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Research article

Study of Multivalent Spirallike Bazilevic Functions

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Abstract: In this paper, we introduce certain new subclasses of multivalent spirallike Bazilevic functions by using the concept of k-uniformly starlikness and k-uniformly convexity. We prove inclusion relations, sufficient condition and Fekete-Szego inequality for these classes of functions. Convolution properties for these classes are also discussed.

Keywords: spirallike function; Bazilevic functions; multivalent function; necessary and sufficient conditions

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{E} = \{z : \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write

$$\mathcal{A}(1) = \mathcal{A}.$$

Furthermore, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in $\mathbb{E}$.

The familiar class of p-valently starlike functions in $\mathbb{E}$, will be denoted by $\mathcal{S}^*(p)$ which consists of function $f \in \mathcal{A}(p)$ that satisfy the following conditions

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\ z \in \mathbb{E}).$$

One can easily see that

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the well-known class of starlike functions.

Moreover, for two functions f and g analytic in $\mathbb{E}$, we say that the function f is subordinate to the function g and write as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathbb{E}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

Furthermore, if the function g is univalent in $\mathbb{E}$ then it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{E}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{E}) \subset g(\mathbb{E}).$$

Next, for a function $f \in \mathcal{A}(p)$ given by (1.1) and another function $f \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=2}^{\infty} b_{n+p} z^{n+p} \qquad (\forall\, z \in \mathbb{E}),$$

the convolution (or the Hadamard product) of f and g is given by

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

Moreover, the subclass of $\mathcal{A}$ consisting of all analytic functions and has positive real part in $\mathbb{E}$ is denoted by $\mathcal{P}$. An analytic description of $\mathcal{P}$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (\forall\, z \in \mathbb{E}).$$

Furthermore, if

$$Re\{h(z)\} > \rho,$$

then we say that h be in the class $\mathcal{P}(\rho)$. Clearly, one can easily observed that

$$\mathcal{P}(0) = \mathcal{P}.$$

Historically in 1955, Bazilevic [2] define the class of Bazilevic functions, which is the subclass of $\mathcal{S}$, firstly, as follows.

Definition 1.1. For $h \in \mathcal{P}$, $g \in \mathcal{S}^*$ and f be given by (1.1) may be represented as

$$f(z) = \left[(\alpha + i\gamma) \int_0^z h(t) g(t)^{\alpha} t^{i\gamma - 1} dt\right]^{\frac{1}{(\alpha+i\gamma)}}$$

where α and γ are real numbers with $\alpha > 0$. The class of all such Bazilevic functions of type γ is denoted by $\mathcal{B}(\alpha, \gamma, h, g)$.

Furthermore, in 1933, Spacek [19] was the first who introduced β-spirallike functions as follows.

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\beta)$ if and only if

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{E}),$$

for

$$\beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2},$$

where $\mathbb{R}$ is the set of real numbers.

In 1967, Libera [12] extended this definition to functions spirallike of order ρ denoted by $\mathcal{S}^*_\rho(\beta)$ as follows.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*_\rho(\beta)$ if and only if

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \rho \qquad (\forall\, z \in \mathbb{E}),$$

for

$$\left(0 \leq \rho < 1, \quad \beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2}\right),$$

where $\mathbb{R}$ is the set of real numbers.

In fact, Kanas and Wiśniowska were the first (see [7, 8]) who defined the conic domain Ω_k, $k \geq 0$, as

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\} \tag{1.2}$$

and subjected to this domain they also introduced and studied the corresponding class k-$\mathcal{ST}$ of k-starlike functions (see Definition 1.4 below).

Moreover for fixed k, Ω_k represent the conic region bounded successively by the imaginary axis $(k = 0)$, for $k = 1$ a parabola, for $0 < k < 1$ the right branch of hyperbola and for $k > 1$ an ellipse. For these conic regions, following functions $p_k(z)$, which are given by (1.3), play the role of extremal functions.

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} = 1 + 2z + 2z^2 + ... & (k = 0) \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k = 1) \\ 1 + \frac{2}{1-k^2}\, \sinh^2\left\{\left(\frac{2}{\pi}\, \arccos k\right) \arctan h\, \sqrt{z}\right\} & (0 \leq k < 1) \\ 1 + \frac{1}{k^2-1}\, \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \frac{1}{k^2-1} & (k > 1), \end{cases} \tag{1.3}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa} z} \qquad (\forall\, z \in \mathbb{E})$$

and $\kappa \in (0,1)$ is chosen such that

$$k = \cosh\left(\pi K'(\kappa)/(4K(\kappa))\right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and

$$K'(\kappa) = K(\sqrt{1-\kappa^2}),$$

that is $K'(\kappa)$ is the complementary integral of $K(\kappa)$. Assume that

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \ldots \qquad (\forall\, z \in \mathbb{E}).$$

Then it was showed in [5] that for (1.3) one can have

$$P_1 = \begin{cases} \frac{2A^2}{1-k^2} & (0 \le k < 1) \\ \frac{8}{\pi^2}, & (k = 1) \\ \frac{\pi^2}{4k^2(\kappa)^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \tag{1.4}$$

and

$$P_2 = D(k) P_1, \tag{1.5}$$

where

$$D(k) = \begin{cases} \frac{A^2+2}{3} & (0 \le k < 1) \\ \frac{8}{\pi^2} & (k = 1) \\ \frac{(4K(\kappa))^2\left(\kappa^2+6\kappa+1\right)-\pi^2}{24K(\kappa)^2(1+\kappa)\sqrt{\kappa}} & (k > 1) \end{cases} \tag{1.6}$$

with $A = \frac{2}{\pi} \arccos k$.

These conic regions are being studied and generalized by several authors, for example see [6, 15, 18].

The class k-$\mathcal{ST}$ is define as follows.

Definition 1.4. A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{ST}$, if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \qquad (\forall\, z \in \mathbb{E} \text{ and } k \ge 0).$$

or equivalently

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|.$$

In the recent years, several interesting subclasses of analytic functions have been introduced and investigated from different viewpoints for example see ([1, 10, 11, 13, 14, 16]). Motivated and inspired by the recent research going on and the above mention work, we here introduce and investigate two new subclasses of analytic functions using the concept of Bazilevic and spirallike functions as follows.

Definition 1.5. Let $f \in \mathcal{A}(p)$. Then for $k \geq 0$ and $(0 < \mu < 1)$, $f \in k\text{-}\mathcal{S}(\beta, \mu)$ if and only if

$$\Re\left[e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu}\right] > k\left|e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu} - 1\right| + \rho\cos\beta \quad \left(\beta \in \mathbb{R} \text{ and } |\beta| < \frac{\pi}{2}\right)$$

Definition 1.6. Let $f \in \mathcal{A}(p)$. Then for $k \geq 0$ and $(0 < \mu < 1)$, $f \in k\text{-}\mathcal{M}(\beta, \lambda, \mu)$ if and only if

$$\Re\left\{\mathcal{L}(\beta, \mu, k, \lambda)\right\} > k\left|\mathcal{L}(\beta, \mu, k, \lambda) - 1\right| + \rho\cos\beta, \quad \left(\beta \in \mathbb{R} \text{ and } |\beta| < \frac{\pi}{2}\right)$$

where

$$\mathcal{L}(\beta, \mu, k, \lambda) = e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) \quad (\lambda \in \mathbb{R}) \tag{1.7}$$

and $\mathbb{R}$ is the set of real numbers.

2. A Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 2.1. (*see [17]*) *If $h(z)$ is analytic in $\mathbb{E}$ with*

$$h(0) = 1 \quad \textit{and} \quad \Re\{h(0)\} > \frac{1}{2} \quad (\forall\, z \in \mathbb{E}),$$

then for any function $\mathcal{F}$ analytic in $\mathbb{E}$, the function $h * \mathcal{F}$ takes values in the convex hull of the image of $\mathbb{E}$ under $\mathcal{F}$.

Lemma 2.2. (*see [4]*) *If a function w, of the form given by*

$$w(z) = c_1 z + c_2 z^2 + \ldots \quad \textit{and} \quad |w(z)| \leq |z| \qquad (\forall\, z \in \mathbb{E}), \tag{2.1}$$

then for every complex number s, we have

$$\left|c_2 - s c_1^2\right| \leq 1 + (|s| - 1)\left|c_1^2\right|.$$

Lemma 2.3. *[12] An analytic $f(z)$ is β- spirallike of order ρ $\left(0 \leq \rho < 1,\ |\beta| < \frac{\pi}{2}\right)$ if and only if there exist an analytic function $w(z)$ satisfying*

$$w(0) = 0 \quad \textit{and} \quad |w(z)| < 1$$

such that

$$e^{i\beta}\frac{zf'(z)}{f(z)} = \rho\cos\beta + (1-\rho)(\cos\beta)\frac{1 - w(z)}{1 + w(z)} + i\sin\beta \quad (\forall z \in \mathbb{E}).$$

Lemma 2.4. *[3] Let $w(z)$ be analytic in $\mathbb{E}$ with*

$$w(0) = 0$$

if there exist a $z_0 \in \mathbb{E}$ such that

$$\max_{|z|<|z_0|}(w|z| = w|z_0|)$$

then

$$z_0 w'(z_0) = m w(z_0)$$

for some $m \geq 1$.

3. Main results and their demonstrations

In this section, we will prove our main results.

Theorem 3.1. *Let the function be defined by* (1.1) *and* $0 \leq k < \infty$ *be a fixed number. If the function* f *is a member of the function class* k-$\mathcal{M}(\beta, \lambda, \mu)$ *then for* $-\infty < v < \infty$

$$\left|a_3 - va_2^2\right| \leq \left|\frac{P_1}{e^{i\beta}(\mu + 2\lambda)}\right| \begin{cases} \left|v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2} - \Lambda(k)\right| & (v > \eta_1) \\ 1 & (\eta_2 \leq v \leq \eta_1) \\ \left|\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right| & (v < \eta_2), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} \Lambda(k) &= \frac{2D(k)e^{i\beta}(\mu+\lambda)^2 - (\mu+2\lambda)(\mu-1)P_1}{2e^{i\beta}(\mu+\lambda)^2} \\ v &= \left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu - \frac{2+3p+p^2}{2+7p-p^3}\right) \\ \eta_1 &= \frac{1+D(k)}{P_1} \\ \eta_2 &= \frac{D(k)-1}{P_1} \end{aligned} \tag{3.2}$$

and P_1, $D(k)$ *are given by* (1.4)*, and* (1.6)*, respectively.*

Proof. If $f(z) \in k$-$\mathcal{M}(\beta, \lambda, \mu)$ then there exists a Schwarz function w in $\mathbb{E}$, such that

$$\mathcal{L}(\beta, \mu, k, \lambda) = p_k(w(z)). \tag{3.3}$$

where $\mathcal{L}(\beta, \mu, k, \lambda)$ is given by (1.7). We find after some simplification that

$$a_2 = \frac{P_1 c_1}{e^{i\beta}(\mu+\lambda)}, \tag{3.4}$$

$$a_3 = \frac{P_1}{e^{i\beta}(\mu+2\lambda)}\left[c_2 + \Lambda(k)c_1^2 - v\frac{P_1^2}{e^{2i\beta}(\mu+\lambda)^2}c_1^2\right], \tag{3.5}$$

where v is given by (3.2).

Making use of (3.4) and (3.5), we have

$$\left(a_3 - va_2^2\right) = \frac{P_1}{e^{i\beta}(\mu+2\lambda)}\left[c_2 + \left\{\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right\}c_1^2\right]. \tag{3.6}$$

Taking the moduli in (3.6), we thus obtain

$$\left|a_3 - va_2^2\right| = \left|\frac{P_1}{e^{i\beta}(\mu+2\lambda)}\right|\left|c_2 - c_1^2 + \left\{1 + \Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right\}c_1^2\right|. \tag{3.7}$$

In order to prove the first inequality in (3.1), we assume that $v > \eta_1$, then using the estimate

$$\left|c_2 - c_1^2\right| \leq 1,$$

from Lemma 2.2 and the known estimate $|c_1| \leq 1$ of the Schwarz Lemma, as a consequence, we have

$$\left|a_3 - va_2^2\right| \leq \left|\frac{p_1}{e^{i\beta}(\mu+2\lambda)}\right| \cdot \left|v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2} - \Lambda(k)\right|$$

and thus the first inequality in (3.1) is now proved.

To prove the last inequality in the (3.1), for this let $v < \eta_2$, then from (3.7), we have

$$\left|a_3 - va_2^2\right| \leq \left|\frac{P_1}{e^{i\beta}(\mu+2\lambda)}\right| \left[|c_2| + \left\{\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right\}\left|c_1^2\right|\right].$$

Applying the estimates

$$|c_2| \leq 1 - \left|c_1^2\right|$$

of Lemma 2.2 and the known estimate $|c_1| \leq 1$, we have

$$\left|a_3 - va_2^2\right| \leq \frac{P_1}{e^{i\beta}(\mu+2\lambda)}\left[1 + \left\{\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2} - 1\right\}\left|c_1^2\right|\right],$$

$$\left|a_3 - va_2^2\right| \leq \left|\frac{P_1}{e^{i\beta}(\mu+2\lambda)}\right| \cdot \left|\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right|.$$

This is the last expression of (3.1).

Finally, if $\eta_2 \leq v \leq \eta_1$ then

$$\left|\left\{\Lambda(k) - v\frac{P_1(\mu+2\lambda)}{e^{i\beta}(\mu+\lambda)^2}\right\}\right| \leq 1. \tag{3.8}$$

Therefore (3.7) yields

$$\begin{aligned}\left|a_3 - va_2^2\right| &\leq \left|\frac{P_1}{e^{i\beta}(\mu+2\lambda)}\right|\left[|c_2| + \left|c_1^2\right|\right] \\ &= \left|\frac{P_1}{e^{i\beta}(\mu+2\lambda)}\right|.\end{aligned}$$

Thus, we have the middle inequality of (3.1). Now, we have completed the proof of our Theorem. □

Theorem 3.2. *Let $\rho > 0$ and $|\beta| < \frac{\pi}{2}$. Then*

$$k\text{-}\mathcal{M}(\beta,\lambda,\mu) \subset 0\text{-}\mathcal{S}(\beta,\mu).$$

Proof. Let $f \in k\text{-}\mathcal{M}(\beta,\lambda,\mu)$ and let

$$e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu} = \rho\cos\beta + (1-\rho)\cos\beta\left(\frac{1-w}{1+w}\right) + i\sin\beta. \tag{3.9}$$

Clearly in view of Lemma 2.3 it is sufficient to show that

$$|w(z)| < 1.$$

From (3.9), we have

$$e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu} = \frac{(2\rho\cos\beta - \cos\beta + i\sin\beta)w(z) + e^{i\beta}}{1 + w(z)}. \tag{3.10}$$

Differentiating (3.10) logarithmically and after some straightforward simplification, we have

$$\begin{aligned} & e^{i\beta}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right] \\ &= \rho\cos\beta + i\sin\beta + \frac{3\lambda m(2\rho\cos\beta - \cos\beta + i\sin\beta)}{4\mu} + \frac{m\lambda e^{i\beta}}{4\mu}. \end{aligned} \tag{3.11}$$

Suppose that there exist $\zeta \in \mathbb{E}$ such that

$$\max_{|z|<|\zeta|}(w\,|z| = w\,|\zeta|)$$

and, from Lemma 2.4,

$$\zeta w'(\zeta) = mw(\zeta)$$

for some $m \geq 1$, so we have

$$\begin{aligned} & \Re\left[e^{i\beta}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right] \\ =\ & \Re\left[\rho\cos\beta + i\sin\beta + \frac{3\lambda m(2\rho\cos\beta - \cos\beta + i\sin\beta)}{4\mu} + \frac{m\lambda e^{i\beta}}{4\mu}\right]. \end{aligned} \tag{3.12}$$

After some simplification, we have

$$\begin{aligned} & \Re\left[e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right] \\ &= \rho\cos\beta - \frac{m\lambda}{2\mu}(1-3\rho)\cos\beta < \rho\cos\beta, \quad (\lambda(1-3\rho) > 0;\ 0 \leq \rho < 1). \end{aligned}$$

Now consider

$$\begin{aligned} & k\left|e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) - 1\right| + \rho\cos\beta \\ &= k\left|\rho\cos\beta + i\sin\beta + \frac{3\lambda m(2\rho\cos\beta - \cos\beta + i\sin\beta)}{4\mu} + \frac{m\lambda e^{i\beta}}{4\mu}\right| + \rho\cos\beta \\ &= k\sqrt{\left[\left(\rho - 1 + \frac{m\lambda(3\rho-1)\cos\beta}{2\mu}\right)^2 + \left(1 + \frac{\lambda m}{\mu}\sin\beta\right)^2\right]} + \rho\cos\beta > \rho\cos\beta. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\Re\left[e^{i\beta}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right]$$

$$< k\left|e^{i\beta}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]-1\right|+\rho\cos\beta.$$

This contradicts the fact that $f(z) \in k\text{-}\mathcal{M}(\beta,\lambda,\mu)$. Thus $|w(z)| < 1$ in $\mathbb{E}$. This implies that $f \in 0\text{-}\mathcal{S}(\beta,\mu)$, which completes the proof. □

Theorem 3.3. *For* $0 \le \lambda_1 < \lambda_2$,

$$k\text{-}\mathcal{M}(\beta,\lambda_1,\mu) \subset 0\text{-}\mathcal{M}(\beta,\lambda_2,\mu).$$

Proof. Let $f(z) \in k\text{-}\mathcal{M}(\beta,\lambda_2,\mu)$
Now

$$\begin{aligned}
& e^{i\beta}\left((1-\lambda_1)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda_1 f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\\
&=\frac{\lambda_1}{\lambda_2}\left[e^{i\beta}\left((1-\lambda_2)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda_2 f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right]\\
&-\left(\frac{\lambda_1-\lambda_2}{\lambda_2}\right)e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu}\\
&=\frac{\lambda_1}{\lambda_2}N_1(z)+\left(1-\frac{\lambda_1}{\lambda_2}\right)N_2(z)=N(z),
\end{aligned}$$

where

$$N_1(z)=e^{i\beta}\left[(1-\lambda_2)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda_2 f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\in\mathcal{P}(h_{k,\rho})\subset\mathcal{P}(\rho)$$

and

$$N_2(z)=^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu}\in\mathcal{P}(\rho).$$

Since $\mathcal{P}(\rho)$ is a convex set (see [9]), therefore $N(z) \in \mathcal{P}(\rho)$. This implies that $\in 0\text{-}\mathcal{M}(\beta,\lambda_2,\mu)$. Thus $k\text{-}\mathcal{M}(\beta,\lambda_1,\mu) \subset 0\text{-}\mathcal{M}(\beta,\lambda_2,\mu)$. □

Theorem 3.4. $f \in \mathcal{A}(p)$*satisfies the condition*

$$\left|\frac{1}{e^{i\chi}F(z)}-\frac{1}{2\rho}\right|<\frac{1}{2\rho}\qquad(0\le\rho<1;\ \chi\in\mathbb{R})\tag{3.14}$$

if and only if, $f \in 0\text{-}\mathcal{M}(0,\lambda,\mu)$,*where*

$$F(z)=\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right].$$

Proof. Suppose f satisfies (3.14) , then we can write

$$\left|\frac{2\rho-e^{i\chi}F(z)}{e^{i\chi}F(z)2\rho}\right|<\frac{1}{2\rho}$$

$$
\begin{aligned}
&\iff \left(\left|\frac{2\rho - e^{i\chi}F(z)}{e^{i\chi}F(z)2\rho}\right|\right)^2 < \left(\frac{1}{2\rho}\right)^2 \\
&\iff \left(2\rho - e^{i\chi}F(z)\right)\left(\overline{2\rho - e^{i\chi}F(z)}\right) < e^{-i\chi}\overline{F(z)}e^{i\chi}F(z) \\
&\iff 4\rho^2 - 2\rho\left[e^{-i\chi}\overline{F(z)} + e^{i\chi}F(z)\right] + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
&\iff 4\rho^2 - 2\rho\left[e^{-i\chi}\overline{F(z)} + e^{i\chi}F(z)\right] < 0 \\
&\iff 2\rho - 2\Re\left[e^{i\chi}F(z)\right] < 0 \\
&\iff \Re\left[e^{i\chi}F\left(z\right)\right] > \rho \\
&\iff \Re\left[e^{i\chi}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right] > \rho.
\end{aligned}
$$

This completes the proof. □

Theorem 3.5. *Let $f \in k\text{-}\mathcal{S}(\beta,\mu)$ and $\phi(z) \in \mathcal{A}(p)$ with*

$$\Re\left(\frac{\phi(z)}{z^p}\right) > \frac{1}{2}. \tag{3.15}$$

Then

$$h(z) = (\phi * f)(z) \in k\text{-}S\,(\beta,\mu)\,.$$

Proof. Since $f \in k\text{-}\mathcal{S}(\beta,\mu)$ therefore

$$\Re\left(e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu}\right) > \frac{k+\rho\cos\beta}{k+1}. \tag{3.16}$$

Moreover we can write

$$e^{i\beta}\left(\frac{h(z)}{z}\right)^{\mu} = \left(\frac{\phi(z)}{z^p}\right) * \left(e^{i\beta}\left(\frac{f(z)}{z}\right)^{\mu}\right). \tag{3.17}$$

Finally, by applying Lemma 2.1 in conjunction with (3.15), (3.16) and (3.17) we obtain the result asserted by Theorem 3.5. □

The following result (Theorem 3.6) can be proved by using arguments similar to those that are already presented in the proof of Theorem 3.5, so we choose to omit the details of our proof of Theorem 3.6.

Theorem 3.6. *Let $f \in k\text{-}\mathcal{M}(\beta,\lambda,\mu)$ and $\phi(z) \in \mathcal{A}(p)$ with*

$$\Re\left(\frac{\phi(z)}{z^p}\right) > \frac{1}{2}. \tag{3.18}$$

Then

$$h(z) = (\phi * f)(z) \in k\text{-}\mathcal{M}(\beta,\lambda,\mu).$$

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Conflict of Interest

The authors declare no conflict of interest.

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