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Research article

Certain subclass of analytic functions related with conic domains and associated with Salagean q-differential operator

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Abstract: In our present investigation, by using Salagean q-differential operator we introduce and define new subclass $k - \mathcal{US}(q, \gamma, m)$, $\gamma \in C\backslash\{0\}$, and studied certain subclass of analytic functions in conic domains. We investigate the number of useful properties of this class such structural formula and coefficient estimates Fekete–Szego problem, we give some subordination results, and some other corollaries.

Keywords: analytic functions; subordination; conic domain; Salagean q-differential operator
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1. Introduction

Let $\mathcal{A}$ denotes the class of all function $f(z)$ which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f^{'}(0) = 1$, so each $f \in A$ has the Maclaurin's series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f : E \to \mathbb{C}$ is called univalent on E if $f(z_1) = f(z_2)$ for all $z_1 = z_2$, $z_1, z_2 \in E$. Let $\mathcal{S} \subset A$ be the class of all functions which are univalent in E (see [3]). Recall $D \subset \mathbb{C}$ is said to be a starlike with respect to the point $d_0 \in D$ if and only if the line segment joining d_0 to every other point $d \in D$ lies entirely in D, while the set D is said to be convex if and only if it is starlike with respect to each of its points. By S^* and K we means the subclasses of S composed of starlike and convex functions. A function $f \in A$ is said to be starlike of order α, $0 \leq \alpha < 1$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E.$$

A function $f \in A$ is said to be convex of order α, $0 \leq \alpha < 1$, if

$$\Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > \alpha, z \in E.$$

In 1991, Goodman [4] introduced the class $\mathcal{UCV}$ of uniformly convex functions which was extensively studied by Ronning and independently by Ma and Minda [1, 2]. A more convenient characterization of class $\mathcal{UCV}$ was given by Ma and Minda as:

$$f(z) \in \mathcal{UCV} \Longleftrightarrow f(z) \in \mathcal{A} \text{ and } \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in E.$$

In 1999, Kanas and Wisniowska [5, 6] introduced the class $k-$uniformly convex functions, $k \geq 0$, denoted by $k - \mathcal{UCV}$ and a related class $k - \mathcal{ST}$ as:

$$f \in k - \mathcal{UCV} \Longleftrightarrow zf' \in k - \mathcal{ST} \Longleftrightarrow f \in A \text{ and } \Re\left\{\frac{\left(zf'(z)\right)'}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \; z \in E.$$

The class $k - \mathcal{UCV}$ was discussed earlier in [7], see also [8] with same extra restriction and without geometrical interpretation by Bharati et.al [8]. In 1985, Nasr et al., studied a natural extension of classical starlikness in order terminology. We say that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*_{k,\gamma}$, $k \geq 0$, $\gamma \in \mathbb{C}\backslash\{0\}$, if and only if

$$\Re\left\{\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right|, \quad z \in E.$$

Several author investigated the properties of the class, $\mathcal{S}^*_{k,\gamma}$ and their generalizations in several directions for detail study see [4, 6, 9, 10, 11, 12, 13]. The convolution or Hadamard product of two function f and g is denoted by $f * g$ is defined as

$$(f * g)z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

where $f(z)$ is given by (1.1) and $g(z) = \sum_{n=2}^{\infty} b_n z^n$, $\quad (z \in E)$.
If $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then we have the following equivalence, see [3, 14].

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E). \quad z \in E.$$

Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum theory, number theory, statistical mechanics, etc. At the beginning of the last century studies on q-difference equations appeared in intensive works especially by Jackson [33], Carmichael [32], Mason [34], Adams [31] and Trjitzinsky [35]. Research work in connection with function theory and q-theory together was first introduced by Ismail et al. [36]. Till now only non-significant interest in this area was shown although it deserves more attention.
Many differential and integral operators can be written in term of convolution, for details we refer [21]. It is worth mentioning that the technique of convolution helps researchers in further investigation of geometric properties of analytic functions.
For any non-negative integer n, the q-integer number n denoted by $[n]_q$, is defined by

$$[n]_q = \frac{1-q^n}{1-q}, \ \ [0]_q = 0.$$

For non-negative integer n the q-number shift factorial is defined by

$$[n]_q! = [1]_q[2]_q[3]_q...[n]_q, \quad \left([0]_q! = 1\right).$$

We note that when $q \to 1$, $[n]!$ reduces to classical definition of factorial. In general, for a non-integer number t, $[t]_q$ is defined by $[t]_q = \frac{1-q^t}{1-q}$, $[0]_q = 0$. Throughout in this paper, we will assume q to be a fixed number between 0 and 1
The q-difference operator related to the q-calculus was introduced by Andrews et al. (see in [30] CH 10). For $f \in A$, the q-derivative operator or q-difference operator is defined as.

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in E, z \neq, q \neq 1.$$

It can easily be seen that for $n \in N = \{1, 2, 3, ...\}$ and $z \in E$.

$$\partial_q z^n = [n]_q z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

Recently, Govindaraj and Sivasubramanian defined Salagean q-differential operator [28] as:
Let $f \in A$, let Salagean q-differential operator

$$S_q^0 f(z) = f(z), \ \ S_q^1 f(z) = z\partial_q f(z), \qquad ,S_q^m f(z) = z\partial_q \left(S_q^{m-1} f(z)\right).$$

A simple calculation implies

$$S_q^m f(z) = f(z) * G_{q,m}(z) \tag{1.2}$$

$$G_{q,m}(z) = z + \sum_{n=2}^{\infty} [n]_q^m z^n, \tag{1.3}$$

Making use of (1.2) and (1.3), the power series of $S_q^m f(z)$ for f of the form (1.1) is given by

$$S_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \tag{1.4}$$

Note that

$$Lim_{q\to 1}G_{q,m}(z) = z + \sum_{n=2}^{\infty} n^m z^n$$

$$Lim_{q\to 1}S_q^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n$$

which is the familiar Salagean derivative [29].
Taking motivation from the work shahid et.al [23], we introduce new subclass $k-\mathcal{US}(q,\gamma,m)$, of analytic functions with the theory of q-calculus by using Salagean q-differential operator.

Definition 1.1. Let $f(z)\in\mathcal{A}$. Then $f(z)$ is in the class $k-\mathcal{US}(q,\gamma,m)$, $\gamma\in C\backslash\{0\}$, if it satisfies the condition

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)}-1\right)\right\} > k\left|\frac{1}{\gamma}\left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)}-1\right)\right|, \quad z\in E.$$

By taking specific values of parameters, we obtain many important subclasses studied by various authors in earlier papers. Here we inlist some of them.
(1) For $m=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathcal{US}(q,\gamma,m)$ reduce into the class $\mathcal{SD}(k,\beta)$ studied by Shams et.al [24].
(2) For $m=0$, $q\to 1$, and $\gamma=\frac{2}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathcal{US}(q,\gamma,m)$ reduces into the class $\mathcal{KD}(k,\beta)$, studied by Owa et.al [26].
(3) For $k=1$, $m=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathcal{US}(q,\gamma,m)$ reduce into the class $\mathcal{S}_p(\beta)$ studied by Ali et.al [27].
(4) For $k=1$, $m=0$, $q\to 1$, and $\gamma=\frac{2}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathcal{US}(q,\gamma,m)$ reduces into the class $\mathcal{K}_p(\beta)$, studied by Ali et.al [27].
(5) For $m=0$, $q\to 1$, the class $k-\mathcal{US}(q,\gamma,m)$ reduce into the class $\mathcal{K}-\mathcal{ST}$, introduced by Kanas and Wisniowska [5].
(6) For $k=0$, $m=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathcal{US}(q,\gamma,m)$ reduce into the class $\mathcal{S}^*(\beta)$, well-known class of starlike of order respectively.

Geometric Interpretation

A function $f(z)\in\mathcal{A}$ is in the class $k-\mathcal{US}(q,\gamma,m)$ if and only if $\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)}$ takes all the values in the conic domain $\Omega_{k,\gamma}=p_{k,\gamma}(E)$, such that

$$\Omega_{k,\gamma}=\gamma\Omega_k+(1-\alpha),$$

where

$$\Omega_k=\left\{u+iv : u>k\sqrt{(u-1)^2+v^2}\right\}.$$

Since $p_{k,\gamma}(z)$ is convex univalent, so above definition can be written as

$$\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} \prec p_{k,\gamma}(z), \tag{1.5}$$

where

$$p_{k,\gamma}(z) = \begin{cases} \frac{1+z}{1-z}, & \text{for } k = 0, \\ 1 + \frac{2\gamma}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{for } k = 1, \\ 1 + \frac{2\gamma}{1-k^2} \sinh^2 \left\{\left(\frac{2}{\pi} \arccos k\right) \arctan h \sqrt{z}\right\}, & \text{for } 0 < k < 1, \\ 1 + \frac{\gamma}{k^2-1} \sin\left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx\right) + \frac{\gamma}{1-k^2}, & \text{for } k > 1. \end{cases} \tag{1.6}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes the imaginary axis when $k = 0$, while a hyperbola when $0 < k < 1$. For $k = 1$ the boundary $\partial\Omega_{k,\gamma}$ becomes a parabola and it is an ellipse when $k > 1$ and in this case where

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{tz}}, \quad z \in E,$$

and $t \in (0, 1)$ is chosen such that $k = \cosh(\pi K'(t)/(4K(t)))$. Here $K(t)$ is Legender's complete elliptic integral of first kind and $K'(t) = K(\sqrt{1-t^2})$ and $K'(t)$ is the complementary integral of $K(t)$ for details see [5, 6, 14, 17]. Moreover, $p_{k,\gamma}(E)$ is convex univalent in E, see [5, 6]. All of these curves have the vertex at the point $\frac{k+\gamma}{k+1}$.

2. Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 2.1. *[18]. Let $p(z) = \sum_{n=1}^{\infty} p_n z^n \prec F(z) = \sum_{n=1}^{\infty} d_n z^n$ in E. If $F(z)$ is convex univalent in E then*

$$|p_n| \le |d_1|, \quad n \ge 1. \tag{2.1}$$

Lemma 2.2. *[19]. Let $k \in [0, \infty)$ be fixed and let $p_{k,\gamma}$ be defined (1.6). If*

$$p_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + \ldots \tag{2.2}$$

$$Q_1 = \begin{cases} \frac{2\gamma A^2}{1-k^2}, & 0 \le k < 1 \\ \frac{8\gamma}{\pi^2}, & k = 1, \\ \frac{\pi^2\gamma}{4(1+t)\sqrt{t}K^2(t)(k^2-1)}, & k > 1, \end{cases} \tag{2.3}$$

$$Q_2 = \begin{cases} \frac{A^2+2}{3} Q_1, & 0 \le k < 1 \\ \frac{2}{3} Q_1, & k = 1, \\ \frac{4K^2(t)(t^2+6t+1)-\pi^2}{24K^2(t)(1+t)\sqrt{t}} Q_1, & k > 1, \end{cases} \tag{2.4}$$

where $A = \frac{2\cos^{-1} k}{\pi}$,and $t \in (0, 1)$ is chosen such that $k = \cosh\left(\frac{\pi K'(t)}{K(t)}\right)$, $K(t)$ is the Legendre's complete elliptic integral of the first kind.

Lemma 2.3. *[20]. Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, let $p(z)$ be analytic in E and satisfy $Re\{p(z)\} > 0$ for z in E, then the following sharp estimate holds*

$$\left|c_2 - \mu c_1^2\right| \le 2 \max\{1, |2\mu - 1|\}, \quad \forall \mu \in \mathbb{C}. \tag{2.5}$$

3. Main Results

In this section, we will prove our main results.

Theorem 3.1. *Let $f(z) \in k - \mathcal{US}(q, \gamma, m)$. Then*

$$S_q^m f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(w(\xi)) - 1}{\zeta} d\xi, \tag{3.1}$$

where $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Moreover, for $|z| = \rho$, we have

$$\exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho\right) \leq \left|\frac{S_q^m f(z)}{z}\right| \leq \exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho\right), \tag{3.2}$$

where $p_{k,\gamma}(z)$ is defined by (1.6).

Proof. If $f(z) \in k - \mathcal{US}(q, \gamma, m)$ then using the identity (1.5), we obtain

$$\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - \frac{1}{z} = \frac{p_{k,\gamma}(w(z)) - 1}{z}. \tag{3.3}$$

For some function $w(z)$ is analytic in E with $w(0) = 0$ and $|w(z)| < 1$. Integrating (3.3) and after some simplification we have

$$S_q^m f(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(w(\xi)) - 1}{\zeta} d\xi. \tag{3.4}$$

This proves (3.1). Noting that the univalent function $p_{k,\gamma}(z)$ maps the disk $|z| < \rho$ $(0 < \rho \leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we see

$$p_{k,\gamma}(-\rho\,|z|) \leq \Re\left\{p_{k,\gamma}(w(\rho z)\right\} \leq p_{k,\gamma}(\rho\,|z|) \quad (0 < \rho \leq 1, \quad z \in E). \tag{3.5}$$

Using (3.4) and (3.5) gives

$$\int_0^1 \frac{p_{k,\gamma}(-\rho\,|z|) - 1}{\rho} d\rho \leq \Re \int_0^1 \frac{p_{k,\gamma}(w\,(\rho\,(z)) - 1}{\rho} d\rho \leq \int_0^1 \frac{p_{k,\gamma}(\rho\,|z|) - 1}{\rho} d\rho,$$

for $z \in E$. Consequently, subordination (3.4) leads us to

$$\int_0^1 \frac{p_{k,\gamma}(-\rho\,|z|) - 1}{\rho} d\rho \leq \log\left|\frac{S_q^m f(z)}{z}\right| \leq \int_0^1 \frac{p_{k,\gamma}(\rho\,|z|) - 1}{\rho} d\rho$$

$$p_{k,\gamma}(-\rho) \leq p_{k,\gamma}(-\rho\,|z|),\ p_{k,\gamma}(\rho\,|z|) \leq p_{k,\gamma}(\rho)$$

implies that

$$\exp \int_0^1 \frac{p_{k,\gamma}(-\rho) - 1}{\rho} d\rho \leq \left|\frac{S_q^m f(z)}{z}\right| \leq \exp \int_0^1 \frac{p_{k,\gamma}(\rho) - 1}{\rho} d\rho.$$

this completes the proof. □

Theorem 3.2. *If $f(z) \in k - \mathcal{US}(q, \gamma, m)$. Then*

$$|a_2| \leq \frac{\delta}{[2]_q^m \left\{[2]_q - 1\right\}}, \tag{3.6}$$

and

$$|a_n| \leq \frac{\delta}{[n]_q^m \left\{[n]_q - 1\right\}} \prod_{j=1}^{n-2} \left(1 + \frac{\delta}{[j+1]_q - 1}\right), \quad \textit{for } n = 3, 4, ... \tag{3.7}$$

where $\delta = |Q_1|$ with Q_1 is given by (2.3).

Proof. Let

$$\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} = p(z). \tag{3.8}$$

where $p(z)$ is analytic in E and $p(0) = 1$.Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $S_q^m f(z)$ is given by (1.4). Then (3.8) becomes

$$z + \sum_{n=2}^{\infty} [n]_q^{m+1} a_n z^n = \left(\sum_{n=0}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n\right).$$

Now comparing the coefficients of z^n, we obtain

$$[n]_q^{m+1} a_n = [n]_q^m a_n + \sum_{j=1}^{n-1} [j]_q^m a_j c_{n-j}.$$

which implies

$$a_n = \frac{1}{[n]_q^m \left\{[n]_q - 1\right\}} \sum_{j=1}^{n-1} [j]_q^m a_j h_{n-j}.$$

Using the results that $|c_n| \leq |Q_1|$ given in ([17]), we have

$$|a_n| \leq \frac{Q_1}{[n]_q^m \left\{[n]_q - 1\right\}} \sum_{j=1}^{n-1} [j]_q^m \left|a_j\right|.$$

Let us take $\delta = |Q_1|$. Then we have

$$|a_n| \leq \frac{\delta}{[n]_q^m \left\{[n]_q - 1\right\}} \sum_{j=1}^{n-1} [j]_q^m \left|a_j\right|. \tag{3.9}$$

For $n = 2$ in (3.9), we have

$$|a_2| \leq \frac{\delta}{[2]_q^m \left\{[2]_q - 1\right\}}, \tag{3.10}$$

which shows that (3.7) holds for $n = 2$. To prove (3.7) we use principle of mathematical induction, for this, consider the case $n = 3$

$$|a_3| \leq \frac{\delta}{[3]_q^m \left\{[3]_q - 1\right\}} \{1 + [2]_q^m |a_2|\}.$$

Using (3.10), we have

$$|a_3| \leq \frac{\delta}{[3]_q^m \left\{[3]_q - 1\right\}}\{1 + \frac{\delta}{[2]_q - 1}\}.$$

which shows that (3.7) holds for $n = 3$. Let us assume that (3.7) is true for $n \leq t$, that is,

$$|a_t| \leq \frac{\delta}{[t]_q^m \left\{[t]_q - 1\right\}} \prod_{j=1}^{t-2} \left(1 + \frac{\delta}{[j+1]_q - 1}\right), \qquad \text{for } n = 3, 4, ...$$

consider

$$\begin{aligned}
|a_{t+1}| &\leq \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\}} \left\{1 + [2]_q^m |a_2| + [3]_q^m |a_3| + [4]_q^m |a_4| + ...[t]_q^m |a_t|\right\} \\
&\leq \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\}} \left\{ \begin{array}{c} 1 + \frac{\delta}{[2]_q - 1} + \frac{\delta}{[3]_q - 1}\left(1 + \frac{\delta}{[2]_q - 1}\right) + ... \\ + \frac{\delta}{[t]_q - 1} \prod_{j=1}^{t-2}\left(1 + \frac{\delta}{[j+1]_q - 1}\right) \end{array} \right\} \\
&= \frac{\delta}{[t+1]_q^m \left\{[t+1]_q - 1\right\}} \prod_{j=1}^{t-1} \left(1 + \frac{\delta}{[j+1]_q - 1}\right).
\end{aligned}$$

which proves the assertion of theorem $n = t + 1$. Hence (3.7) holds for all n, $n \geq 3$.
This completes the proof.

□

Theorem 3.3. *Let $0 \leq k < \infty$ be fixed and let $f(z) \in k - \mathcal{US}(q, \gamma, m)$ with the form (1.1) then for a complex number μ*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{d_1}{2\,[3]_q^m \left\{[3]_q - 1\right\}} \max\left[1, |2v - 1|\right], \tag{3.11}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{d_2}{d_1} - d_1\left(\frac{1}{\left\{[2]_q - 1\right\}} - \mu\frac{[3]_q^m \left\{[3]_q - 1\right\}}{2\,[2]_q^m \left\{[2]_q - 1\right\}}\right)\right\}. \tag{3.12}$$

Q_1 and Q_2 are given by (2.3) and (2.4).

Proof. Let $f(z) \in k{-}\mathcal{US}(q, \gamma, m)$, then there exists Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} = p_{k,\gamma}(w(z)) \quad z \in E. \tag{3.13}$$

Let $p(z) \in \mathcal{P}$ be a function defined as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ...$$

This gives

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}(c_2 - \frac{c_1^2}{2})z^2 + ...$$

and

$$p_{k,\gamma}(w(z)) = 1 + \frac{Q_1 c_1}{2} z + \left\{ \frac{Q_2 c_1^2}{4} + \frac{1}{2}(c_2 - \frac{c_1^2}{2}) Q_1 \right\} z^2 + ... \tag{3.14}$$

$$\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} = 1 + [2]_q^m \left\{ [2]_q - 1 \right\} a_2 z + \left\{ [3]_q^m \left\{ [3]_q - 1 \right\} a_3 - \left([2]_q^m \right)^2 \left\{ [2]_q - 1 \right\} a_2^2 \right\} z^2 \tag{3.15}$$

Using (3.14) in (3.13) and coparing with (3.15), we obtain

$$a_2 = \frac{Q_1 c_1}{2\, [2]_q^m \left\{ [2]_q - 1 \right\}}.$$

and

$$a_3 = \frac{1}{[3]_q^m \left\{ [3]_q - 1 \right\}} \left\{ \frac{Q_1 c_2}{2} + \frac{c_1^2}{4} \left(Q_2 - Q_1 + \frac{Q_1^2}{\left\{ [2]_q - 1 \right\}} \right) \right\}.$$

For any complex number μ and after some calculation we have

$$a_3 - \mu a_2^2 = \frac{Q_1}{2\, [3]_q^m \left\{ [3]_q - 1 \right\}} \left\{ c_2 - v c_1^2 \right\}, \tag{3.16}$$

where

$$v = \frac{1}{2} \left\{ 1 - \frac{Q_2}{Q_1} - Q_1 \left(\frac{1}{\left\{ [2]_q - 1 \right\}} - \mu \frac{[3]_q^m \left\{ [3]_q - 1 \right\}}{2\, [2]_q^m \left\{ [2]_q - 1 \right\}} \right) \right\}.$$

Using a lemm(2.5) on (3.16) we have the required results.

□

Theorem 3.4. *If a function $f(z) \in \mathcal{A}$ has the form (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} \left\{ \left\{ [n]_q - 1 \right\} (k+1) + |\gamma| \right\} \left| [n]_q^m \right| |a_n| \leq |\gamma| \tag{3.17}$$

then $f(z) \in k - \mathcal{US}(q, \gamma, m)$.

Proof. Let we note that

$$\begin{aligned} \left| \frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right| &= \left| \frac{z\partial_q S_q^m f(z) - S_q^m f(z)}{S_q^m f(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} [n]_q^m \left\{ [n]_q - 1 \right\} a_n z^n}{z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} \left| [n]_q^m \left\{ [n]_q - 1 \right\} \right| |a_n|}{1 - \sum_{n=2}^{\infty} \left| [n]_q^m \right| |a_n|}. \end{aligned} \tag{3.18}$$

From (3.17) it follows that

$$1 - \sum_{n=2}^{\infty} \left| [n]_q^m \right| |a_n| > 0.$$

To show that $f(z) \in k - \mathcal{US}(q, \gamma, m)$ it is suffies that

$$\left| \frac{k}{\gamma} \left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right) \right| - \Re \left\{ \frac{1}{\gamma} \left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right) \right\} \leq 1.$$

From (3.18), we have

$$\begin{aligned}
& \left| \frac{k}{\gamma} \left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right) \right| - \Re \left\{ \frac{1}{\gamma} \left(\frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right) \right\} \\
\leq\ & \frac{k}{|\gamma|} \left| \frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right| + \frac{1}{|\gamma|} \left| \frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right| \\
\leq\ & \frac{(k+1)}{|\gamma|} \left| \frac{z\partial_q S_q^m f(z)}{S_q^m f(z)} - 1 \right| = \left| \frac{z\partial_q S_q^m f(z) - S_q^m f(z)}{S_q^m f(z)} \right| \\
\leq\ & \frac{(k+1)}{|\gamma|} \frac{\sum_{n=2}^{\infty} \left| [n]_q^m \left\{ [n]_q - 1 \right\} \right| |a_n|}{1 - \sum_{n=2}^{\infty} \left| [n]_q^m \right| |a_n|} \\
\leq\ & 1.
\end{aligned}$$

Because from (3.8).

□

When $q \to 1$, $m = 0$, $\gamma = 1 - \alpha$, with $0 \leq \alpha < 1$, then we have the following known result, proved by Shams et-al. in [24].

Corollary 3.1. *A function $f \in A$ and of the form (1.1) is in the class $k - \mathcal{US}\,(1 - 2\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\alpha)\} |a_n| \leq 1 - \alpha$$

where $0 \leq \alpha < 1$ and $k \geq 0$.
When $q \to 1$, $m = 0$, $\gamma = 1 - \alpha$, with $0 \leq \alpha < 1$ and $k = 0$, then we have the following known result, proved by Selverman in [25]

Corollary 3.2. *A function $f \in A$ and of the form (1.1) is in the class $0 - \mathcal{US}\,(1 - \alpha,)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n - \alpha\} |a_n| \leq 1 - \alpha, \qquad 0 \leq \alpha < 1.$$

Theorem 3.5. *Let $f(z) \in k - \mathcal{US}(q, \gamma, m)$. Then $f(E)$ contains an open disk of radius*

$$\frac{[2]^m \left\{ [2]_q - 1 \right\}}{2[2]_q^m \left\{ [2]_q - 1 \right\} + \delta}.$$

where Q_1 is given by (2.3)

Proof. Let $w_0 \neq 0$ be a complex number such that $f(z) \neq w_0$ for $z \in E$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + ...$$

since $f_1(z)$ is univalent, so

$$\left|a_2 + \frac{1}{w_0}\right| \leq 2.$$

know using (3.6), we have

$$\left|\frac{1}{w_0}\right| \leq \frac{2[2]_q^m \left\{[2]_q - 1\right\} + \delta}{[2]_q^m \left\{[2]_q - 1\right\}},$$

hence we have.

$$|w_0| \geq \frac{[2]_q^m \left\{[2]_q - 1\right\}}{2[2]_q^m \left\{[2]_q - 1\right\} + \delta}.$$

□

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Conflict of Interest

No potential conflict of interest was reported by the authors.

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