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Research article

Time periodic solutions of Cahn-Hilliard systems with dynamic boundary conditions

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Abstract: The existence problem for Cahn-Hilliard systems with dynamic boundary conditions and time periodic conditions is discussed. We apply the abstract theory of evolution equations with viscosity approach and the Schauder fixed point theorem in the level of approximate problems. One of the key points is the assumption for maximal monotone graphs with respect to their effective domains. Thanks to this, we obtain the existence result of periodic solutions by using the passage to the limit.

Keywords: Cahn-Hilliard system; dynamic boundary condition; time periodic solution; perturbation term; growth condition
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1. Introduction

In this paper, we consider the following Cahn-Hilliard systems with dynamic boundary conditions and time periodic conditions, say (P), which consists of the following equations:

$$\frac{\partial u}{\partial t} - \Delta\mu = 0 \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

$$\mu = -\kappa_1 \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q, \tag{1.2}$$

$$u_\Gamma = u_{|_\Gamma}, \quad \mu_\Gamma = \mu_{|_\Gamma} \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.3}$$

$$\frac{\partial u_\Gamma}{\partial t} + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma, \tag{1.4}$$

$$\mu_\Gamma = \kappa_1 \partial_\nu u - \kappa_2 \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) - f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma, \tag{1.5}$$

$$u(0) = u(T) \quad \text{in } \Omega, \quad u_\Gamma(0) = u_\Gamma(T) \quad \text{on } \Gamma \tag{1.6}$$

where $0 < T < +\infty$, Ω is a bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary $\Gamma := \partial\Omega$, κ_1, κ_2 are positive constants, ∂_ν is the outward normal derivative on Γ, $u_{|_\Gamma}, \mu_{|_\Gamma}$ stand for the trace of u

and μ to Γ, respectively, Δ is the Laplacian, Δ_Γ is the Laplace-Beltrami operator (see, e.g., [21]), and $f : Q \to \mathbb{R}$, $f_\Gamma : \Sigma \to \mathbb{R}$ are given data. Moreover, $\beta, \beta_\Gamma : \mathbb{R} \to 2^{\mathbb{R}}$ are maximal monotone operators and $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are Lipschitz perturbations.

The Cahn-Hilliard equation [8] is a description of mathematical model for phase separation, e.g., the phenomenon of separating into two phases from homogeneous composition, the so-called spinodal decomposition. In (1.1)–(1.2), u is the order parameter and μ is the chemical potential. Moreover, it is well known that the Cahn-Hilliard equation is characterized by the nonlinear term $\beta + \pi$. It plays an important role as the derivative of the double-well potential W. The well-known example of nonlinear terms is $W(r) = (1/4)(r^2 - 1)^2$, namely $W'(r) = r^3 - r$ for $r \in \mathbb{R}$, this is called the prototype double well potential. Other examples are stated later. As the abstract mathematical result, Kenmochi, Niezgódka and Pawłow study the Cahn-Hilliard equation with constraint by subdifferential operator approach [24] (see also [25]). Essentially we apply the same method in this paper.

In terms of (1.3)–(1.5), we consider the dynamic boundary condition as being u_Γ, μ_Γ unknown functions on the boundary. The dynamic boundary condition is treated in recent years, for example, for the Stefan problem [1, 2, 14], wider the degenerate parabolic equation [3, 15, 16] and the Cahn-Hilliard equation [11, 12, 17, 18, 19, 20, 22, 29]. To the best our knowledge, the type of dynamic boundary conditions on the Cahn-Hilliard equation like (P) is formulated in [17, 20]. Recently, the well-posedness with singular potentials is discussed in [11]; the maximal L_p regularity in bounded domains is treated in [22]; the related new model is also introduced in [29]. Based on the result [11], we also used the property of dynamic boundary conditions, more precisely, we set up the function space which satisfies that the total mass is equal to 0. At the sight of (P), we consider the same type of equations (1.1)–(1.2) on the boundary. In other words, (1.1)–(1.5) is a transmission problem connecting Ω and Γ. The nonlinear term $\beta_\Gamma + \pi_\Gamma$ on boundary is also the derivative of the double-well potential W_Γ, that is, we treat different nonlinear terms W' and W'_Γ in Ω and on Γ, respectively. In this case, it is necessary to assume some compatibility condition (see, e.g., [9, 11]), stated (A4).

Focusing on (1.6), the study of time periodic problems of the Cahn-Hilliard equation is treated in [26, 27, 28, 31]. In particular, Wang and Zheng discuss the existence of time periodic solutions of the Cahn-Hilliard equation with the Neumann boundary condition [31]. The authors employ the method of [4]. Note that the authors impose two assumptions for a maximal monotone graph, specifically, a restriction of effective domains and the following growth condition for the maximal monotone graph β:

$$\widehat{\beta}(r) \geq cr^2 \quad \text{for all } r \in \mathbb{R},$$

for some positive constant c. However, the above assumption is too restrictive for some physical applications. In this paper, we follow the method of [31] and apply the abstract theory of evolution equations by using the viscosity approach and the Schauder fixed point theorem in the level of approximate problems. Moreover, by virtue of the viscosity approach, we also can apply the abstract result [4]. Note that, the growth condition is not need to solve the Cahn-Hilliard equation (see, e.g., [11]), therefore, setting the appropriate convex functional and using the Poincaré-Wirtinger inequality, we can relax the growth condition for the time periodic problem. Thanks to this, we can choose various kinds of nonlinear diffusion terms $\beta + \pi$ and $\beta_\Gamma + \pi_\Gamma$. On the other hand, a restriction of effective domains is essential to show the existence of solutions of (P).

The present paper proceeds as follows.

In Section 2, a main theorem and a definition of solutions are stated. At first, we prepare the notation used in this paper and set appropriate function spaces. Next, we introduce the definition of periodic solutions of (P) and the main theorems are given there. Also, we give examples of double-well potentials.

In Section 3, in order to pass to the limit, we set convex functionals and consider approximate problems. Next, we obtain the solution of $(\mathrm{P})_\varepsilon$ by using the Schauder fixed point theorem. Finally, we deduce uniform estimates for the solution of $(\mathrm{P})_\varepsilon$.

In Section 4, we prove the existence of periodic solutions by passing to the limit $\varepsilon \to 0$.

A detailed index of sections and subsections follows.

2. Main results

2.1. Notation

We introduce the spaces $H := L^2(\Omega)$, $H_\Gamma := L^2(\Gamma)$, $V := H^1(\Omega)$, $V_\Gamma := H^1(\Gamma)$ with standard norms $|\cdot|_H$, $|\cdot|_{H_\Gamma}$, $|\cdot|_V$, $|\cdot|_{V_\Gamma}$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_{H_\Gamma}$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{V_\Gamma}$, respectively. Moreover, we set $\boldsymbol{H} := H \times H_\Gamma$ and

$$\boldsymbol{V} := \{\boldsymbol{z} := (z, z_\Gamma) \in V \times V_\Gamma \ : \ z_{|_\Gamma} = z_\Gamma \text{ a.e. on } \Gamma\}.$$

$\boldsymbol{H}$ and $\boldsymbol{V}$ are then Hilbert spaces with inner products

$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}} := (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma), \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{H},$$
$$(\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{V}} := (u, z)_V + (u_\Gamma, z_\Gamma)_{V_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma), \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}.$$

Note that $\boldsymbol{z} \in \boldsymbol{V}$ implies that the second component z_Γ of $\boldsymbol{z}$ is equal to the trace of the first component z of $\boldsymbol{z}$ on Γ, and $\boldsymbol{z} \in \boldsymbol{H}$ implies that $z \in H$ and $z_\Gamma \in H_\Gamma$ are independent. Throughout this paper, we use the bold letter $\boldsymbol{u}$ to represent the pair corresponding to the letter; i.e., $\boldsymbol{u} := (u, u_\Gamma)$.

Let $m : \boldsymbol{H} \to \mathbb{R}$ be the mean function defined by

$$m(\boldsymbol{z}) := \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma \right\} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $|\Omega| := \int_\Omega 1 dx, |\Gamma| := \int_\Gamma 1 d\Gamma$. Then, we define $\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$, $\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$. Moreover, $\boldsymbol{V}^*, \boldsymbol{V}_0^*$ denote the dual spaces of $\boldsymbol{V}, \boldsymbol{V}_0$, respectively; the duality pairing between $\boldsymbol{V}_0^*$ and $\boldsymbol{V}_0$

is denoted by $\langle\cdot,\cdot\rangle_{V_0^*,V_0}$. We define the norm of $\boldsymbol{H}_0$ by $|z|_{\boldsymbol{H}_0} := |z|_{\boldsymbol{H}}$ for all $z \in \boldsymbol{H}_0$ and the bilinear form $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u}, \boldsymbol{z}) := \kappa_1 \int_\Omega \nabla u \cdot \nabla z dx + \kappa_2 \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{V}.$$

Then, for all $\boldsymbol{z} \in \boldsymbol{V}_0$, $|\boldsymbol{z}|_{\boldsymbol{V}_0} := \sqrt{a(\boldsymbol{z},\boldsymbol{z})}$ becomes a norm of $\boldsymbol{V}_0$. Also, we let $\boldsymbol{F} : \boldsymbol{V}_0 \to \boldsymbol{V}_0^*$ be the duality mapping, namely,

$$\langle \boldsymbol{F}\boldsymbol{z}, \tilde{\boldsymbol{z}}\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} := a(\boldsymbol{z}, \tilde{\boldsymbol{z}}) \quad \text{for all } \boldsymbol{z}, \tilde{\boldsymbol{z}} \in \boldsymbol{V}_0.$$

We note that the following the Poincaré-Wirtinger inequality holds: There exists a positive constant c_{P} such that

$$|\boldsymbol{z}|_{\boldsymbol{V}}^2 \le c_{\mathrm{P}} |\boldsymbol{z}|_{\boldsymbol{V}_0}^2 \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}_0 \tag{2.1}$$

(see [11, Lemma A]). Moreover, we define the inner product of $\boldsymbol{V}_0^*$ by

$$(\boldsymbol{z}^*, \tilde{\boldsymbol{z}}^*)_{\boldsymbol{V}_0^*} := \langle \boldsymbol{z}^*, \boldsymbol{F}^{-1}\tilde{\boldsymbol{z}}^*\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} \quad \text{for all } \boldsymbol{z}^*, \tilde{\boldsymbol{z}}^* \in \boldsymbol{V}_0^*.$$

Also, we define the projection $\boldsymbol{P} : \boldsymbol{H} \to \boldsymbol{H}_0$ by

$$\boldsymbol{P}\boldsymbol{z} := \boldsymbol{z} - m(\boldsymbol{z})\mathbf{1} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $\mathbf{1} := (1, 1)$. Then, since $\boldsymbol{P}$ is a linear bounded operator, the following property holds: Let $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{H}$ such that $\boldsymbol{z}_n \to \boldsymbol{z}$ weakly in $\boldsymbol{H}$ for some $\boldsymbol{z}$, then we infer that

$$\boldsymbol{P}\boldsymbol{z}_n \to \boldsymbol{P}\boldsymbol{z} \quad \text{weakly in } \boldsymbol{H}_0 \quad \text{as } n \to \infty. \tag{2.2}$$

Then, we have $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$, where "$\hookrightarrow\hookrightarrow$" stands for compact embedding (see [11, Lemmas A and B]).

2.2. Definition of the solution and main theorem

In this subsection, we define our periodic solutions for (P) and then we state the main theorem.

Firstly, from (1.1) and (1.4), the following total mass conservation holds:

$$m(\boldsymbol{u}(t)) = m(\boldsymbol{u}(0)) \quad \text{for all } t \in [0, T].$$

Therefore, for any given $m_0 \in \mathrm{int}D(\beta_\Gamma)$, we define the periodic solution satisfying the total mass conservation $m(\boldsymbol{u}(t)) = m_0$ for all $t \in [0, T]$. We use the following notation: the variable $\boldsymbol{v} := \boldsymbol{u} - m_0\mathbf{1}$; the datum $\boldsymbol{f} := (f, f_\Gamma)$; the function $\boldsymbol{\pi}(\boldsymbol{z}) := (\pi(z), \pi_\Gamma(z_\Gamma))$ for $\boldsymbol{z} \in \boldsymbol{H}$. Moreover, we set the space $\boldsymbol{W} := H^2(\Omega) \times H^2(\Gamma)$.

Definition 2.1. *For any given $m_0 \in \mathrm{int}D(\beta_\Gamma)$, the triplet $(\boldsymbol{v}, \boldsymbol{\mu}, \boldsymbol{\xi})$ is called the periodic solution of* (P) *if*

$$\boldsymbol{v} \in H^1(0, T; \boldsymbol{V}_0^*) \cap L^\infty(0, T; \boldsymbol{V}_0) \cap L^2(0, T; \boldsymbol{W}),$$
$$\boldsymbol{\mu} \in L^2(0, T; \boldsymbol{V}),$$
$$\boldsymbol{\xi} = (\xi, \xi_\Gamma) \in L^2(0, T; \boldsymbol{H}),$$

and they satisfy

$$\langle \boldsymbol{v}'(t), z \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} + a(\boldsymbol{\mu}(t), z) = 0 \quad \textit{for all } z \in \boldsymbol{V}_0, \tag{2.3}$$

$$(\boldsymbol{\mu}(t), z)_{\boldsymbol{H}} = a(\boldsymbol{v}(t), z) + (\boldsymbol{\xi}(t) - m(\boldsymbol{\xi}(t))\mathbf{1} + \boldsymbol{\pi}(\boldsymbol{v}(t) + m_0\mathbf{1}) - \boldsymbol{f}(t), z)_{\boldsymbol{H}} \quad \textit{for all } z \in \boldsymbol{V} \tag{2.4}$$

for a.a. $t \in (0, T)$*, and*

$$\xi \in \beta(v + m_0) \quad \textit{a.e. in } Q, \qquad \xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \textit{a.e. on } \Sigma$$

with

$$\boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \textit{in } \boldsymbol{H}_0. \tag{2.5}$$

Remark 2.1. We can see that $\boldsymbol{\mu} := (\mu, \mu_\Gamma)$ satisfies

$$\mu = -\kappa_1 \Delta u + \xi - m(\boldsymbol{\xi}) + \pi(u) - f \quad \text{a.e. in } Q,$$
$$\mu_\Gamma = \kappa_1 \partial_\nu u - \kappa_2 \Delta_\Gamma u_\Gamma + \xi_\Gamma - m(\boldsymbol{\xi}) + \pi_\Gamma(u_\Gamma) - f_\Gamma \quad \text{a.e. on } \Sigma,$$

where $u = v + m_0$ and $u_\Gamma = v_\Gamma + m_0$, because of the regularity $\boldsymbol{v} \in L^2(0, T; \boldsymbol{W})$.

Remark 2.2. In (2.4), this is different from the following definition of [11, Definition 2.1]:

$$(\boldsymbol{\mu}(t), z)_{\boldsymbol{H}} = a(\boldsymbol{v}(t), z) + (\boldsymbol{\xi}(t) + \boldsymbol{\pi}(\boldsymbol{v}(t) + m_0\mathbf{1}) - \boldsymbol{f}(t), z)_{\boldsymbol{H}} \quad \text{for all } z \in \boldsymbol{V} \tag{2.6}$$

for a.a. $t \in (0, T)$. However, by setting $\widetilde{\boldsymbol{\mu}} := \boldsymbol{\mu} + m(\boldsymbol{\xi})\mathbf{1}$, $\widetilde{\boldsymbol{\mu}}$ satisfies $\widetilde{\boldsymbol{\mu}} \in L^2(0, T; \boldsymbol{V})$ and (2.6). Hence, in other words, we can employ (2.6) as definition of (P) replaced by (2.4).

We assume that

(A1) $\boldsymbol{f} \in L^2(0, T; \boldsymbol{V})$ and $\boldsymbol{f}(t) = \boldsymbol{f}(t + T)$ for a.a. $t \in [0, T]$;
(A2) $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions;
(A3) $\beta, \beta_\Gamma : \mathbb{R} \to 2^{\mathbb{R}}$ are maximal monotone operators, which is the subdifferential

$$\beta = \partial_{\mathbb{R}} \widehat{\beta}, \quad \beta_\Gamma = \partial_{\mathbb{R}} \widehat{\beta}_\Gamma$$

of some proper lower semicontinuous convex functions $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ with domains $D(\beta)$ and $D(\beta_\Gamma)$, respectively;
(A4) $D(\beta_\Gamma) \subseteq D(\beta)$ and there exist positive constants ρ and c_0 such that

$$|\beta^\circ(r)| \le \rho |\beta_\Gamma^\circ(r)| + c_0 \quad \text{for all } r \in D(\beta_\Gamma); \tag{2.7}$$

(A5) $D(\beta), D(\beta_\Gamma)$ are bounded domains with non-empty interior, i.e., $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ and $\overline{D(\beta_\Gamma)} = [\sigma_{\Gamma*}, \sigma_\Gamma^*]$ for some constants $\sigma_*, \sigma^*, \sigma_{\Gamma*}$ and σ_Γ^* with $-\infty < \sigma_* \le \sigma_{\Gamma*} < \sigma_\Gamma^* \le \sigma^* < \infty$.

The minimal section β° of β is defined by $\beta^\circ(r) := \{q \in \beta(r) : |q| = \min_{s \in \beta(r)} |s|\}$ for $r \in \mathbb{R}$. Also, β_Γ° is defined similarly. In particular, (A3) yields $0 \in \beta(0)$. The assumption (A5) is not imposed in [11]. However, it is essential to obtain uniform estimates in Section 3. This is a difficulty of time periodic problems. Also, the assumption of compatibility of β and β_Γ (A4) is the same as in [9, 11].

Now, we give some examples of the nonlinear perturbation terms which satisfies the above assumptions:

- $\beta(r) = \beta_\Gamma(r) = (\alpha_1/2)\ln((1+r)/(1-r))$, $\pi(r) = \pi_\Gamma(r) = -\alpha_2 r$ for all $r \in D(\beta) = D(\beta_\Gamma) = (-1, 1)$ and $0 < \alpha_1 < \alpha_2$ for the logarithmic double well potential $W(r) = W_\Gamma(r) = (\alpha_1/2)\{(1-r)\ln((1-r)/2) + (1+r)\ln((1+s)/2)\} + (\alpha_2/2)(1-r^2)$. The condition $\alpha_1 < \alpha_2$ ensures that W, W_Γ have double-well forms (see, e.g., [10]).
- $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r)$, $\pi(r) = \pi_\Gamma(r) = -r$ for all $r \in D(\beta) = D(\beta_\Gamma) = [-1, 1]$ for the singular potential $W(r) = W_\Gamma(r) = I_{[-1,1]}(r) - r^2/2$, where $\partial I_{[-1,1]}$ is the subdifferential of the indicator function $I_{[-1,1]}$ of the interval $[-1, 1]$ (namely, $I_{[-1,1]}(r) = 0$ if $r \in [-1, 1]$ and $I_{[-1,1]}(r) = +\infty$ otherwise).
- $\beta(r) = \beta_\Gamma(r) = \partial I_{[-1,1]}(r) + r^3$, $\pi(r) = \pi_\Gamma(r) = -r$ for all $r \in D(\beta) = D(\beta_\Gamma) = [-1, 1]$ for the modified prototype double well potential $W(r) = W_\Gamma(r) = I_{[-1,1]}(r) + (1/4)(r^2-1)^2 - r^2/2$.

Our main theorem is given now.

Theorem 2.1. *Under the assumptions* (A1)–(A5)*, for any given* $m_0 \in \mathrm{int}D(\beta_\Gamma)$*, there exist at least one periodic solution of* (P) *such that* $m(\boldsymbol{u}(t)) = m_0$ *for all* $t \in [0, T]$.

Remark 2.3. We note that periodic solutions of (P) is not uniquely determined. It is due to the usage of the Gronwall inequality. Indeed, in [11, Theorem 2.1], the continuous dependent on the data is proved, that is, the uniqueness of the solution to a Cauchy problem is obtained. However, in this periodic problem (P), even if we use the same method, the continuous dependent can not be obtained because of Lipschitz perturbations π and π_Γ. Without the perturbations, we can obtain the uniqueness (see Section 3).

3. Approximate problems and uniform estimates

In this section, we consider approximate problems and obtain uniform estimates to show the existence of periodic solutions of (P). Hereafter, we fix a given constant $m_0 \in \mathrm{int}D(\beta_\Gamma)$.

3.1. Abstract formulation

In order to prove the main theorem, we apply the abstract theory of evolution equations. To do so, we define a proper lower semicontinuous convex functional $\varphi : \boldsymbol{H}_0 \to [0, +\infty]$ by

$$\varphi(z) := \begin{cases} \dfrac{\kappa_1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{\kappa_2}{2}\displaystyle\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma \\ \qquad + \displaystyle\int_\Omega \widehat{\beta}(z+m_0)dx + \displaystyle\int_\Gamma \widehat{\beta}_\Gamma(z_\Gamma + m_0)d\Gamma \\ \qquad\qquad \text{if} \quad z \in \boldsymbol{V}_0 \text{ with } \widehat{\beta}(z+m_0) \in L^1(\Omega),\ \widehat{\beta}_\Gamma(z_\Gamma+m_0) \in L^1(\Gamma), \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

Next, for each $\varepsilon \in (0, 1]$, we define a proper lower semicontinuous convex functional $\varphi_\varepsilon : \boldsymbol{H}_0 \to$

$[0, +\infty]$ by

$$\varphi_\varepsilon(z) := \begin{cases} \dfrac{\kappa_1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{\kappa_2}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma \\ \qquad + \displaystyle\int_\Omega \widehat{\beta}_\varepsilon(z + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma + m_0)d\Gamma & \text{if} \quad z \in \boldsymbol{V}_0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widehat{\beta}_\varepsilon, \widehat{\beta}_{\Gamma,\varepsilon}$ are Moreau-Yosida regularizations of $\widehat{\beta}, \widehat{\beta}_\Gamma$ defined by

$$\widehat{\beta}_\varepsilon(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon}|r-s|^2 + \widehat{\beta}(s) \right\} = \frac{1}{2\varepsilon}|r - J_\varepsilon(r)|^2 + \widehat{\beta}(J_\varepsilon(r)),$$

$$\widehat{\beta}_{\Gamma,\varepsilon}(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\varepsilon\rho}|r-s|^2 + \widehat{\beta}_\Gamma(s) \right\} = \frac{1}{2\varepsilon\rho}|r - J_{\Gamma,\varepsilon}(r)|^2 + \widehat{\beta}_\Gamma(J_{\Gamma,\varepsilon}(r)),$$

for all $r \in \mathbb{R}$, where ρ is a constant as in (2.7) and $J_\varepsilon, J_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ are resolvent operators given by

$$J_\varepsilon(r) := (I + \varepsilon\beta)^{-1}(r), \quad J_{\Gamma,\varepsilon}(r) := (I + \varepsilon\rho\beta_\Gamma)^{-1}(r)$$

for all $r \in \mathbb{R}$. Moreover, $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ are Yosida approximations for maximal monotone operators β, β_Γ, respectively:

$$\beta_\varepsilon(r) := \frac{1}{\varepsilon}(r - J_\varepsilon(r)), \quad \beta_{\Gamma,\varepsilon}(r) := \frac{1}{\varepsilon\rho}(r - J_{\Gamma,\varepsilon}(r))$$

for all $r \in \mathbb{R}$. Then, we easily see that $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$ holds from the definition of the subdifferential. It is well known that $\beta_\varepsilon, \beta_{\Gamma,\varepsilon}$ are Lipschitz continuous with Lipschitz constants $1/\varepsilon, 1/(\varepsilon\rho)$, respectively. Here, we have following properties:

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \quad 0 \le \widehat{\beta}_{\Gamma,\varepsilon}(r) \le \widehat{\beta}_\Gamma(r) \quad \text{for all } r \in \mathbb{R}.$$

Hence, $0 \le \varphi_\varepsilon(z) \le \varphi(z)$ holds for all $z \in \boldsymbol{H}_0$. These properties of Yosida approximation and Moreau-Yosida regularizations are as in [5, 6, 23]. Moreover, thanks to [9, Lemma 4.4], we have

$$\left|\beta_\varepsilon(r)\right| \le \rho\left|\beta_{\Gamma,\varepsilon}(r)\right| + c_0 \quad \text{for all } r \in \mathbb{R} \tag{3.1}$$

with the same constants ρ and c_0 as in (2.7).

Now, for each $\varepsilon \in (0, 1]$, we also define two proper lower semicontinuous convex functionals $\widetilde{\varphi}, \psi_\varepsilon : \boldsymbol{H}_0 \to [0, +\infty]$ by

$$\widetilde{\varphi}(z) := \begin{cases} \dfrac{\kappa_1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{\kappa_2}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma & \text{if} \quad z \in \boldsymbol{V}_0, \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\psi_\varepsilon(z) := \int_\Omega \widehat{\beta}_\varepsilon(z + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(z_\Gamma + m_0)d\Gamma$$

for all $z \in \boldsymbol{H}_0$, respectively. Then, from [11, Lemma C], the subdifferential $\boldsymbol{A} := \partial_{\boldsymbol{H}_0}\widetilde{\varphi}$ on $\boldsymbol{H}_0$ is characterized by

$$\boldsymbol{A}\boldsymbol{z} = (-\kappa_1\Delta z, \kappa_1\partial_{\boldsymbol{\nu}} z - \kappa_2\Delta_\Gamma z_\Gamma) \quad \text{with } \boldsymbol{z} = (z, z_\Gamma) \in D(\boldsymbol{A}) = \boldsymbol{W} \cap \boldsymbol{V}_0.$$

Moreover, the representation of the subdifferential $\partial_{\boldsymbol{H}_0}\psi_\varepsilon$ is given by

$$\partial_{\boldsymbol{H}_0}\psi_\varepsilon(\boldsymbol{z}) = \boldsymbol{P}\boldsymbol{\beta}_\varepsilon(\boldsymbol{z} + m_0\mathbf{1}) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H}_0,$$

where $\boldsymbol{\beta}_\varepsilon(\boldsymbol{z} + m_0\mathbf{1}) := (\beta_\varepsilon(z + m_0), \beta_{\Gamma,\varepsilon}(z_\Gamma + m_0))$ for $\boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{H}_0$. This is proved by the same way as in [16, Lemma 3.2]. Noting that it holds that $D(\partial_{\boldsymbol{H}_0}\psi_\varepsilon) = \boldsymbol{H}_0$ and $\boldsymbol{A}$ is a maximal monotone operator; indeed it follows from the abstract monotonicity methods (see, e.g., [5, Sect. 2.1]) that $\boldsymbol{A} + \partial_{\boldsymbol{H}_0}\psi_\varepsilon$ is also a maximal monotone operator. Moreover, by a simple calculation, we deduce that $(\boldsymbol{A} + \partial_{\boldsymbol{H}_0}\psi_\varepsilon) \subset \partial_{\boldsymbol{H}_0}\varphi_\varepsilon$. Hence,

$$\partial_{\boldsymbol{H}_0}\varphi_\varepsilon(\boldsymbol{z}) = (\boldsymbol{A} + \partial_{\boldsymbol{H}_0}\psi_\varepsilon)(\boldsymbol{z}) \tag{3.2}$$

for any $\boldsymbol{z} \in \boldsymbol{H}_0$ (see, e.g., [13]).

3.2. Approximate problems for (P)

Now, we consider the following approximate problem, say (P)$_\varepsilon$: for each $\varepsilon \in (0, 1]$ find $\boldsymbol{v}_\varepsilon := (v_\varepsilon, v_{\Gamma,\varepsilon})$ satisfying

$$\begin{aligned} &\varepsilon\boldsymbol{v}_\varepsilon'(t) + \boldsymbol{F}^{-1}\boldsymbol{v}_\varepsilon'(t) + \partial_{\boldsymbol{H}_0}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(t)) \\ &\qquad + \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(t) + m_0\mathbf{1})) = \boldsymbol{P}\boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}_0 \quad \text{for a.a. } t \in (0, T), && (3.3) \\ &\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_\varepsilon(T) \quad \text{in } \boldsymbol{H}_0. && (3.4) \end{aligned}$$

where, for all $\boldsymbol{z} \in \boldsymbol{H}$, $\widetilde{\boldsymbol{\pi}}(\boldsymbol{z}) := (\widetilde{\pi}(z), \widetilde{\pi}_\Gamma(z_\Gamma))$ is a cut-off function of π, π_Γ given by

$$\widetilde{\pi}(r) := \begin{cases} 0 & \text{if } r \le \sigma_* - 1, \\ \pi(\sigma_*)(r - \sigma_* + 1) & \text{if } \sigma_* - 1 \le r \le \sigma_*, \\ \pi(r) & \text{if } \sigma_* \le r \le \sigma^*, \\ -\pi(\sigma^*)(r - \sigma^* - 1) & \text{if } \sigma^* \le r \le \sigma^* + 1, \\ 0 & \text{if } r \ge \sigma^* + 1 \end{cases} \tag{3.5}$$

and

$$\widetilde{\pi}_\Gamma(r) := \begin{cases} 0 & \text{if } r \le \sigma_{\Gamma*} - 1, \\ \pi_\Gamma(\sigma_{\Gamma*})(r - \sigma_* + 1) & \text{if } \sigma_{\Gamma*} - 1 \le r \le \sigma_{\Gamma*}, \\ \pi_\Gamma(r) & \text{if } \sigma_{\Gamma*} \le r \le \sigma_\Gamma^*, \\ -\pi_\Gamma(\sigma_\Gamma^*)(r - \sigma^* - 1) & \text{if } \sigma_\Gamma^* \le r \le \sigma_\Gamma^* + 1, \\ 0 & \text{if } r \ge \sigma_\Gamma^* + 1 \end{cases} \tag{3.6}$$

for all $r \in \mathbb{R}$, respectively. We establish the above cut-off function by referring to [31].

From now, we show the next proposition of the existence of the periodic solution for (P)$_\varepsilon$.

Proposition 3.1. *Under the assumptions* (A1)–(A5)*, for each* $\varepsilon \in (0, 1]$*, there exist at least one function*

$$\boldsymbol{v}_\varepsilon \in H^1(0, T; \boldsymbol{H}_0) \cap L^\infty(0, T; \boldsymbol{V}_0) \cap L^2(0, T; \boldsymbol{W})$$

such that $\boldsymbol{v}_\varepsilon$ *satisfies* (3.3) *and* (3.4)*.*

The proof of Proposition 3.1 is given later. In order to show the Proposition 3.1, we use the method in [31], that is, we employ the fixed point argument. To do so, we consider the following problem: for each $\varepsilon \in (0, 1]$ and $\boldsymbol{g} \in L^2(0, T; \boldsymbol{V}_0)$,

$$(\boldsymbol{F}^{-1} + \varepsilon I)\boldsymbol{v}'_\varepsilon(t) + \partial\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(t)) = \boldsymbol{g}(t) \quad \text{in } \boldsymbol{H}_0 \quad \text{for a.a. } t \in (0, T), \tag{3.7}$$

$$\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_\varepsilon(T) \quad \text{in } \boldsymbol{H}_0. \tag{3.8}$$

Now, we can apply the abstract theory of doubly nonlinear evolution equations respect to the time periodic problem [4] for (3.7), (3.8) because the operator $\varepsilon\boldsymbol{I} + \boldsymbol{F}^{-1}$ and $\partial\varphi_\varepsilon$ are coercive in $\boldsymbol{H}_0$. It is an important assumption to apply Theorem 2.2 in [4]. Moreover, the function $\boldsymbol{v}_\varepsilon$ satisfying (3.7) and (3.8) is uniquely determined. Indeed, let $\boldsymbol{v}_{1\varepsilon}, \boldsymbol{v}_{2\varepsilon}$ be periodic solutions of the problem (3.7) and (3.8). Then, at the time $t \in (0, T)$, taking the difference (3.7) for $\boldsymbol{v}_{1\varepsilon}$ and $\boldsymbol{v}_{2\varepsilon}$, respectively, we have

$$\varepsilon(\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)) + \boldsymbol{F}^{-1}(\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)) + \partial\varphi_\varepsilon(\boldsymbol{v}_{1\varepsilon}(t)) - \partial\varphi_\varepsilon(\boldsymbol{v}_{2\varepsilon}(t)) = \boldsymbol{0} \quad \text{in } \boldsymbol{H}_0 \tag{3.9}$$

for a.a. $t \in (0, T)$. Now, we test (3.9) at time $t \in (0, T)$ by $\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)$. Then, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\left|\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)\right|^2_{\boldsymbol{H}_0} + \left|\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)\right|^2_{\boldsymbol{V}_0^*}\right) + \frac{1}{2}\left|\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)\right|^2_{\boldsymbol{V}_0} \le 0$$

for a.a. $t \in (0, T)$, because of (3.2) and the monotonicity of β, β_Γ. Therefore, by integrating it over $[0, T]$ with respect to t, it follows from (2.1) that

$$\int_0^T \left|\boldsymbol{v}_{1\varepsilon}(t) - \boldsymbol{v}_{2\varepsilon}(t)\right|^2_{\boldsymbol{V}} dt \le 0.$$

It implies that the function $\boldsymbol{v}_\varepsilon$ satisfying (3.7) and (3.8) is unique.

Hence, we obtain the next proposition.

Proposition 3.2. *For each* $\varepsilon \in (0, 1]$ *and* $\boldsymbol{g} \in L^2(0, T; \boldsymbol{V}_0)$*, there exists a unique function* $\boldsymbol{v}_\varepsilon$ *such that* (3.7) *and* (3.8) *are satisfied.*

We apply the Schauder fixed point theorem to prove Proposition 3.1. To this aim, we set

$$\boldsymbol{Y}_1 := \{\bar{\boldsymbol{v}}_\varepsilon \in H^1(0, T; \boldsymbol{H}_0) \cap L^\infty(0, T; \boldsymbol{V}_0) : \bar{\boldsymbol{v}}_\varepsilon(0) = \bar{\boldsymbol{v}}_\varepsilon(T)\}.$$

Firstly, for each $\bar{\boldsymbol{v}}_\varepsilon \in \boldsymbol{Y}_1$, we consider the following problem, say $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$:

$$\varepsilon\boldsymbol{v}'_\varepsilon(s) + \boldsymbol{F}^{-1}\boldsymbol{v}'_\varepsilon(s) + \partial\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon(s) + m_0\boldsymbol{1})) = \boldsymbol{P}\boldsymbol{f}(s) \quad \text{in } \boldsymbol{H}_0 \tag{3.10}$$

for a.a. $s \in (0, T)$, with

$$\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_\varepsilon(T) \quad \text{in } \boldsymbol{H}_0.$$

Next, we obtain estimates of the solution of $(\mathrm{P}_\varepsilon;\bar{\boldsymbol{v}}_\varepsilon)$ to apply the Schauder fixed point theorem. Note that we can allow the dependent of $\varepsilon \in (0,1]$ for estimates of Lemma 3.1 because we use the Schauder fixed point theorem in the level of approximation.

Lemma 3.1. *Let $\boldsymbol{v}_\varepsilon$ be the solution of problem $(\mathrm{P}_\varepsilon;\bar{\boldsymbol{v}}_\varepsilon)$. Then, there exist positive constants $C_{1\varepsilon}, C_2, C_{3\varepsilon}$ such that*

$$\varepsilon\int_0^T \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{H}_0}^2 ds + \int_0^T \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds \le C_{1\varepsilon}, \tag{3.11}$$

$$\int_0^T \bigl|\boldsymbol{v}_\varepsilon(s)\bigr|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)ds \le C_2 \tag{3.12}$$

and

$$\frac{1}{2}\bigl|\boldsymbol{v}_\varepsilon(t)\bigr|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t)+m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \le C_{3\varepsilon} \tag{3.13}$$

for all $t \in [0,T]$.

Proof. At first, for each $\bar{\boldsymbol{v}}_\varepsilon \in Y_1$, there exists a positive constant M, depending only on $\sigma_*, \sigma_{\Gamma*}, \sigma^*$ and σ_Γ^*, such that

$$\bigl|\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon(t)+m_0\mathbf{1})\bigr|_{\boldsymbol{H}_0}^2 \le M \quad \text{for all } t \in [0,T]. \tag{3.14}$$

Now, testing (3.10) at time $s \in (0,T)$ by $\boldsymbol{v}_\varepsilon'(s)$ and using the Young inequality, we infer that

$$\begin{aligned}
&\varepsilon\bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{H}_0}^2 + \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{V}_0^*}^2 + \frac{d}{ds}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \\
&\quad = \bigl(\boldsymbol{P}\boldsymbol{f}(s) - \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon(s)+m_0\mathbf{1})), \boldsymbol{v}_\varepsilon'(s)\bigr)_{\boldsymbol{H}_0} \\
&\quad \le \frac{1}{2}\bigl|\boldsymbol{f}(s)\bigr|_{\boldsymbol{V}}^2 + \frac{1}{2}\bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{V}_0^*}^2 + \frac{M}{2\varepsilon} + \frac{\varepsilon}{2}\bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{H}_0}^2
\end{aligned}$$

for a.a. $s \in (0,T)$. Therefore, we have that

$$\varepsilon\bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{H}_0}^2 + \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{V}_0^*}^2 + 2\frac{d}{ds}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \le \bigl|\boldsymbol{f}(s)\bigr|_{\boldsymbol{V}}^2 + \frac{M}{\varepsilon} \tag{3.15}$$

for a.a. $s \in (0,T)$. Then, integrating it over $(0,T)$ with respect to s and using the periodic property, we see that

$$\varepsilon\int_0^T \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{H}_0}^2 ds + \int_0^T \bigl|\boldsymbol{v}_\varepsilon'(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds \le \int_0^T \bigl|\boldsymbol{f}(s)\bigr|_{\boldsymbol{V}}^2 ds + \frac{MT}{\varepsilon},$$

which implies the first estimate (3.11).

Next, testing (3.10) at time $s \in (0,T)$ by $\boldsymbol{v}_\varepsilon(s)$ and from (2.1), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\bigl|\boldsymbol{v}_\varepsilon(s)\bigr|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{ds}\bigl|\boldsymbol{v}_\varepsilon(s)\bigr|_{\boldsymbol{H}_0}^2 + \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \\
&\quad \le \bigl(\boldsymbol{P}\boldsymbol{f}(s) - \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon(s)+m_0\mathbf{1})), \boldsymbol{v}_\varepsilon(s)\bigr)_{\boldsymbol{H}_0} + \varphi_\varepsilon(0)
\end{aligned}$$

$$
\begin{aligned}
&\le 2c_{\mathrm{P}}\left|\boldsymbol{f}(s)\right|_{\boldsymbol{H}_0}^2 + \frac{1}{4c_{\mathrm{P}}}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{H}_0}^2 + 2c_{\mathrm{P}}M + \varphi(0) \\
&\le 2c_{\mathrm{P}}\left|\boldsymbol{f}(s)\right|_{\boldsymbol{H}_0}^2 + \frac{1}{4}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 + 2c_{\mathrm{P}}M + \varphi(0) \\
&\le 2c_{\mathrm{P}}\left|\boldsymbol{f}(s)\right|_{\boldsymbol{H}_0}^2 + \frac{1}{2}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + 2c_{\mathrm{P}}M + \varphi(0)
\end{aligned}
$$

for a.a. $s \in (0,T)$, thanks to the definition of the subdifferential. From the definition of φ_ε, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{ds}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{H}_0}^2 + \frac{1}{4}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 \\
&\quad + \frac{1}{2}\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dx + \frac{1}{2}\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma \\
&\le 2c_{\mathrm{P}}\left|\boldsymbol{f}(s)\right|_{\boldsymbol{H}_0}^2 + 2c_{\mathrm{P}}M + \int_\Omega \widehat{\beta}_\varepsilon(m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(m_0)d\Gamma
\end{aligned}
$$

for a.a. $s \in (0,T)$. Integrating it over $(0,T)$ and using the periodic property, we see that

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma ds \\
&\le 4c_{\mathrm{P}}\left|\boldsymbol{f}\right|_{L^2(0,T;\boldsymbol{H}_0)}^2 + 4c_{\mathrm{P}}TM + T|\Omega|\left|\widehat{\beta}(m_0)\right| + T|\Gamma|\left|\widehat{\beta}_\Gamma(m_0)\right|.
\end{aligned}
$$

Hence, there exist a positive constant C_2 such that the second estimate (3.12) holds.

Next, for each $s,t \in [0,T]$ such that $s \le t$, we integrate (3.15) over $[s,t]$ with respect to s. Then, by neglecting the first two positive terms, we have

$$
\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(t)) \le \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT}{2\varepsilon}
$$

for all $s,t \in [0,T]$, namely,

$$
\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t)+m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \\
&\le \frac{1}{2}\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma \\
&\quad + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT}{2\varepsilon} \qquad (3.16)
\end{aligned}
$$

for all $s,t \in [0,T]$. Now, integrating it over $(0,t)$ with respect to s, we deduce that

$$
\begin{aligned}
&\frac{t}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + t\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t)+m_0)dx + t\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \\
&\le \frac{1}{2}\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma ds \\
&\quad + \frac{T}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT^2}{2\varepsilon} \qquad (3.17)
\end{aligned}
$$

for all $t \in [0, T]$. In particular, putting $t := T$ and dividing (3.17) by T, it follows that

$$\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(T)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(T) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0)d\Gamma \\
&\le \frac{1}{2T}\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \frac{1}{T}\int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds \\
&\quad + \frac{1}{T}\int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT}{2\varepsilon}.
\end{aligned} \tag{3.18}$$

Hence, combining the second estimate (3.12) and (3.18), we see that

$$\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(T)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(T) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0)d\Gamma \\
&\le \frac{C_2}{T} + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT}{2\varepsilon}.
\end{aligned}$$

Moreover, from the periodic property, we infer that

$$\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(0)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(0) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(0) + m_0)d\Gamma \\
&\le \frac{C_2}{T} + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}}^2 ds + \frac{MT}{2\varepsilon}.
\end{aligned} \tag{3.19}$$

Now, let s be 0 in (3.16). Then, owing to (3.19), we deduce that

$$\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t) + m_0)d\Gamma \\
&\le \frac{C_2}{T} + \left|\boldsymbol{f}\right|_{L^2(0,T;\boldsymbol{V})}^2 + \frac{MT}{\varepsilon}
\end{aligned}$$

for all $t \in [0, T]$. Thus, there exists a positive constant $C_{3\varepsilon}$ such that the final estimate (3.13) holds. □

In terms of (3.11), one key point to prove the estimate is exploiting (3.14). The estimate (3.14) is arised from the form of cut-off functions (3.5) and (3.6). The form of cut-off functions depends on the assumption (A5) essentially. However, considered the same estimate in [11, Lemma 4.1], it is not imposed the assumption. They use the Gronwall inequality to obtain the estimate because the initial value is given data. On the other hand, we can not obtain it even though we use the Gronwall inequality, because the initial value is not given. For this reason, it is necessary to impose (A5). This is a difficult point to solve this time periodic problem (P).

Now, we show the existence of solutions of the approximate problem (P)$_\varepsilon$.

Proof of Proposition 3.1. We apply the Schauder fixed point theorem. To do so, we set

$$\boldsymbol{Y}_2 := \left\{ \bar{\boldsymbol{v}}_\varepsilon \in \boldsymbol{Y}_1 : \sup_{t\in[0,T]} \left|\bar{\boldsymbol{v}}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + \varepsilon\left|\bar{\boldsymbol{v}}_\varepsilon\right|_{H^1(0,T;\boldsymbol{H}_0)}^2 \le M_\varepsilon \right\},$$

where M_ε is a positive constant and be determined by Lemma 3.1. Then, the set $\boldsymbol{Y}_2$ is non-empty compact convex on $C([0, T]; \boldsymbol{H}_0)$. Now, from Proposition 3.2, for each $\bar{\boldsymbol{v}}_\varepsilon \in \boldsymbol{Y}_2$, there exists a unique solution $\boldsymbol{v}_\varepsilon$ of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$. Moreover, from Lemma 3.1, it holds $\boldsymbol{v}_\varepsilon \in \boldsymbol{Y}_2$. Here, we define the mapping $\boldsymbol{S} : \boldsymbol{Y}_2 \to \boldsymbol{Y}_2$ such that, for each $\bar{\boldsymbol{v}}_\varepsilon \in \boldsymbol{Y}_2$, corresponding $\bar{\boldsymbol{v}}_\varepsilon$ to the solution $\boldsymbol{v}_\varepsilon$ of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$. Then, the mapping $\boldsymbol{S}$ is continuous on $\boldsymbol{Y}_2$ with respect to topology of $C([0,T]; \boldsymbol{H}_0)$. Indeed, let $\{\bar{\boldsymbol{v}}_{\varepsilon,n}\}_{n\in\mathbb{N}} \subset \boldsymbol{Y}_2$ be $\bar{\boldsymbol{v}}_{\varepsilon,n} \to \bar{\boldsymbol{v}}_\varepsilon$ in $C([0,T]; \boldsymbol{H}_0)$ and $\{\boldsymbol{v}_{\varepsilon,n}\}_{n\in\mathbb{N}}$ be the sequence of the solution of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_{\varepsilon,n})$. From Lemma 3.1, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}$, with $n_k \to \infty$ as $k \to \infty$, and $\boldsymbol{v}_\varepsilon \in H^1(0,T; \boldsymbol{H}_0) \cap L^\infty(0,T; \boldsymbol{V}_0)$ such that

$$\boldsymbol{v}_{\varepsilon,n_k} \to \boldsymbol{v}_\varepsilon \quad \text{weakly star in } H^1(0,T; \boldsymbol{H}_0) \cap L^\infty(0,T; \boldsymbol{V}_0). \tag{3.20}$$

Hence, from (3.20) and the Ascoli-Arzelà theorem (see, e.g., [30]), there exists a subsequence (not relabeled) such that

$$\boldsymbol{v}_{\varepsilon,n_k} \to \boldsymbol{v}_\varepsilon \quad \text{in } C([0,T]; \boldsymbol{H}_0) \tag{3.21}$$

as $k \to \infty$. Also, we have

$$\boldsymbol{v}'_{\varepsilon,n_k} \to \boldsymbol{v}'_\varepsilon \quad \text{weakly in } L^2(0,T; \boldsymbol{H}_0) \tag{3.22}$$

as $k \to \infty$. Because we have $\boldsymbol{v}_{\varepsilon,n_k}(0) = \boldsymbol{v}_{\varepsilon,n_k}(T)$, it implies $\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_\varepsilon(T)$ in $\boldsymbol{H}_0$. Hereafter, we show that $\boldsymbol{v}_\varepsilon$ is the solution of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$. Since $\boldsymbol{v}_{\varepsilon,n_k}$ is the solution of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_{\varepsilon,n_k})$, we see that

$$\begin{aligned}\int_0^T &\left(\boldsymbol{P}\boldsymbol{f}(s) - \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_{\varepsilon,n_k}(s) + m_0\mathbf{1})) - \varepsilon\boldsymbol{v}'_{\varepsilon,n_k}(s) - \boldsymbol{F}^{-1}\boldsymbol{v}'_{\varepsilon,n_k}(s), \boldsymbol{\eta}(s) - \boldsymbol{v}_{\varepsilon,n_k}(s)\right)_{\boldsymbol{H}_0} ds \\ &\le \int_0^T \varphi_\varepsilon(\boldsymbol{\eta}(s))ds - \int_0^T \varphi_\varepsilon(\boldsymbol{v}_{\varepsilon,n_k}(s))ds\end{aligned} \tag{3.23}$$

for all $\boldsymbol{\eta} \in L^2(0,T; \boldsymbol{H}_0)$, thanks to the definition of the subdifferential $\partial\varphi_\varepsilon$. Moreover, it follows from $\bar{\boldsymbol{v}}_{\varepsilon,n_k} \to \bar{\boldsymbol{v}}_\varepsilon$ in $C([0,T]; \boldsymbol{H}_0)$ that

$$\boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_{\varepsilon,n_k} + m_0\mathbf{1})) \to \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon + m_0\mathbf{1})) \quad \text{in } C([0,T]; \boldsymbol{H}_0). \tag{3.24}$$

Thus, on account of (3.20)–(3.24), taking the upper limit as $k \to \infty$ in (3.23) and using

$$\liminf_{k\to\infty} \int_0^T \varphi_\varepsilon(\boldsymbol{v}_{\varepsilon,n_k}(s))ds \ge \int_0^T \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s))ds,$$

we infer that

$$\begin{aligned}\int_0^T &\left(\boldsymbol{P}\boldsymbol{f}(s) - \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\bar{\boldsymbol{v}}_\varepsilon(s) + m_0\mathbf{1})) - \varepsilon\boldsymbol{v}'_\varepsilon(s) - \boldsymbol{F}^{-1}\boldsymbol{v}'_\varepsilon(s), \boldsymbol{\eta}(s) - \boldsymbol{v}_\varepsilon(s)\right)_{\boldsymbol{H}_0} ds \\ &\le \int_0^T \varphi_\varepsilon(\boldsymbol{\eta}(s))ds - \int_0^T \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s))ds\end{aligned}$$

for all $\boldsymbol{\eta} \in L^2(0,T; \boldsymbol{H}_0)$. Hence, we see that the function $\boldsymbol{v}_\varepsilon$ is the solution of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$. As a result, it follows from the uniqueness of the solution of $(\mathrm{P}_\varepsilon; \bar{\boldsymbol{v}}_\varepsilon)$ that

$$\boldsymbol{S}(\bar{\boldsymbol{v}}_{\varepsilon,n_k}) = \boldsymbol{v}_{\varepsilon,n_k} \to \boldsymbol{v}_\varepsilon = \boldsymbol{S}(\bar{\boldsymbol{v}}_\varepsilon) \quad \text{in } C([0,T]; \boldsymbol{H}_0)$$

as $k \to \infty$. Therefore, the mapping $\boldsymbol{S}$ is continuous with respect to $C([0,T]; \boldsymbol{H}_0)$. Thus, from the Schauder fixed point theorem, there exists a fixed point on $\boldsymbol{Y}_2$, namely, the problem $(\mathrm{P})_\varepsilon$ admits a solution $\boldsymbol{v}_\varepsilon$. Finally, from the fact that $\partial\varphi_\varepsilon(\boldsymbol{v}_\varepsilon) \in L^2(0,T;\boldsymbol{H}_0)$, which implies $\boldsymbol{v}_\varepsilon \in L^2(0,T;\boldsymbol{W})$. $\square$

Now, we consider the chemical potential $\boldsymbol{\mu} := (\mu, \mu_\Gamma)$ by approximating. For each $\varepsilon \in (0,1]$, we set the approximate sequence

$$\boldsymbol{\mu}_\varepsilon(s) := \varepsilon \boldsymbol{v}_\varepsilon'(s) + \partial\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + \widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}) - \boldsymbol{f}(s) \tag{3.25}$$

for a.a. $s \in (0,T)$. From (3.2), we can rewrite (3.25) as

$$\boldsymbol{\mu}_\varepsilon(s) = \varepsilon \boldsymbol{v}_\varepsilon'(s) + \boldsymbol{A}\boldsymbol{v}_\varepsilon(s) + \boldsymbol{P}\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}) + \widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}) - \boldsymbol{f}(s) \tag{3.26}$$

for a.a. $s \in (0,T)$. Then, we rewrite (3.3) as

$$\boldsymbol{F}^{-1}\boldsymbol{v}_\varepsilon'(s) + \boldsymbol{\mu}_\varepsilon(s) - \omega_\varepsilon(s)\mathbf{1} = \mathbf{0} \quad \text{in } \boldsymbol{V}$$

for a.a. $s \in (0,T)$, where

$$\omega_\varepsilon(s) := m(\widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}) - \boldsymbol{f}(s))$$

for a.a. $s \in (0,T)$. Therefore, we have $\boldsymbol{P}\boldsymbol{\mu}_\varepsilon = \boldsymbol{\mu}_\varepsilon - \omega_\varepsilon\mathbf{1} \in L^2(0,T;\boldsymbol{V}_0)$ and $\omega_\varepsilon \in L^2(0,T)$. Then, it holds $\boldsymbol{\mu}_\varepsilon \in L^2(0,T;\boldsymbol{V})$ and

$$\boldsymbol{v}_\varepsilon'(s) + \boldsymbol{F}\boldsymbol{P}\boldsymbol{\mu}_\varepsilon(s) = \mathbf{0} \quad \text{in } \boldsymbol{V}_0^* \tag{3.27}$$

for a.a. $s \in (0,T)$.

3.3. Uniform estimates

In this subsection, we obtain uniform estimates independent of $\varepsilon \in (0,1]$. We refer to [31] to obtain uniform estimates.

Lemma 3.2. *There exists a positive constant M_1, independent of $\varepsilon \in (0,1]$, such that*

$$\frac{1}{2}\int_0^T |\boldsymbol{v}_\varepsilon(s)|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds \le M_1. \tag{3.28}$$

Proof. From (3.5), (3.6) and the assumption (A3), note that $\widetilde{\pi}, \widetilde{\pi}_\Gamma$ are globally Lipschitz continuous on $\mathbb{R}$. We denote Lipschitz constants of $\widetilde{\pi}, \widetilde{\pi}_\Gamma$ by $\widetilde{L}, \widetilde{L}_\Gamma$, respectively. Moreover, we can take the primitive function $\widehat{\widetilde{\pi}}$ of $\widetilde{\pi}$ satisfying

$$\int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(s))dx \ge 0$$

for a.a. $s \in (0,T)$. Analogously, we define $\widehat{\widetilde{\pi}}_\Gamma$. Now, we test (3.3) at time $s \in (0,T)$ by $\boldsymbol{v}_\varepsilon(s)$ and use the Young inequality. Then, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}|\boldsymbol{v}_\varepsilon(s)|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{ds}|\boldsymbol{v}_\varepsilon(s)|_{\boldsymbol{H}_0}^2 + \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \\
&\quad \le (\boldsymbol{P}\boldsymbol{f}(s) - \boldsymbol{P}(\widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1})), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}_0} + \varphi_\varepsilon(0) \\
&\quad \le c_{\mathrm{P}}|\boldsymbol{f}(s)|_{\boldsymbol{H}}^2 + \frac{1}{4c_{\mathrm{P}}}|\boldsymbol{v}_\varepsilon(s)|_{\boldsymbol{H}_0}^2 + c_{\mathrm{P}}M + \varphi(0)
\end{aligned}$$

$$\le \frac{1}{4}\big|\boldsymbol{v}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 + c_{\mathrm{P}}\big|\boldsymbol{f}(s)\big|_{\boldsymbol{H}}^2 + c_{\mathrm{P}}M + \varphi(0)$$
$$\le \frac{1}{2}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + c_{\mathrm{P}}\big|\boldsymbol{f}(s)\big|_{\boldsymbol{H}}^2 + c_{\mathrm{P}}M + \varphi(0)$$

for a.a. $s \in (0, T)$. Namely, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\big|\boldsymbol{v}_\varepsilon(s)\big|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\frac{d}{ds}\big|\boldsymbol{v}_\varepsilon(s)\big|_{\boldsymbol{H}_0}^2 + \frac{1}{4}\big|\boldsymbol{v}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 \\ &\qquad + \frac{1}{2}\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dx + \frac{1}{2}\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma \\ &\le c_{\mathrm{P}}\big|\boldsymbol{f}(s)\big|_{\boldsymbol{H}}^2 + c_{\mathrm{P}}M + \varphi(0)\end{aligned}$$

for a.a. $s \in (0, T)$. Integrating it over $(0, T)$ and using the periodic property, we see that

$$\begin{aligned}&\frac{1}{2}\int_0^T \big|\boldsymbol{v}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds \\ &\le 2c_{\mathrm{P}}\big|\boldsymbol{f}\big|_{L^2(0,T;\boldsymbol{H})}^2 + 2c_{\mathrm{P}}TM + 2T\varphi(0).\end{aligned}$$

This yields that the estimate (3.28) holds. □

Lemma 3.3. *There exists a positive constant M_2, independent of $\varepsilon \in (0, 1]$, such that*

$$\varepsilon\int_0^T \big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{H}_0}^2 ds + \frac{1}{2}\int_0^T \big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{V}_0^*}^2 ds \le M_2.$$

Proof. We test (3.3) at time $s \in (0, T)$ by $\boldsymbol{v}_\varepsilon'(s)$. Then, by using the Young inequality, we see that

$$\begin{aligned}&\varepsilon\big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{H}_0}^2 + \big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{V}_0^*}^2 + \frac{d}{ds}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \\ &\qquad + \frac{d}{ds}\int_\Omega \widehat{\widehat{\pi}}(v_\varepsilon(s) + m_0)dx + \frac{d}{ds}\int_\Gamma \widehat{\widehat{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma \\ &= \big(\boldsymbol{P}\boldsymbol{f}(s), \boldsymbol{v}_\varepsilon'(s)\big)_{\boldsymbol{H}_0} \\ &\le \frac{1}{2}\big|\boldsymbol{f}(s)\big|_{\boldsymbol{V}}^2 + \frac{1}{2}\big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{V}_0^*}^2\end{aligned}$$

for a.a. $s \in (0, T)$. This implies that

$$\begin{aligned}&\varepsilon\big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{H}_0}^2 + \frac{1}{2}\big|\boldsymbol{v}_\varepsilon'(s)\big|_{\boldsymbol{V}_0^*}^2 + \frac{d}{ds}\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) \\ &\qquad + \frac{d}{ds}\int_\Omega \widehat{\widehat{\pi}}(v_\varepsilon(s) + m_0)dx + \frac{d}{ds}\int_\Gamma \widehat{\widehat{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma \\ &\le \frac{1}{2}\big|\boldsymbol{f}(s)\big|_{\boldsymbol{V}}^2\end{aligned} \tag{3.29}$$

for a.a. $s \in (0, T)$. Therefore, by integrating it over $(0, T)$ with respect to s and using the periodic property, we can conclude. □

Lemma 3.4. *There exists a positive constant M_3, independent of $\varepsilon \in (0,1]$, such that*

$$\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(t)+m_0)dx + \int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \leq M_3 \tag{3.30}$$

for all $t \in [0,T]$.

Proof. For each $s,t \in [0,T]$ such that $s \leq t$, we integrate (3.29) over $[s,t]$. Then, by neglecting the first two positive terms, we see that

$$\begin{aligned}
&\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(t)) + \int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(t)+m_0)dx + \int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \\
&\quad \leq \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(s)) + \int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(s)+m_0)dx + \int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds
\end{aligned}$$

for all $s,t \in [0,T]$. Now, integrating it over $(0,t)$ with respect to s, it follows that

$$\begin{aligned}
&\frac{t}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + t\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t)+m_0)dx + t\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t)+m_0)d\Gamma \\
&\leq \frac{1}{2}\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_0^T\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s)+m_0)dxds + \int_0^T\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma ds \\
&\quad + \int_0^T\int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(s)+m_0)dxds + \int_0^T\int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma ds \\
&\qquad + \frac{T}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds
\end{aligned} \tag{3.31}$$

for all $t \in [0,T]$. Here, note that we have

$$\begin{aligned}
\left|\widehat{\widetilde{\pi}}(r)\right| &\leq \int_0^r \left|\widetilde{\pi}(\tau)\right|d\tau \\
&\leq \widetilde{L}\int_0^{|r|}|\tau|d\tau + \int_0^r \left|\widetilde{\pi}(0)\right|d\tau \\
&\leq \frac{\widetilde{L}}{2}r^2 + \left|\widetilde{\pi}(0)\right||r|
\end{aligned} \tag{3.32}$$

for $r > 0$. Then we can easily show that (3.32) holds for any $r \in \mathbb{R}$. Similarly, we have

$$\left|\widehat{\widetilde{\pi}}_\Gamma(r)\right| \leq \frac{\widetilde{L}_\Gamma}{2}r^2 + \left|\widetilde{\pi}_\Gamma(0)\right||r| \quad \text{for all } r \in \mathbb{R}.$$

Then, by using the Young inequality, we infer that

$$\begin{aligned}
\int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(s)+m_0)dx &\leq \int_\Omega \left(\frac{\widetilde{L}}{2}\left|v_\varepsilon(s)+m_0\right|^2 + \left|\widetilde{\pi}(0)\right|\left|v_\varepsilon(s)+m_0\right|\right)dx \\
&\leq \widetilde{L}\int_\Omega \left|v_\varepsilon(s)+m_0\right|^2 dx + \frac{1}{2\widetilde{L}}\left|\widetilde{\pi}(0)\right|^2|\Omega|
\end{aligned}$$

$$\leq \ 2\widetilde{L}\int_\Omega \left|v_\varepsilon(s)\right|^2 dx + 2m_0^2|\Omega| + \frac{1}{2\widetilde{L}}\left|\widetilde{\pi}(0)\right|^2|\Omega| \tag{3.33}$$

for a.a. $s \in [0, T]$. Similarly, we have

$$\int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma \leq 2\widetilde{L}_\Gamma \int_\Gamma \left|v_{\Gamma,\varepsilon}(s)\right|^2 d\Gamma + 2m_0^2|\Gamma| + \frac{1}{2\widetilde{L}_\Gamma}\left|\widetilde{\pi}_\Gamma(0)\right|^2|\Gamma| \tag{3.34}$$

for a.a. $s \in [0, T]$. Thus, on account of (3.31)–(3.34), we deduce that

$$\begin{aligned}
&\frac{t}{2}\left|\boldsymbol{v}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 + t\int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(t) + m_0)dx + t\int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(t) + m_0)d\Gamma \\
&\leq \frac{1}{2}\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds \\
&\quad + 2\widetilde{L}\int_\Omega \left|v_\varepsilon(s)\right|^2 dx + 2\widetilde{L}_\Gamma \int_\Gamma \left|v_{\Gamma,\varepsilon}(s)\right|^2 d\Gamma + \frac{T}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds + \tilde{M}_4 \\
&\leq \left(\frac{1}{2} + \widehat{L}c_{\mathrm{P}}\right)\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds \\
&\quad + \int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds + \frac{T}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds + \tilde{M}_4
\end{aligned}$$

for all $t \in [0, T]$, where $\widehat{L} := \max\{2\widetilde{L}, 2\widetilde{L}_\Gamma\}$ and

$$\tilde{M}_4 := 2m_0^2|\Omega| + \frac{1}{2\widetilde{L}}\left|\widetilde{\pi}(0)\right|^2|\Omega| + 2m_0^2|\Gamma| + \frac{1}{2\widetilde{L}_\Gamma}\left|\widetilde{\pi}_\Gamma(0)\right|^2|\Gamma|.$$

In particular, putting $t := T$ and dividing it by T, it follows that

$$\begin{aligned}
&\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(T)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(T) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0)d\Gamma \\
&\leq \frac{1}{T}\left(\frac{1}{2} + \widehat{L}c_{\mathrm{P}}\right)\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \frac{1}{T}\int_0^T \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(s) + m_0)dxds \\
&\quad + \frac{1}{T}\int_0^T \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0)d\Gamma ds + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds + \frac{\tilde{M}_4}{T}.
\end{aligned} \tag{3.35}$$

Combining (3.28) and (3.35), there exists a positive constant $\tilde{M}_3$ such that

$$\frac{1}{2}\left|\boldsymbol{v}_\varepsilon(T)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(T) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(T) + m_0)d\Gamma \leq \tilde{M}_3.$$

From the periodic property, we have

$$\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(0)) = \frac{1}{2}\left|\boldsymbol{v}_\varepsilon(0)\right|_{\boldsymbol{V}_0}^2 + \int_\Omega \widehat{\beta}_\varepsilon(v_\varepsilon(0) + m_0)dx + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(0) + m_0)d\Gamma \leq \tilde{M}_3. \tag{3.36}$$

Now, integrating (3.29) by $(0, t)$ with respect to s, it follows from (3.33)–(3.34) that

$$\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(t)) + \int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(t) + m_0)dx + \int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(t) + m_0)d\Gamma$$

$$\leq \varphi_\varepsilon(\boldsymbol{v}_\varepsilon(0)) + \int_\Omega \widehat{\widetilde{\pi}}(v_\varepsilon(0)+m_0)dx + \int_\Gamma \widehat{\widetilde{\pi}}_\Gamma(v_{\Gamma,\varepsilon}(0)+m_0)d\Gamma + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds$$
$$\leq (1+2\widehat{L}c_{\mathrm{P}})\varphi_\varepsilon(\boldsymbol{v}_\varepsilon(0)) + \frac{1}{2}\int_0^T \left|\boldsymbol{f}(s)\right|_{\boldsymbol{V}_0}^2 ds + \tilde{M}_4 \tag{3.37}$$

for all $t \in [0,T]$. Therefore, by virtue of (3.36)–(3.37), there exists a positive constant M_3 such that the estimate (3.30) holds. □

Lemma 3.5. *There exists a positive constant M_4, independent of $\varepsilon \in (0,1]$, such that*

$$\delta_0 \int_0^T \left|\beta_\varepsilon(v_\varepsilon(s)+m_0)\right|_{L^1(\Omega)}^2 ds + \delta_0 \int_0^T \left|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\right|_{L^1(\Gamma)}^2 ds \leq M_4 \tag{3.38}$$

for some positive constants δ_0.

Proof. We employ the method of [11, Lemmas 4.1, 4.3], indeed we impose same assumptions as [11] for β, β_Γ and being $m_0 \in \mathrm{int}D(\beta_\Gamma)$. Therefore, we can also exploit the following inequalities stated in [18, Sect. 5]: for each $\varepsilon \in (0,1]$, there exist two positive constants δ_0 and c_1 such that

$$\beta_\varepsilon(r)(r-m_0) \geq \delta_0\left|\beta_\varepsilon(r)\right| - c_1, \quad \beta_{\Gamma,\varepsilon}(r)(r-m_0) \geq \delta_0\left|\beta_{\Gamma,\varepsilon}(r)\right| - c_1$$

for all $r \in \mathbb{R}$. Hence, it follows that

$$(\boldsymbol{\beta}_\varepsilon(\boldsymbol{u}_\varepsilon(s)), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}} \geq \delta_0 \int_\Omega \left|\beta_\varepsilon(u_\varepsilon(s))\right| dx - c_1|\Omega| + \delta_0 \int_\Gamma \left|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))\right| d\Gamma - c_1|\Gamma| \tag{3.39}$$

for a.a. $s \in (0,T)$. On the other hand, we test (3.3) at time $s \in (0,T)$ by $\boldsymbol{v}_\varepsilon(s)$. Then, from (3.2), we see that

$$\begin{aligned}&(\varepsilon \boldsymbol{v}_\varepsilon'(s), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}_0} + (\boldsymbol{v}_\varepsilon'(s), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{V}_0^*} + (\boldsymbol{A}\boldsymbol{v}_\varepsilon(s), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}_0} + (\boldsymbol{P}\boldsymbol{\beta}_\varepsilon(\boldsymbol{u}_\varepsilon(s)), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}_0}\\ &\leq (\boldsymbol{f}(s) - \widetilde{\boldsymbol{\pi}}(\boldsymbol{u}_\varepsilon(s)), \boldsymbol{v}_\varepsilon(s))_{\boldsymbol{H}}.\end{aligned} \tag{3.40}$$

Hence, from (3.39)–(3.40) and the maximal monotonicity of $\boldsymbol{A}$, by squaring we have

$$\begin{aligned}\left(\delta_0 \int_\Omega \left|\beta_\varepsilon(u_\varepsilon(s))\right| dx + \delta_0 \int_\Gamma \left|\beta_{\Gamma,\varepsilon}(u_{\Gamma,\varepsilon}(s))\right| d\Gamma\right)^2 &\leq 3c_1^2(|\Omega|+|\Gamma|)^2\\ +9\left(\left|\boldsymbol{f}(s)\right|_{\boldsymbol{H}}^2 + \left|\boldsymbol{\pi}(\boldsymbol{u}_\varepsilon(s))\right|_{\boldsymbol{H}}^2 + \varepsilon^2\left|\boldsymbol{v}_\varepsilon'(s)\right|_{\boldsymbol{H}_0}^2\right)&\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{H}_0}^2 + 3\left|\boldsymbol{v}_\varepsilon'(s)\right|_{\boldsymbol{V}_0^*}^2\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0^*}^2\end{aligned}$$

for a.a. $s \in (0,T)$. Therefore, from the Lipschitz continuity of $\widetilde{\pi}, \widetilde{\pi}_\Gamma$ and Lemma 3.4, by integrating it over $(0,T)$ with respect to s, there exists a positive constant M_4 such that the estimate (3.38) holds. □

Lemma 3.6. *There exists a positive constants M_5, independent of $\varepsilon \in (0,1]$, such that*

$$\int_0^T \left|\boldsymbol{\mu}_\varepsilon(s)\right|_{\boldsymbol{V}}^2 ds \leq M_5. \tag{3.41}$$

Proof. Firstly, by using the Lipschitz continuity of $\widetilde{\pi}, \widetilde{\pi}_\Gamma$ and the Hölder inequality, it follows from (2.1) and Lemma 3.4 that there exists a positive constant M_5^* such that

$$\begin{aligned}
&\left|m(\widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}))\right| \\
&\leq \frac{1}{|\Omega| + |\Gamma|}\left\{\int_\Omega \left|\widetilde{\pi}(v_\varepsilon(s) + m_0)\right| dx + \int_\Gamma \left|\widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(s) + m_0)\right| d\Gamma\right\} \\
&\leq \frac{1}{|\Omega| + |\Gamma|}\left\{\widetilde{L}|\Omega|^{\frac{1}{2}}\left|v_\varepsilon(s)\right|_H^2 + \widetilde{L}|\Omega||m_0| + |\Omega|\left|\widetilde{\pi}(0)\right|\right. \\
&\qquad \left. + \widetilde{L}_\Gamma|\Gamma|^{\frac{1}{2}}\left|v_{\Gamma,\varepsilon}(s)\right|_{H_\Gamma}^2 + \widetilde{L}_\Gamma|\Gamma||m_0| + |\Gamma|\left|\widetilde{\pi}_\Gamma(0)\right|\right\} \\
&\leq \frac{1}{|\Omega| + |\Gamma|} M_5^*\left\{\left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 + 1\right\} \\
&\leq \frac{1}{|\Omega| + |\Gamma|} M_5^*(M_3 + 1) =: \tilde{M}_5
\end{aligned} \tag{3.42}$$

for a.a. $s \in (0, T)$. Therefore, owing to (3.42) we deduce that

$$\begin{aligned}
\left|m(\boldsymbol{\mu}_\varepsilon(s))\right|^2 &= \left|m(\widetilde{\boldsymbol{\pi}}(\boldsymbol{v}_\varepsilon(s) + m_0\mathbf{1}) - \boldsymbol{f}(s))\right|^2 \\
&\leq 2\tilde{M}_5^2 + \frac{4}{(|\Omega| + |\Gamma|)^2}\left(\left|f(s)\right|_{L^1(\Omega)} + \left|f_\Gamma(s)\right|_{L^1(\Gamma)}\right) =: \hat{M}_5
\end{aligned}$$

for a.a. $s \in (0, T)$. Next, from (2.1), (3.27) and the fact $\boldsymbol{P\mu}_\varepsilon(s) = \boldsymbol{\mu}_\varepsilon(s) - m(\boldsymbol{\mu}_\varepsilon(s))\mathbf{1}$ for a.a. $s \in (0, T)$, we deduce that

$$\begin{aligned}
\int_0^T \left|\boldsymbol{\mu}_\varepsilon(s)\right|_{\boldsymbol{V}}^2 ds &\leq 2\int_0^T \left|\boldsymbol{P\mu}_\varepsilon(s)\right|_{\boldsymbol{V}}^2 ds + 2\int_0^T \left|m(\boldsymbol{\mu}_\varepsilon(s))\mathbf{1}\right|_{\boldsymbol{V}}^2 ds \\
&\leq 2c_{\mathrm{P}}\int_0^T \left|\boldsymbol{P\mu}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + 2(|\Omega| + |\Gamma|)\int_0^T \left|m(\boldsymbol{\mu}_\varepsilon(s))\right|^2 ds \\
&\leq 2c_{\mathrm{P}}\int_0^T \left|\boldsymbol{v}_\varepsilon'(s)\right|_{\boldsymbol{V}_0^*}^2 ds + 2T(|\Omega| + |\Gamma|)\hat{M}_5^2.
\end{aligned}$$

Thus, from Lemma 3.3, there exists a positive constant M_5 such that the estimate (3.41) holds. □

Lemma 3.7. *There exists a positive constant M_6, independent of $\varepsilon \in (0, 1]$, such that*

$$\frac{1}{2}\int_0^T \left|\beta_\varepsilon(v_\varepsilon(s) + m_0)\right|_H^2 ds + \frac{1}{4\rho}\int_0^T \left|\beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0)\right|_{H_\Gamma}^2 ds \leq M_6. \tag{3.43}$$

Proof. From the definition of $\boldsymbol{\mu}_\varepsilon$, we can infer that

$$\mu_\varepsilon = \varepsilon\partial_t v_\varepsilon - \kappa_1\Delta v_\varepsilon + \beta_\varepsilon(v_\varepsilon + m_0) - m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon + m_0\mathbf{1})) + \widetilde{\pi}(v_\varepsilon + m_0) - f \quad \text{a.e. in } Q, \tag{3.44}$$

$$\begin{aligned}
\mu_{\Gamma,\varepsilon} = {}& \varepsilon\partial_t v_{\Gamma,\varepsilon} + \kappa_1\partial_\nu v_\varepsilon - \kappa_2\Delta_\Gamma v_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon} + m_0) - m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon + m_0\mathbf{1})) \\
& + \widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon} + m_0) - f_\Gamma \quad \text{a.e. on } \Sigma.
\end{aligned} \tag{3.45}$$

Now, it follows from (3.38) that there exists a positive constant $\tilde{M}_6$ such that

$$\begin{aligned}
&\left|m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1}))\right|^2 \\
&\le \frac{2}{(|\Omega|+|\Gamma|)^2}\left(\left|\beta_\varepsilon(v_\varepsilon+m_0)\right|_{L^1(\Omega)}+\left|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\right|_{L^1(\Gamma)}\right) \\
&\le \tilde{M}_6 \qquad (3.46)
\end{aligned}$$

for a.a. $s\in(0,T)$. Moreover, we test (3.44) at time $s\in(0,T)$ by $\beta_\varepsilon(v_\varepsilon(s)+m_0)$ and exploit (3.45). Then, on account of the fact $(\beta_\varepsilon(v_\varepsilon+m_0))_{|_\Gamma}=\beta_\varepsilon(v_{\Gamma,\varepsilon}+m_0)$, by integrating over Ω we deduce that

$$\begin{aligned}
&\kappa_1\int_\Omega \beta_\varepsilon'(v_\varepsilon(s)+m_0)\left|\nabla v_\varepsilon(s)\right|^2dx+\kappa_2\int_\Gamma \beta_\varepsilon'(v_{\Gamma,\varepsilon}(s)+m_0)\left|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\right|^2d\Gamma \\
&\quad +\left|\beta_\varepsilon(v_\varepsilon(s)+m_0)\right|_H^2+\int_\Gamma \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma \\
&\le \left(f(s)+\mu_\varepsilon(s)-\varepsilon v_\varepsilon'(s)-\widetilde{\pi}(v_\varepsilon(s)+m_0),\beta_\varepsilon(v_\varepsilon(s)+m_0)\right)_H \\
&\quad +\left(m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1})),\beta_\varepsilon(v_\varepsilon(s)+m_0)\right)_H \\
&\quad +\left(f_\Gamma(s)+\mu_{\Gamma,\varepsilon}(s)-\varepsilon v_{\Gamma,\varepsilon}'(s)-\widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0),\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\right)_{H_\Gamma} \\
&\quad +\left(m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1})),\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\right)_{H_\Gamma} \qquad (3.47)
\end{aligned}$$

for a.a. $s\in(0,T)$. Now, from (3.1), since the both signs of $\beta_\varepsilon(r)$ and $\beta_{\Gamma,\varepsilon}(r)$ are same for all $r\in\mathbb{R}$, we infer that

$$\begin{aligned}
\int_\Gamma \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)d\Gamma &= \int_\Gamma \left|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\right|\left|\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\right|d\Gamma \\
&\ge \frac{1}{2\rho}\int_\Gamma \left|\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\right|^2d\Gamma-\frac{c_0^2}{2\rho}|\Gamma|. \qquad (3.48)
\end{aligned}$$

Also, it holds

$$\int_\Omega \beta_\varepsilon'(v_\varepsilon(s)+m_0)\left|\nabla v_\varepsilon(s)\right|^2dx\ge 0,\qquad \int_\Gamma \beta_{\Gamma,\varepsilon}'(v_{\Gamma,\varepsilon}(s)+m_0)\left|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\right|^2d\Gamma\ge 0. \qquad (3.49)$$

Moreover, by using the Young inequality, the Lipschitz continuity of $\widetilde{\pi},\widetilde{\pi}_\Gamma$ and (3.46), there exists a positive constant $\hat{M}_6$ such that

$$\begin{aligned}
&\left(f(s)+\mu_\varepsilon(s)-\varepsilon v_\varepsilon'(s)-\widetilde{\pi}(v_\varepsilon(s)+m_0),\beta_\varepsilon(v_\varepsilon(s)+m_0)\right)_H \\
&\quad +\left(m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1})),\beta_\varepsilon(v_\varepsilon(s)+m_0)\right)_H \\
&\le \frac{1}{2}\left|\beta_\varepsilon(v_\varepsilon(s)+m_0)\right|_H^2+4\left|f(s)\right|_H^2+4\left|\mu_\varepsilon(s)\right|_H^2+4\varepsilon^2\left|v_\varepsilon'(s)\right|_H^2+4\left|\widetilde{\pi}(v_\varepsilon(s)+m_0)\right|_H^2 \\
&\quad +\left|m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1}))\right|_H^2 \\
&\le \frac{1}{2}\left|\beta_\varepsilon(v_\varepsilon(s)+m_0)\right|_H^2+\hat{M}_6\left(\left|f(s)\right|_H^2+\left|\mu_\varepsilon(s)\right|_H^2+\varepsilon^2\left|v_\varepsilon'(s)\right|_H^2+\left|v_\varepsilon(s)\right|_H^2+1\right) \\
&\quad +|\Omega|\tilde{M}_6 \qquad (3.50)
\end{aligned}$$

and

$$\left(f_\Gamma(s)+\mu_{\Gamma,\varepsilon}(s)-\varepsilon v_{\Gamma,\varepsilon}'(s)-\widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0),\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\right)_{H_\Gamma}$$

$$
\begin{aligned}
&\quad +\big(m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1})),\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\big)_H\\
&\le \frac{1}{4\rho}\big|\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2+2\rho\big|f_\Gamma(s)\big|_H^2+2\rho\big|\mu_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2+2\rho\varepsilon^2\big|v_{\Gamma,\varepsilon}'(s)\big|_{H_\Gamma}^2\\
&\quad +2\rho\big|\widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2+2\rho\big|m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1}))\big|_{H_\Gamma}^2\\
&\le \frac{1}{4\rho}\big|\beta_\varepsilon(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2+2\rho|\Gamma|\tilde{M}_6\\
&\quad +\rho\hat{M}_6\big(\big|f_\Gamma(s)\big|_{H_\Gamma}^2+\big|\mu_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2+\varepsilon^2\big|v_{\Gamma,\varepsilon}'(s)\big|_{H_\Gamma}^2+\big|v_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2+1\big)
\end{aligned}
\tag{3.51}
$$

for a.a. $s \in (0,T)$. Thus, from Lemmas 3.3, 3.4 and (2.1), by combining from (3.47)–(3.51) and integrating it over $(0,T)$, we can conclude the existence of the constant M_6 satisfying (3.43). □

Lemma 3.8. *There exists a positive constant M_7, independent of $\varepsilon \in (0,1]$, such that*

$$
\kappa_1\int_0^T\big|\Delta v_\varepsilon(s)\big|_H^2ds+\int_0^T\big|v_\varepsilon(s)\big|_{H^{\frac{3}{2}}(\Omega)}^2ds+\int_0^T\big|\partial_{\boldsymbol{\nu}}v_\varepsilon(s)\big|_{H_\Gamma}^2ds\le M_7. \tag{3.52}
$$

This lemma is proved exactly the same as in [11, Lemmas 4.4] because the necessary uniform estimates to prove it is obtained by Lemmas 3.3, 3.4, 3.6 and 3.7. Sketching simply, comparing in (3.44) we deduce that $|\Delta v_\varepsilon|_{L^2(0,T;H)}$ is uniformly bounded. Moreover, by using the theory of the elliptic regularity (see, e.g., [7, Theorem 3.2, p. 1.79]), we see that $|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))}$ is also uniformly bounded. Thus, using both uniformly boundeds, we can conclude that (3.52) holds.

Lemma 3.9. *There exists a positive constant M_8, independent of $\varepsilon \in (0,1]$, such that*

$$
\int_0^T\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2ds\le M_8. \tag{3.53}
$$

Proof. We test (3.45) at time $s \in (0,T)$ by $\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)$ and integrating it over Γ. Then, by using the Young inequality and the Lipschitz continuity of $\widetilde{\pi}_\Gamma$, there exists a positive constant $\tilde{M}_8$ such that

$$
\begin{aligned}
&\kappa_2\int_\Gamma \beta_{\Gamma,\varepsilon}'(v_{\Gamma,\varepsilon}(s)+m_0)\big|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\big|^2d\Gamma+\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2\\
&=\big(f_\Gamma(s)+\mu_\Gamma(s)-\varepsilon v_{\Gamma,\varepsilon}'(s)-\partial_{\boldsymbol{\nu}}v_\varepsilon(s)-\widetilde{\pi}_\Gamma(v_{\Gamma,\varepsilon}(s)+m_0),\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\big)_{H_\Gamma}\\
&\le \frac{1}{2}\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2+\big|m(\boldsymbol{\beta}_\varepsilon(\boldsymbol{v}_\varepsilon(s)+m_0\mathbf{1}))\big|_{H_\Gamma}^2\\
&\quad +\tilde{M}_8\big(\big|f_\Gamma(s)\big|_{H_\Gamma}^2+\big|\mu_\Gamma(s)\big|_{H_\Gamma}^2+\varepsilon^2\big|v_{\Gamma,\varepsilon}'(s)\big|_{H_\Gamma}^2+\big|\partial_{\boldsymbol{\nu}}v_\varepsilon(s)\big|_{H_\Gamma}^2+\big|v_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2+1\big)\\
&\le \frac{1}{2}\big|\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s)+m_0)\big|_{H_\Gamma}^2+|\Gamma|\tilde{M}_6\\
&\quad +\tilde{M}_8\big(\big|f_\Gamma(s)\big|_{H_\Gamma}^2+\big|\mu_\Gamma(s)\big|_{H_\Gamma}^2+\varepsilon^2\big|v_{\Gamma,\varepsilon}'(s)\big|_{H_\Gamma}^2+\big|\partial_{\boldsymbol{\nu}}v_\varepsilon(s)\big|_{H_\Gamma}^2+\big|v_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2+1\big)
\end{aligned}
\tag{3.54}
$$

for a.a. $s \in (0,T)$. Note that it holds

$$
\kappa_2\int_\Gamma \beta_{\Gamma,\varepsilon}'(v_{\Gamma,\varepsilon}(s)+m_0)\big|\nabla_\Gamma v_{\Gamma,\varepsilon}(s)\big|^2d\Gamma\ge 0.
$$

Thus, on account of Lemmas 3.3, 3.4, 3.6 and 3.8, by integrating (3.54) over $(0, T)$, we can find a positive constant M_7 such that the estimate (3.53) holds. □

Lemma 3.10. *There exists a positive constant M_9, independent of $\varepsilon \in (0, 1]$, such that*

$$\int_0^T \left|\boldsymbol{v}_\varepsilon(s)\right|_{\boldsymbol{W}}^2 ds \le M_9.$$

This lemma is also proved the same as in [11, Lemmas 4.5]. The key point to prove it is that we can obtain the uniform estimate of $|\Delta_\Gamma v_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma)}$ by comparing in (3.45). We omit the proof.

4. Proof of convergence theorem

In this section, we obtain the existence of periodic solutions of (P) by performing passage to the limit for the approximate problem $(\mathrm{P})_\varepsilon$. The convergence theorem is also nearly the same [11, Sect. 4]. The different point from [11] is that the component of the periodic solution of (P) satisfies (2.4) and the periodic property (2.5).

Thanks to the previous estimates in Lemmas from 3.3 to 3.10, there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and some limits functions $\boldsymbol{v} \in H^1(0, T; \boldsymbol{V}_0^*) \cap L^\infty(0, T; \boldsymbol{V}_0) \cap L^2(0, T; \boldsymbol{W})$, $\boldsymbol{\mu} \in H^1(0, T; \boldsymbol{V})$, $\xi \in L^2(0, T; H)$ and $\xi_\Gamma \in L^2(0, T; H_\Gamma)$ such that

$$\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{v} \quad \text{weakly star in } H^1(0, T; \boldsymbol{V}_0^*) \cap L^\infty(0, T; \boldsymbol{V}_0) \cap L^2(0, T; \boldsymbol{W}), \tag{4.1}$$

$$\varepsilon_k \boldsymbol{v}_{\varepsilon_k} \to 0 \quad \text{strongly in } H^1(0, T; \boldsymbol{H}_0),$$

$$\boldsymbol{\mu}_{\varepsilon_k} \to \boldsymbol{\mu} \quad \text{weakly in } L^2(0, T; \boldsymbol{V}),$$

$$\beta_{\varepsilon_k}(u_{\varepsilon_k}) \to \xi \quad \text{weakly in } L^2(0, T; H), \tag{4.2}$$

$$\beta_{\Gamma,\varepsilon_k}(u_{\Gamma,\varepsilon_k}) \to \xi_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma) \tag{4.3}$$

as $k \to \infty$. Owing to (4.1) and a well-known compactness results (see, e.g., [30]), we obtain

$$\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{v} \quad \text{strongly in } C([0, T]; \boldsymbol{H}_0) \cap L^2(0, T; \boldsymbol{V}_0) \tag{4.4}$$

as $k \to \infty$. This yeilds that

$$\boldsymbol{u}_{\varepsilon_k} \to \boldsymbol{u} := \boldsymbol{v} + m_0\mathbf{1} \quad \text{strongly in } C([0, T]; \boldsymbol{H}_0) \cap L^2(0, T; \boldsymbol{V}_0) \tag{4.5}$$

as $k \to \infty$. Therefore, from (4.5) and the Lipschitz continuity of $\widetilde{\pi}, \widetilde{\pi}_\Gamma$, we deduce that

$$\widetilde{\boldsymbol{\pi}}(\boldsymbol{u}_{\varepsilon_k}) \to \widetilde{\boldsymbol{\pi}}(\boldsymbol{u}) \quad \text{strongly in } C([0, T]; \boldsymbol{H})$$

as $k \to \infty$. Hence, by passing to the limit in (3.26) and (3.27), we obtain (2.3) and the following weak formulation:

$$(\boldsymbol{\mu}(t), z)_{\boldsymbol{H}} = a(\boldsymbol{v}(t), z) + (\boldsymbol{\xi}(t) - m(\boldsymbol{\xi}(t))\mathbf{1} + \widetilde{\boldsymbol{\pi}}(\boldsymbol{u}(t)) - \boldsymbol{f}(t), z)_{\boldsymbol{H}} \quad \text{for all } z \in \boldsymbol{V} \tag{4.6}$$

for a.a. $t \in (0,T)$, where $\boldsymbol{\xi} := (\xi, \xi_\Gamma)$, because of the property (2.2) of linear bounded operator $\boldsymbol{P}$. Now, we can infer $v + m_0 \in D(\beta)$ and $v_\Gamma + m_0 \in D(\beta_\Gamma)$. Hence, from the form (3.5) and (3.6), we deduce that $\widetilde{\pi}(v + m_0) = \pi(v + m_0)$ a.e. in Q and $\widetilde{\pi}_\Gamma(v_\Gamma + m_0) = \pi_\Gamma(v_\Gamma + m_0)$ a.e. on Σ. This implies that we obtain (2.4) replaced by (4.6). Moreover, it follows from (4.4) that

$$\boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \text{in } \boldsymbol{H}_0.$$

Also, due to (4.2), (4.3), (4.5) and the monotonicity of β, from the fact [5, Prop. 2.2, p. 38] we obtain

$$\xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma.$$

Thus, we complete the proof of Theorem 2.1.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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