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Research Article

Results on spirallike p-valent functions

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Abstract: In this paper, we introduce two new subclasses of p-valent spirallike functions of order α. We prove necessary and sufficient conditions for these newly defined classes and also point out some known consequences of our results.

Keywords: spirallike function; p-valent function; necessary and sufficient conditions
Mathematics Subject Classification: Primary 30C45; Secondary 30C50

1. Introduction

Let $A(p)$ denote the class of all functions f defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N = 1, 2, 3...) \tag{1.1}$$

which are analytic and p-valent in the unit disk

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}.$$

For a real number α $(0 \leq \alpha < p)$ the well-known subclasses $S_p^*(\alpha)$, p-valently starlike functions of order α and $C_p(\alpha)$, p-valently convex functions of order α of $A(p)$ are given by

$$\begin{aligned} S^*(\alpha, p) &= \left\{ f \in A(p) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{E}) \right\}, \\ C(\alpha, p) &= \left\{ f \in A(p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{E}) \right\}. \end{aligned}$$

For $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, a function $f \in A$ is said to be β-spirallike of order α in $\mathbb{E}$ if

$$\Re\left\{ e^{\iota\beta} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos\beta \quad (z \in \mathbb{E}). \tag{1.2}$$

The class of all such functions is denoted by $S_\beta(\alpha)$ [3], (also see [5, 14, 15]). In recent years many interesting subclasses of analytic univalent, multivalent and spirallike functions and their many special cases were investigated, see for example [1, 2, 6, 7, 8, 9, 10].

Motivated and inspired by the above mention work, we here define the following.

Definition 1.1. A function $f \in A(p)$ belongs to the class $S_\beta(\alpha, p)$ if it satisfies the inequality

$$\left|\frac{f^{(p-1)}(z)}{e^{\iota\beta} z f^{(p)}(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \qquad z \in \mathbb{E},$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the p^{th} derivative of $f(z)$.

Remark 1.2. First of all, it is easily seen that, for

$$p = 1 \quad \text{and} \quad \beta = 0, S_0(\alpha, 1) = M(\alpha),$$

where $M(\alpha)$ is a function class introduced and studied in [12]. Secondly, we have

$$p = 1, \quad S_\beta(\alpha, 1) = S_\beta(\alpha),$$

where $S_\beta(\alpha)$ is a function class introduced by Owa and Kamali [13].

Definition 1.3. A function $f \in A(p)$ is said to be in the class $C_\beta(\alpha, p)$ if it satisfies the inequality

$$\left|\frac{f^{(p)}(z)}{e^{\iota\beta}\left(z f^{(p)}(z)\right)'} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \qquad z \in \mathbb{E}, \tag{1.3}$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the pth derivative of $f(z)$.

As a special case, the class $C_\beta(\alpha, 1) = K_\beta(\alpha)$, is introduced by Owa and Kamali [13]. Using essentially their technique, we prove the main results for the classes $S_\beta(\alpha, p)$ and $C_\beta(\alpha, p)$ which is the main motivation of this paper.

2. Preliminary results

Lemma 2.1. *[4]. Let $\phi(u, v)$ be a complex-valued function such that*

$$\phi : D \to \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C}$$

$\mathbb{C}$ being the complex plane and let $u = u_1 + \iota u_2$ and $v = v_1 + \iota v_2$. Suppose that the function $\phi(u, v)$ satisfies each of the following conditions

1. *$\phi(u, v)$ is continuous in D;*
2. *$(1, 0) \in D$ and $\Re\{\phi(1, 0)\} > 0$;*
3. *$\Re\{\phi(\iota u_2, v_1)\} \leq 0$ for all $(\iota u_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$. Let*

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

be analytic (regular) in the unit disk $\mathbb{E}$ such that

$$(p(z), zp'(z)) \in D, \quad \textit{for all} \ \ z \in \mathbb{E}.$$

If

$$\Re\left\{\phi\left(p(z), zp'(z)\right)\right\} > 0, \quad \textit{then} \quad \Re\{p(z)\} > 0 \ \ (z \in \mathbb{E}).$$

3. Main results

Theorem 3.1. *A function $f \in S_\beta(\alpha, p)$ if and only if,* $\Re\left(e^{\iota\beta}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right) > \alpha$.

Proof. Let $f(z) \in S_\beta(\alpha, p)$, then we can write

$$\left|\frac{1}{e^{\iota\beta}F(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in E)$$

where $F(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$. From above, we have

$$\begin{aligned}
\left|\frac{2\alpha - e^{\iota\beta}F(z)}{2\alpha e^{\iota\beta}F(z)}\right| &< \left(\frac{1}{2\alpha}\right) \\
&\Leftrightarrow \left|2\alpha - e^{\iota\beta}F(z)\right|^2 < \left(e^{\iota\beta}F(z)\right)^2 \\
&\Leftrightarrow \left[2\alpha - e^{\iota\beta}F(z)\right]\left[\overline{2\alpha - e^{\iota\beta}F(z)}\right] < \left(e^{\iota\beta}F(z)\right)\left[\overline{e^{\iota\beta}F(z)}\right] \\
&\Leftrightarrow \left[2\alpha - e^{\iota\beta}F(z)\right]\left[2\alpha - e^{-\iota\beta}\overline{F(z)}\right] < \left(e^{\iota\beta}F(z)\right)\left[e^{-\iota\beta}\overline{F(z)}\right] \\
&\Leftrightarrow 4\alpha^2 - 2\alpha e^{-\iota\beta}\overline{F(z)} - 2\alpha e^{\iota\beta}F(z) + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
&\Leftrightarrow 4\alpha^2 - 2\alpha\left(e^{-\iota\beta}\overline{F(z)} + e^{\iota\beta}F(z)\right) < 0 \\
&\Leftrightarrow 2\alpha - 2\Re(e^{\iota\beta}F(z)) < 0 \\
&\Leftrightarrow -2\Re(e^{\iota\beta}F(z)) < -2\alpha \\
&\Leftrightarrow \Re\left(e^{\iota\beta}\frac{zf^{(P)}(z)}{f^{(P-1)}(z)}\right) > \alpha.
\end{aligned}$$

This complete the proof. □

When $p = 1$ we have the following known result proved by Owa and Kamali [13].

Corollary 3.2. $f(z) \in S_\beta(\alpha)$ *iff* $\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha$.

Theorem 3.3. *If $f(z) \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left(n + 1 + \left|(n+1) - 2\alpha e^{-\iota\beta}\right|\right)\left|a_{n+p}\right| \le 1 - \left|1 - 2\alpha e^{-\iota\beta}\right| \tag{3.1}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in S_\beta(\alpha, p)$.

Proof. If $f(z) \in S_\beta(\alpha, p)$ then, it suffices to show that

$$\left|\frac{2\alpha - e^{\iota\beta}F(z)}{e^{\iota\beta}F(z)}\right| < 1$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, where $F(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$.

Now we have

$$\left|\frac{2\alpha - e^{\iota\beta}F(z)}{e^{\iota\beta}F(z)}\right| = \left|\frac{2\alpha e^{-\iota\beta} f^{(p-1)}(z) - z f^{(p)}(z)}{z f^{(p)}(z)}\right|$$

$$= \left|\frac{2\alpha e^{-\iota\beta} - 1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left(2\alpha e^{-\iota\beta} - (n+1)\right) a_{n+p} z^n}{1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}(n+1)\, a_{n+p} z^n}\right|$$

$$\leq \frac{\left|2\alpha e^{-\iota\beta} - 1\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left|\left(2\alpha e^{-\iota\beta} - (n+1)\right)\right| \left|a_{n+p}\right| |z^n|}{1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}(n+1)\left|a_{n+p}\right| |z^n|}$$

$$< \frac{\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left|(n+1) - 2\alpha e^{-\iota\beta}\right| \left|a_{n+p}\right|}{1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}(n+1)\left|a_{n+p}\right|}. \tag{3.2}$$

The last expression in (3.2) is bounded above by 1 if

$$\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left|(n+1) - 2\alpha e^{-\iota\beta}\right| \left|a_{n+p}\right| \leq 1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left|(n+1)\, a_{n+p}\right|. \tag{3.3}$$

After simplification of (3.3) we have

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left\{\left(n + 1 + \left|(n+1) - 2\alpha e^{-\iota\beta}\right|\right)\right\} \left|a_{n+p}\right| \leq 1 - \left|1 - 2\alpha e^{-\iota\beta}\right|.$$

Therefore, $f(z) \in S_\beta(\alpha, p)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$. □

When $\beta = 0$ and $p = 1$, we have the following result proved by Owa *et al.* [12].

Corollary 3.4. *Let* $0 < \alpha < 1$. *If* $f(z) \in A$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} (n - \alpha)\, |a_n| \leq \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha; & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1 - \alpha; & \left(\frac{1}{2} < \alpha < 1\right) \end{cases}$$

then $f(z) \in M(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in above Theorem we have the following result.

Corollary 3.5. *If* $f(z) \in A(p)$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left(n + 1 + \sqrt{(n+1)^2 - 2\sqrt{2}\alpha(n+1) + 4\alpha^2}\right)\left|a_{n+p}\right| \leq 1 - \sqrt{1 - 2\sqrt{2}\alpha + 4\alpha^2}$$

for some $0 < \alpha < \frac{\sqrt{2}}{2}$, *then* $f(z) \in S_{\frac{\pi}{4}}(\alpha, p)$.

Theorem 3.6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_\beta(\alpha, p)$ *and let*

$$0 < \lambda \leq \frac{1}{2(\cos\beta - \alpha)}, \quad 0 < \alpha < \cos\beta. \tag{3.4}$$

Then we have

$$\Re\left\{\left(\frac{f^{(p-1)}(z)}{z}\right)^{\lambda e^{\iota\beta}}\right\} > \frac{(p!)^{-\lambda e^{\iota\beta}}}{2\lambda(\cos\beta - \alpha) + 1} \; (z \in E). \tag{3.5}$$

Proof. If we put
$$A = \frac{1}{2\lambda(\cos\beta - \alpha) + 1}$$
and
$$\left(\frac{f^{(p-1)}(z)}{p!z}\right)^{\lambda e^{\iota\beta}} = (1 - A)p(z) + A \tag{3.6}$$
where λ satisfies (3.4) then $p(z)$ is regular in the unit disk E and $p(z) = 1 + p_1 z + p_2 z^2 + ...$
Logarithmic differentiation of (3.6) yields
$$\lambda e^{\iota\beta}\left[\frac{f^{(p)}(z)}{f^{(p-1)}(z)} - \frac{1}{z}\right] = (1 - A)\frac{p'(z)}{(1 - A)p(z) + A}.$$
This can be written as
$$e^{\iota\beta}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - e^{\iota\beta} = (1 - A)\frac{zp'(z)}{\lambda\{(1 - A)p(z) + A\}},$$
equivalently
$$e^{\iota\beta}\frac{zf^{(p)}(z)}{f^{p-1}(z)} - \alpha = e^{\iota\beta} - \alpha + (1 - A)\frac{zp'(z)}{\lambda\{(1 - A)p(z) + A\}}. \tag{3.7}$$
Since $f(z) \in S_\beta(\alpha, p)$ then from (3.7) we have
$$\Re\left\{e^{\iota\beta} - \alpha + (1 - A)\frac{zp'(z)}{\lambda\{(1 - A)p(z) + A\}}\right\} > 0, \quad (z \in E, 0 < \alpha < \cos\beta).$$
Let us consider the functional $\theta(u, v)$ defined by
$$\theta(u, v) = e^{\iota\beta} - \alpha + (1 - A)\frac{v}{\lambda\{(1 - A)u + A\}},$$
where $u = p(z)$ and $v = zp'(z)$. Then $\theta(u, v)$ is continuous in $D = \left(\mathbb{C} - \left\{\frac{A}{A-1}\right\}\right) \times \mathbb{C}$.
Also, $(1, 0) \in D$ and $\Re\{\theta(1, 0)\} = \cos\beta - \alpha > 0$. Furthermore, for all $(\iota u_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$, we have
$$\begin{aligned}
\Re\{\theta(\iota u_2, v_1)\} &= \cos\beta - \alpha + \Re\left\{(1 - A)\frac{v_1}{[\lambda(1 - A)\iota u_2 + A]}\right\} \\
&= \cos\beta - \alpha + \frac{A(1 - A)v_1}{\lambda\left[(1 - A)^2 u_2^2 + A^2\right]} \\
&< \cos\beta - \alpha - \frac{A(1 - A)\left(1 + u_2^2\right)}{2\lambda\left[(1 - A)^2 u_2^2 + A^2\right]} \\
&= (\cos\beta - \alpha)\frac{A^2\left[4\lambda^2(\cos\beta - \alpha)^2 - 1\right]u_2^2}{\left[(1 - A)^2 u_2^2 + A^2\right]} \\
&\leq 0,
\end{aligned}$$

because $0 < \alpha < \cos\beta$ and $4\lambda^2(\cos\beta - \alpha)^2 - 1 \leq 0$ implies that $\lambda \leq \frac{1}{2(\cos\beta-\alpha)}$.
Therefore, the functional $\theta(u, v)$ satisfies all the conditions of Lemma 2.1.This proves that $\Re\{p(z)\} > 0$, that is from (3.6)

$$\begin{aligned}
\Re\left(\frac{f^{(p-1)}(z)}{p!z}\right)^{\lambda e^{\iota\beta}} &> A \\
\Re\left(\frac{f^{(p-1)}(z)}{z}\right)^{\lambda e^{\iota\beta}} &> \frac{(p!)^{-\lambda e^{\iota\beta}}}{2\lambda(\cos\beta - \alpha) + 1}.
\end{aligned}$$

This completes the proof. □

For $\beta = 0$ and $p = 1$, in above theorem we have the following known result given by [11].

Corollary 3.7. *Let $f \in A$ be in the class $S_0(\alpha, 1)$ and $0 < \lambda \leq \frac{1}{2(1-\alpha)}$, $0 < \alpha < 0$ then*

$$\Re\left(\frac{f(z)}{z}\right)^{\lambda} > \frac{1}{2\lambda(1-\alpha)+1}, \quad z \in E.$$

Theorem 3.8. *A function $f \in C_\beta(\alpha, p)$ if and only if*

$$\Re\left\{e^{\iota\beta}\left(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} > \alpha.$$

Proof. Let $f(z) \in C_\beta(\alpha, p)$, then we can write

$$\left|\frac{1}{e^{\iota\beta}G(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}.$$

This can be written as

$$\begin{aligned}
\left|\frac{1}{e^{\iota\beta}G(z)} - \frac{1}{2\alpha}\right| &< \frac{1}{2\alpha} \Leftrightarrow \left|\frac{2\alpha - e^{\iota\beta}G(z)}{2\alpha e^{\iota\beta}G(z)}\right| < \frac{1}{2\alpha} \\
&\Leftrightarrow \left|2\alpha - e^{\iota\beta}G(z)\right|^2 < \left(e^{\iota\beta}G(z)\right)^2 \\
&\Leftrightarrow \left(2\alpha - e^{\iota\beta}G(z)\right)\overline{(2\alpha - e^{\iota\beta}G(z))} < \left(e^{\iota\beta}G(z)\right)\overline{(e^{\iota\beta}G(z))} \\
&\Leftrightarrow \left(2\alpha - e^{\iota\beta}G(z)\right)\left(2\alpha - e^{-\iota\beta}\overline{G(z)}\right) < \left(e^{\iota\beta}G(z)\right)\left(e^{-\iota\beta}\overline{G(z)}\right) \\
&\Leftrightarrow 4\alpha^2 - 2\alpha\left[e^{-\iota\beta}\overline{G(z)} + e^{\iota\beta}G(z)\right] + G(z)\overline{G(z)} < G(z)\overline{G(z)} \\
&\Leftrightarrow 4\alpha^2 - 2\alpha\left[e^{-\iota\beta}\overline{G(z)} + e^{\iota\beta}G(z)\right] < 0 \\
&\Leftrightarrow 2\alpha\left[2\alpha - \Re\left(e^{\iota\beta}G(z)\right)\right] < 0 \\
&\Leftrightarrow 2\alpha - 2\Re\left(e^{\iota\beta}G(z)\right) < 0 \\
&\Leftrightarrow \Re\left\{e^{\iota\beta}\left(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} > \alpha.
\end{aligned}$$

This completes the proof. □

Theorem 3.9. *If $f(z) \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{n!} \left\{n+1+\left|(n+1)-2\alpha e^{-\iota\beta}\right|\right\} \left|a_{n+p}\right| \leq 1-\left|1-2\alpha e^{-\iota\beta}\right| \tag{3.8}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in C_{\beta}(\alpha, p)$.

Proof. To prove that $f(z) \in C_{\beta}(\alpha, p)$ we need to prove that

$$\left|\frac{2\alpha - e^{\iota\beta} G(z)}{e^{\iota\beta} G(z)}\right| < 1 \tag{3.9}$$

for some $|\beta| < \frac{\pi}{2}, 0 < \alpha < 1$, where $G(z) = 1 + \frac{z f^{(p+1)}(z)}{f^{(p)}(z)}$.
For this consider the left hand side of (3.9), we have

$$\begin{aligned}
\left|\frac{2\alpha - e^{\iota\beta} G(z)}{e^{\iota\beta} G(z)}\right| &= \left|\frac{2\alpha e^{-\iota\beta} f^{(p)}(z) - \left(f^{(p)}(z) + z f^{(p+1)}(z)\right)}{\left(f^{(p)}(z) + z f^{(p+1)}(z)\right)}\right| \\
&= \left|\frac{2\alpha e^{-\iota\beta} - 1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1)\, a_{n+p} 2\alpha e^{-\iota\beta} - (n+1)}{1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1)^2 \left|a_{n+p}\right|}\right| \\
&= \frac{\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|(n+1)\, a_{n+p}\right| \left|(n+1) - 2\alpha e^{-\iota\beta}\right|}{1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1)^2 \left|a_{n+p}\right|}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$\begin{aligned}
&\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|(n+1)\, a_{n+p}\right| \left|(n+1) - 2\alpha e^{-\iota\beta}\right| \\
\leq\ & 1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1)^2 \left|a_{n+p}\right|
\end{aligned} \tag{3.10}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < 1$. After simplification, inequality (3.10) can be written as

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{n!} \left\{n+1+\left|(n+1)-2\alpha e^{-\iota\beta}\right|\right\} \left|a_{n+p}\right| \leq 1-\left|1-2\alpha e^{-\iota\beta}\right|.$$

This completes the proof. □

When we take $p = 1$, we have the following known result given in [13].

Corollary 3.10. *If $f \in A$ satisfies*

$$\sum_{n=2}^{\infty} \left\{n\left(n+\left|n-2\alpha e^{-\iota\beta}\right|\right)\right\} |a_n| \leq 1-\left|1-2\alpha e^{-\iota\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f \in K_{\beta}(\alpha)$.

Taking $p = 1, \beta = 0$ in above theorem we have the following result given in [12].

Corollary 3.11. *Let* $0 < \alpha < 1$. *If* $f \in A$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \frac{1}{2}(1 - |1-2\alpha|) = \begin{cases} \alpha; & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1-\alpha; & \left(\frac{1}{2} < \alpha < 1\right) \end{cases}$$

then $f(z) \in N(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in above Theorem we have the following result.

Corollary 3.12. *If* $f(z) \in A(p)$ *satisfies*

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(p+n)!}{n!} \left(n + 1 + \sqrt{(n+1)^2 - 2\sqrt{2}\alpha(n+1) + 4\alpha^2} \right) \left| a_{n+p} \right| \\ \leq \; & 1 - \sqrt{1 - 2\sqrt{2}\alpha + 4\alpha^2} \end{aligned}$$

for some $0 < \alpha < \frac{\sqrt{2}}{2}$, *then* $f(z) \in S_{\frac{\pi}{4}}(\alpha, p)$.

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Conflict of Interest

No potential conflict of interest was reported by the authors.

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