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Research article

On partition dimension of fullerene graphs

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Abstract: Let $G = (V(G), E(G))$ be a connected graph and $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a k-partition of $V(G)$. The representation $r(v|\Pi)$ of a vertex v with respect to Π is the vector $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, s_i) \mid s_i \in S_i\}$. The partition Π is called a resolving partition of G if $r(u|\Pi) \neq r(v|\Pi)$ for all distinct $u, v \in V(G)$. The partition dimension of G, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G. In this paper, we calculate the partition dimension of two $(4, 6)$-fullerene graphs. We also give conjectures on the partition dimension of two $(3, 6)$-fullerene graphs.

Keywords: partition dimension; fullerene graphs
Mathematics Subject Classification: 05C12

1. Introduction

Slater [13] and Harary et al. [6] introduced the notions of resolvability and locating number in graphs. Chartrand et al. [4] introduced the partition dimension of a graph. These concepts have some applications in Chemistry for representing chemical compounds [2] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [10].

Kroto et al. [9] discovered fullerene molecule and since then, scientists took a great interest in the fullerene graphs. A $(k, 6)$-fullerene graph is a connected cubic plane graph whose faces have sizes k and 6. There are only three types of fullerene graphs, that is, $(3, 6)$, $(4, 6)$ and $(5, 6)$-fullerene graphs. A $(5, 6)$-fullerene is the usual fullerene as the molecular graph of sphere carbon fullerene. A $(3, 6)$-fullerene graph has cycles of order three and six. The Euler's formula implies that a $(3, 6)$-fullerene graph has exactly four faces of size 3 and $(n/2) - 2$ hexagons. Similarly $(4, 6)$ and $(5, 6)$-fullerene graphs has cycles of order four and six, and five and six, respectively. The Euler's formula implies that a $(4, 6)$-fullerene graph has exactly six square faces and $(n/2) - 4$ hexagons.

Chartrand et al. [3] gave useful definitions and results related to the partition dimension of a connected graph. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. If S is a subset of $V(G)$ and $v \in V(G)$ then the distance between v and S, denoted by $d(v, S)$, is defined as $d(v, S) = \min\{d(v, x) \mid x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex v of G, the representation of v with respect to Π is defined as the k-vector $r(v \mid \Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. The partition Π is called a resolving partition if $r(u \mid \Pi) \neq r(v \mid \Pi)$ for each $u, v \in V(G)$, $u \neq v$. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension of G and is denoted by $pd(G)$.

Many authors determined the partition dimension of specific classes of graphs. Rodríguez-Velázquez et al. [14, 15] find the bounds on the partition dimension of trees and unicyclic graphs. Tomescu et al. [16] calculated the partition dimension of a wheel graph and Tomescu [17] discussed the metric and partition dimension of a connected graph. Grigorious et al. [5] and Javaid et al. [7] calculated the partition dimension of some classes of circulant graphs.

The following result is a useful property in determining partition dimension.

Lemma 1.1. [3] *Let Π be a resolving partition of vertex set $V(G)$ of a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$ then u and v belong to different classes of Π.*

The partition dimension of some families of graphs is given in next lemma.

Lemma 1.2. [3] *Let G be a connected graph. Then*

1. *$pd(G) = 2$ if and only if $G = P_n$ for $n \geq 2$,*
2. *$pd(G) = n$ if and only if $G = K_n$,*
3. *$pd(G) = 3$ if $G = C_n$ for $n \geq 3$.*

Above results are useful in computing the partition dimension of connected graphs. Ashrafi et al. [1] studied the topological indices of $(3, 6)$ and $(4, 6)$-fullerene graphs. Moftakhar et al. [8] calculated the automorphism group and fixing number of $(3, 6)$ and $(4, 6)$-fullerene graphs. Siddiqui et al. [11, 12] calculated the metric dimension and partition dimension of nanotubes. In this paper, we calculate the partition dimension of two $(4, 6)$-fullerene graphs. Also we give conjectures on the partition dimension of two $(3, 6)$-fullerene graphs.

2. Partition dimension of $(4, 6)$-fullerene graphs

In this section, we consider two $(4, 6)$-fullerene graphs $G_1[n]$ and $G_2[n]$ shown in Figure 1 and Figure 2, respectively. It is easily seen that the order of $G_1[n]$ and $G_2[n]$ is $8n$ and $8n + 4$, respectively. We calculate the partition dimension of $G_1[n]$ and $G_2[n]$ graphs.

Theorem 2.1. *The partition dimension of fullerene graph $G_1[n]$ is 3.*

Proof. Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{x_{2n}, x_{2n+1}\}$, $S_2 = \{y_{2n}\}$ and $S_3 = V(G_1[n]) \setminus \{x_{2n}, x_{2n+1}, y_{2n}\}$, be a partition of $V(G_1[n])$. We show that Π is a resolving partition of $G_1[n]$ with minimum cardinality. The representation of each vertex of $G_1[n]$ with respect to Π is given as follows:

$$r(x_{2n} \mid \Pi) = (0, 1, 1), \quad r(x_{2n+1} \mid \Pi) = (0, 2, 1), \quad r(y_{2n} \mid \Pi) = (1, 0, 1).$$

$$r(x_i \mid \Pi) = \begin{cases} (2n - i, 2n - i + 1, 0) & \text{if } 1 \leq i \leq 2n - 1, \\ (i - 2n - 1, i - 2n + 1, 0) & \text{if } 2n + 2 \leq i \leq 4n. \end{cases}$$

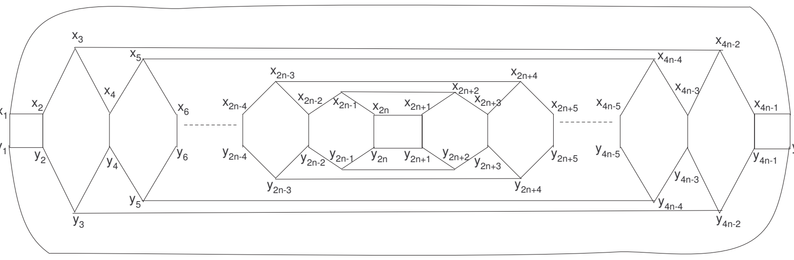

Figure 1. Graph $G_1[n]$

and

$$r(y_i \mid \Pi) = \begin{cases} (2n-i+1, 2n-i, 0) & \text{if } 1 \leq i \leq 2n-1, \\ (i-2n, i-2n, 0) & \text{if } 2n+1 \leq i \leq 4n. \end{cases}$$

Therefore, it is easily seen that the representation of each vertex with respect to Π is distinct. This shows that Π is a resolving partition of $G_1[n]$. Thus $pd(G_1[n]) \leq 3$.

On the other hand, by Lemma 1.2, it follows that $pd(G_1[n]) \geq 3$. Hence $pd(G_1[n]) = 3$. □

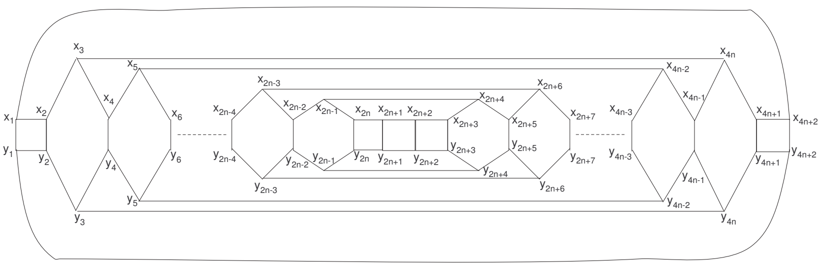

Figure 2. Graph $G_2[n]$

In next theorem, we calculate the partition dimension of $G_2[n]$.

Theorem 2.2. *The partition dimension of fullerene graph* $G_2[n]$ *is* 3.

Proof. Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{x_{2n+1}, x_{2n+2}\}$, $S_2 = \{y_{2n+1}\}$ and $S_3 = V(G_2[n]) \setminus \{x_{2n+1}, x_{2n+2}, y_{2n+1}\}$, be a partition of $V(G_2[n])$. We show that Π is a resolving partition of $G_2[n]$ with minimum cardinality. The representation of each vertex of $G_2[n]$ with respect to Π is given as follows:

$$r(x_{2n+1} \mid \Pi) = (0, 1, 1), \quad r(x_{2n+2} \mid \Pi) = (0, 2, 1), \quad r(y_{2n+1} \mid \Pi) = (1, 0, 1).$$

$$r(x_i \mid \Pi) = \begin{cases} (2n+1-i, 2n+2-i, 0) & \text{if } 1 \leq i \leq 2n, \\ (i-2n-2, i-2n, 0) & \text{if } 2n+3 \leq i \leq 4n+2. \end{cases}$$

and

$$r(y_i \mid \Pi) = \begin{cases} (2n+2-i, 2n+1-i, 0) & \text{if } 1 \leq i \leq 2n, \\ (i-2n-1, i-2n-1, 0) & \text{if } 2n+2 \leq i \leq 4n+2. \end{cases}$$

All pairs of vertices can easily be resolved by the partitioning set Π. Therefore Π is a resolving partition of $G_2[n]$ and $pd(G_2[n]) \leq 3$.

On the other hand, by Lemma 1.2, it follows that $pd(G_2[n]) \geq 3$. Hence $pd(G_2[n]) = 3$. □

3. Conjectures on partition dimension of two $(3,6)$-fullerene graphs

In this section, we consider two $(3,6)$-fullerene graphs $F_3[n]$ and $F_4[n]$ shown in Figure 3 and Figure 4, respectively. We can see that order of $F_3[n]$ and $F_4[n]$ is $16n-32$, $n \geq 4$ and $24n$, $n \geq 1$, respectively.

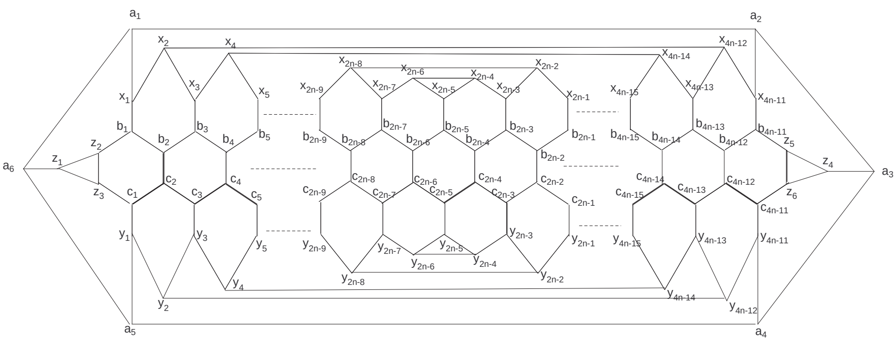

Figure 3. Graph $F_3[n]$

Firstly we consider the fullerene graph $F_3[n]$ and give a conjecture on the partition dimension of $F_3[n]$. The set of vertices $V(F_3[n])$, $n \geq 5$, is divided into the following sets:

$$
\begin{array}{lll}
X_1 = \{x_i \mid 1 \leq i \leq 2n-6\}, & X_2 = \{x_i \mid 2n-4 \leq i \leq 4n-11\}, & Y_1 = \{y_i \mid 1 \leq i \leq 2n-6\}, \\
Y_2 = \{y_i \mid 2n-4 \leq i \leq 4n-11\}, & Z_1 = \{z_1, z_2, z_3\}, & Z_2 = \{z_4, z_5, z_6\}, \\
A = \{a_i \mid 1 \leq i \leq 6\}, & B_1 = \{b_i \mid 1 \leq i \leq 2n-6\}, & B_2 = \{b_i \mid 2n-4 \leq i \leq 4n-11\}, \\
C_1 = \{c_i \mid 1 \leq i \leq 2n-6\}, & C_2 = \{c_i \mid 2n-4 \leq i \leq 4n-11\}. &
\end{array}
\tag{3.1}
$$

The relations of distances of vertices of $F_3[n]$ are given by:

$$d(a_4, x) = d(y_1, x), \quad \text{for all } x \in X_1, \tag{3.2}$$
$$d(a_5, x) = d(y_{4n-11}, x), \quad \text{for all } x \in X_2, \tag{3.3}$$
$$d(a_2, y) = d(x_1, y), \quad \text{for all } y \in Y_1, \tag{3.4}$$
$$d(a_1, y) = d(x_{4n-11}, y), \quad \text{for all } y \in Y_2, \tag{3.5}$$
$$d(z_2, z) = d(z_3, z), \quad \text{for all } z \in Z_2, \tag{3.6}$$
$$d(z_5, z) = d(z_6, z), \quad \text{for all } z \in Z_1, \tag{3.7}$$
$$d(z_4, x) = d(z_6, x), \quad \text{for all } x \in X_2 \cup \{x_{2n-5}\}, \tag{3.8}$$
$$d(z_4, y) = d(z_5, y), \quad \text{for all } y \in Y_2 \cup \{y_{2n-5}\}, \tag{3.9}$$
$$d(z_1, x) = d(z_3, x), \quad \text{for all } x \in X_1 \cup \{x_{2n-5}, x_{2n-4}\}, \tag{3.10}$$
$$d(z_1, y) = d(z_2, y), \quad \text{for all } y \in Y_1 \cup \{y_{2n-5}, y_{2n-4}\}, \tag{3.11}$$
$$d(a_1, x) = d(x_{4n-11}, x), \quad \text{for all } x \in X_1 \setminus \{x_1\}, \tag{3.12}$$
$$d(a_5, y) = d(y_{4n-11}, y), \quad \text{for all } y \in Y_1 \setminus \{y_1\}, \tag{3.13}$$
$$d(a_2, x) = d(x_1, x), \quad \text{for all } x \in X_2 \setminus \{x_{2n-4}, x_{4n-11}\}, \tag{3.14}$$
$$d(a_4, y) = d(y_1, y), \quad \text{for all } y \in Y_2 \setminus \{y_{2n-4}, y_{4n-11}\}, \tag{3.15}$$

$$d(a_6,b)=d(a_5,b), \qquad \text{for all } b\in B_1\cup B_2\cup\{b_{2n-5}\}\setminus\{b_1\}, \tag{3.16}$$
$$d(a_6,c)=d(a_1,c), \qquad \text{for all } c\in C_1\cup C_2\cup\{c_{2n-5}\}\setminus\{c_1\}, \tag{3.17}$$
$$d(a_1,b)=d(x_2,b), \qquad \text{for all } b\in \{b_1,b_2,b_{4n-12},b_{4n-11}\}, \tag{3.18}$$
$$d(a_5,c)=d(y_2,c), \qquad \text{for all } b\in \{c_1,c_2,c_{4n-12},c_{4n-11}\}. \tag{3.19}$$

The relations of distances of vertices of $C_1\cup\{c_{2n-5}\}$, $C_2\cup\{c_{2n-5}\}$, $B_1\cup\{b_{2n-5}\}$ and $B_2\cup\{b_{2n-5}\}$ are given by:

$$d(z_1,c)=d(z_2,c),\ \ d(z_1,c)=d(a_5,c),\ \ d(z_2,c)=d(a_5,c) \ \ \text{for all } c\in C_1\cup\{c_{2n-5}\}, \tag{3.20}$$
$$d(z_4,c)=d(z_5,c),\ \ d(z_4,c)=d(a_4,c),\ \ d(z_5,c)=d(a_4,c) \ \ \text{for all } c\in C_2\cup\{c_{2n-5}\}, \tag{3.21}$$
$$d(z_1,b)=d(z_3,b),\ \ d(z_1,b)=d(a_1,b),\ \ d(z_3,b)=d(a_1,b) \ \ \text{for all } b\in B_1\cup\{b_{2n-5}\}, \tag{3.22}$$
$$d(z_4,b)=d(z_6,b),\ \ d(z_4,b)=d(a_2,b),\ \ d(z_6,b)=d(a_2,b) \ \ \text{for all } b\in B_2\cup\{b_{2n-5}\}. \tag{3.23}$$

The relations of distances of the pair of vertices of $Z_1\cup Z_2$, A, $X_1\cup X_2\cup\{x_{2n-5}\}$ and $Y_1\cup Y_2\cup\{y_{2n-5}\}$ are given by:

$$d(a_1,z)=d(a_5,z),\ \ d(a_2,z)=d(a_4,z), \ \ \text{for all } z\in Z_1\cup Z_2, \tag{3.24}$$
$$d(z_2,a)=d(z_3,a),\ \ d(z_5,a)=d(z_6,a), \ \ \text{for all } a\in A, \tag{3.25}$$
$$d(z_1,x)=d(a_5,x),\ \ d(z_4,x)=d(a_4,x), \ \ \text{for all } x\in X_1\cup X_2\cup\{x_{2n-5}\}, \tag{3.26}$$
$$d(z_1,y)=d(a_1,y),\ \ d(z_4,y)=d(a_2,y), \ \ \text{for all } y\in Y_1\cup Y_2\cup\{y_{2n-5}\}. \tag{3.27}$$

The distance between the vertices $b_i\in B_1\cup B_2$ and $c_i\in C_1\cup C_2$ is given as:

$$d(b_i,c_i)=\begin{cases}1 & \text{for } i \text{ is even},\\ 3 & \text{for } i \text{ is odd}.\end{cases} \tag{3.28}$$

The distance between the vertices $b_i\in B_1\cup B_2$ and $x_i\in X_1\cup X_2$ is given as:

$$d(x_i,b_i)=\begin{cases}1 & \text{for } i \text{ is even},\\ 3 & \text{for } i \text{ is odd}.\end{cases} \tag{3.29}$$

The distance between the vertices $c_i\in C_1\cup C_2$ and $y_i\in Y_1\cup Y_2$ is given as:

$$d(y_i,c_i)=\begin{cases}1 & \text{for } i \text{ is even},\\ 3 & \text{for } i \text{ is odd}.\end{cases} \tag{3.30}$$

Lemma 3.1. *Let $F_3[n]$ be a fullerene graph shown in Figure 3. Then $3\le pd(F_3[n])\le 4$, where $n\ge 5$.*

Proof. Let $\{z_1,z_2,z_3\}$ and $\{z_4,z_5,z_6\}$ be the vertices of outer triangles and $\{a_1,a_2,a_3,a_4,a_5,a_6\}$ be the vertices of outer hexagon of $F_3[n]$. Let $\Pi=\{S_1,S_2,S_3,S_4\}$, where $S_1=\{a_5\}$, $S_2=\{z_2\}$, $S_3=\{z_5\}$ and $S_4=V(F_3[n])\setminus\{a_5,z_2,z_5\}$, be a partition of $V(F_3[n])$. We show that Π is a resolving partition of $F_3[n]$ with minimum cardinality. For this we give the representation of each vertex of $F_3[n]$ other than $a_5,\ z_2,\ z_5$ with respect to Π. The representation of vertices of A with respect to Π is given by:

$$r(a_1\mid\Pi)=(2,3,4,0),\quad r(a_2\mid\Pi)=(3,4,3,0),\quad r(a_3\mid\Pi)=(2,5,2,0),$$
$$r(a_4\mid\Pi)=(1,4,3,0),\quad r(a_6\mid\Pi)=(1,2,5,0).$$

The representation of vertices of $(Z_1 \cup Z_2) \setminus \{z_2, z_5\}$ with respect to Π is given by:

$$r(z_1 \mid \Pi) = (2,1,6,0), \quad r(z_3 \mid \Pi) = (3,1,7,0), \quad r(z_4 \mid \Pi) = (3,6,1,0), \quad r(z_6 \mid \Pi) = (4,7,1,0).$$

The representation of vertices of $X_1 \cup X_2$ with respect to Π is given by:

$$r(x_i \mid \Pi) = \begin{cases} (3,2,5,0) & \text{if } i = 1, \\ (i+2, i+1, i+2, 0) & \text{if } 2 \le i \le 2n-6, \\ (2n-3, 2n-4, 2n-4, 0) & \text{if } i = 2n-5, \\ (4n-i-7, 4n-i-8, 4n-i-9, 0) & \text{if } 2n-4 \le i \le 4n-12, \\ (4,5,2,0) & \text{if } i = 4n-11. \end{cases}$$

The representation of vertices of $B_1 \cup B_2$ and $C_1 \cup C_2$ with respect to Π is given by:

$$r(b_i \mid \Pi) = \begin{cases} (4, i, i+5, 0) & \text{if } i \in \{1,2\}, \\ (i+2, i, 4n-i-10, 0) & \text{if } 3 \le i \le 2n-5, \\ (2n-3, 2n-4, 2n-6, 0) & \text{if } i = 2n-4, \\ (4n-i-7, 4n-i-7, 4n-i-10, 0) & \text{if } 2n-3 \le i \le 4n-13, \\ (5, 4n-i-5, 4n-i-10, 0) & \text{if } i \in \{4n-12, 4n-11\}. \end{cases}$$

$$r(c_i \mid \Pi) = \begin{cases} (i+1, i+1, i+5, 0) & \text{if } i \in \{1,2\}, \\ (i+1, i+1, 4n-i-9, 0) & \text{if } 3 \le i \le 2n-5, \\ (2n-4, 2n-3, 2n-5, 0) & \text{if } i = 2n-4, \\ (4n-i-8, 4n-i-6, 4n-i-9, 0) & \text{if } 2n-3 \le i \le 4n-13, \\ (4n-i-8, 4n-i-5, 4n-i-9, 0) & \text{if } i \in \{4n-12, 4n-11\}. \end{cases}$$

The representation of vertices of $Y_1 \cup Y_2$ with respect to Π is given by:

$$r(y_i \mid \Pi) = \begin{cases} (1,3,5,0) & \text{if } i = 1, \\ (i, i+2, i+3, 0) & \text{if } 2 \le i \le 2n-6, \\ (2n-5, 2n-3, 2n-3, 0) & \text{if } i = 2n-5, \\ (4n-i-9, 4n-i-7, 4n-i-8, 0) & \text{if } 2n-4 \le i \le 4n-12, \\ (2,5,3,0) & \text{if } i = 4n-11. \end{cases}$$

It is easily seen that the representation of each vertex with respect to Π is distinct. This shows that Π is a resolving partition of $F_3[n]$. Thus $pd(F_3[n]) \le 4$. Also by Lemma 1.2, we have $pd(F_3[n]) \ge 3$. □

Suppose that there exists a partition $\widetilde{\Pi}$ of $F_3[n]$, $n \ge 5$, such that $|\widetilde{\Pi}| = 3$. Let $\widetilde{\Pi} = \{\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3\}$. Consider the following cases:

Case I: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either Z_1 or Z_2 then from (3.6) and (3.7), it is clear that either $r(z_5 \mid \widetilde{\Pi}) = r(z_6 \mid \widetilde{\Pi})$ or $r(z_2 \mid \widetilde{\Pi}) = r(z_3 \mid \widetilde{\Pi})$.

Case II: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either A or X_1 or B_1 then (3.25), (3.2), (3.10) and (3.22) implies that either $r(z_2 \mid \widetilde{\Pi}) = r(z_3 \mid \widetilde{\Pi})$ or $r(a_4 \mid \widetilde{\Pi}) = r(y_1 \mid \widetilde{\Pi})$ or $r(z_1 \mid \widetilde{\Pi}) = r(z_3 \mid \widetilde{\Pi})$.

Case III: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either Y_1 or C_1 then (3.4), (3.11) and (3.20) implies that either $r(z_1 \mid \widetilde{\Pi}) = r(z_2 \mid \widetilde{\Pi})$ or $r(a_2 \mid \widetilde{\Pi}) = r(x_1 \mid \widetilde{\Pi})$ or $r(z_1 \mid \widetilde{\Pi}) = r(a_5 \mid \widetilde{\Pi})$.

Case IV: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either X_2 or B_2 then from (3.3), (3.8) and (3.23) we obtain either $r(z_4 \mid \widetilde{\Pi}) = r(z_6 \mid \widetilde{\Pi})$ or $r(a_5 \mid \widetilde{\Pi}) = r(y_{4n-11} \mid \widetilde{\Pi})$ or $r(z_4 \mid \widetilde{\Pi}) = r(a_2 \mid \widetilde{\Pi})$.

Case V: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either Y_2 or C_2 then from (3.4), (3.9) and (3.21) we obtain either $r(z_4 \mid \widetilde{\Pi}) = r(z_5 \mid \widetilde{\Pi})$ or $r(a_1 \mid \widetilde{\Pi}) = r(x_{4n-11} \mid \widetilde{\Pi})$.
Case VI: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either $Z_1 \cup Z_2$ or $X_1 \cup X_2$ or $Y_1 \cup Y_2$ then from (3.24), (3.26) and (3.27), we can easily be seen that either $r(a_1 \mid \widetilde{\Pi}) = r(a_5 \mid \widetilde{\Pi})$ or $r(z_1 \mid \widetilde{\Pi}) = r(a_5 \mid \widetilde{\Pi})$ or $r(z_1 \mid \widetilde{\Pi}) = r(a_1 \mid \widetilde{\Pi})$.
Case VII: If two partitioning sets of $\widetilde{\Pi}$ are subsets of $B_1 \cup B_2$ then from (3.16), (3.18), (3.22) and (3.23) we see that some either a_i, a_j or a_i, x_j or z_i, z_j have same representations with respect to $\widetilde{\Pi}$.
Case VIII: If two partitioning sets of $\widetilde{\Pi}$ are subsets of $C_1 \cup C_2$ then from (3.17), and (3.19)-(3.21) we conclude that either a_i, a_j or a_i, x_j or z_i, z_j have same representations with respect to $\widetilde{\Pi}$.
Case IX: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either $(X_1 \cup B_1 \cup \{x_{2n-5}, b_{2n-5}\})$ or $(X_2 \cup B_2 \cup \{x_{2n-5}, b_{2n-5}\})$ then from (3.8), (3.10), (3.22) and (3.23) it is clear that either $r(z_1 \mid \widetilde{\Pi}) = r(z_3 \mid \widetilde{\Pi})$ or $r(z_4 \mid \widetilde{\Pi}) = r(z_6 \mid \widetilde{\Pi})$.
Case X: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either $(Y_1 \cup C_1 \cup \{y_{2n-5}, c_{2n-5}\})$ or $(Y_2 \cup C_2 \cup \{y_{2n-5}, c_{2n-5}\})$ then (3.9), (3.11), (3.20) and (3.21) implies that either $r(z_1 \mid \widetilde{\Pi}) = r(z_2 \mid \widetilde{\Pi})$ or $r(z_4 \mid \widetilde{\Pi}) = r(z_5 \mid \widetilde{\Pi})$.
Case XI: Also If two partite sets of $\widetilde{\Pi}$ are subsets of $(C_1 \cup C_2 \cup B_1 \cup B_2)$ then there exists some $x_i \in X_1 \cup X_2$ and $y_j \in Y_1 \cup Y_2$ with same representations.
Case XII: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either $(X_1 \cup X_2 \cup C_1 \cup \{c_{2n-5}\})$ or $(X_1 \cup X_2 \cup C_2 \cup \{c_{2n-5}\})$ then by (3.20), (3.21)and (3.26) we obtain either $r(z_1 \mid \widetilde{\Pi}) = r(a_5 \mid \widetilde{\Pi})$ or $r(z_4 \mid \widetilde{\Pi}) = r(a_4 \mid \widetilde{\Pi})$.
Case XIII: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either $(Y_1 \cup Y_2 \cup B_2 \cup \{b_{2n-5}\})$ or $(Y_1 \cup Y_2 \cup B_1 \cup \{c_{2n-5}\})$ then from (3.22), (3.23) and (3.27) either $r(z_4 \mid \widetilde{\Pi}) = r(a_2 \mid \widetilde{\Pi})$ or $r(z_1 \mid \widetilde{\Pi}) = r(a_1 \mid \widetilde{\Pi})$.

Note that there are total 2047 possible combinations of subsets of vertex set of $F_3[n]$ shown in (3.1), we guess that no two partite sets of $\widetilde{\Pi}$ can be subsets of combinations of X_1, X_2, Y_1, Y_2, Z_1, Z_2, A, B_1, B_2, C_1 and C_2. Thus, we have the following conjecture.

Conjecture 3.1. *The partition dimension of $F_3[n]$, $n \geq 5$, is* 4.

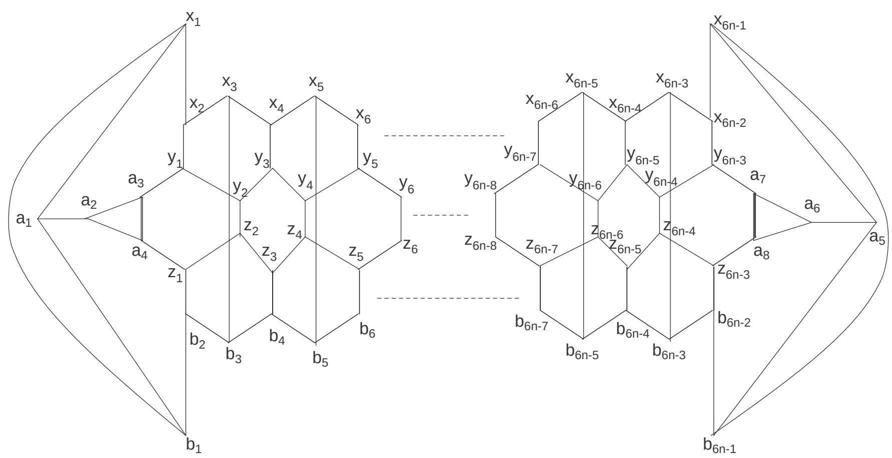

Figure 4. Graph $F_4[n]$

Next, we give the conjecture on the partition dimension of fullerene graph $F_4[n]$. The set of vertices of $F_4[n]$ is divided into the following sets:

$$\begin{array}{lll} A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}, & X = \{x_i \mid 1 \leq i \leq 6n-1\}, & B = \{b_i \mid 1 \leq i \leq 6n-1\}, \\ Y = \{y_i \mid 1 \leq i \leq 6n-3\}, & Z = \{z_i \mid 1 \leq i \leq 6n-3\}. & \end{array} \tag{3.31}$$

The relations of distances of the vertices of $F_4[n]$ are as follows:

$$d(x_1, a) = d(b_1, a), \quad \text{for all } a \in A \setminus \{a_7, a_8\}, \tag{3.32}$$
$$d(x_{6n-1}, a) = d(b_{6n-1}, a), \quad \text{for all } a \in A \setminus \{a_3, a_4\}, \tag{3.33}$$
$$d(a_2, x) = d(a_4, x), \quad \text{for all } x \in X \setminus \{x_1\}, \tag{3.34}$$
$$d(a_4, y) = d(b_2, y), \quad \text{for all } y \in Y \setminus \{y_1\}, \tag{3.35}$$
$$d(a_3, z) = d(x_2, z), \quad \text{for all } z \in Z \setminus \{z_1\}, \tag{3.36}$$
$$d(a_2, b) = d(a_3, b), \quad \text{for all } b \in B \setminus \{b_1\}, \tag{3.37}$$
$$d(a_1, x) = d(b_1, x), \quad \text{for all } x \in X, \tag{3.38}$$
$$d(a_1, b) = d(x_1, b), \quad \text{for all } b \in B, \tag{3.39}$$
$$d(a_7, z) = d(x_{6n-2}, z), \quad \text{for all } z \in Z \setminus \{z_{6n-3}\}, \tag{3.40}$$
$$d(a_8, y) = d(b_{6n-2}, y), \quad \text{for all } y \in Y \setminus \{y_{6n-3}\}. \tag{3.41}$$

The relations of distances of the vertices of Z and $F_4[n]$ are as follows:

$$d(a_1, z) = d(x_1, z), \quad d(a_4, z) = d(b_2, z), \quad d(a_8, z) = d(b_{6n-2}, z), \quad d(a_2, z) = d(a_3, z). \tag{3.42}$$

The relations of distances of the vertices of Y and $F_4[n]$ are as follows:

$$d(a_1, y) = d(b_1, y), \quad d(a_3, y) = d(x_2, y), \quad d(a_7, y) = d(x_{6n-2}, y), \quad d(a_2, y) = d(a_4, y). \tag{3.43}$$

Lemma 3.2. *Let $F_4[n]$ be a fullerene graph shown in Figure 4. Then $3 \leq pd(F_4[n]) \leq 4$, where $n \geq 1$.*

Proof. Let $\Pi = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{a_3\}$, $S_2 = \{a_7\}$, $S_3 = \{a_8\}$ and $S_4 = V(F_4[n]) \setminus \{a_3, a_7, a_8\}$, be a partition of $V(F_4[n])$. We show that Π is a resolving partition of $F_4[n]$ with minimum cardinality. The representation of each vertex of A other than a_3, a_7, a_8 with respect to Π is given as:

$$\begin{array}{lll} r(a_1 \mid \Pi) = (2, 6n, 6n, 0), & r(a_2 \mid \Pi) = (1, 6n-1, 6n-1, 0), & r(a_4 \mid \Pi) = (1, 6n-1, 6n-2, 0), \\ r(a_5 \mid \Pi) = (6n, 2, 2, 0), & r(a_6 \mid \Pi) = (6n-1, 1, 1, 0). & \end{array}$$

The representation of each vertex of X with respect to Π is given as:

$$r(x_i \mid \Pi) = \begin{cases} (3, 6n-1, 6n, 0) & \text{if } i = 1, \\ (i, 6n-i, 6n+1-i, 0) & \text{if } 2 \leq i \leq 6n-2, \\ (6n-1, 3, 3, 0) & \text{if } i = 6n-1. \end{cases}$$

The representation of each vertex B with respect to Π is given as:

$$r(b_i \mid \Pi) = \begin{cases} (3, 6n, 6n-1, 0) & \text{if } i = 1, \\ (i-1, 6n+1-i, 6n-i, 0) & \text{if } 2 \leq i \leq 6n-2, \\ (6n, 3, 3, 0) & \text{if } i = 6n-1. \end{cases}$$

The representation of each vertex of Y and Z with respect to Π is given as:

$$\begin{aligned} r(y_i \mid \Pi) &= (i, 6n-2-i, 6n-1-i, 0) && \text{if } 1 \le i \le 6n-3, \\ r(z_i \mid \Pi) &= (i+1, 6n-1-i, 6n-2-i, 0) && \text{if } 1 \le i \le 6n-3. \end{aligned}$$

From above representations of vertices with respect to Π it can be easily seen that representations are distinct. This implies that Π is a resolving partition of $F_4[n]$. Thus $pd(F_4[n]) \le 4$. Also by Lemma 1.2, we note that $pd(F_4[n]) \ge 3$. □

Suppose that there exists partition $\widetilde{\Pi}$ of $F_4[n]$, $n \ge 1$, such that $|\widetilde{\Pi}| = 3$. Let $\widetilde{\Pi} = \{\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3\}$. Consider the following cases:

Case I: If two partitioning sets of $\widetilde{\Pi}$ are subsets of X then by (3.38), we have $r(a_1|\widetilde{\Pi}) = r(b_1|\widetilde{\Pi})$ and if two partitioning sets of $\widetilde{\Pi}$ are subsets of Y then by (3.43), we have $r(a_1 \mid \widetilde{\Pi}) = r(b_1 \mid \widetilde{\Pi})$.

Case II: If two partitioning sets of $\widetilde{\Pi}$ are subsets of A except $\{a_7, a_8\}$ then by (3.32), we have $r(b_1 \mid \widetilde{\Pi}) = r(x_1 \mid \widetilde{\Pi})$. If two partitioning sets of $\widetilde{\Pi}$ are subsets of A except $\{a_3, a_4\}$ then by (3.33), we have and $r(x_{6n-1} \mid \widetilde{\Pi}) = r(b_{6n-1} \mid \widetilde{\Pi})$.

Case III: If two partitioning sets of $\widetilde{\Pi}$ are subsets of either B or Z then by (3.39) and (3.42), we have $r(a_1 \mid \widetilde{\Pi}) = r(x_1 \mid \widetilde{\Pi})$.

Case IV: Similarly, from equations (3.38) and (3.43) we observe that if two partitioning sets of $\widetilde{\Pi}$ are subsets of $X \cup Y$ then $r(a_1 \mid \widetilde{\Pi}) = r(b_1 \mid \widetilde{\Pi})$.

Case V: If two partitioning sets of $\widetilde{\Pi}$ are subsets of $B \cup Z$ then from (3.39) and (3.42), we see that $r(a_1 \mid \widetilde{\Pi}) = r(x_1 \mid \widetilde{\Pi})$.

Case VI: We notice that if two partitioning sets of $\widetilde{\Pi}$ are subsets of $Y \cup Z$ then there exists either some a_i, x_j or a_i, b_j with same representations with respect to $\widetilde{\Pi}$.

Note that there are total 31 possible combinations of subsets of vertex set of $F_4[n]$, shown in (3.31). Thus because of unique structural properties of $F_4[n]$, we can observe that no two partitioning sets of $\widetilde{\Pi}$ can be subsets of combinations of A, B, X, Y and Z. Thus, we have the following conjecture.

Conjecture 3.2. *The partition dimension of $F_4[n]$ is* 4.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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