.1) in conservation form.

Similarly to (3.4), to define the solution to (4.5)–(4.6), introduce the following notation. Define  $(\check{\rho}, \check{w}) \in F$  and  $(\hat{\rho}, \hat{w}) \in C$  by the unique solution to:

$$\begin{cases} f_1(\check{\rho}, \check{w}) = F_\alpha(\check{w}) + \check{\rho} V_b \\ f_1(\hat{\rho}, \hat{w}) = F_\alpha(\hat{w}) + \hat{\rho} V_b \\ \check{w} = \hat{w} = w_l. \end{cases} \quad (4.8)$$

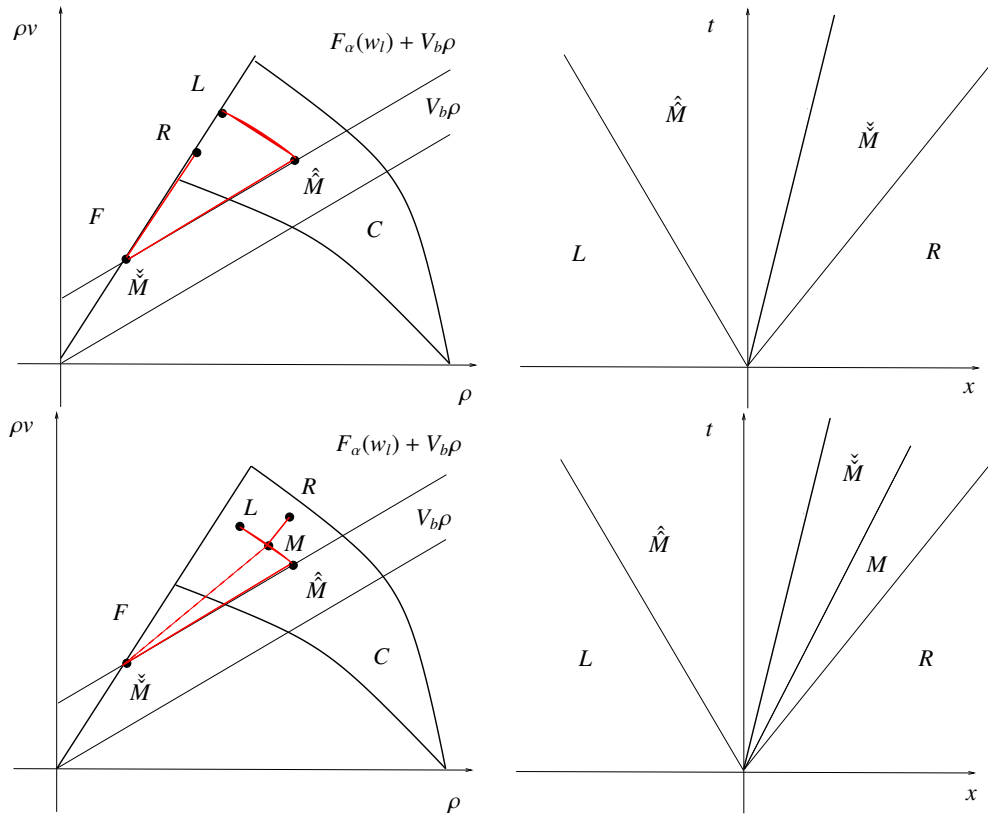
Fix  $(\rho_l, w_l), (\rho_r, w_r) \in F \cup C$ . As in the previous section, by  $\mathcal{RS}$  we denote the unconstrained Riemann solver of conservation laws (2.1), right; see **(RS1)–(RS2)–(RS3)–(RS4)** and see also [1]. By  $\mathcal{RS}_\rho$ , respectively  $\mathcal{RS}_w$  and  $\mathcal{RS}_v$ , we denote the  $\rho$ , respectively  $w$  and  $v$ , values computed on the classical Lax solution  $\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(\cdot)$ .

The Riemann solver  $\mathcal{RS}^\alpha$  for (4.5) with initial datum (4.6)–(4.7) is defined as follows.

**Definition 4.1.** *The constrained Riemann solver*

$$\mathcal{RS}^\alpha : (F \cup C)^2 \rightarrow \mathbf{L}_{\text{loc}}^1(\mathbb{R}; F \cup C)$$

is defined as follows.



**Figure 7.** The constrained Riemann Problem in the first case of Definition 4.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . Above, if the initial data  $(\rho_l, w_l), (\rho_r, w_r) \in F$  (in the present drawing,  $F \cap C$ ). Below, if the initial data  $(\rho_l, w_l), (\rho_r, w_r) \in C$ .

1. If  $f_1(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b)) > F_\alpha(w_l) + V_b \mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_r))(V_b)$ , then

$$\mathcal{RS}^\alpha((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) = \begin{cases} \mathcal{RS}((\rho_l, w_l), (\hat{\rho}, \hat{w}))\left(\frac{x}{t}\right) & \text{if } \frac{x}{t} < V_b \\ \mathcal{RS}(\check{\rho}, \check{w}), (\rho_r, w_r)\left(\frac{x}{t}\right) & \text{if } \frac{x}{t} > V_b, \end{cases}$$

$$y(t) = V_b t.$$

with  $(\check{\rho}, \check{w}) \in F$  and  $(\hat{\rho}, \hat{w}) \in C$  defined in (4.8).

2. If  $f_1(\mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))(V_b)) \leq F_\alpha(w_l) + V_b \mathcal{RS}_\rho((\rho_l, w_l), (\rho_r, w_r))(V_b)$  and  $V_b < \mathcal{RS}_v((\rho_l, w_l), (\rho_r, w_r))(V_b)$ , then

$$\mathcal{RS}^\alpha((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) = \mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right)$$

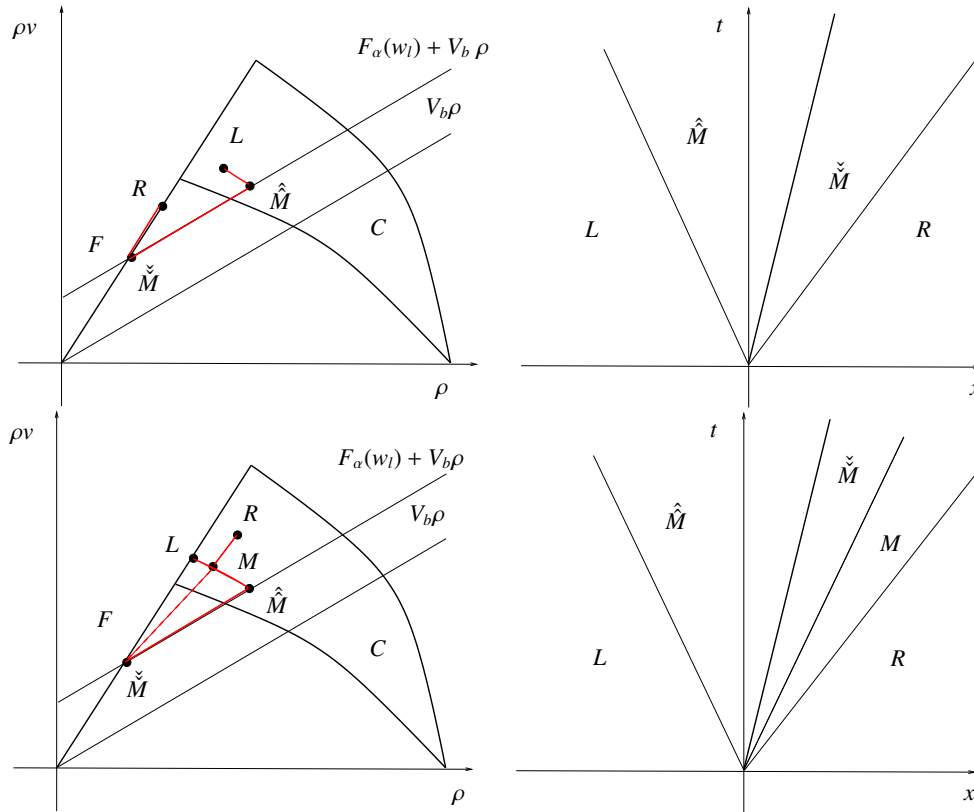
$$y(t) = V_b t.$$

3. If  $V_b \geq \mathcal{RS}_v((\rho_l, w_l), (\rho_r, w_r))(V_b)$ , then

$$\mathcal{RS}^\alpha((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right) = \mathcal{RS}((\rho_l, w_l), (\rho_r, w_r))\left(\frac{x}{t}\right)$$

$$y(t) = \mathcal{RS}_v((\rho_l, w_l), (\rho_r, w_r))(V_b) t.$$

Note that the first case refers to the case when the constraint is enforced; traffic is influenced by the bus which travels with its own velocity. In the second case, there is essentially no interaction between the bus and the traffic, thanks to a low traffic density. The third case refers to a situation in which the traffic is so heavy that the bus has to slow down and adapt its speed. In the latter two cases the standard Lax solution to the Riemann problem satisfies the constraint.



**Figure 8.** The constrained Riemann Problem in the first case of Definition 4.1 in the coordinates, from left to right,  $(\rho, \rho v)$  and  $(x, t)$ . We denote  $L \equiv (\rho_l, w_l)$ ,  $M \equiv (\rho_m, w_m)$ ,  $\check{M} \equiv (\check{\rho}, \check{w})$ ,  $\hat{M} \equiv (\hat{\rho}, \hat{w})$  and  $R \equiv (\rho_r, w_r)$ . Above, if the initial data  $(\rho_l, w_l) \in C$  and  $(\rho_r, w_r) \in F$ . Below, if the initial data  $(\rho_l, w_l) \in F$  and  $(\rho_r, w_r) \in C$ .

The construction outlined in Section 3 can now be extended to cover the case of a moving and  $w$  dependent constraint on the flow. We detail only the first case, corresponding to an enforced constraint, since the other ones (namely cases 2. and 3. in Definition 4.1) are essentially covered by the standard construction described in **(RS1)–(RS2)–(RS3)–(RS4)**, see also [1].

- (1) If  $(\rho_l, w_l), (\rho_r, w_r) \in F$ , then the solution consists of a phase transition wave separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$  and a linear wave between  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$ . The phase transition has negative speed. See Figure 7, above.
- (2) If  $(\rho_l, w_l), (\rho_r, w_r) \in C$ , then the solution consists of a first family wave separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$ , a phase transition between  $(\check{\rho}, \check{w})$  and  $(\rho_m, w_m)$  and a (possibly null) second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . Note that the

middle state  $(\rho_m, w_m)$  is uniquely characterized by the two conditions  $v(\rho_m, w_m) = v(\rho_r, w_r)$  and  $w_m = w_l$ . The phase transition has positive speed. See Figure 7, below.

- (3) If  $(\rho_l, w_l) \in C$  and  $(\rho_r, w_r) \in F$ , then the solution consists of first family wave separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$  and a linear wave between  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$ , see Figure 8, above.
- (4) If  $(\rho_l, w_l) \in F$  and  $(\rho_r, w_r) \in C$ , then the solution consists of a phase transition wave with negative speed separating  $(\rho_l, w_l)$  from  $(\hat{\rho}, \hat{w})$ , a non-classical shock between  $(\hat{\rho}, \hat{w})$  and  $(\check{\rho}, \check{w})$ , a phase transition wave between  $(\check{\rho}, \check{w})$  and  $(\rho_m, w_m)$  and a (possibly null) second family wave between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ . Note that the middle state  $(\rho_m, w_m)$  is uniquely characterized by the two conditions  $v(\rho_m, w_m) = v(\rho_r, w_r)$  and  $w_m = w_l$ . The phase transition has positive speed. See Figure 8, below.

**Remark 4.2.** We note that the states  $(\rho_l, w_l)$  and  $(\hat{\rho}, \hat{w})$  are connected by waves of the first family or phase transition waves with negative speed. The states  $(\check{\rho}, \check{w})$  and  $(\rho_r, w_r)$  are connected by linear waves or by phase transition waves between  $(\check{\rho}, \check{w})$  and  $(\rho_m, w_m)$  with positive speed followed by waves of the second family between  $(\rho_m, w_m)$  and  $(\rho_r, w_r)$ .

## 5. Conclusions

The Riemann solvers introduced above open the way to a through analytic investigation of their properties, aiming at the well posedness of the general Cauchy problem under fixed and moving constraints. Suitable stability estimates might then enable the consideration of optimal management problems. For instance, whenever the fixed constraint represents a toll gate, one can seek the optimal threshold on the through flow that avoids the overcrowding of roads after the gate.

A further direction not yet considered is the interplay between flux constraints and crossroads or junctions. Analytically, this amounts to correlate the present techniques with the ones more typical in the case of networks, see [33].

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## Conflict of interest

The author declares there is no conflict of interest.

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