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Research article

Osgood type blow-up criterion for the 3D Boussinesq equations with partial viscosity

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Abstract: This paper is dedicated to studying the blow-up criterion of smooth solutions to the three-dimensional Boussinesq equations with partial viscosity. By means of the Littlewood-Paley decomposition, we give an improved logarithmic Sobolev inequality and through this, we obtain the corresponding blow-up criterion in a space larger than $\dot{B}^0_{\infty,\infty}$, which extends several previous works.

Keywords: Boussinesq equations; blow-up criterion; Besov space
Mathematics Subject Classification: 35B44, 35B65, 76D03

1. Introduction

In this paper we consider the following Cauchy problem of 3D Boussinesq system:

$$\begin{cases} u_t - \nu\Delta u + u\cdot\nabla u + \nabla P = \theta e_3, \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where, $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ is a vector field denoting the velocity, $\theta = \theta(x,t)$ is a scalar function denoting either the temperature in the content of thermal convection, or the density in the modeling of geophysical fluids, $P = P(x,t)$ the scalar pressure and $e_3 = (0,0,1)$ is the unit vector in the x_3 direction. The parameters $\nu, \kappa \geqslant 0$ represent the kinematic viscosity and molecular diffusion coefficients, respectively, while u_0 and b_0 are the given initial data.

The Boussinesq system has widely been used in atmospheric sciences and oceanic fluids[13, 18]. Local existence and uniqueness theories of solutions have been studied by many mathematicians and physicists (see, e.g., [1, 2, 14]). Chae established the global regularity criteria for the 2D Boussinesq equations with partial dissipation in celebrated paper [3]. Recently, Ye [23] considered the case with horizontal dissipation in the horizontal velocity equation and vertical dissipation in the temperature equation. Similar results about global regularity for 2D incompressible fluid models please refer to [11, 12], and the reference therein. However, for the 3D Boussinesq equations, whether the unique local strong solution can exists globally for general large initial data is an outstanding challenging open problem. Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong or smooth solutions to the system (1.1). Ishimura and Morimoto [16] proved the following blow-up criterion

$$\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^3)). \tag{1.2}$$

Fan and Ozawa [9] and Fan and Zhou [10] established the following refined blow-up criterion for system (1.1) as

$$\mathrm{curl} u \in L^1(0, T; \dot{B}^0_{\infty,\infty}), \tag{1.3}$$

with $\nu = 1, \kappa = 0$ (zero-diffusive case) and $\nu = 0, \kappa = 1$ (zero-viscosity case), respectively. By means of the Littlewood-Paley theory and Bony's paradifferential calculus, Qiu-Du-Yao [19] extended the condition to

$$u \in L^q\left(0, T; B^s_{p,\infty}(\mathbb{R}^3)\right), \frac{2}{q} + \frac{3}{p} = 1 + s, \frac{3}{1+s} < p \leqslant \infty, -1 < s \leqslant 1, (p, s) \neq (\infty, 1), \tag{1.4}$$

and further studied by Dong-Jia-Zhang[5] in the case of $\kappa = 0$. Zhang and Gala [24] gave an Osgood type regularity criterion for the Newton-Boussinesq equation, that is,

$$\sup_{q \geq 2} \int_0^T \frac{\|\bar{S}_q \nabla u(\tau)\|_{L^\infty}}{q \ln q} d\tau = \infty, \tag{1.5}$$

where $\bar{S}_q = \sum_{l=-q}^{q} \dot{\Delta}_l$, $\dot{\Delta}_l$ being the homogenous Fourier localization operator.

Recently, Wu-Hu-Liu [21] established the blow-up criterion (1.5) for the Boussinesq equation with full viscosities ($\nu = 1, \kappa = 1$) and Ren[20] obtained the following blow-up criterion

$$\int_0^T \sup_{2 \leqslant q < \infty} \frac{\|\Delta_q \mathrm{curl} u(\tau)\|_{L^\infty}}{\log q} d\tau = \infty, \tag{1.6}$$

with the zero-viscosity constant. Here, Δ_q stands for nonhomegenous Littlewood-Paley projector operator. For more results about blow-up criterion for the system (1.1), we refer to [4, 6, 7, 15] and the reference therein.

Motivated by above-mentioned results, the purpose of this paper is to establish a blow-up criterion in a space (see definition 2.1) larger than $\dot{B}^0_{\infty,\infty}$. We should point out that, in the thesis [22], the author gave the blow-up criterion (1.3) for the fractional Boussinesq equations in n-dimensions($n \geq 2$) by using commutator operator estimate and the inequality

$$\|\nabla\theta(t)\|_{L^p} \leqslant \|\nabla\theta(T_0)\|_{L^p} exp\left\{\int_{T_0}^t \|\nabla u(\tau)\|_\infty \, d\tau\right\}, \quad p \in [2, \infty], \tag{1.7}$$

where L^∞-norm of the gradient of velocity can be controlled as

$$\|\nabla u\|_\infty \leq C\left(1 + \|u\|_{L^2} + \|\nabla \times u\|_{\dot{B}^0_{\infty,\infty}} \log\left(e + \|\Lambda^s u\|_{L^2}\right)\right), \quad s > 1 + \frac{n}{2}.$$

However, for our cases, the logarithmic Sobolev inequality becomes

$$\|u\|_{L^\infty} \leqslant C\Big(1 + \|u\|_{\dot{V}_\Theta} \ln(\|u\|_{H^m} + e) \ln(\ln(\|u\|_{H^m} + e) + e)\Big), \quad m > \frac{3}{2}, \tag{1.8}$$

then we can not apply the inequality (1.7) in [22]. Therefore, some new estimates need to be developed. In this paper, we will take use of the L^∞-norm of the temperature due to the special structure of the temperature equation. Through the boundness of $\|\theta\|_{L^\infty}$ and elaborate energy estimate which together with the logarithmic Sobolev inequality, we obtain our main results. As a consequence, we improve the results in reference [9, 10, 20, 21].

The paper is organized as follows. In section 2, we recall the definition of Besov space and state our main results. The commutator operator estimate and the logarithmic Sobolev inequality are presented in Lemma 2.1 and Lemma 2.2, respectively. Section 3 is devoted to proving Theorem 2.1.

Through this paper, C stands for some real positive constants which may be different in each occurrence.

2. Preliminaries and main results

Before presenting our results, we introduce some function spaces and some notations. First, we are going to recall some basic facts on Littlewood-Paley theory. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = 2\pi^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(\xi) dx.$$

Choose two nonnegative radial functions χ and φ , valued in the interval [0,1], supported in $B = \{\xi \in \mathbb{R}^3, |\xi| \leqslant \frac{4}{3}\}$, $C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, respectively, such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x-y) dy,$$

and

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}(2^j y) u(x-y) dy.$$

Formally, $\dot{\Delta}_j$ is a frequency projection to the annulus$\{|\xi| \approx 2^j\}$, and $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. By using of Littlewood-Paley's decomposition, we give the definition of the homogenous Besov space.

Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. $\mathcal{S}_h' = \{u \in \mathcal{S}'(\mathbb{R}^3); \lim_{j \to -\infty} \dot{S}_j = 0\}$ which can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$. Then the homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{u \in \mathcal{S}_h'(\mathbb{R}^3); \|u\|_{\dot{B}^s_{p,q}} < \infty\}$$

where

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j=-\infty}^{\infty} 2^{jsq} \|\dot{\Delta}_j u\|^q_{L^p})^{\frac{1}{q}}, & for \quad q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p}, & for \quad q = \infty. \end{cases}$$

Next, we introduce the space of Besov type, see [17].

Definition 2.1. *Let $\Theta(\alpha)(\geqslant 1)$ be a nondecreasing function on $[1, \infty]$. We denote by $\dot{V}_\Theta$ the set of tempered distributions u such that $\{u \in \mathcal{S}_h'(\mathbb{R}^3); \|u\|_{V_\Theta} < \infty\}$ and the norm is defined by*

$$\|u\|_{\dot{V}_\Theta} = \sup_{N=1,2\cdots} \frac{\| \sum_{j=-N}^{N} \dot{\Delta}_j u\|_{L^\infty}}{\Theta(N)}.$$

Remark 2.1. *We can easily see that $\|u\|_{\dot{V}_\Theta} \leqslant C\|u\|_{\dot{B}^0_{\infty,\infty}} \leqslant C\|u\|_{BMO} \leqslant C\|u\|_{L^\infty}$, provided $\Theta(N) \geqslant N$. In this paper, we will take $\Theta(N) = N\ln(N+e)$.*

Next, we present the following well-know commutator estimate and we can find the details in [8] for example.

Lemma 2.1. *Suppose that $s > 0$ and $p \in (1, \infty)$. Let f, g be two smooth functions such that $\nabla f \in L^{p_1}$, $\Lambda^s f \in L^{p_3}$, $\Lambda^{s-1} g \in L^{p_2}$ and $g \in L^{p_4}$, then there exist a constant C independent of f and g such that*

$$\|[\Lambda^s, f]g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}) \tag{2.1}$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Here $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

Next, we give the following logarithmic Sobolev inequality which plays an important role in the proof of the blow-up criterion for the classic solutions.

Lemma 2.2. *Let $m > \frac{3}{2}$, then there exists constant C depending only on s, p and Θ such that*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant C(1 + \|u\|_{\dot{V}_\Theta} \Theta(\ln(\|u\|_{H^m(\mathbb{R}^3)} + e))) \tag{2.2}$$

for all $u \in H^m(\mathbb{R}^3)$.

Proof. By using Littlewood-Paley theory, we decompose the function into three parts. More precisely, we write

$$u(x) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u = u_l(x) + u_m(x) + u_h(x) \tag{2.3}$$

where

$$u_l(x) = \sum_{j<-N} \dot{\Delta}_j u, \quad u_m(x) = \sum_{-N\leqslant j\leqslant N} \dot{\Delta}_j u \quad \text{and} \quad u_m(x) = \sum_{j>N} \dot{\Delta}_j u. \tag{2.4}$$

For the low-frequency part $u_l(x)$, we can show that

$$\|u_l(x)\|_\infty \leqslant \sum_{j<-N} \|\dot{\Delta}_j u\|_\infty \leqslant \sum_{j<-N} C2^{3j/2}\|\dot{\Delta}_j u\|_{L^2} \leqslant C \sum_{j<-N} 2^{3j/2}\|u\|_{L^2} \leqslant C2^{-\frac{3}{2}N}\|u\|_{H^s}. \tag{2.5}$$

For the high-frequency part

$$\begin{aligned}\|u_h(x)\|_\infty &\leqslant \sum_{j>N} \|\dot{\Delta}_j u\|_\infty \\ &\leqslant \sum_{j>N} C2^{3j/2}\|\dot{\Delta}_j u\|_{L^2} \\ &= C\sum_{j>N} 2^{sj}\|\dot{\Delta}_j u\|_{L^2} 2^{-sj+3j/2} \\ &\leqslant C2^{-(s-3/2)N}\|u\|_{H^s},\end{aligned} \tag{2.6}$$

for $s > \frac{3}{2}$, where we have used the following Bernstein estimate

$$\|\dot{\Delta}_j u\|_{L^{p_2}} \leqslant C2^{jd(\frac{1}{p_1}-\frac{1}{p_2})}\|\dot{\Delta}_j u\|_{L^{p_1}}, \quad \text{for} \quad 1 \leqslant p_1 \leqslant p_2 \leqslant \infty,$$

and the norm equivalent between $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}^s_{2,2}}$.

Next, we consider $u_m(x)$, by definition 2.1 we have

$$\|u_m(x)\|_\infty \leqslant \sum_{j=-N}^{N} \|\dot{\Delta}_j u\|_\infty \leqslant \Theta(N)\|u\|_{\dot{V}_\Theta}. \tag{2.7}$$

Taking from (2.3) to (2.7) into consideration, we get

$$\|u(x)\|_\infty \leqslant C(2^{-\frac{3}{2}N}\|u\|_{H^s} + 2^{-(s-3/2)N}\|u\|_{H^s} + \Theta(N)\|u\|_{\dot{V}_\Theta}). \tag{2.8}$$

If we take $N = [\frac{\ln(\|u\|_{H^s}+e)}{min(s-3/2,3/2)\ln 2}] + 1$, where $[\cdot]$ denotes Gausss symbol, then we have the desired estimate (2.2). □

Now, we state our main results.

Theorem 2.1. *Suppose $\nu > 0$, $\kappa = 0$ with $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Let $T > 0$ be the maximum time such that (u, θ) be a local smooth solution to the system (1.1). If $T < \infty$, then*

$$\limsup_{t\nearrow T}(\|u(t)\|_{H^s} + \|\theta(t)\|_{H^s}) = \infty, \tag{2.9}$$

if and only if

$$\int_0^T \|\nabla u\|_{\dot{V}_\Theta} d\tau = \infty. \tag{2.10}$$

Remark 2.2. *For the zero-diffusion case, that is, $\nu > 0$, $\kappa = 0$, the stiuation becomes more difficult. The main obstacle is that the temperature $\theta(x,t)$ in the transport equation does not gain any smoothness. Hence, the blow-up issue of the zero-diffusive Boussinesq equations is more difficult than that of full viscous Boussinesq system. The result (2.10) is an improvement of (1.5) in [21].*

Remark 2.3. *After some slight modifications, our methods can also be applied to the zero-viscosity case, that is, $\nu = 0, \kappa = 1$. More precisely, J_2 and K_2 in (3.10) and (3.15) can be estimated as*

$$\begin{aligned} J_2 &\leqslant C\left(\|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^4}\|\nabla^3\theta\|_{L^2}\right) \\ &\leqslant \|\nabla^3\theta\|_{L^2}^2 + C\left(\|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2}\right) \\ &\leqslant \|\nabla^3\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\left(\|\nabla^2\theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + 1\right), \end{aligned} \tag{2.11}$$

similarily

$$\begin{aligned} K_2 &\leqslant C\left(\|\nabla^3 u\|_{L^2}\|\nabla\theta\|_{L^3}\|\nabla^3\theta\|_{L^6} + \|\nabla^2 u\|_{L^2}\|\nabla^2\theta\|_{L^3}\|\nabla^3\theta\|_{L^6} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right) \\ &\leqslant \|\nabla^4\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\left(\|\nabla^2\theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + 1\right). \end{aligned} \tag{2.12}$$

Remark 2.4. *From the Biot-Savart law, we have $\|\nabla u\|_{\dot V_\Theta} \leqslant C\|\nabla u\|_{\dot B^0_{\infty,\infty}} \leqslant C\|w\|_{\dot B^0_{\infty,\infty}}$, which refined the results in [9, 10].*

Remark 2.5. *For the nonhomogenous case, our space becomes $V_\Theta = \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{V_\Theta} < \infty\}$, where*

$$\|u\|_{V_\Theta} = \sup_{N\geq 2}\frac{\left\|\sum_{j=-1}^{N}\Delta_j u\right\|_{L^\infty}}{\Theta(N)}.$$

Then our results is an improvement of (1.6) in [20], since

$$\sup_{N\geq 2}\frac{\left\|\sum_{j=-1}^{N}\Delta_j u\right\|_{L^\infty}}{N\ln(N+e)} \leqslant C\sup_{2\leqslant q<\infty}\frac{\|\Delta_q u\|_{L^\infty}}{\ln q} \leqslant C\|u\|_{B^0_{\infty,\infty}}.$$

Moreover, we give the following function for example

$$f(x) = \log\left(\frac{1}{|x|} + e\right)\log\log\left(\frac{1}{|x|} + e\right) \in V_\Theta,$$

but does not belongs to $B^0_{\infty,\infty}$, please refer to [17, 20] for details.

3. Proof of theorem 2.1

Proof. We consider the 3D Boussinesq equations (1.1) with $\nu > 0$, $\kappa = 0$. It is easy to see that (2.10) implies (2.9), since $\|\nabla u\|_{\dot V_\Theta} \leqslant C\|\nabla u\|_\infty \leqslant C\|u(t)\|_{H^3}$. Hence, we only need to prove that (2.9) implies (2.10). If (2.10) is false, then there exists a constant C such that

$$\int_0^T \|\nabla u\|_{\dot V_\Theta} d\tau < C. \tag{3.1}$$

Multiplying the second equation of (1.1) by u and θ, respectively, integrating and using the divergence-free condition $\nabla \cdot u = 0$, we immediately have

$$\|u(t)\|_{L^2} \leqslant C\left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}\right), \quad \|\theta(t)\|_{L^2} \leqslant \|\theta_0\|_{L^2}, \tag{3.2}$$

for any $t \in [0, T]$. Furthermore, we have $\|\theta(t)\|_{L^p} \leqslant \|\theta_0\|_{L^p}$, for $p \in [2, \infty]$.

Next, we are going to give H^1, H^2 and H^3 estimates to complete our proof.

(H^1 estimate). Multiplying the first two equations of (1.1) by Δu and $\Delta\theta$, respectively, after integrating by parts and taking the divergence-free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx + \int_{\mathbb{R}^3} (u \cdot \nabla\theta) \cdot \Delta\theta \, dx - \int_{\mathbb{R}^3} (\theta e_n) \cdot \Delta u \, dx \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.3}$$

After integrating by parts several times, one can conclude that the three terms above can be bounded as

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx \\
&= -\int_{\mathbb{R}^3} \partial_k(u_i \partial_i u_j) \partial_k u_j \, dx \\
&= -\int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j + u_i \partial_i \partial_k u_j \partial_k u_j \, dx \\
&\leqslant C\|\nabla u\|_\infty \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.4}$$

Similarly, we have

$$I_2 = \int_{\mathbb{R}^3} (u \cdot \nabla\theta) \cdot \Delta\theta \, dx \leqslant C\|\nabla u\|_\infty \|\nabla\theta\|_{L^2}^2, \tag{3.5}$$

and for the third term, we can show as

$$I_3 = -\int_{\mathbb{R}^3} (\theta e_n) \cdot \Delta u \, dx \leqslant \|\nabla\theta\|_{L^2} \|\nabla u\|_{L^2} \leqslant \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \tag{3.6}$$

Plugging the estimates (3.4), (3.5), (3.6) into (3.3), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 \\
&\leqslant C\|\nabla u\|_\infty \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_\infty \|\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \\
&\leqslant C(1 + \|\nabla u\|_\infty)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned} \tag{3.7}$$

(H^2 estimate) applying ∇^2 to the first two equations of (1.1), respectively, integrating and adding the resulting equations together it follows that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) + \|\nabla^3 u(t)\|_{L^2}^2$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla u)\cdot\nabla^2 u\,dx - \int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla\theta)\cdot\nabla^2\theta\,dx + \int_{\mathbb{R}^3} \nabla^2(\theta e_n)\cdot\nabla^2 u\,dx \\
&:= J_1 + J_2 + J_3. \qquad (3.8)
\end{aligned}
$$

In what follows, we will deal with each term on the right-hand side of (3.8) separately below

$$
\begin{aligned}
J_1 &= -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla u)\cdot\nabla^2 u\,dx = -\int_{\mathbb{R}^3} [\nabla^2, u\cdot\nabla]u\cdot\nabla^2 u\,dx \\
&\leqslant \|[\nabla^2, u\cdot\nabla]u\|_{L^2}\|\nabla^2 u\|_{L^2} \\
&\leqslant C\|\nabla u\|_\infty\|\nabla^2 u\|_{L^2}^2, \qquad (3.9)
\end{aligned}
$$

where we have used the Lemma 2.1 commutator estimate.

$$
\begin{aligned}
J_2 &= -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla\theta)\cdot\nabla^2\theta\,dx \\
&\leqslant \|\nabla^3 u\|_{L^2}\|\nabla\theta\|_{L^4}^2 + \|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2 \\
&\leqslant C\left(\|\nabla^3 u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2\right) \\
&\leqslant \|\nabla^3 u\|_{L^2}^2 + C\left(1 + \|\nabla u\|_\infty\right)\|\nabla^2\theta\|_{L^2}, \qquad (3.10)
\end{aligned}
$$

where we have used the following Gagliardo-Nirenberg inequality

$$
\|\nabla\theta\|_{L^4}^2 \leqslant C\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2}.
$$

The third term can be estimated as

$$
J_3 = \int_{\mathbb{R}^3} \nabla^2(\theta e_n)\cdot\nabla^2 u\,dx \leqslant \|\nabla^2\theta\|_{L^2}\|\nabla^2 u\|_{L^2}. \qquad (3.11)
$$

Combining the above estimates (3.9), (3.10), (3.11) with (3.8), we get

$$
\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) \leqslant C\left(1 + \|\nabla u\|_\infty)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2\right). \qquad (3.12)
$$

(H^3 estimate) applying ∇^3 to the first two equations of (1.1), respectively, integrating and adding the resulting equations together, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2) + \|\nabla^4 u(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx - \int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla b)\cdot\nabla^3 b\,dx + \int_{\mathbb{R}^3} \nabla^3(\theta e_n)\cdot\nabla^3 u\,dx \\
&:= K_1 + K_2 + K_3. \qquad (3.13)
\end{aligned}
$$

$K_i (i = 1, 2, 3)$ can be bounded as

$$
\begin{aligned}
K_1 &= -\int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx = -\int_{\mathbb{R}^3} [\nabla^3, u\cdot\nabla]u\cdot\nabla^3 u\,dx \\
&\leqslant \|[\nabla^3, u\cdot\nabla]u\|_{L^2}\|\nabla^3 u\|_{L^2}
\end{aligned}
$$

$$\leqslant C\|\nabla u\|_{\infty}\|\nabla^3 u\|_{L^2}^2, \tag{3.14}$$

and

$$\begin{aligned}
K_2 &= -\int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla\theta)\cdot\nabla^3\theta\,dx \\
&\leqslant C\left(\|\nabla^4 u\|_{L^2}\|\nabla\theta\|_{L^4}\|\nabla^2\theta\|_{L^4} + \|\nabla^3 u\|_{L^6}\|\nabla^2\theta\|_{L^2}\|\nabla^2\theta\|_{L^3} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right) \\
&\leqslant C\left(\|\nabla^4 u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla^3\theta\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right) \\
&\leqslant \|\nabla^4 u\|_{L^2}^2 + C\left(1+\|\nabla u\|_{\infty}\right)\|\nabla^3\theta\|_{L^2},
\end{aligned} \tag{3.15}$$

where we have used Young's inequality and the following Gagliardo-Nirenberg inequality in the three dimension

$$\|\nabla\theta\|_{L^4} \leqslant C\|\theta\|_{L^\infty}^{\frac{5}{6}}\|\nabla^3\theta\|_{L^2}^{\frac{1}{6}}, \quad \|\nabla^2\theta\|_{L^4} \leqslant C\|\theta\|_{L^\infty}^{\frac{1}{6}}\|\nabla^3\theta\|_{L^2}^{\frac{5}{6}},$$

$$\|\nabla^2\theta\|_{L^2} \leqslant C\|\theta\|_{L^\infty}^{\frac{2}{3}}\|\nabla^3\theta\|_{L^2}^{\frac{1}{3}}, \quad \|\nabla^2\theta\|_{L^3} \leqslant C\|\theta\|_{L^\infty}^{\frac{1}{3}}\|\nabla^3\theta\|_{L^2}^{\frac{2}{3}},$$

and

$$\|\nabla^3 u\|_{L^6} \leqslant C\|\nabla^4 u\|_{L^2}.$$

The third term can be estimated as

$$K_3 = \int_{\mathbb{R}^3} \nabla^3(\theta e_n)\cdot\nabla^3 u\,dx \leqslant \|\nabla^3\theta\|_{L^2}\|\nabla^3 u\|_{L^2}. \tag{3.16}$$

Combining the above estimates into (3.13), we get

$$\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2) \leqslant C\left(1+\|\nabla u\|_{\infty})(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2\right). \tag{3.17}$$

Taking (3.2), (3.7), (3.12), (3.17) into account and adding them up, integrating the resulting inequality from 0 to T, which together with Lemma 2.2, we can infer that

$$\ln\left(M(T)+e\right) \leqslant \ln\left(M(0)+e\right) + C\int_0^T (1+\|\nabla u\|_{\dot{V}_\Theta})\ln\ln(M(\tau)+e)\ln(M(\tau)+e)d\tau, \tag{3.18}$$

where $M(t) := \max_{\tau\in[0,t]}(\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2)$ for any $t \in [0,T]$. Then, we take use of Gronwall's inequality, finally we have

$$\ln\left(M(T)+e\right) \leqslant \exp\left\{\exp C\int_0^T \|\nabla u\|_{\dot{V}_\Theta}\,d\tau\right\}.$$

Therefore, we get the boundness of $H^3 \times H^3$-norm of (u,θ) for all $t \in [0,T]$ which leads a contradiction, this completes the proof of Theorem 2.1. □

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Conflict of Interest

The author declare no conflicts of interest in this paper.

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