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Research article

A geometric formulation of Lax integrability for nonlinear equations in two independent variables

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Abstract: A geometric formulation of Lax integrability is introduced which makes use of a Pfaffian formulation of Lax integrability. The Frobenius theorem gives a necessary and sufficient condition for the complete integrability of a distribution, and provides a powerful way to study nonlinear evolution equations. This permits an examination of the relation between complete integrability and Lax integrability. The prolongation method is formulated in this context and gauge transformations can be examined in terms of differential forms as well as the Frobenius theorem.

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1. Introduction

When a nonlinear evolution equation can be generated from a pair of linear partial differential equations of first order by means of a compatibility condition, the nonlinear evolution equation is said to be Lax integrable and the nonlinear system is called a Lax pair [7, 8]. However, if such a pair of equations can be determined, then such objects such as gauge and Darboux transformations as well as an infinite number of conservation laws can be constructed for the equation. Complete integrability is another property associated with such systems. This concept is usually formulated for a distribution on a manifold [9].

Definition 1. (i) Let m, n be integers with $1 \leq m \leq n$, then an m-dimensional distribution $\mathcal{D}$ on an n-dimensional manifold M is a selection of an m-dimensional subspace $\mathcal{D}_p$ of T_pM for each $p \in M$. The distribution is C^∞ if for $p \in M$, there exists a neighborhood U of p and m vector fields $X_1, \dots, X_m$ on U which span $\mathcal{D}$ at each point in U. A vector field is said to lie in the distribution $\mathcal{D}$ if $X_p \in \mathcal{D}_p$, for each $p \in M$. (ii) A C^∞ distribution is called involutive, or completely integrable, if $[X, Y] \in \mathcal{D}$ whenever $X, Y \in \mathcal{D}$. (iii) We say D is integrable if for any $p \in M$ there is a (local) submanifold $N \subset M$

called a (local) integral manifold of $\mathcal{D}$ at p containing p whose tangent bundle is exactly $\mathcal{D}$ restricted to N. □

An equivalent way of stating this is that a distribution $\mathcal{D}$ is completely integrable if through each point $p \in M$, there exists an integral manifold whose dimension is equal to the dimension of $\mathcal{D}$. The Frobenius theorem gives the necessary and sufficient condition for the integrability of a distribution. The theorem can be expressed in several different but equivalent forms, in terms of vector fields as well as differential forms.

The exterior differential version of the Frobenius theorem states that a distribution $\mathcal{D}$ is completely integrable if and only if the ideal $I(\mathcal{D})$ generated by $\mathcal{D}$ is a closed differential ideal, hence $d\,I(\mathcal{D}) \subset I(\mathcal{D})$. This is a very practical form of the theorem.

The objective here is to investigate $1+1$ dimensional nonlinear evolution equations which possess Lax integrability by means of a geometric approach. It will be seen that the relation between complete and Lax integrability can be formulated in this way when the Lax integrability of the nonlinear evolution equation is given in terms of exterior differential forms [11]. This permits geometric interpretations of the Lax equation to be given. Exterior differential systems also provides an efficient way to formulate the concept of gauge and Darboux transformations for such equations. The terminology nonlinear evolution equation will be understood to signify an equation in one space and one time variable everywhere, that is, in one+one dimensions [2–4].

2. Lax Representations in Two Independent Variables

Historically in the theory of nonlinear evolution equations of soliton type, it is well known that these equations may be considered as sufficient conditions for the integrability of an eigenvalue problem involving a set of linear partial differential equations which contain the solution of the corresponding evolution equation. As a particular example [5], the integrability of the linear system

$$y_x = \lambda + u, \qquad y_t = -u_{xx} + \frac{6}{n+1}(-u)^{n+1} + \mu,$$

leads to the nonlinear system

$$u_t + u_{xxx} + 6(-u)^n\, u_x = 0.$$

If we eliminate u from the pair, it is found that $y = y(x,t)$ satisfies the equation

$$y_t + y_{xxx} - \frac{6}{n+1}(\lambda - y_x)^{n+1} - \mu = 0.$$

This is often called the potential equation when $\lambda = \mu = 0$, and is an example of a Bäcklund transformation as well

The Lax representation of a nonlinear evolution equation in two independent variables considered here is introduced as a matrix system,

$$\psi_x = M(\lambda)\psi, \qquad \psi_t = N(\lambda)\psi. \tag{2.1}$$

The parameter λ in (2.1) is usually called a spectral parameter, ψ is referred to as an eigenfunction associated with the spectral parameter λ, and M, N are particular matrices whose elements depend on

λ. The sizes of the matrices may be left arbitrary for the moment. Collectively (2.1) is referred to as a Lax pair. The nonlinear equation which corresponds to M and N is obtained by differentiating the first equation with respect to t and the second with respect to x and equating the mixed partials of ψ to produce the compatibility condition for (2.1)

$$M_t - N_x + [M, N] = 0. \tag{2.2}$$

The bracket in (2.2) is defined as $[M, N] = MN - NM$.

To obtain a geometric picture of this type of integrability, define a matrix of one-forms ω as

$$\omega = M\,dx + N\,dt. \tag{2.3}$$

System (2.1) results as a consequence of the following relation

$$\sigma = d\psi - \psi\omega = 0. \tag{2.4}$$

Exterior differentiation of (2.3) produces the result,

$$d\sigma = -d\omega\,\psi + \omega \wedge\, d\psi = 0. \tag{2.5}$$

Solving (2.4) for $d\psi$ and putting this in (2.5), we obtain,

$$d\sigma = -d\omega\,\psi + \omega \wedge (\sigma + \omega\,\psi) = \omega \wedge \sigma - (d\omega - \omega \wedge \omega)\,\psi. \tag{2.6}$$

The Frobenius theorem implies system (2.3) is completely integrable if and only if the system of one-forms ω satisfies the condition

$$\Omega = d\omega - \omega \wedge \omega = 0. \tag{2.7}$$

In (2.7), Ω is a square matrix of two-forms usually referred to as the curvature. It is straightforward to verify that (2.7) is equivalent to (2.2). These equations, (2.2) and (2.7), are referred to as zero curvature conditions, and (2.7) is equivalent under (2.3) to the nonlinear evolution equation whose Lax representation is (2.1).

Theorem 1. The nonlinear evolution equation in (x, t) whose Lax representation is given by (2.1) must be completely integrable.

Proof: Differentiating the curvature Ω in (2.7) gives

$$\begin{aligned} d\Omega &= -d\omega \wedge \omega + \omega \wedge d\omega = -(\Omega + \omega \wedge \omega) \wedge \omega + \omega \wedge (\Omega + \omega \wedge \omega) \\ &= -\Omega \wedge \omega + \omega \wedge \Omega. \end{aligned} \tag{2.8}$$

Therefore, system (2.7) is completely integrable according to the Frobenius theorem. Hence (2.7) corresponds to the nonlinear evolution equation whose Lax representation is (2.1) as required. □

There is an equivalent approach to the definition of Lax integrability which is discussed now. Let $U \subset \mathbb{R}^2$ be coordinatized by x and t and let $V = U \times \mathbb{R}^n \subset \mathbb{R}^{n+2}$ be a fiber bundle with U as the base manifold and U carries coordinates related to a particular nonlinear evolution equation.

Suppose the equation can be represented by a system of two-forms defined on V. As a specific example, consider the system of two-forms

$$\alpha_1 = du \wedge dt - p\,dx \wedge dt,$$

$$\alpha_2 = dp \wedge dt - q\,dx \wedge dt, \tag{2.9}$$
$$\alpha_3 = -du \wedge dx + dq \wedge dx - u\,dq \wedge dt + u\,du \wedge dt + \beta\,(u-q)\,du \wedge dt.$$

The parameter β which appears is a constant. If the map $s : U \to V$ is a cross section of V with the property

$$s^*\,\alpha_i = 0, \tag{2.10}$$

where the α_i are a system of forms such as (2.9), and mapping s^* denotes the pull-back of s, it can be verified that $u(x,t)$ is a solution of the associated equation. Conversely, for any given solution $u(x,t)$ of the equation, the map $s : U \to V$ which is given by $s(x,t) = (x, t, u(x,t), u_x(x,t), \cdots)$ is a cross section of V satisfying $s^*\alpha_i = 0$.

Differentiating the forms in (2.9) gives

$$d\alpha_1 = -dp \wedge dx \wedge dt = dx \wedge \alpha_2,$$

$$d\alpha_2 = dx \wedge (-\alpha_3 + u((1+\beta)u - q)\,\alpha_1),$$
$$d\alpha_3 = (1-\beta)[dq \wedge \alpha_1 + p\,dt \wedge \alpha_3 - p\,dx \wedge \alpha_1].$$

Clearly, all of the $d\alpha_j$ vanish modulo (2.9) so the $\{\alpha_i\}$ represent a closed differential ideal. Sectioning α_1 and α_2 gives $p = u_x$ and $q = u_{xx}$ and the nonlinear equation results from evaluating

$$0 = \alpha_3|_s = (u_t - q_t + u(u_x - q_x) + \beta(u-q)u_x)\,dx \wedge dt.$$

These results imply the partial differential equation

$$(u-q)_t + u(u-q)_x + \beta(u-q)u_x = 0.$$

Introducing $\rho = u - u_{xx}$, then for the case $\beta = 3$, the Degasperis-Procesi equation results,

$$\rho_t + \rho_x u + 3\rho u_x = 0,$$

and for $\beta = 2$, the Camassa-Holm equation appears

$$\rho_t + \rho_x u + 2\rho\,u_x = 0.$$

It is important to note that if a set of forms $\{\alpha_i\}$ can be determined which correspond to a particular evolution equation these forms can be used to construct both Lax pairs and Bäcklund transformations for that equation.

For any nonlinear evolution equation, a corresponding set of two forms can be constructed. In fact, two systems of two-forms $\{\alpha_i\}$ and $\{\beta_i\}$ are said to be equivalent if one set can be expressed in terms of the other as $\beta_i = f_i^j\,\alpha_j$, and the rank of the matrix (f_i^j) is maximal.

Theorem 2. (Lax Integrability) A nonlinear evolution equation in (x,t) is Lax integrable if and only if there exists a square matrix ω of one-forms in (dx, dt) such that the system (2.7) is equivalent to $\{\alpha_i = 0\}$.

Proof: Assume the Lax representation for the nonlinear equation is given by (2.1). If ω is defined to be the one-form (2.3), then ω is exactly what is required according to the demonstrated equivalence of relations starting with (2.7) implying (2.2) which then yields the nonlinear evolution equation (2.1) which finally can be put in the equivalent differential form $\{\alpha_i = 0\}$.

Second suppose the matrix of one-forms is ω, then ω can be identified as a combination of a dx and dt term, so (2.3) holds. Then (M,N) can be identified as the Lax pair demanded by $\Omega = 0$. □

3. Prolongation and Differential Systems

The Prolongation Method emerges in a natural way out of this theory. Prolongations are important for constructing solutions of evolution equations and also for deriving Bäcklund and auto-Bäcklund transformations which transform a solution of an equation into a solution of another or the same equation. The main ideas seem to originate with Wahlquist and Estabrook [10, 12, 13]. The nonlinear equation with Lax representation given by (2.1) can be replaced by an equivalent system of two-forms $\{\alpha_i = 0\}$ defined on the manifold V. Now introduce an additional system of Pfaff forms φ_i on the vector bundle $E = V \times \mathbb{R}^n$ which are defined as follows

$$\varphi_i = dy_i - F_i\, dx - G_i\, dt. \qquad i = 1, \ldots, n, \tag{3.1}$$

The new variables y_i are often called pseudopotentials and provide a coordinate system for $\mathbb{R}^n$. The φ_i are one-forms on E and the F_i, G_i are a set of functions which may depend on all the coordinates of E. The prolongation method requires the ideal which is generated by the system $\{\alpha_i = 0, \varphi_i = 0\}$ to be a closed differential ideal. This will provide constraints which serve to determine the form of the unknown functions F_i and G_i in (3.1). Thus, the prolongation condition has to have the following structure: the derivative of (3.1) must be of the following form,

$$d\varphi_i = \beta_i^j\, \alpha_j + \gamma_i^j \wedge \varphi_j, \qquad i = 1, \ldots, n, \tag{3.2}$$

where (β_i^j) is a matrix of functions and (γ_i^j) is a matrix of one-forms. Formally comparing σ with φ_i, the pseudo-potentials y_i can be identified with the eigenfunction ψ, while F_i, G_i correspond to ω. There is also a formal correspondence between (3.2) and (2.6). Out of this correspondence, it may be seen that the prolongation condition guarantees that (3.1) is completely integrable on the solution manifold $\{\alpha_i = 0\}$ of the equation whose Lax representation is (2.1). The Lax system (2.1) for the equation can be obtained with the complete integrability condition, although Lax integrability is a stronger property than complete integrability.

A geometric interpretation comes directly from differential geometry for the Lax equation (2.4) [1, 6]. Assume there is a connection ∇ which is defined on the vector bundle E, and the sections $e_1, \cdots, e_n$ form a frame of sections of E. By using the frame of sections $\mathcal{S} = (e_1, \cdots, e_n)^T$ and the connection, and $n \times n$ connection matrix ω can be defined by the formula

$$\nabla \mathcal{S} = \omega\, \mathcal{S}. \tag{3.3}$$

Elements of the connection matrix ω has elements which depend on the coordinates of the manifold V. If a section $s = \eta^i e_i$ is a parallel section of E with the η^i functions on V, from $\nabla s = 0$, the η^i will satisfy the following equations

$$d\eta^i + \eta^j \omega_j^i = 0, \qquad i = 1, \cdots, n. \tag{3.4}$$

If we set $\eta = (\eta^1, \cdots, \eta^n)$, then (3.4) can be put in matrix form as

$$d\eta + \eta\, \omega = 0. \tag{3.5}$$

The connection ∇ on the vector bundle E induces a connection ∇^* on the dual vector bundle $E^* = V \times (\mathbb{R}^n)^*$, where $(\mathbb{R}^n)^*$ denotes the dual space of $\mathbb{R}^n$. If a dual frame of sections $\mathcal{S}^* = (e^{1*}, \cdots, e^{n*})^T$ of

E^* is established, then $\langle e_i, e^{j*}\rangle = \delta_i^j$ with $\langle , \rangle$ the inner product in the vector bundles E and E^*. On the dual space, the connection is written $\nabla^* \mathcal{S}^* = -\omega \mathcal{S}^*$, so the induced connection matrix on the dual E^* is $-\omega$. If the section $s^* = \theta_i e^{i*}$ is a parallel section of E^* where the θ_i are functions on V with $\nabla^* s^* = 0$, then the θ_i satisfy the equation

$$d\theta^i - \theta_j \omega_i^j = 0, \qquad i = 1, \cdots, n.$$

Defining $\theta = (\theta_1, \cdots, \theta_n)$, then this can be put in matrix form is

$$d\theta - \omega\,\theta = 0. \tag{3.6}$$

Equation (3.6) is exactly Lax equation (2.4), and this Lax equation may be thought of as the parallel section equation on the dual vector bundle E^* with connection matrix equal to $-\omega = -M\,dx - N\,dt$. The eigenfunctions ψ correspond to the vector formed by the coordinates of the parallel section of E^* under the dual frame of sections of S^*. Then the equation for Ω represents the curvature matrix where ω plays the role of connection matrix. Pfaff system (2.3) is completely integrable then if and only if curvature matrix Ω vanishes. Also, if the zero curvature condition $\Omega = 0$ is satisfied, there exists n linearly independent parallel sections, or alternatively, the Lax equation (2.4) has n linearly independent solutions.

4. Lax Systems and Gauge Transformations in Terms of Differential Systems

Consider two Lax systems defined by

$$\Psi_x = M\Psi, \qquad \Psi_t = N\Psi, \tag{4.1}$$

$$\Phi_x = \tilde{M}\Phi, \qquad \Phi_t = \tilde{N}\Phi. \tag{4.2}$$

If a gauge transformation τ exists such that

$$\Phi = \tau\,\Psi, \tag{4.3}$$

where τ is a matrix that transforms (4.1) into (4.2), a system of equations satisfied by τ can be obtained. Differentiating (4.3) with respect to x gives

$$\Phi_x = \tau_x\Psi + \tau\Psi_x = \tau_x\Psi + \tau\,M\Psi.$$

This must be equal to $\tilde{M}\Phi = \tilde{M}\tau\Psi$, and so isolating $\tau_x\Psi$

$$\tau_x\Psi = (\tilde{M}\tau - \tau M)\Psi. \tag{4.4}$$

This yields an equation for τ_x. Similarly, evaluating Ψ_t gives

$$\tau_t\Psi = (\tilde{N}\tau - \tau N)\Psi. \tag{4.5}$$

Therefore, the matrix τ satisfies the following system,

$$\tau_x = \tilde{M}\tau - \tau M, \qquad \tau_t = \tilde{N}\tau - \tau N. \tag{4.6}$$

In fact, system (4.6) can be expressed in the form [10]

$$\tilde{\omega} = \tau\omega\tau^{-1} + d\tau\,\tau^{-1} \tag{4.7}$$

upon noting that $\omega = M\,dx + N\,dt$ and $\tilde{\omega} = \tilde{M}\,dx + \tilde{N}\,dt$. Now (4.7) is just the transformation formula for the connection matrix under the transformation of the basis of sections, namely,

$$\tilde{S} = \tau\,S. \tag{4.8}$$

The transformation equation for the curvature matrix is given by

$$\tilde{\Omega} = \tau\,\Omega\,\tau^{-1}, \tag{4.9}$$

where Ω and $\tilde{\Omega}$ are the respective curvature forms. This can be summarized as the following statement.

Theorem 3. A nonlinear evolution equation which is Lax integrable is equivalent to the system $\tau\Omega\tau^{-1} = 0$ for some $n \times n$ invertible matrix τ whose elements depend on the coordinates of V.

Proof: If an equation with Lax integrability is equivalent to system (4.9), then the Lax equation corresponding to that equation is $d\tilde{\psi} = \tilde{\omega}\,\tilde{\psi}$, with $\tilde{\omega}$ given by (4.7).

Conversely, it has been noted that every nonlinear evolution equation with Lax integrability is equivalent to a representative equation under gauge transformation. Therefore, there is an invertible matrix τ such that the given equation is equivalent to the system $\tau\Omega\tau^{-1} = 0$. □

System (4.7) is clearly equivalent to the following Pfaff system

$$\Theta = d\tau - \tilde{\omega}\,\tau + \tau\,\omega = 0. \tag{4.10}$$

Expanding out (4.10) more fully, it is

$$\Theta = \tau_x\,dx + \tau_t\,dt - (\tilde{M}\,dx + \tilde{N}dt)\tau + \tau(M\,dx + N\,dt) = 0.$$

Theorem 4. Pfaff system (4.10) is completely integrable.

Proof: Differentiating (4.10), it is found that

$$d\Theta = -d\tilde{\omega}\tau + \tilde{\omega} \wedge d\tau + d\tau \wedge \omega + \tau\,d\omega$$

$$= -d\tilde{\omega}\tau + \tilde{\omega} \wedge (\Theta + \tilde{\omega}\tau - \tau\omega) + (\Theta + \tilde{\omega}\tau - \tau\omega) \wedge \omega + \tau\,d\omega$$

$$= -d\tilde{\omega}\tau + \tilde{\omega} \wedge \Theta + \tilde{\omega} \wedge \tilde{\omega}\tau - \tilde{\omega} \wedge \tau\omega + \Theta \wedge \omega + \tilde{\omega}\tau \wedge \omega - \tau\omega \wedge \omega + \tau\,d\omega$$

$$= -(d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega})\tau + \tau(d\omega - \omega \wedge \omega) + \tilde{\omega} \wedge \Theta + \Theta \wedge \omega. \tag{4.11}$$

Given that $\Omega = 0$ and $\tilde{\Omega} = 0$, the Frobenius theorem implies Pfaff system (4.11) is completely integrable and moreover,

$$d\Theta = (\tilde{\omega} - \omega) \wedge \Theta. \tag{4.12}$$

□

Theorem 4 says that every nonlinear evolution equation in one+one dimensions with Lax integrability is gauge equivalent to any other. For example, if the KdV equation is taken as a typical representative, assuming the n-dimensional Lax representation for it is $d\psi = \omega\,\psi$, the KdV equation is equivalent to $\Omega = 0$.

Theorem 5. The nonlinear equation in (x, t) with Lax integrability is equivalent to the system $\tau\Omega\tau^{-1} = 0$ for some $n \times n$ invertible matrix τ whose elements are functions on the manifold V.

Proof: If a nonlinear equation with Lax integrability is equivalent to the system $\tau\Omega\tau^{-1} = 0$, the Lax equation corresponding to that equation is $d\tilde{\psi} = \tilde{\omega}\,\tilde{\psi}$, where $\tilde{\omega}$ is given by (4.7). Then that nonlinear evolution equation is Lax integrable.

Conversely, it is known that every nonlinear evolution equation in one+one dimensions with Lax integrability is gauge equivalent to one of the equations, say the representative equation. Thus there is an invertible matrix τ such that the given nonlinear equation with Lax integrability is equivalent to the system $\tau\Omega\tau^{-1} = 0$. □

If two systems $\Omega = 0$ and $\tau\Omega\tau^{-1} = 0$ correspond to the same nonlinear evolution equation with Lax integrability, it is said they are equivalent. The set of gauge transformations forms a group referred to as the gauge group. Since these transformations preserve the solution manifold it may also be called the symmetry group.

Suppose the form $\omega(\lambda) = M(\lambda)dx + N(\lambda)\,dt$ is expanded in powers of λ in the form $\omega(\lambda) = \sum_{i=0}^{m} \omega_i\lambda^i$, then if S is an $n \times n$ matrix independent of λ, $\omega(S)$ is defined to be

$$\omega(S) = \sum_{i=0}^{m} \omega_i\, S^i. \tag{4.13}$$

Theorem 6. If there exists a gauge transformation of the form $\tau = \lambda I - S$ for the nonlinear evolution equation whose Lax representation is (2.1), then an $n\times n$ matrix S independent of the parameter λ must satisfy the following equation

$$dS + [S, \omega(S)] = 0. \tag{4.14}$$

Proof: Substitute the forms $\omega(\lambda) = \sum_{i=0}^{m} \omega_i\lambda^i$ and $\tilde{\omega}(\lambda) = \sum_{i=0}^{m} \tilde{\omega}_i\lambda^i$ and $\tau = \lambda I - S$ into Θ from (4.10). Collecting coefficients of the same powers of the spectral parameter λ

$$-\,dS - \sum_{i=1}^{m} \tilde{\omega}_i\lambda^{i+1} + \sum_{i=1}^{m} \tilde{\omega}_i\lambda^i S + \sum_{i=0}^{m} \omega_i\lambda^{i+1} - S\sum_{i=0}^{m} \omega_i\lambda^i = 0. \tag{4.15}$$

Collecting like powers of λ in (4.15), there results,

$$-\,dS + \tilde{\omega}_0 S - S\,\omega_0 + \sum_{i=1}^{m} (-\tilde{\omega}_{i-1} + \tilde{\omega}_i\,S - \omega_{i-1} - S\,\omega_i)\,\lambda^i + (-\tilde{\omega}_m + \omega_m)\lambda^{m+1} = 0. \tag{4.16}$$

Equating the coefficients of each power λ^i to zero, then if i refers to the power on λ, the following system of equations results,

$$dS + S\omega_0 - \tilde{\omega}_0 S = 0, \quad i = 0, \quad -\tilde{\omega}_{i-1} + \tilde{\omega}_i S + \omega_{i-1} - S\,\omega_i = 0, \quad i = 1, \dots, m,$$

$$-\,\tilde{\omega}_m + \omega_m = 0, \quad i = m+1. \tag{4.17}$$

Applying these equations recursively, we can write

$$-S\,\omega_j S^{j-1} + \tilde{\omega}_j S^j = -S\,\omega_j S^{j-1} + (\omega_j - S\,\omega_{j+1} + \tilde{\omega}_{j+1} S)S^j$$

$$= -S\,\omega_j S^{j-1} + \omega_j S^j - S\,\omega_{j+1} S^j + \tilde{\omega}_{j+1} S^{j+1} = [\omega_j, S]S^{j-1} - S\,\omega_{j+1} S^j + \tilde{\omega}_{j+1} S^{j+1}. \tag{4.18}$$

Thus iterating (4.18), a relationship between ω_0 and $\tilde{\omega}_0$ can be obtained,

$$\tilde{\omega}_0 = \omega_0 + [\omega_1, S] - S\omega_2 S + \tilde{\omega}_2 S^2 = \omega_0 + [\omega_1, S] + (\omega_2 S - S\omega_2)S + S\omega_3 S^2 + \tilde{\omega}_3 S^3$$
$$= \omega_0 + \sum_{k=1}^{m} [\omega_k, S] S^{k-1}. \quad (4.19)$$

Solving for dS from the $i = 0$ equation of (4.17) and substituting $\tilde{\omega}_0$ from (4.19), we get

$$dS = \tilde{\omega}_0 S - S\omega_0 = \omega_0 S - S\omega_0 + \sum_{k=1}^{m} [\omega_k, S] S^k = -[S, \omega_0] - \sum_{k=1}^{m} [S, \omega_k] S^k$$
$$= -\sum_{k=0}^{m} [S, \omega_k] S^k.$$

Thus the result is exactly equation (4.14). □

Substituting $\omega(S) = M(S)\,dx + N(S)\,dt$ into (4.14), it breaks up into two equations as follows,

$$S_x + [S, M(S)] = 0, \qquad S_t + [S, N(S)] = 0. \quad (4.20)$$

Theorem 7. System (4.14) is completely integrable if and only if the following equation is satisfied,

$$d\omega(S) - \omega(S) \wedge \omega(S) = 0, \quad (4.21)$$

or equivalently, if and only if

$$M_t(S) - N_x(S) + [M(S), N(S)] = 0. \quad (4.22)$$

Proof: Define the Pfaff system ξ to be

$$\xi = dS + [S, \omega(S)] = dS + [S, M(S)]\,dx + [S, N(S)]\,dt = 0. \quad (4.23)$$

Differentiating ξ

$$d\xi = [dS, M(S)] \wedge dx + [S, M_t(S) - N_x(S)]\,dt \wedge dx + [dS, N(S)] \wedge dt. \quad (4.24)$$

Obtaining dS from (4.23) and replacing it in (4.24) gives,

$$d\xi = [\xi, M(S)] \wedge dx + [\xi, N(S)] \wedge dt + [S, M_t(S) - N_x(S)]\,dt \wedge dx$$
$$- [[S, N(S)], M(S)]\,dt \wedge dx - [[S, M(S)], N(S)]\,dx \wedge dt. \quad (4.25)$$

Using the Jacobi identity, equation (4.25) simplifies to the form

$$d\xi = [\xi, M(S)] \wedge dx + [\xi, N(S)] \wedge dt + [S, M_t(S) - N_x(S) + [M(S), N(S)]]\,dt \wedge dx.$$

Therefore, by the Frobenius Theorem, the Pfaff system $\xi = 0$ is completely integrable if and only if (4.22) holds. □

Let matrix S be constructed in the following way. Suppose Λ is an $n \times n$ matrix formed by putting the n complex parameters $\lambda_1, \dots, \lambda_n$ on the main diagonal and zero everywhere else. Then B is an $n \times n$ invertible matrix such that $B = \mathrm{diag}(\psi_1, \dots, \psi_n)$, where the ψ_i, $i = 1, \dots, n$ satisfy the equations

$$d\psi_i = \omega(\lambda_i)\,\psi_i, \qquad i = 1, \dots, n. \tag{4.26}$$

Define the matrix S to be

$$S = B\Lambda\, B^{-1}. \tag{4.27}$$

Theorem 8. The matrix S from (4.27) satisfies the following relation,

$$dS^i = \omega(S)\,S^i - S^i\,\omega(S). \tag{4.28}$$

Proof: Differentiating B, there results

$$\begin{aligned} dB &= \mathrm{diag}(d\psi_1, \dots, d\psi_n) = \mathrm{diag}(\omega(\lambda_1)\psi_1, \dots, \omega(\lambda_n)\psi_n) \\ &= \mathrm{diag}(\sum_{i=1}^{m} \omega_i \lambda_1^i \psi_1, \dots, \sum_{i=0}^{m} \omega_i \lambda_n^i \psi_n) = \sum_{i=0}^{m} \omega_i B \Lambda^i. \end{aligned} \tag{4.29}$$

This can be used to evaluate dS,

$$\begin{aligned} dS &= d(B\Lambda B^{-1}) = dB\Lambda B^{-1} + B\Lambda dB^{-1} = dB\Lambda B^{-1} - B\Lambda B^{-1} \cdot dBB^{-1} \\ &= (\sum_{i=0}^{m} \omega_i B\Lambda^i)\Lambda B^{-1} - (B\Lambda B^{-1})(\sum_{i=0}^{m} \omega_i B\Lambda^i)B^{-1} \\ &= (\sum_{i=0}^{m} \omega_i B\Lambda^i B^{-1})(B\Lambda B^{-1}) - (B\Lambda B^{-1})(\sum_{i=0}^{m} \omega_i B\Lambda^i B^{-1}) \\ &= (\sum_{i=0}^{m} \omega_i S^i)S - S(\sum_{i=0}^{m} \omega_i S^i) = \omega(S)S - S\,\omega(S). \end{aligned} \tag{4.30}$$

Differentiating one S at a time sequentially in S^i, we obtain

$$dS^i = dS\, S^{i-1} + S\, dS\, S^{i-2} + \dots + S^{i-1}\, dS. \tag{4.31}$$

Substituting dS from (4.30) into each of these terms one after the other, the following system is obtained,

$$\begin{aligned} dS\,S^{i-1} &= (\omega(S)S - S\,\omega(S))S^{i-1} = \omega(S)S^i - S\,\omega(S)S^{i-1}, \\ S\,dS\,S^{i-2} &= S(\omega(S)S - S\,\omega(S))S^{i-2} = S\,\omega(S)S^{i-1} - S^2\omega(S)S^{i-2}, \\ &\vdots \\ S^{i-1}dS &= S^{i-1}(\omega(S)S - S\,\omega(S)) = S^{i-1}\omega(S)S - S^i\omega(S). \end{aligned}$$

Adding these together vertically most terms cancel out, as the sum telescopes, and the result (4.28) follows. □

Theorem 9. Matrix S satisfies the equation

$$\sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge \omega(S) S^i = 0. \tag{4.32}$$

Proof: The compatibility condition for the system (2.4) is

$$d\omega(\lambda)\psi - \omega(\lambda) \wedge d\psi = 0.$$

The curvature can be expanded into a series in powers of the spectral parameter λ as

$$\Omega(\lambda) = \sum_{i=0}^{m} d\omega_i \lambda^i - \sum_{i=0}^{m} \omega_i \lambda^i \wedge \omega(\lambda) = 0. \tag{4.33}$$

Replacing λ by the spectral parameters $\lambda_1, \dots, \lambda_n$ one after the other yields the system of equations,

$$\begin{array}{ccccc} \sum_{i=0}^{m} d\omega_i \lambda_1^i & = & \sum_{i=0}^{m} \omega_i \wedge \omega(\lambda_1)\lambda_1^i & = & \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_1^{i+j}, \\ \vdots & & \vdots & & \vdots \\ \sum_{i=0}^{m} d\omega_i \lambda_n^i & = & \sum_{i=0}^{m} \omega_i \wedge \omega(\lambda_n)\lambda_n^i & = & \sum_{i,j=0}^{m} \omega_i \wedge \omega_j \lambda_n^{i+j}. \end{array}$$

Multiply the j-th equation in this set by ψ_j, so if B represents the $n \times n$ matrix defined above, this system of equations can be written in the following way,

$$\sum_{i=0}^{m} d\omega_i B \Lambda^i = \sum_{i,j=1}^{m} \omega_i \wedge \omega_j B \Lambda^{i+j}. \tag{4.34}$$

Finally, multiply (4.34) from the right on both sides by B^{-1} and use the definition of S in (4.27), and the result follows. □

Theorem 10. Matrix S in (4.27) satisfies the relation

$$d\omega(S) - \omega(S) \wedge \omega(S) = 0.$$

Proof: Differentiating (4.13) and using Theorem 8 to replace dS^i, we obtain

$$d\omega(S) = d(\sum_{i=0}^{m} \omega_i S^i) = \sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge dS^i = \sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge (\omega(S) S^i - S^i \omega(S))$$

$$= \sum_{i=0}^{m} \omega_i \wedge S^i \omega(S) + [\sum_{i=0}^{m} d\omega_i S^i - \sum_{i=0}^{m} \omega_i \wedge \omega(S) S^i] = \omega(S) \wedge \omega(S).$$

In the second line, (4.32) has been used to eliminate the second term in the square brackets to finish the proof. □

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