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Research Article

The Minkowski's inequality by means of a generalized fractional integral

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Abstract: We use the definition of a fractional integral, recently proposed by Katugampola, to establish a generalization of the reverse Minkowski's inequality. We show two new theorems associated with this inequality, as well as state and show other inequalities related to this fractional operator.

Keywords: Minkowski's inequality; generalized fractional integral
Mathematics Subject Classification: 26A33, 26D10

Studies involving integral inequalities are important in several areas of science: mathematics, physics, engineering, among others, in particular we mention: initial value problem, linear transformation stability, integral-differential equations, and impulse equations [1, 2].

The space of p-integrable functions $L^p(a,b)$ plays a relevant role in the study of inequalities involving integrals and sums. Further, it is possible to extend this space of p-integrable functions, to the space of the measurable Lebesgue functions, denoted by $X_c^p(a,b)$, in which the space $L^p(a,b)$ is contained [3]. Thus, new results involving integral inequalities have been possible and consequently, some applications have been made [1, 2]. We mention few of them, the inequalities of: Minkowski, Hölder, Hardy, Hermite-Hadamard, Jensen, among others [4, 5, 6, 7, 12, 13, 14].

On the other hand, when we consider the non-integer order calculus or fractional calculus, as widely known, we are able in some cases adapt the theoretical model to the experimental data, in addition, it is used to generalize integrals and derivatives, in integrating inequalities. There are many definitions of fractional integrals, for example: Riemann-Liouville, Hadamard, Liouville, Weyl, Erdélyi-Kober and Katugampola [3, 15, 16, 17, 18]. Recently, Khalil et al. [19] and Adeljawad [20], introduced the local conformable fractional integrals and derivatives. From such fractional integrals, one obtains generalizations of the inequalities: Hadamard, Hermite-Hadamard, Opial, Grüss, Ostrowski, Gronwall among others [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Recently, Katugampola [33] proposed a fractional integral unifying other well known ones: Riemann-Liouville, Hadamard, Weyl, Liouville and Erdélyi-Kober. Motivated by this formulation, we present a generalization of the reverse Minkowski's inequality [34, 35, 36], using the fractional integral introduced by Katugampola. We point out that studies in this direction, involving fractional integrals, are growing in several branches of mathematics [23, 37, 38].

The work is organized as follows: In section 1, we present the definition of the fractional integral, as well as its particular cases. We present the main theorems involving the reverse Minkowski's inequality, as well as the suitable spaces for such definitions. In section 2, our main result, we propose the reverse Minkowski's inequality using the fractional integral. In section 3, we discuss other inequalities involving this fractional integral. Concluding remarks close the article.

1. Prelimiaries

In this section, we present the reverse Minkowski's inequality theorem associated with the classical Riemann integral and its respective generalization via Riemann-Liouville and Hadamard fractional integrals. In addition, we present the fractional integral introduced by Katugampola, and we conclude with a theorem in order to recover particular cases.

Erhan et al. [5] address the inequalities of Hermite-Hadamard and reverse Minkowski for two functions f and g by means of the classical Riemann integral. On the other hand, Lazhar [7] also proposed a work related to the inequality involving integrals, that is, Hardy's inequality and the reverse Minkowski's inequality. Two theorems below have been motivation for the works performed so far, via the Riemann-Liouville and Hadamard integrals, involving the reverse Minkowski's inequality.

Definition 1. [3] *The space $X_c^p(a,b)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ consists of those complex-valued Lebesgue measurable functions f on (a,b), for which $\|f\|_{X_c^p} < \infty$ with*

$$\|f\|_{X_c^p} = \left(\int_a^b |x^c f(x)|^p \, \frac{dx}{x}\right)^{1/p} \quad (1 \leq p < \infty)$$

and

$$\|f\|_{X_c^\infty} = \sup \mathrm{ess}_{x\in(a,b)} \left[x^c |f(x)|\right].$$

In particular, when $c = 1/p$ the space $X_c^p(a,b)$ coincides with the space $L^p(a,b)$. Note that, the regularity of the function, can be obtained by the inequality of the norms below,

$$\|f\|_{X_c^p(a,b)} \leq C\|f\|_{L^p(a,b)}.$$

On the other hand, we can also note that space $X_c^p(a,b)$, implicitly involves the fractional Sobolev space $W_X^{1,p}$ [8, 9, 10, 11].

Theorem 1. [5] *Let $f, g \in L^p(a,b)$ be two positive functions, with $1 \leq p \leq \infty$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a,b]$, then*

$$\left(\int_a^b f^p(t)\, dt\right)^{\frac{1}{p}} + \left(\int_a^b g^p(t)\, dt\right)^{\frac{1}{p}} \leq c_1 \left(\int_a^b (f^p + g^p)(t)\, dt\right)^{\frac{1}{p}}, \tag{1}$$

with $c_1 = \dfrac{M(m+1) + (M+1)}{(m+1)(M+1)}$.

Theorem 2. [5] *Let $f, g \in L^p(a,b)$ be two positive functions, with $1 \le p \le \infty$. If $0 < m \le \dfrac{f(t)}{g(t)} \le M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a,b]$, then*

$$\left(\int_a^b f^p(t)\,dt\right)^{\frac{2}{p}} + \left(\int_a^b g^p(t)\,dt\right)^{\frac{2}{p}} \ge c_2 \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b g^p(t)\,dt\right)^{\frac{1}{p}}, \tag{2}$$

with $c_2 = \dfrac{(M+1)(m+1)}{M} - 2$.

We present the definitions of the fractional integrals that will be useful in the development of the article: Riemann-Liouville fractional integral, Hadamard integral, Erdélyi-Kober integral, Katugampola integral, Weyl integral and Liouville integral.

Definition 2. [3, 16] *Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real-axis $\mathbb{R}$. The Riemann-Liouville fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ of a real function $f \in L^p(a,b)$, are defined by*

$$J_{a^+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \ a < x < b \tag{3}$$

and

$$J_{b^-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \ a < x < b, \tag{4}$$

respectively.

Definition 3. [3, 16] *Let (a,b) $(0 \le a < b < \infty)$ be a finite or infinite interval on the half-axis $\mathbb{R}^+$. The Hadamard fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ of a real function $f \in L^p(a,b)$ are defined by*

$$H_{a^+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \ \ a < x < b \tag{5}$$

and

$$H_{b^-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \ a < x < b \tag{6}$$

respectively.

Definition 4. [3, 16] *Let (a,b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval or half-axis $\mathbb{R}^+$. Also let $\mathrm{Re}(\alpha) > 0$, $\sigma > 0$ and $\eta \in \mathbb{C}$. The Erdélyi-Kober fractional integrals (left-sided and right-sided) of order $\alpha \in \mathbb{C}$ of a real function $f \in L^p(a,b)$ are defined by*

$$I_{a^+,\sigma,\eta}^{\alpha} f(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma(\eta+1)-1}}{(x^\sigma - t^\sigma)^{1-\alpha}} f(t)\,dt, \ a < x < b \tag{7}$$

and

$$I_{b^-,\sigma,\eta}^{\alpha} f(x) := \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1}}{(t^\sigma - x^\sigma)^{1-\alpha}} f(t)\,dt, \ a < x < b, \tag{8}$$

respectively.

Definition 5. [17] *Let* $[a, b] \subset \mathbb{R}$ *be a finite interval. The Katugampola fractional integrals* (*left-sided and right-sided*) *of order* $\alpha \in \mathbb{C}$, $\rho > 0$, $\mathrm{Re}(\alpha) > 0$ *of a real function* $f \in X_c^p(a, b)$ *are defined by*

$$^{\rho}I_{a^+}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)\, dt,\ x > a \tag{9}$$

and

$$^{\rho}I_{b^-}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)\, dt,\ x < b, \tag{10}$$

respectively.

Definition 6. [39] *The Weyl fractional integrals of order* $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ *of a real function* f *locally integrable into* $(-\infty, \infty)$ *being* $-\infty \leq x \leq \infty$ *are defined by*

$${}_xW_{\infty}^{\alpha} = {}_x I_{\infty}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \tag{11}$$

and

$${}_{-\infty}W_x^{\alpha} = {}_{-\infty} I_x^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt, \tag{12}$$

respectively.

Definition 7. [3, 16] *Let a continuous function by parts in* $\mathbb{R} = (-\infty, \infty)$. *The Liouville fractional integrals* (*left-sided and right-sided*) *of order* $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ *of a real function* f, *are defined by*

$$I_+^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \tag{13}$$

and

$$I_-^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt, \tag{14}$$

respectively.

Zoubir [35] established the reverse Minkowski's inequality and another result that refers to the inequality via Riemann-Liouville fractional integral according to the following two theorems.

Theorem 3. [35] *Let* $\alpha > 0$, $p \geq 1$ *and* f, g *two positive functions in* $[0, \infty)$, *such that* $\forall x > 0$, $J^\alpha f^p(x) < \infty$ *and* $J^\alpha g^p(x) < \infty$. *If* $0 < m \leq \frac{f(t)}{g(t)} \leq M$, *for* $m, M \in \mathbb{R}_+^*$ *and* $\forall t \in [0, x]$, *then*

$$(J^\alpha f^p(x))^{\frac{1}{p}} + (J^\alpha g^p(x))^{\frac{1}{p}} \leq c_1 (J^\alpha (f+g)^p(x))^{\frac{1}{p}}, \tag{15}$$

where $c_1 = \dfrac{M(m+1) + (M+1)}{(m+1)(M+1)}$.

Theorem 4. [35] *Let* $\alpha > 0$, $p \geq 1$ *and* f, g *two positive functions in* $[0, \infty)$, *such that* $\forall x > 0$, $J^\alpha f^p(x) < \infty$ *and* $J^\alpha g^p(x) < \infty$. *If* $0 < m \leq \frac{f(t)}{g(t)} \leq M$, *for* $m, M \in \mathbb{R}_+^*$ *and* $\forall t \in [0, x]$, *then*

$$(J^\alpha f^p(x))^{\frac{2}{p}} + (J^\alpha g^p(x))^{\frac{2}{p}} \geq c_2 (J^\alpha f^p(x))^{\frac{1}{p}} (J^\alpha g^p(x))^{\frac{1}{p}}, \tag{16}$$

where $c_2 = \dfrac{(M+1)(m+1)}{M} - 2$.

In 2014, Chinchane et al. [36] and Taf et al. [40] also established the reverse Minkowski's inequality via Hadamard fractional integral as in two theorems below.

Theorem 5. [36, 40] *Let $\alpha > 0$, $p \geq 1$ and f, g two positive functions in $[0, \infty)$, such that $\forall x > 0$, $H_1^\alpha f^p(x) < \infty$ and $H_1^\alpha g^p(x) < \infty$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [0, x]$, then*

$$\left(H_1^\alpha f^p(x)\right)^{\frac{1}{p}} + \left(H_1^\alpha g^p(x)\right)^{\frac{1}{p}} \leq c_1 \left(H_1^\alpha (f+g)^p(x)\right)^{\frac{1}{p}}, \tag{17}$$

where $c_1 = \dfrac{M(m+1) + (M+1)}{(m+1)(M+1)}$.

Theorem 6. [36, 40] *Let $\alpha > 0$, $p \geq 1$ and f, g two positive functions in $[0, \infty)$, such that $\forall x > 0$, $H_1^\alpha f^p(x) < \infty$ and $H_1^\alpha g^p(x) < \infty$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [0, x]$, then*

$$\left(H_1^\alpha f^p(x)\right)^{\frac{2}{p}} + \left(H_1^\alpha g^p(x)\right)^{\frac{2}{p}} \geq c_2 \left(H_1^\alpha f^p(x)\right)^{\frac{1}{p}} \left(H_1^\alpha g^p(x)\right)^{\frac{1}{p}} \tag{18}$$

where $c_2 = \dfrac{(M+1)(m+1)}{M} - 2$.

In 2014 Chinchane et al. [41] and recently Chinchane [42], established the reverse Minkowski's inequality via fractional integral of Saigo and the k-fractional integral, respectively.

In 2017, Katugampola [33] introduced a fractional integral that unifies the six fractional integrals above mentioned. Finally, we introduce this integral and with a theorem we study their respective particular cases.

Definition 8. [33] *Let $\varphi \in X_c^p(a, b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then, the fractional integrals of a function f, left and right, are given by*

$${}^\rho\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\varphi(x) := \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau)\, d\tau, \ 0 \leq a < x < b \leq \infty \tag{19}$$

and

$${}^\rho\mathcal{I}_{b-,\eta,\kappa}^{\alpha,\beta}\varphi(x) := \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau)\, d\tau, \ \ 0 \leq a < x < b \leq \infty \tag{20}$$

respectively, if integrals exist.

From now on, let's work only with the left integral, Eq.(19), because with the right integral we have a similar treatment.

Remark 1. *Let $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then for $f \in X_c^p(a, b)$, with $a < x < b$, we have* [33]*:*

1. *For $\kappa = 0$, $\eta = 0$ and the limit $\rho \to 1$, in* Eq.(19)*, we get the Riemann-Liouville fractional integral, i.e; Eq.(3).*
2. *With $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, we take the limit $\rho \to 0^+$ and using the ℓ'Hospital role, in* Eq.(19)*, we get the Hadamard fractional integral, i.e; Eq.(5).*
3. *In the case $\beta = 0$ and $\kappa = -\rho(\alpha + \eta)$, in* Eq.(19)*, we get the Erdélyi-Kober fractional integral, i.e; Eq.(7).*

4. *For $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$, in* Eq.(19)*, we get Katugampola fractional integral, i.e; Eq.(9).*
5. *With $\kappa = 0$, $\eta = 0$, $a = -\infty$ and take the limit $\rho \to 1$, in* Eq.(19)*, we get Weyl fractional integral, i.e; Eq.(11).*
6. *With $\kappa = 0$, $\eta = 0$, $a = 0$ and take the limit $\rho \to 1$, in* Eq.(19)*, we get Liouville fractional integral, i.e; Eq.(13).*

2. Reverse Minkowski fractional integral inequality

In this section, our main contribution, we establish and prove the reverse Minkowski's inequality via generalized fractional integral Eq.(19) and a theorem that refers to the reverse Minkowski's inequality.

Theorem 7. *Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$ and $p \geq 1$. Let $f, g \in X_c^p(a, x)$ be two positive functions in $[0, \infty)$, $\forall x > a$. If $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a, x]$, then*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \leq c_1 \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x)\right)^{\frac{1}{p}}, \tag{21}$$

with $c_1 = \dfrac{M(m+1) + (M+1)}{(m+1)(M+1)}$.

Proof 1. *Using the condition $\frac{f(t)}{g(t)} \leq M$, $t \in [a, x]$, we can write*

$$f(t) \leq M(f(t) + g(t)) - Mf(t),$$

which implies,

$$(M+1)^p f^p(t) \leq M^p (f(t) + g(t))^p. \tag{22}$$

Multiplying by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}}$ both sides of Eq.(22) *and integrating with respect to the variable t, we have*

$$\frac{(M+1)^p \rho^{1-\beta} x^{\kappa}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f^p(t)\, dt \leq \frac{M^p \rho^{1-\beta} x^{\kappa}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} (f+g)^p(t)\, dt. \tag{23}$$

Consequently, we can write

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} \leq \frac{M}{M+1} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x)\right)^{\frac{1}{p}}. \tag{24}$$

On the other hand, as $mg(t) \leq f(t)$, follows

$$\left(1 + \frac{1}{m}\right)^p g^p(t) \leq \left(\frac{1}{m}\right)^p (f(t) + g(t))^p. \tag{25}$$

Further, multiplying by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}}$ both sides of Eq.(25) *and integrating with respect to the variable t, we have*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(t)\right)^{\frac{1}{p}} \leq \frac{1}{m+1} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(t)\right)^{\frac{1}{p}}. \tag{26}$$

From Eq.(24) *and* Eq.(26)*, the result follows.*

Eq.(21) is the so-called reverse Minkowski's inequality associated with the Katugampola fractional integral.

Theorem 8. *Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$ and $p \geq 1$. Let $f, g \in X_c^p(a,x)$ be two positive functions in $[0,\infty)$, $\forall x > a$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a,x]$, then*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{2}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{2}{p}} \geq c_2 \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \tag{27}$$

with $c_2 = \dfrac{(M+1)(m+1)}{M} - 2$.

Proof 2. *Carrying out the product between* Eq.(24) *and* Eq.(26)*, we have*

$$\frac{(M+1)(m+1)}{M} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x)\right)^{\frac{2}{p}}. \tag{28}$$

Using the Minkowski's inequality, on the right side of Eq.(28)*, we have*

$$\begin{aligned} & \frac{(M+1)(m+1)}{M} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \\ \leq \; & \left(\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}}\right)^2. \end{aligned} \tag{29}$$

So, from Eq.(29)*, we conclude that*

$$\begin{aligned} & \left(\frac{(M+1)(m+1)}{M} - 2\right) \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \\ \leq \; & \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{2}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{2}{p}}. \end{aligned}$$

Note that, if $\beta = \alpha$, $\kappa = 0$, $\eta = 0$ and the limit $\rho \to 1$, in Eq.(19), we recover Riemann-Liouville fractional integral, Eq.(3). In this sense, choosing $a+ = 0$, and substituting in Theorem 8 and Theorem 9, we obtain, as particular cases, the respective Theorem 3 and Theorem 4, which correspond to the inequality via Riemann-Liouville fractional integral. On the other hand, if $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, and the limit $\rho \to 0+$ and using the L'Hôpital rule, in Eq.(19), we obtain the Hadamard fractional integral, Eq.(5). Similarly, choosing $a = 1$ and substituting in Theorem 8 and Theorem 9, we obtain, as particular cases, the Theorem 5 and Theorem 6, respectively.

3. Other fractional integral inequalities

In this section we generalize the results discussed by Chinchane [42], Sulaiman [43] and Sroysang [44] on the reverse Minkowski's inequality via Riemann integral, using the fractional integral proposed by Katugampola [33].

Theorem 9. *Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$, $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in X_c^p(a,x)$ be two positive functions in $[0,\infty)$, $\forall x > a$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a,x]$, then*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g(x)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{\frac{1}{p}}(x)\, g^{\frac{1}{q}}(x)\right). \tag{30}$$

Proof 3. *Using the condition* $\frac{f(t)}{g(t)} \leq M$*,* $t \in [a, x]$ *with* $x > a$*, we have*

$$f(t) \leq Mg(t) \Rightarrow g^{\frac{1}{q}}(t) \geq M^{-\frac{1}{q}} f^{\frac{1}{q}}(t). \tag{31}$$

Multiplying by $f^{\frac{1}{p}}(t)$ *both sides of* Eq.(31)*, we can rewrite it as follows*

$$f^{\frac{1}{p}}(t)\, g^{\frac{1}{q}}(t) \geq M^{-\frac{1}{q}} f(t). \tag{32}$$

Now, multiplying by $\frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(32) *and integrating with respect to the variable t, we have*

$$\int_a^x \frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}} M^{-\frac{1}{q}} f(t)\, dt \leq \int_a^x \frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}} f^{\frac{1}{p}}(t)\, g^{\frac{1}{q}}(t)\, dt. \tag{33}$$

So, the inequality follows

$$M^{-\frac{1}{pq}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\right)^{\frac{1}{p}} \leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{\frac{1}{p}}(x)\, g^{\frac{1}{q}}(x)\right)^{\frac{1}{p}}. \tag{34}$$

On the order hand, we have

$$m^{\frac{1}{p}} g^{\frac{1}{p}}(t) \leq f^{\frac{1}{p}}(t), \ x > a. \tag{35}$$

Multiplying by $g^{\frac{1}{q}}(t)$ *both sides of* Eq.(35) *and using the relation* $\frac{1}{p} + \frac{1}{q} = 1$*, we have*

$$m^{\frac{1}{p}} g(t) \leq f^{\frac{1}{p}}(t)\, g^{\frac{1}{q}}(t). \tag{36}$$

Multiplying by $\frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho} - t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(36) *and integrating with respect to the variable t, we have*

$$m^{\frac{1}{pq}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g(x)\right)^{\frac{1}{q}} \leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{\frac{1}{p}}(x)\, g^{\frac{1}{q}}(x)\right)^{\frac{1}{q}}. \tag{37}$$

Evaluating the product between Eq.(34) *and* Eq.(37) *and using the relation* $\frac{1}{p} + \frac{1}{q} = 1$*, we conclude that*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\right)^{\frac{1}{p}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g(x)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{\frac{1}{p}}(x)\, g^{\frac{1}{q}}(x)\right)^{\frac{1}{p}}.$$

Theorem 10. *Let* $\alpha > 0$*,* $\rho, \eta, \kappa, \beta \in \mathbb{R}$*,* $p \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* $f, g \in X^p_c(a, x)$ *be two positive functions in* $[0, \infty)$*,* $\forall x > a$*. If* $0 < m \leq \frac{f(t)}{g(t)} \leq M$*, for* $m, M \in \mathbb{R}^*_+$ *and* $\forall t \in [a, x]$*, then*

$${}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\, g(x) \leq c_3 \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f^p + g^p)(x)\right) + c_4 \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f^q + g^q)(x)\right), \tag{38}$$

with $c_3 = \frac{2^{p-1} M^p}{p(M+1)^p}$ *and* $c_4 = \frac{2^{p-1}}{q(m+1)^q}$.

Proof 4. *Using the hypothesis, we have the following identity*

$$(M+1)^p f^p(t) \leq M^p (f+g)^p(t). \tag{39}$$

Multiplying by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(39) *and integrating with respect to the variable t, we get*

$$\int_a^x \frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}} (M+1)^p f^p(t)\, dt \leq \int_a^x \frac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}} M^p (f+g)^p(t)\, dt.$$

In this way, we have

$$ {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x) \leq \frac{M^p}{(M+1)^p}\, {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x). \tag{40}$$

On the other hand, as $0 < m < \frac{f(t)}{g(t)}$, $t \in (a,x)$, *we have*

$$(m+1)^q g^q(t) \leq (f+g)^q(t). \tag{41}$$

Again, multiplying by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(41) *and integrating with respect to the variable t, we get*

$$ {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^q(x) \leq \frac{1}{(m+1)^q}\, {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^q(x). \tag{42}$$

Considering Young's inequality, [45]

$$f(t)\, g(t) \leq \frac{f^p(t)}{p} + \frac{g^q(t)}{q}, \tag{43}$$

multiplying by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(43) *and integrating with respect to the variable t, we have*

$$ {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (fg)(x) \leq \frac{1}{p}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right) + \frac{1}{q}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^q(x)\right). \tag{44}$$

Thus, using Eq.(40), Eq.(42) *and* Eq.(44), *we get*

$$\begin{aligned}
{}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (fg)(x) &\leq \frac{1}{p}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right) + \frac{1}{q}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^q(x)\right) \\
&\leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right) + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^q(x)\right) \\
&\leq \frac{M^p}{p(M+1)^p}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x)\right) \\
&\quad + \frac{1}{q(m+1)^q}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^q(x)\right).
\end{aligned} \tag{45}$$

Using the following inequality, $(a+b)^r \leq 2^{p-1}(a^r + b^r)$, $r > 1$, $a, b \geq 0$, *we get*

$$ {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f+g)^p(x) \leq 2^{p-1}\, {}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} (f^p + g^p)(x) \tag{46}$$

and

$$ {}^{\rho}\mathcal{I}a+,\eta,\kappa^{\alpha,\beta}\left(f+g\right)^{q}\left(x\right)\leq 2^{q-1\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f^{q}+g^{q}\right)\left(x\right). \tag{47} $$

Thus, replacing Eq.(46) *and* Eq.(47) *at* Eq.(45)*, we conclude that*

$$ {}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(fg\right)\left(x\right)\leq\frac{2^{p-1}M^{p}}{p\left(M+1\right)^{p}}\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f^{p}+g^{p}\right)\left(x\right)\right)+\frac{2^{q-1}}{q\left(m+1\right)^{q}}\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f^{q}+g^{q}\right)\left(x\right)\right). $$

Theorem 11. *Let $\alpha>0$, $\rho,\eta,\kappa,\beta\in\mathbb{R}$ and $p\geq 1$. Let $f,g\in X_{c}^{p}(a,x)$ be two positive functions in $[0,\infty)$, $\forall x>a$. If $0<m\leq\dfrac{f\left(t\right)}{g\left(t\right)}\leq M$, for $m,M\in\mathbb{R}_{+}^{*}$ and $\forall t\in[a,x]$, then*

$$ \begin{aligned} \frac{M+1}{M-c}\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f\left(x\right)-cg\left(x\right)\right)\right)^{\frac{1}{p}} &\leq \left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}f^{p}\left(x\right)\right)^{\frac{1}{p}}+\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}g^{p}\left(x\right)\right)^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f\left(x\right)-cg\left(x\right)\right)\right)^{\frac{1}{p}} \end{aligned} \tag{48} $$

Proof 5. *Considering $0<c<m\leq M$, we can write*

$$ mc\leq Mc\Rightarrow mc+m\leq mc+M\leq Mc+M\Rightarrow\left(M+1\right)\left(m-c\right)\leq\left(m+1\right)\left(M-c\right). $$

Thus, we conclude that

$$ \frac{M+1}{M-c}\leq\frac{m+1}{m-c}. $$

Also, we have

$$ m-c\leq\frac{f\left(t\right)-cg\left(t\right)}{g\left(t\right)}\leq M-c $$

which implies,

$$ \frac{\left(f\left(t\right)-cg\left(t\right)\right)^{p}}{\left(M-c\right)^{p}}\leq g^{p}\left(t\right)\leq\frac{\left(f\left(t\right)-cg\left(t\right)\right)^{p}}{\left(m-c\right)^{p}}. \tag{49} $$

Again, we have

$$ \frac{1}{M}\leq\frac{g\left(t\right)}{f\left(t\right)}\leq\frac{1}{m}\Rightarrow\frac{m-c}{cm}\leq\frac{f\left(t\right)-cg\left(t\right)}{cf\left(t\right)}\leq\frac{M-c}{cM}, $$

which implies,

$$ \left(\frac{M}{M-c}\right)^{p}\left(f\left(t\right)-cg\left(t\right)\right)^{p}\leq f^{p}\left(t\right)\leq\left(\frac{m}{m-c}\right)^{p}\left(f\left(t\right)-cg\left(t\right)\right)^{p}. \tag{50} $$

Multiplying by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}$ both sides of Eq.(49) *and integrating with respect to the variable t, we have*

$$ \begin{aligned} \int_{a}^{x}\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}\frac{\left(f\left(t\right)-cg\left(t\right)\right)^{p}}{\left(M-c\right)^{p}}dt &\leq \int_{a}^{x}\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}g^{p}\left(t\right)dt \\ &\leq \int_{a}^{x}\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}\frac{\left(f\left(t\right)-cg\left(t\right)\right)^{p}}{\left(m-c\right)^{p}}dt. \end{aligned} $$

In this way, we obtain

$$ \frac{1}{M-c}\left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}} \leq \left({}^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta}g^{p}\left(x\right)\right)^{\frac{1}{p}} \tag{51} $$

$$\leq \frac{1}{m-c}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}}.$$

Realizing the same procedure as in Eq.(50)*, we have*

$$\begin{aligned}\frac{M}{M-c}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}} &\leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}f^{p}\left(x\right)\right)^{\frac{1}{p}} \\ &\leq \frac{m}{m-c}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}}.\end{aligned} \tag{52}$$

Adding Eq.(51) *and* Eq.(52)*, we conclude that*

$$\begin{aligned}\frac{M+1}{M-c}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}} &\leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}f^{p}\left(x\right)\right)^{\frac{1}{p}}+\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}g^{p}\left(x\right)\right)^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f\left(x\right)-cg\left(x\right)\right)^{p}\right)^{\frac{1}{p}}.\end{aligned}$$

Theorem 12. *Let $\alpha > 0$, $\rho,\eta,\kappa,\beta \in \mathbb{R}$ and $p \geq 1$. Let $f, g \in X^{p}_{c}(a,x)$ be two positive functions in $[0,\infty)$, $\forall x > a$. If $0 \leq a \leq f(t) \leq A$ and $0 \leq b \leq g(t) \leq B$, $\forall t \in [a,x]$, then*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}f^{p}\left(x\right)\right)^{\frac{1}{p}}+\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}g^{p}\left(x\right)\right)^{\frac{1}{p}} \leq c_{5}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f+g\right)^{p}\left(x\right)\right)^{\frac{1}{p}}, \tag{53}$$

with $c_5 = \dfrac{A\left(a+B\right)+B\left(A+b\right)}{\left(A+b\right)\left(a+B\right)}$.

Proof 6. *By hypothesis, it follows that*

$$\frac{1}{B} \leq \frac{1}{g\left(t\right)} \leq \frac{1}{b}. \tag{54}$$

Realizing the product between Eq.(54) *and $0 < a \leq f\left(t\right) \leq A$, we have*

$$\frac{a}{B} \leq \frac{f\left(t\right)}{g\left(t\right)} \leq \frac{A}{b}. \tag{55}$$

From Eq.(55)*, we get*

$$g^{p}\left(t\right) \leq \left(\frac{B}{a+B}\right)^{p}\left(f\left(t\right)+g\left(t\right)\right)^{p} \tag{56}$$

and

$$f^{p}\left(t\right) \leq \left(\frac{A}{b+A}\right)^{p}\left(f\left(t\right)+g\left(t\right)\right)^{p}. \tag{57}$$

Multiplying by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}$ *both sides of* Eq.(56) *and integrating with respect to the variable t, we have*

$$\int_{a}^{x}\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}g^{p}\left(t\right)dt \leq \int_{a}^{x}\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}\left(\frac{B}{a+B}\right)^{p}\left(f\left(t\right)+g\left(t\right)\right)^{p}dt.$$

Thus, it follows that

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}g^{p}\left(x\right)\right)^{\frac{1}{p}} \leq \frac{B}{a+B}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}\left(f+g\right)^{p}\left(x\right)\right)^{\frac{1}{p}}. \tag{58}$$

Similarly, we perform the calculations for Eq.(57)*, we get*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{p}(x)\right)^{\frac{1}{p}} \leq \frac{A}{b+A}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}(f+g)^{p}(x)\right)^{\frac{1}{p}}. \tag{59}$$

Adding Eq.(58) *and* Eq.(59)*, we conclude that*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^{p}(x)\right)^{\frac{1}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^{p}(x)\right)^{\frac{1}{p}} \leq \frac{A(a+B)+B(b+A)}{(a+B)(b+A)}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}(f+g)^{p}(x)\right)^{\frac{1}{p}}.$$

Theorem 13. *Let $\alpha > 0$ and $\rho,\eta,\kappa,\beta \in \mathbb{R}$. Let $f,g \in X^{p}_{c}(a,x)$ be two positive functions in $[0,\infty)$, $\forall x > a$. If $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}^{*}_{+}$ and $\forall t \in [a,x]$, then*

$$\begin{aligned} \frac{1}{M}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\,g(x)\right) &\leq \frac{1}{(m+1)(M+1)}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}(f+g)^{2}(x)\right) \\ &\leq \frac{1}{m}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\,g(x)\right). \end{aligned} \tag{60}$$

Proof 7. *Taking $0 < m \leq \dfrac{f(t)}{g(t)} \leq M$, $\forall t \in [a,x]$, we have*

$$g(t)(m+1) \leq g(t) + f(t) \leq g(t)(M+1). \tag{61}$$

Also, it follows that $\dfrac{1}{M} \leq \dfrac{g(t)}{f(t)} \leq \dfrac{1}{m}$, which implies,

$$g(t)\left(\frac{M+1}{M}\right) \leq g(t) + f(t) \leq g(t)\left(\frac{m+1}{m}\right). \tag{62}$$

Evaluating the product between Eq.(61) *and* Eq.(62)*, we have*

$$\frac{f(t)\,g(t)}{M} \leq \frac{(g(t)+f(t))^{2}}{(m+1)(M+1)} \leq \frac{f(t)\,g(t)}{m}. \tag{63}$$

Multiplying by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ both sides of Eq.(63) *and integrating with respect to the variable t, we have*

$$\begin{aligned} \frac{\rho^{1-\beta}x^{\kappa}}{M\Gamma(\alpha)}\int_{a}^{x}\frac{t^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} f(t)\,g(t)\,dt &\leq c_{6}\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_{a}^{x}\frac{t^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}(g(t)+f(t))^{2}\,dt \\ &\leq \frac{\rho^{1-\beta}x^{\kappa}}{m\Gamma(\alpha)}\int_{a}^{x}\frac{t^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}} f(t)\,g(t)\,dt, \end{aligned}$$

with $c_{6} = \dfrac{1}{(m+1)(M+1)}$.

Thus, we conclude that

$$\begin{aligned} \frac{1}{M}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\,g(x)\right) &\leq \frac{1}{(m+1)(M+1)}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}(g(x)+f(x))^{2}\right) \\ &\leq \frac{1}{m}\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f(x)\,g(x)\right). \end{aligned}$$

Theorem 14. *Let $\alpha > 0$, $\rho, \eta, \kappa, \beta \in \mathbb{R}$ and $p \geq 1$. Let $f, g \in X_c^p(a, x)$ be two positive functions in $[0, \infty)$, $\forall x > a$. If $0 < m \leq \frac{f(t)}{g(t)} \leq M$, for $m, M \in \mathbb{R}_+^*$ and $\forall t \in [a, x]$, then*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} f^p(x)\right)^{\frac{1}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} g^p(x)\right)^{\frac{1}{p}} \leq 2\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa} h^p(f(x), g(x))\right)^{\frac{1}{p}},$$

with $h(f(x), g(x)) = \max\left\{M\left[\left(\frac{M}{m}+1\right)f(x) - Mg(x)\right], \frac{(m+M)g(x) - f(x)}{m}\right\}$.

Proof 8. *From the hypothesis,* $0 < m \leq \frac{f(t)}{g(t)} \leq M$, $\forall t \in [a, x]$, *we have*

$$0 < m \leq M + m - \frac{f(t)}{g(t)} \tag{64}$$

and

$$M + m - \frac{f(t)}{g(t)} \leq M. \tag{65}$$

Thus, using Eq.(64) *and* Eq.(65)*, we get*

$$g(t) < \frac{(M+m)g(t) - f(t)}{m} \leq h(f(t), g(t)), \tag{66}$$

where $h(f(t), g(t)) = \max\left\{M\left[\left(\frac{M}{m}+1\right)f(t) - Mg(t)\right], \frac{(M+m)g(t) - f(t)}{m}\right\}$.

Using the hypothesis, it follows that $0 < \frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{1}{m}$. *In this way, we obtain*

$$\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{g(t)}{f(t)} \tag{67}$$

and

$$\frac{1}{M} + \frac{1}{m} - \frac{g(t)}{f(t)} \leq \frac{1}{m}. \tag{68}$$

Then, from Eq.(67) *and* Eq.(68)*, we have*

$$\frac{1}{M} \leq \frac{\left(\frac{1}{m} + \frac{1}{M}\right)f(t) - g(t)}{f(t)} \leq \frac{1}{m},$$

which can be rewritten as

$$\begin{aligned} f(t) &\leq M\left(\frac{1}{m} + \frac{1}{M}\right)f(t) - Mg(t) \\ &= \frac{M(M+m)f(t) - M^2 mg(t)}{Mm} \\ &= \left(\frac{M}{m} + 1\right)f(t) - Mg(t) \end{aligned}$$

$$\begin{aligned} &\leq M\left[\left(\frac{M}{m}+1\right)f(t)-Mg(t)\right] \\ &\leq h(f(t),g(t)). \end{aligned} \tag{69}$$

Thus, using Eq.(66) *and* Eq.(69)*, we can write*

$$f^p(t) \leq h^p(f(t),g(t)) \tag{70}$$

and

$$g^p(t) \leq h^p(f(t),g(t)). \tag{71}$$

Multiplying by $\dfrac{\rho^{1-\alpha}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ *both sides of* Eq.(70) *and integrating with respect to the variable t, we have*

$$\int_a^x \frac{\rho^{1-\alpha}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}} f^p(t)\,dt \leq \int_a^x \frac{\rho^{1-\alpha}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}} h^p(f(t),g(t))\,dt.$$

In this way, we obtain

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}f^p(x)\right)^{\frac{1}{p}} \leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}h^p(f(x),g(x))\right)^{\frac{1}{p}}. \tag{72}$$

Using the same procedure as above, for Eq.(71)*, we have*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}g^p(x)\right)^{\frac{1}{p}} \leq \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}h^p(f(x),g(x))\right)^{\frac{1}{p}}. \tag{73}$$

Thus, using Eq.(72) *and* Eq.(73)*, we conclude that*

$$\left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}f^p(x)\right)^{\frac{1}{p}} + \left({}^{\rho}\mathcal{I}^{\alpha,\beta}_{a+,\eta,\kappa}g^p(x)\right)^{\frac{1}{p}} \leq 2\left({}^{\rho}I^{\alpha,\beta}_{a+,\eta,\kappa}h^p(f(x),g(x))\right)^{\frac{1}{p}}.$$

Using Eq.(19) and Theorem 7 with the convenient conditions for each respective fractional integral, we have the previous theorems, that is, Theorem 10 to Theorem 15 introduced and demonstrated above, contain as particular cases, each result involving the following fractional integrals: Riemann-Liouville, Hadamard, Liouville, Weyl, Erdélyi-Kober, and Katugampola.

4. Concluding remarks

After a brief introduction to the fractional integral, proposed by Katugampola and fractional integrals in the sense of Riemann-Liouville and Hadamard, we generalize the reverse Minkowski's inequality obtaining, as a particular case, the inequality involving the fractional integral in the Riemann-Liouville sense and Hadamard sense [33]. We also show other inequalities using the Katugampola fractional integral. The application of this fractional integral can be used to generalize several inequalities, among them, we mention the Grüss-type inequality, recently introduced and proved [46]. A continuation of this work, with this formulation of fractional integral, consists in generalize the inequalities of Hermite-Hadamard and Hermite-Hadamard-Féjer [27, 28, 29, 30]. Moreover, we will discuss inequalities via *M*-fractional integral according to [47].

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

References

1. C. Bandle, A. Gilányi, L. Losonczi, et al. *Inequalities and Applications: Conference on Inequalities and Applications Noszvaj (Hungary)*, vol. **157**, Springer Science & Business Media, 2008.
2. D. Bainov, P. Simeonov, *Integral Inequalities and Applications*, vol. **57**, Springer Science & Business Media, 2013.
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. **204**, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.
4. O. Hutník, *On Hadamard type inequalities for generalized weighted quasi-arithmetic means*, Journal of Inequalities in Pure and Applied Mathematics, **7** (2006), 1–10.
5. E. Set, M. Özdemir, S. Dragomir, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., **2010** (2010), 148102.
6. W. Szeligowska, M. Kaluszka, *On Jensen's inequality for generalized Choquet integral with an application to risk aversion*, eprint arXiv:1609.00554, 2016.
7. L. Bougoffa, *On Minkowski and Hardy integral inequalities*, Journal of Inequalities in Pure and Applied Mathematics, **7** (2006), 1–3.
8. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, vol. **140**, Academic press, 2003.
9. D. Idczak, S. Walczak, *Fractional Sobolev spaces via Riemann-Liouville derivatives*, J. Funct. Space. Appl., **2013** (2013), 1–15.
10. T. Krainer, B. W. Schulze, *Weighted Sobolev spaces*, vol. **138**, Springer, 1985.
11. V. Gol'dshtein, A. Ukhlov, *Weighted Sobolev spaces and embedding theorems*, T. Am. Math. Soc., **361** (2009), 3829–3850.
12. O. Hutnık, *Some integral inequalities of Hölder and Minkowski type*, Colloq. Math-Warsaw, **108** (2007), 247–261.
13. P. R. Beesack, *Hardys inequality and its extensions*, Pac. J. Math., **11** (1961), 39–61.
14. C. O. Imoru, *New generalizations of Hardy's integral inequality*, J. Math. Anal. Appl., **241** (1987), 73–82.
15. R. Herrmann, *Fractional calculus: An Introduction for Physicists*, World Scientific Publishing Company, Singapore, 2001.
16. I. Podlubny, *Fractional Differential Equation*, vol. **198**, Mathematics in Science and Engineering, Academic Press, San Diego, 1999.

17. U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl., **6** (2014), 1–15.

18. J. Vanterler da C. Sousa, E. Capelas de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci., **60** (2018), 72–91.

19. R. Khalil, M. Al Horani, A. Yousef, et al. *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.

20. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.

21. M. Bohner, T. Matthews, *The Grüss inequality on time scales*, Commun. Math. Anal., **3** (2007), 1–8.

22. D. R. Anderson, *Taylors Formula and Integral Inequalities for Conformable Fractional Derivatives*, Contr. Math. Eng., **2014** (2016), 25–43.

23. M. E. Özdemir, M. Avci, H. Kavurmaci, *Hermite-Hadamard-type inequalities via (α, m)-convexity*, Comput. Math. Appl., **61** (2011), 2614–2620.

24. F. Chen, *Extensions of the Herminte–Hadamard inequality for convex functions via fractional integrals*, J. Math. Inequal., **10** (2016), 75–81.

25. J. Wang, X. Li, M. Fekan, et al. *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal., **92** (2013), 2241–2253.

26. H. Chen, U. N. Katugampola, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl., **446** (2017), 1274–1291.

27. I. İşcan, *On generalization of different type inequalities for harmonically quasi-convex functions via fractional integrals*, Appl. Math. Comput., **275** (2016), 287–298.

28. I. İşcan, S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **238** (2014), 237–244.

29. F. Chen, *Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **268** (2015), 121–128.

30. H. Shiow-Ru, Y. Shu-Ying and T. Kuei-Lin, *Refinements and similar extensions of Hermite-Hadamard inequality for fractional integrals and their applications*, Appl. Math. Comput., **249** (2014), 103–113.

31. M. Z. Sarikaya, H. Budak, *New inequalities of Opial type for conformable fractional integrals*, Turk. J. Math., **41** (2017), 1164–1173.

32. J. Vanterler da C. Sousa and E. Capelas de Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator*, arXiv:1709.03634, 2017.

33. U. N. Katugampola, *New fractional integral unifying six existing fractional integrals*, arxiv.org/abs/1612.08596, 2016.

34. E. Set, M. Özdemir, S. Dragomir, *On the Hermite-Hadamard Inequality and Other Integral Inequalities Involving Two Functions*, J. Inequal. Appl., **2010** (2010), 148102.

35. Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., **1** (2010), 51–58.

36. V. L. Chinchane, D. B. Pachpatte, *New fractional inequalities via Hadamard fractional integral*, Internat. J. Functional Analysis, Operator Theory and Application, **5** (2013), 165–176.

37. S. S. Dragomir, *Hermite-Hadamards type inequalities for operator convex functions*, Appl. Math. Comput., **218** (2011), 766–772.

38. H. Yildirim, Z. Kirtay, *Ostrowski inequality for generalized fractional integral and related inequalities*, Malaya Journal of Matematik, **2** (2014), 322–329.

39. R. F. Camargo, E. Capelas de Oliveira, *Fractional Calculus (In Portuguese)*, Editora Livraria da Física, São Paulo, 2015.

40. S. Taf, K. Brahim, *Some new results using Hadamard fractional integral*, Int. J. Nonlinear Anal. Appl., **7** (2015), 103–109.

41. V. L. Chinchane, D. B. Pachpatte, *New fractional inequalities involving Saigo fractional integral operator*, Math. Sci. Lett., **3** (2014), 133–139.

42. V. L. Chinchane, *New approach to Minkowski's fractional inequalities using generalized k-fractional integral operator*, arXiv:1702.05234, 2017.

43. W. T. Sulaiman, *Reverses of Minkowski's, Hölders, and Hardys integral inequalities*, Int. J. Mod. Math. Sci., **1** (2012), 14–24.

44. B. Sroysang, *More on Reverses of Minkowski's Integral Inequality*, Math. Aeterna, **3** (2013), 597–600.

45. E. Kreyszig, *Introductory Functional Analysis with Applications*, vol. **1**, Wiley, New York, 1989.

46. J. Vanterler da C. Sousa, D. S. Oliveira, E. Capelas de Oliveira, *Grüss-type inequality by mean of a fractional integral*, arXiv:1705.00965, 2017.

47. J. Vanterler da C. Sousa, E. Capelas de Oliveira, *A new truncated M-fractional derivative unifying some fractional derivatives with classical properties*, International Journal of Analysis and Applications, **16** (2018), 83–96.

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