Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 3(2): 316–321
DOI:10.3934/Math.2018.2.316
Received: 04 May 2018
Accepted: 05 June 2018
Published: 21 June 2018

Research Article

A renormalization approach to the Riemann zeta function at -1, $1+2+3+.... \sim -1/12$.

Gunduz Caginalp*

Mathematics Department, University of Pittsburgh, Pittsburgh, PA 15260, USA

* **Correspondence:** Email: caginalp@pitt.edu; Tel: +14126248339.

Abstract: A scaling and renormalization approach to the Riemann zeta function, ζ, evaluated at -1 is presented in two ways. In the first, one takes the difference between $U_n := \sum_{q=1}^{n} q$ and $4U_{\lfloor \frac{n}{2} \rfloor}$ where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer function. Using the Cesaro mean twice, i.e., $(C, 2)$, yields convergence to the appropriate value. For values of z for which the zeta function is represented by a *convergent* infinite sum, the double Cesaro mean also yields $\zeta(z)$, suggesting that this could be used as an alternative method for extension from the convergent region of z. In the second approach, the difference $U_n - k^2 \bar{U}_{n/k}$ between U_n and a particular average, $\bar{U}_{n/k}$, involving terms up to $k < n$ and scaled by k^2 is shown to equal exactly $-\frac{1}{12}\left(1-k^2\right)$ for all $k < n$. This leads to another perspective for interpreting $\zeta(-1)$.

Keywords: Riemann zeta function; sum of natural numbers; $\zeta(-1)$; 1+2+3+...; Cesaro mean or sum
Mathematics Subject Classification: 11M99, 40C99

---

1. Introduction

The Riemann zeta function is defined as the analytic continuation of the infinite sum, $\zeta(z) = \sum_{q=1}^{\infty} q^{-z}$ where $z = x + iy \in \mathbb{C}$ for $\operatorname{Re} z = x > 1$. For $x > 1$ the series converges absolutely to an analytic function. For all other values of z it diverges. Riemann showed [1] that it can be continued analytically for complex values of $z \in \mathbb{C} \setminus \{1\}$, i.e., except for the value corresponding to the harmonic series. For $z = -1$ one has the (divergent) sum of natural numbers. The analytic continuation of the series yields the result $\zeta(-1) = -1/12$, with the formal representation that appears to be an obvious contradiction:

$$1+2+3+\ ...\ \sim \zeta(-1) = -\frac{1}{12}\,. \tag{1}$$

In this note we examine this relation using an approach that involves scaling the truncated (finite) sum and renormalizing in order to obtain a finite result as one takes the infinite limit of the sum.

Renormalization consists of a set of methodologies constituting a philosophy and approach to problems exhibiting a divergence in some form. Originally introduced for statistical mechanics and quantum field theory by Ken Wilson in the 1970's, renormalization was able to yield the exponents with which key physical properties diverge (see for example, [2, 3]). The basic idea is first to average spins within a particular geometric configuration, thereby reducing the size of the system by a factor greater than unity. The reduction in size must be compensated by adjusting the interaction strengths. If this were not done, then iteration of this process would yield a trivial fixed point of zero or infinity. With the appropriate renormalization, however, one can iterate the procedure repeatedly. The key ansatz is that the exponent of the divergent quantity should not change due to this averaging process (with the interactions appropriately renormalized) since the singularity is due to the divergence of the "correlation length" which is the a measure of the distance at which spins can influence one another.

This approach to statistical mechanics revolutionized many calculations, as very simple calculations yielded the results previously obtained by a *tour de force*, and led to its adaptation in a number of other areas. The text by Creswick, Poole and Farach [3] describes the implementation of this approach to classical mathematical problems such as fractals and random walk. For example, in random walk, the classical result under robust conditions is that after n steps the random walk has mean distance $\sim n^{1/2}$ from the original point. This result can also be obtained by averaging sets of k steps, and readjusting (i.e., renormalizing) the step size so that one considers a walk of n/k steps with the new step size. The unique renormalization (i.e., setting the new step size) that leads to a non-trivial result (i.e., not 0 or ∞) yields the exponent $1/2$ in $n^{1/2}$.

In this paper we describe methodology along the lines of this approach to obtain an analog of (1) that is well-defined.

The expression (1) has been of interest in applications such as string theory [4]. In addition to this perspective, two physicists [5] have also provided an explanation of (1) based on shifting infinite sums.

2. Averaging and re-scaling (Method 1)

Using the notation $U_n = \sum_{q=1}^{n} q$ and $\lfloor r \rfloor$ as the greatest integer less than or equal to r we define

$$Y_n = U_n - 4U_{\lfloor \frac{n}{2} \rfloor}$$

One has from a simple calculation,

$$Y_n = \begin{cases} -n/2 & if \quad n\ even \\ (n+1)/2 & if \quad n\ odd \end{cases} .$$

Now, let Z_N be the Cesaro mean of $\{Y_n\}$, which is also known as the Cesaro sum and plays an important role in Fourier analysis (see for example, [6], p.52), i.e.,

$$Z_N := Avg\{Y_n : n \leq N\} = \frac{1}{N}\sum_{n=1}^{N} Y_n.$$

Considering the odd and even terms separately, one can readily observe that

$$Z_{2K+1} = \frac{1}{2K+1}\left\{1 - 1 + 2 - 2 + ... + \frac{2K+2}{2}\right\}$$

$$
\begin{aligned}
&= \frac{1}{2} + \frac{1}{4K+2} \, . \\
Z_{2K} &= 0. \qquad (2)
\end{aligned}
$$

Let $X_M := Avg\{Z_N : N \le M\}$, i.e., the Cesaro mean of Z_N, which is the Cesaro mean of the Cesaro mean, i.e., $(C, 2)$, of the original $\{Y_n\}$.

Theorem 1. *For $K \in \mathbb{N}$ one has the following:*

$$X_{2K} = \frac{1}{4} + \frac{1}{8K} \sum_{j=1}^{K} \frac{1}{j - 1/2}, \quad X_{2K+1} = \frac{K+1}{4K+2} + \frac{1}{8K+4} \sum_{j=1}^{K+1} \frac{1}{j - 1/2} \qquad (3)$$

$$\left| X_{2K} - \frac{1}{4} \right| \le \frac{1}{3K} + \frac{\log (K+1)}{8K} \qquad (4)$$

$$\left| X_{2K+1} - \frac{1}{4} \right| \le \frac{16/3 + \log (K+1)}{8K+4} \qquad (5)$$

and thus the limit

$$\lim_{M \to \infty} X_M = \frac{1}{4}, \qquad (6)$$

which can also be expressed as

$$\lim_{M \to \infty} Avg_{N \le M} \left\{ Avg_{n \le N} \left\{ \frac{U_n - 4U_{\lfloor \frac{n}{2} \rfloor}}{1-4} \right\} \right\} = -\frac{1}{12} \, . \qquad (7)$$

Proof. A computation using (2) for even and odd values of M results in (4) and (5). The inequalities

$$\log (K+1) \le \sum_{j=1}^{K+1} \frac{1}{j - 1/2} = \frac{8}{3} + \sum_{j=3}^{K+1} \frac{1}{j - 1/2} \le \frac{8}{3} + \log (K+1)$$

yield the result (6). □

Remark 1. *One can summarize this heuristically as*

$$
\begin{aligned}
E\left[U_n - 4U_{\lfloor \frac{n}{2} \rfloor}\right] &= P\{n = odd\} \left(\frac{n+1}{2} \right) + P\{n = even\} \left(-\frac{n}{2} \right) \\
&= \frac{1}{2} \left(\frac{n+1}{2} - \frac{n}{2} \right) = \frac{1}{4}. \qquad (8)
\end{aligned}
$$

In order to make (8) precise, one would need to invoke some basic probabilistic ideas, primarily the Kolmogorov extension, or existence, theorem (see for example, [7], p. 514) whereby the probability on a finite subset of $\mathbb{N}$ can be extended to all of $\mathbb{N}$. Formally identifying U_n and $U_{\lfloor \frac{n}{2} \rfloor}$ as $n \to \infty$ as though they were convergent leads to (1).

One can define the analogous relations $U_n(z) := \sum_{q=1}^{n} q^{-z}$ which, as noted above, converges to $\zeta(z)$ for $\operatorname{Re} z > 1$. Note that if a series that converges in the ordinary sense, the Cesaro mean must converge to the same limit. Moreover, if a sequence $\{c_j\}$ is convergent to c, the Cesaro mean also converges to

c. Thus, for values of $z \in \mathbb{C}$ in the convergent region, the infinite sum is equal to the Cesaro mean, so that the Cesaro mean can be used for both convergent and nonconvergent values.

In particular for a value z for which $U_n(z)$ converges in the usual sense to $\zeta(z)$, one has

$$\lim_{n\to\infty} Y_n(z) = \lim_{n\to\infty} \left\{ U_n(z) - 4U_{\lfloor \frac{n}{2} \rfloor}(z) \right\} = -3\zeta(z)$$

as $n \to \infty$. Since $Y_n(z)$ is convergent for this value of z, it follows that the Cesaro mean $Z_N(z) := N^{-1} \sum_{n=1}^{N} Y_n(x)$ also converges to the same limit, $-3\zeta(z)$. Similarly, the Cesaro mean, $X_M(z) := M^{-1} \sum_{N=1}^{M} Z_N(z)$ also converges to $-3\zeta(z)$. Thus, the interpretation that double Cesaro mean of

$$\frac{U_n(z) - 4U_{\lfloor \frac{n}{2} \rfloor}(z)}{1-4}$$

converges to $\zeta(z)$ is maintained for values of z for which $U_n(z) = \sum_{q=1}^{n} q^{-z}$ is convergent.

For example, setting $z = 2$, so that one has a convergent series,

$$\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}, \ \textit{i.e.}, \ \lim_{n\to\infty} Y_n(2) = (-3)\frac{\pi^2}{6},$$

and consequently $\lim_{M\to\infty} X_M(z) / (-3) = \pi^2/6$.

Hence, this approach using the double Cesaro mean may present another avenue to extend the sum from the convergent to the nonconvergent regions, and offer other ways to study the Riemann zeta function.

3. Averaging and re-scaling (Method 2)

An alternate approach to averaging (without using the greatest integer concept) can be implemented by fixing k and letting $n = mk + j$.

Note that for $r \in \mathbb{N}$ one has $U_r = \sum_{q=1}^{r} q = \frac{1}{2} r(r+1)$. Define a continuous extension of U_r by $U_r := \frac{1}{2} r(r+1)$ to $r \in \mathbb{R}$. Then define the average of over the values of $j \in \{0, ..., k-1\}$ as

$$\bar{U}_{\frac{n}{k}} := \frac{1}{k} \sum_{j=0}^{k-1} U_{\frac{n-j}{k}}. \tag{9}$$

Theorem 2. *For $n, k \in \mathbb{N}$ and $k < n$ one has the exact relation*

$$\frac{U_n - k^2 \bar{U}_{\frac{n}{k}}}{1-k^2} = -\frac{1}{12}. \tag{10}$$

Proof. From the definition (9) one observes

$$\begin{aligned} U_n - k^2 \bar{U}_{\frac{n}{k}} &= \frac{1}{2} n(n+1) - k^2 \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{2} \left(\frac{n-j}{k} \right) \left(\frac{n-j}{k} + 1 \right) \\ &= -\frac{1}{12} \left(1 - k^2 \right). \end{aligned}$$

Thus, the quotient in (10) is thus independent of both k and n. This averaging and scaling of U_n by any factor k together with renormalization by k^2 yields the unique number $-1/12$. □

Formally, the identification of U_n and $\bar{U}_{n/k}$ with U in the limit as $n \to \infty$ leads to $\sum_{q=1}^{\infty} q \sim \zeta(-1) = -1/12$.

Remark 2. *(a) The number k can be regarded as the analog of the "subwalk" of k steps in the random walk problem discussed in the introduction, with n/k the new walk with the mean step size increased by a factor k^2.*

(b) Setting $k := n^p$ with $p \in (0, 1)$ one can write expression (10) *in the form*

$$\frac{n^{-p} U_n - n^p \bar{U}_{n^{1-p}}}{n^{-p} - n^p} = -\frac{1}{12}.$$

(c) Applying this approach for $k = 2$ yields,

$$\bar{U}_{\frac{n}{2}} = \frac{1}{2}\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2} + \frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)}{2}\right) = \frac{1}{8}n + \frac{1}{8}n^2 - \frac{1}{16}$$

$$U_n - 4\bar{U}_{\frac{n}{2}} = \frac{1}{4}$$

so that formally identifying U_n and $\bar{U}_{\frac{n}{2}}$ as U in the limit $n \to \infty$ yields $U \sim -\frac{1}{12}$.

(d) To extend this result to other values of $z \in \mathbb{C}$, one can utilize again the analogous quantity, $U_n^{(z)} = \sum_{q=1}^{n} q^{-z}$, determine whether it is possible to formulate a definition analogous to (9)*, and consider values of z for which these converges. The left hand side of* (10) *can be considered in the same manner as described for Method 1.*

Acknowledgments

The author thanks Dr. Alban Deniz and Prof. Bogdan Ion for useful discussions.

Conflict of Interest

The author declares no conflict of interest.

References

1. E. C. Titchmarsh, D. R. Heath-Brown, *The theory of the Riemann zeta-function*, Oxford University Press, 1986.
2. K. G. Wilson, J. B. Kogut, *The renormalization group and the ϵ expansion*, Phys. Rep., **12** (1974), 75–200.
3. R. J. Creswick, C. P. Poole and H. A. Farach, *Introduction to renormalization group methods in physics*, 1992.

4. J. G. Polchinski, String Theory, Volume I, *An Introduction to the Bosonic String*, Cambridge University Press, 1998.
5. A. Padilla, E. Copeland, Available from: `https://www.youtube.com/watch?v=w-I6XTVZXww`.
6. E. M. Stein, R. Shakarchi, *Fourier Analysis an introduction*, Princeton University Press, 2003.
7. P. Billingsley, *Probability and Measure*, Wiley: Anniversary Edition, 2012.

AIMS Press