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Research article

The Jordan decomposition of bounded variation functions valued in vector spaces

Francisco J. Mendoza-Torres*, **Juan A. Escamilla-Reyna and Daniela Rodríguez-Tzompantzi**

Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, Mexico

* **Correspondence:** francisco.mendoza@correo.buap.mx

Abstract: In this paper we show the Jordan decomposition for bounded variation functions with values in Riesz spaces. Through an equivalence relation, we prove that this decomposition is satisfied for functions valued in Hilbert spaces. This result is a generalization of the real case. Moreover, we prove that, in general, the Jordan decomposition is not satisfied for vector-valued functions.

Keywords: Jordan decomposition; bounded variation function; Hilbert spaces; Riesz spaces; normed spaces
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1. Introduction

The concept of bounded variation function was introduced in 1881 by Camille Jordan [1] for real functions defined in a closed interval $I = [a, b] \subset \mathbb{R}$. He proves that a function is of bounded variation if and only if it can be represented as the difference of two increasing functions. This representation is known as the *Jordan decomposition*.

Because of the existence of several kinds of functions, mainly due to variations of domain and codomain, it has been necessary to define different types of bounded variation. We can mention Vitali, Hardy, Arzela, Pierpont, Frechet, and Tonelli who give different definitions of bounded variation for real functions of two variables. C. R. Adams [2, 3] studied the relation between the concepts defined by the previous authors.

For bounded variation functions $f : [a, b] \to X$, where X is a metric space, V. V. Chistyakov studies many aspects around those functions [4, 5, 6, 7]. In the first reference, he proves an alternative result to the Jordan decomposition, affirming that for bounded variation functions valued in metric spaces the decomposition as the difference of two monotone functions is inapplicable. On the other hand, Bianchini and Tonon [8] assert that there is no hope for a further generalization of this decomposition to vector valued BV functions, apart from the case of a function $f : \mathbb{R} \to \mathbb{R}^m$ where the analysis is straightforward.

Defining the bounded variation with respect to the order in the first part of this paper we show that the Jordan decomposition is possible for functions valued in Riesz spaces. Additionally, as an alternative to affirmations of Chistyakov and Bianchini-Tonon, we prove that for functions valued in Hilbert spaces, proposition 2.10, the Jordan decomposition is satisfied in a generalized sense from an equivalence relation, being the decomposition for real-valued functions a particular case. This result allows us to give a negative answer to the Jordan decomposition problem of a bounded variation function $f : I \to (\mathcal{H}, \mathcal{H}_+)$, where the Hilbert space $\mathcal{H}$ is ordered by a given extensible cone $\mathcal{H}_+$.

1.1. Preliminaries

There are vector spaces in which is possible to define a natural order relation, for instance for continuous real functions defined on a compact interval $[a, b]$, denoted by $C([a, b], \mathbb{R})$. In this case: $f \leq g$, if $f(t) \leq g(t)$; for all $t \in [a, b]$. Nevertheless, there are some vector spaces where a natural order relation cannot be defined. This has led to creating mechanisms that permit comparison vectors associated with the order.

We listed some concepts that will be useful in our exposition and that are linked to order in vector spaces. The notation $[a, b]$ always will be reserved for compact intervals in $\mathbb{R}$.

- Let X be a partially ordered set. We say that X is a *lattice* if every subset consisting of two points has a supremum and an infimum.
- A vector space X is called *ordered* if it is partially ordered in such a manner that the structure of vector space and the order structure are compatible, that is to say:

i) $x \leq y$ implies $x + z \leq y + z$, for every $z \in X$,

ii) $x \geq 0$ implies $\alpha x \geq 0$, for every $\alpha \geq 0$ in $\mathbb{R}$.

If, in addition, X is a lattice with respect to the partial order, then X is called a *Riesz space.*

- Let X be a Riesz space. A function $f : [a, b] \to X$ is *bounded above* if there exists $M \in X$ such that $f(t) \leq M$, for all $t \in [a, b]$. f is *bounded below* if there exists $m \in X$ such that $m \leq f(t)$, for all $t \in [a, b]$. We say that f is *bounded with respect to the order* if it is at the same time bounded above and bounded below.
- Let X be a Riesz space. $f : [a, b] \to X$ is an *increasing (decreasing) function* if $f(t_1) \leq f(t_2)$ ($f(t_1) \geq f(t_2)$), when $t_1 \leq t_2$.
- Let X be a normed space. X_+ a closed subset of X is called a *cone* if $X_+ + X_+ \subseteq X_+$, $X_+ \bigcap(-X_+) = \{0\}$ and $cX_+ \subseteq X_+$, for all $c \geq 0$. The order relation $\leq$ defined by
$$x \leq y \text{ if and only if } y - x \in X_+$$
is an order partial in X. The pair (X, X_+) is called *ordered normed space*.
- Let X be an ordered normed space with a cone X_+. We say that $f : [a, b] \to X$ is an *increasing (decreasing) function* if $f(t_2) - f(t_1) \in X_+$ $(f(t_1) - f(t_2) \in X_+)$, provided that $t_1 \leq t_2$.
- Assume that X_+ is a cone in X. If there exists a cone X_1 in X and $b > 0$ such that for any $x \in X_+$: $B(x, b\|x\|) \subset X_1$, then X_+ is called an *extensible cone*.
The following characterization of extensible cones is useful for our purposes.

Theorem 1.1. *[9] Assume that X_+ is a cone in X. Then X_+ is extensible if and only if there exists $g \in X^*$ and a constant $\alpha > 0$ such that $g(x) \geq \alpha\|x\|$, for all $x \in X_+$.*

A *partition* of $[a, b]$ is a finite ordered set of points in $[a, b]$:

$$a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b,$$

which determine subintervals $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, such that $\bigcup_i [t_{i-1}, t_i] = [a, b]$. The set of partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 1.2. *Let X be a normed space. We say that $f : [a, b] \to X$ is of bounded variation on $[a, b]$ if*

$$\sup\{\sum_P \|f(t_i) - f(t_{i-1})\|_X : P \in \mathcal{P}[a, b]\} \in \mathbb{R}^+ \cup \{0\}\,. \tag{1}$$

The expression (1) is the variation *of f on $[a, b]$ and it is denoted by $V_a^b(f, X)$.*

The set of bounded variation functions defined on $[a, b]$, with values in X, is denoted by $BV([a, b], X)$. For $X = \mathbb{R}$, we will use the notation $BV([a, b])$. For $t \in [a, b]$, $V_a^t(f, X)$ will be the *variation function*.

Remark 1.3. *If $f(t) = (f_1(t), f_2(t), ..., f_m(t))$, where each $f_j \in BV([a, b])$, then $f \in BV([a, b], \mathbb{R}^m)$ and the next inequality is satisfied*

$$\sqrt{\sum_{j=1}^{m} V_a^b(f_j, \mathbb{R})^2} \leq V_a^b(f, \mathbb{R}^m) \leq \sum_{j=1}^{m} V_a^b(f_j, \mathbb{R}). \tag{2}$$

This can be seen examining, for instance, the case $m = 2$. Let $f(t) = (f_1(t), f_2(t))$ and $P = \{a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b\}$ any partition of $[a, b]$. By the inequality $\sqrt{\alpha_1 + \alpha_2} \leq \sqrt{\alpha_1} + \sqrt{\alpha_2}$, where $\alpha_i, \alpha_2 \geq 0$; we have

$$\begin{aligned}
\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| &= \sum_{i=1}^{n} \sqrt{(f_1(t_i) - f_1(t_{i-1}))^2 + (f_2(t_i) - f_2(t_{i-1}))^2} \\
&\leq \sum_{i=1}^{n} |f_1(t_i) - f_1(t_{i-1})| + \sum_{i=1}^{n} |f_2(t_i) - f_2(t_{i-1})| \\
&\leq V_a^b(f_1, \mathbb{R}) + V_a^b(f_2, \mathbb{R}).
\end{aligned}$$

Thus, $f \in BV([a, b], \mathbb{R}^2)$ and

$$V_a^b(f, \mathbb{R}^2) \leq V_a^b(f_1, \mathbb{R}) + V_a^b(f_2, \mathbb{R}).$$

By the inequality

$$\sqrt{\left(\sum_{i=1}^{n} \alpha_{1i}\right)^2 + \left(\sum_{i=1}^{n} \alpha_{2i}\right)^2} \leq \sum_{i=1}^{n} \sqrt{\alpha_{1i}^2 + \alpha_{2i}^2},$$

where $\alpha_{ji} \geq 0$ *for* $i = 1, 2, ..., n$; $j = 1, 2$; *and taking* $\alpha_{ji} = \left|f_j(t_i) - f_j(t_{i-1})\right|$, *we have*

$$\begin{aligned}
&\sqrt{\left(\sum_{i=1}^{n} |f_1(t_i) - f_1(t_{i-1})|\right)^2 + \left(\sum_{i=1}^{n} |f_2(t_i) - f_2(t_{i-1})|\right)^2} \\
\leq\; & \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| \\
\leq\; & V_a^b(f, \mathbb{R}^2).
\end{aligned}$$

Evaluating the supremum over all the partitions in $[a, b]$, *we obtain the left-hand inequality of (2). This inequality may be strict. For example, let* $f(t) = (f_1(t), f_2(t)) = \left(t^2, t\right)$, $t \in [-1, 1]$. *We have that*

$$V_{-1}^1(f_1, \mathbb{R}) = 2\int_{-1}^{1} |t|\, dt = 2, \quad V_{-1}^1(f_2, \mathbb{R}) = 2$$

and

$$\sqrt{V_{-1}^1(f_1, \mathbb{R})^2 + V_{-1}^1(f_2, \mathbb{R})^2} = 2\sqrt{2}.$$

The affirmation is proved observing that for the partition $P_0 = \{-1, 0, 1/2, 1\}$:

$$\sum_{i=1}^{3} \sqrt{(f_1(t_i) - f_1(t_{i-1}))^2 + (f_2(t_i) - f_2(t_{i-1}))^2} = \sqrt{2} + \frac{\sqrt{5} + \sqrt{13}}{4} > 2\sqrt{2}.$$

Since $V_a^b(f, \mathbb{R}^m)$ *is a scalar, then one could think that it must be equal to* $\left\|\left(V_a^b(f_1, \mathbb{R}), V_a^b(f_2, \mathbb{R}), ..., V_a^b(f_{m-1}, \mathbb{R}), V_a^b(f_m, \mathbb{R})\right)\right\|$. *The above example shows that this fact may not be possible.*

Definition 1.4. *Let* X *be a normed space and let* X^* *be its dual space.* $f : [a, b] \to X$ *is of weakly bounded variation if for every* $\varphi \in X^*$, *the function* $\varphi(f)$ *belongs to* $BV([a, b])$.

We know that if X is a normed space and $\varphi \in X^*$, then $\|\varphi(x)\| \leq \|\varphi\|\|x\|_X$. Therefore, we can observe that the following result is satisfied.

Lemma 1.5. *If* $f : [a, b] \to X$ *is a bounded variation function, then it is of weakly bounded variation.*

The converse of this lemma is not true, see [[10], Example 7.1.8].

Theorem 1.6. *Let* X *be an ordered normed space with an extensible cone* X_+. *Then every monotone (increasing or decreasing) function is of bounded variation.*

Proof. We prove the case when f is an increasing function, the proof for decreasing functions is similar. Let $\{[t_{k-1}, t_k] : k = 1, \ldots, n\}$ be a partition of $[a, b]$. Since X_+ is an extensible cone, then, by theorem 1.1, there exists $g \in X^*$ and a constant $\alpha > 0$ such that $\|f(t_k) - f(t_{k-1})\| \leq \alpha g(f(t_k) - f(t_{k-1}))$, $k = 1, \ldots, n$. Therefore we conclude the proof with the following inequalities.

$$\begin{aligned}
\sum_{k=1}^{n} \|f(t_k) - f(t_{k-1})\|_X &\leq \alpha g\left(\sum_{k=1}^{n}(f(t_k) - f(t_{k-1}))\right) \\
&= \alpha g(f(b) - f(a)) \leq \alpha\|g\|\|f(b) - f(a)\|_X.
\end{aligned}$$

□

As a consequence of the previous theorem we have that the difference of two increasing functions is of bounded variation. In subsection 2.2 we will see that the reciprocal of this result may not be true.

Supposing that "$\leq$" is a natural order in the normed space X, then $X_+ = \{x \in X : 0 \leq x\}$ will be a cone. In this case we have that a monotone function in the ordered normed space (X, X_+) may not be of bounded variation. This fact contrasts with theorem 1.6. In the following example we use the function defined in [[10], Example 7.1.8].

Example 1.7. *Let F be a function from $[0,1]$ into $L_\infty[0,1]$ defined by*

$$F(t)(x) = \begin{cases} 1 \quad if \qquad 0 \leq x \leq t \\ \\ 0 \quad if \quad t < x \leq 1, \; 0 \leq t < 1. \end{cases}$$

$L_\infty[0,1]$ has a natural order in the following sense. If $h_1, h_2 \in L_\infty[0,1]$, then $h_1 \leq h_2$ if and only if $h_1(x) \leq h_2(x)$ a.e. on $[0,1]$. Suppose that $0 \leq t_1 < t_2 \leq 1$. If $x \in [0, t_1]$, then $F(t_1)(x) = F(t_2)(x) = 1$. If $x \in (t_1, t_2)$, then, since $\chi_{[0,t_1]} \leq \chi_{[0,t_2]}$, it follows that $F(t_1)(x) \leq F(t_2)(x)$. The case when $x \in [t_2, 1]$ is obvious.
For any $P = \{0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = 1\} \in \mathcal{P}[a,b]$, we have $\|F(t_k) - F(t_{k-1})\|_{L_\infty[0,1]} = 1$. Thus

$$\sum_{k=1}^{n} \|F(t_k) - F(t_{k-1})\|_{L_\infty[0,1]} = n.$$

We conclude that $F(t)$ is an increasing function in the natural sense but is not of bounded variation.

2. The Jordan decomposition of functions with values...

2.1. ...in Riesz spaces

In this subsection X will be a Riesz space. We denote

$$x^+ := \sup\{x, 0\}; \quad x^- := \sup\{-x, 0\} \quad \text{and} \quad |x|_o := x^+ + x^-.$$

The previous notation makes sense because a Riesz space is a lattice with respect to the partial order. We define the concept of bounded variation with respect to the order as follows.

Definition 2.1. *A function $f : [a,b] \to X$ is of bounded variation with respect to the order on $[a,b]$ if there exists a $M \in X$ such that*

$$V_f^o[a,b] := \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|_o \leq M,$$

for all partition $P \in \mathcal{P}([a,b])$.

By analogy with the case $X = \mathbb{R}$, it is not difficult to be convinced of the validity of the following results.

Theorem 2.2. *If $f : [a,b] \to X$ is monotone on $[a,b]$, then f is of bounded variation with respect to the order on $[a,b]$*

Theorem 2.3. *Let X be a Riesz space. If $f : [a,b] \to X$ is of bounded variation with respect to the order on $[a,b]$, then f is bounded above on $[a,b]$.*

Theorem 2.4. *Assume that $f, g : [a,b] \to X$ are two bounded variation functions with respect to the order on $[a,b]$. Then also the addition and difference of f and g are of bounded variation with respect to the order. Moreover, we have*

$$V^o_{f\pm g}\,[a,b] \leq V^o_f\,[a,b] + V^o_g\,[a,b]\,.$$

Theorem 2.5. *Let $f : [a,b] \to X$ be a bounded variation function with respect to the order on $[a,b]$, and assume that $c \in (a,b)$. Then f is of bounded variation with respect to the order on $[a,c]$ and on $[c,b]$. Moreover, we have*

$$V^o_f[a,b] = V^o_f[a,c] + V^o_f[c,b].$$

Theorem 2.6. *Let f be a bounded variation function with respect to the order on $[a,b]$. Let V^o be defined on $[a,b]$ as follows: $V^o(t) = V^o_f\,[a,t]$ if $a < t \leq b$, and $V^o(a) = 0$. Then:*

i) *V^o is an increasing function on $[a,b]$.*

ii) *$V^o - f$ is an increasing function on $[a,b]$.*

Proof. If $a < t_1 < t_2 \leq b$, we can write $V^o_f[a,t_2] = V^o_f[a,t_1] + V^o_f[t_1,t_2]$. This implies that $V^o(t_2) - V^o(t_1) = V^o_f[t_1,t_2] \geq 0$. Hence $V^o(t_1) \leq V^o(t_2)$ and *i*) holds. To prove *ii*), let $D(t) = V^o(t) - f(t)$ if $t \in [a,b]$. Then, if $a \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned} D(t_2) - D(t_1) &= V^o(t_2) - f(t_2) - (V^o(t_1) - f(t_1)) \\ &= V^o(t_2) - V^o(t_1) - [f(t_2) - f(t_1)] \\ &= V^o[t_1,t_2] - [f(t_2) - f(t_1)]. \end{aligned}$$

From the definition of $V^o_f[t_1,t_2]$, it follows that

$$f(t_2) - f(t_1) \leq V^o_f[t_1,t_2].$$

This means that $D(t_2) - D(t_1) \geq 0$, and *ii*) holds. □

Theorem 2.7. *Let $f : [a,b] \to X$ be. Then f is of bounded variation with respect to the order on $[a,b]$ if and only if f can be expressed as the difference of two increasing functions.*

Proof. If f is of bounded variation with respect to the order in $[a,b]$, we can write $f = V^o - D$, where V^o is the function of the previous theorem and $D = V^o - f$. Both V^o and D are increasing functions.

The converse is immediately deduced by theorems 2.2 and 2.4. □

Example 2.8. *Let $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2$ be given by $f(t) = (\cos t, t)$, where $\mathbb{R}^2$ is considered as a Riesz space with the order $(x_1,x_2) \leq (y_1,y_2)$, whenever $x_1 \leq y_1$ and $x_2 \leq y_2$. Let $P \in \mathcal{P}\,[a,b]$.*

If P contains to 0, then

$$|(\cos t_i - \cos t_{i-1}, t_i - t_{i-1})|_o \;=\; \sup\{(\cos t_i - \cos t_{i-1}, t_i - t_{i-1}), (0,0)\}$$

$$
\begin{aligned}
&+\sup\{(\cos t_{i-1}-\cos t_i, t_{i-1}-t_i),(0,0)\}\\
&= (g(i,t), t_i - t_{i-1})\\
&= (|\cos t_i - \cos t_{i-1}|, t_i - t_{i-1}),
\end{aligned}
$$

where

$$
g(i,t)=\begin{cases}\cos t_i-\cos t_{i-1} & \text{if } -\frac{\pi}{2}\le t\le 0\\[2ex] \cos t_{i-1}-\cos t_i & \text{if } 0<t\le\frac{\pi}{2}\end{cases}.
$$

Suppose that exist $t_{i-1}, t_i \in P$ *such that* $0 \in (t_{i-1}, t_i)$. *Then*

$$
\begin{aligned}
|(\cos t_i-\cos t_{i-1}, t_i-t_{i-1})|_o &= \sup\{(\cos t_i-\cos t_{i-1}, t_i-t_{i-1}),(0,0)\}\\
&\quad+\sup\{(\cos t_{i-1}-\cos t_i, t_{i-1}-t_i),(0,0)\}\\
&= (h(i,t), t_i-t_{i-1})\\
&= (|\cos t_i-\cos t_{i-1}|, t_i-t_{i-1}),
\end{aligned}
$$

where

$$
h(i,t)=\begin{cases}\cos t_i-\cos t_{i-1} & \text{if } t_i<|t_{i-1}|\\[2ex] \cos t_{i-1}-\cos t_i & \text{if } t_i\ge|t_{i-1}|\end{cases}.
$$

Thus,

$$
\begin{aligned}
\sup_{P\in\mathcal{P}[a,b]}\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|_o &= \sup_{P\in\mathcal{P}[a,b]}\sum_{i=1}^{n}(|\cos t_i-\cos t_{i-1}|, t_i-t_{i-1})\\
&= \sup_{P\in\mathcal{P}[a,b]}\left(\sum_{i=1}^{n}|\cos t_i-\cos t_{i-1}|, \pi\right)\\
&= \left(V_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t,\mathbb{R}), \pi\right).
\end{aligned}
$$

Therefore, f *is of bounded variation with respect to the order and*

$$
V_f^o\left[-\frac{\pi}{2},\frac{\pi}{2}\right]=\left(V_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t,\mathbb{R}), \pi\right).
$$

2.2. ...in Hilbert spaces

We will analyze the Jordan decomposition for functions with values in a Hilbert space.

Definition 2.9. *Let* $\mathcal{H}$ *be a Hilbert space on* $\mathbb{R}$ *and let* x_0 *be fixed in* $\mathcal{H}$. *We say that* $x, y \in \mathcal{H}$ *are related with respect to* x_0, *and we use the notation*

$$
x\sim_{x_0} y,
$$

if $\langle x-y, x_0\rangle = 0$.

The previous relation is of equivalence, so that, for each $x_0 \in \mathcal{H}$, we can divide $\mathcal{H}$ in disjoint classes. Since any two elements $x, y \in \mathcal{H}$ are related with respect to 0, then the only equivalence class will be $\mathcal{H}$.

Using this relation, in the following proposition we prove a generalization of the Jordan decomposition.

Proposition 2.10. *Let $\mathcal{H}$ be a Hilbert space on $\mathbb{R}$. If $f : [a, b] \to \mathcal{H}$ is a bounded variation function, then, for each $x_0 \neq 0$ in $\mathcal{H}$, there exists an extensible cone $\mathcal{H}_{x_0+}$ in $\mathcal{H}$ and $f_{x_01}, f_{x_02} : [a, b] \to \mathcal{H}$ increasing functions in $(\mathcal{H}, \mathcal{H}_{x_0+})$ such that $f \sim_{x_0} [f_{x_01} - f_{x_02}]$.*

Proof. By the Riesz lemma, for $x_0 \in \mathcal{H}$ there is only one $h_0 \in \mathcal{H}^*$ such that $h_0(x) = \langle x, x_0 \rangle$. By lemma 1.5, $h_0 \circ f : [a, b] \to \mathbb{R}$ is of bounded variation. Therefore, there exist $g_1, g_2 : [a, b] \to \mathbb{R}$ increasing such that $h_0 \circ f = g_1 - g_2$. Because of $x_0 \neq 0$, the functional h_0 is not identically zero. Let $\alpha \in (0, \|x_0\|)$ and

$$\mathcal{H}_{x_0+} = \{x \in \mathcal{H} : \langle x, x_0 \rangle \geq \alpha \|x\|\}. \tag{3}$$

This set is a cone because if x and $-x \in \mathcal{H}_{x_0+}$, then

$$\alpha \|x\| \leq \langle -x, x_0 \rangle \leq -\alpha \|x\|,$$

thus $x = 0$. Also, if $\lambda \geq 0$ and $x \in \mathcal{H}_{x_0+}$, we have that $\langle \lambda x, x_0 \rangle \geq \alpha \|\lambda x\|$. By theorem 1.1, $\mathcal{H}_{x_0+}$ is an extensible cone.

Since

$$h_0(x_0) = \|x_0\|^2 > \alpha \|x_0\|,$$

x_0 belongs to $\mathcal{H}_{x_0+}$.

Let $f_{x_01}(t) = g_1(t)\, x_0$ and $f_{x_02}(t) = g_2(t)\, x_0$. Because $0 \leq g_1(t_2) - g_1(t_1)$ for $t_1 < t_2$, then we have

$$f_{x_01}(t_2) - f_{x_01}(t_1) = [g_1(t_2) - g_1(t_1)]\, x_0 \in \mathcal{H}_{x_0+}.$$

Making a similar observation for f_{x_02}, we have that $f_{x_01}(t)$ and $f_{x_02}(t)$ are increasing functions.

On the other hand, considering that $h_0 \circ f = g_1 - g_2$:

$$\begin{aligned}
(h_0 \circ f)(t) &= \langle f(t), x_0 \rangle \qquad (4)\\
&= [g_1(t) - g_2(t)] \frac{1}{\|x_0\|^2} \langle x_0, x_0 \rangle \\
&= \frac{1}{\|x_0\|^2} \langle [g_1(t) - g_2(t)]\, x_0, x_0 \rangle \\
&= \frac{1}{\|x_0\|^2} \langle f_{x_01}(t) - f_{x_02}(t), x_0 \rangle. \qquad (5)
\end{aligned}$$

Since $\mathcal{H}_{x_0+}$ is a cone and $\frac{1}{\|x_0\|^2} > 0$, then the functions $\frac{1}{\|x_0\|^2} f_{x_0i}(t)$, $i = 1, 2$, are increasing in $(\mathcal{H}, \mathcal{H}_{x_0+})$. Redefining $f_{x_01}(t)$ and $f_{x_02}(t)$, respectively by the previous multiple functions, then, by (4) and (5), we get

$$f \sim_{x_0} [f_{x_01} - f_{x_02}].$$

□

Due to Riesz's lemma and taking $x_0 \in \mathcal{H}$ associated with the functional h_0, we have the following corollary.

Corollary 2.11. *Let $\mathcal{H}$ be a Hilbert space on $\mathbb{R}$. If $f : [a,b] \to \mathcal{H}$ is a bounded variation function, then, for each $h_0 \in \mathcal{H}^*$, there exists an extensible cone $\mathcal{H}_{h_0+}$, in $\mathcal{H}$ and $f_{h_0 1}, f_{h_0 2} : [a,b] \to \mathcal{H}$ increasing functions in $(\mathcal{H}, \mathcal{H}_{h_0+})$ such that $f \sim_{x_0} [f_{h_0 1} - f_{h_0 2}]$.*

Remark 2.12. *Because g_1 and g_2 may be $V_a^t(h_0 \circ f, \mathbb{R})$ and $V_a^t(h_0 \circ f, \mathbb{R}) - h_0 \circ f(t)$, respectively, then we have*

$$f_{h_0 1}(t) = V_a^t(h_0 \circ f, \mathbb{R}) x_0 \tag{6}$$

and

$$f_{h_0 2}(t) = \left[-h_0 \circ f(t) + V_a^t(h_0 \circ f, \mathbb{R})\right] x_0.$$

This opens the possibility of defining the right side of (6) as the variation function of f with respect to x_0.

Remark 2.13. *We can make a variant of the proof of proposition 2.10, if we consider the cone*

$$\mathcal{H}_{x_{00}} = \{x \in \mathcal{H} : \langle x, x_0 \rangle \geq 0\}.$$

$\mathcal{H}_{x_0+}$ should be extend to $\mathcal{H}_{x_{00}}$, and this last one is non-extensible.

Remark 2.14. *We can observe that the proposition 2.10 generalizes the case $\mathcal{H} = \mathbb{R}$. The cone $\mathcal{H}_{x_0+}$ associated to each $x_0 \in \mathbb{R}$, $x_0 \neq 0$, has the form*

$$\begin{aligned} \mathcal{H}_{x_0+} &= \{x \in \mathbb{R} : x x_0 \geq \alpha |x|\} \\ &= \begin{cases} [0,\infty) & \text{if} \quad x_0 \geq \alpha > 0 \\ \emptyset & \text{if} \quad x_0 < \alpha. \end{cases} \end{aligned} \tag{7}$$

We have: $x, y \in \mathbb{R}$ are x_0–related if and only if $x = y$. Thus $f \sim_{x_0} [f_{x_0 1} - f_{x_0 2}]$ if and only if $f = f_{x_0 1} - f_{x_0 2}$. We observe that for any other $x_1 \neq 0$, we have by (7) that $\mathcal{H}_{x_0+} = \mathcal{H}_{x_1+}$; although $x_0 \neq x_1$.

Example 2.15. *Let $[a,b] = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\mathcal{H} = \mathbb{R}^2$, $\overline{x}_0 = (1,1)$, and $\alpha = 1 \in \left(0, \sqrt{2}\right)$. It follows that the cone associated to $\overline{x}_0$ is*

$$\begin{aligned} \mathcal{H}_{\overline{x}_0+} &= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq \sqrt{x_1^2 + x_2^2}\right\} \\ &= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\right\} \\ &= \mathbb{R}_+^2. \end{aligned}$$

By remark 1.3, the function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^2$ given by $f(t) = (\cos t, t)$ is of bounded variation from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $\mathcal{H}$, therefore the function

$$\begin{aligned} (h_0 \circ f)(t) &= \langle f(t), \overline{x}_0 \rangle \\ &= \cos t + t, \end{aligned}$$

is of bounded variation from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *to* $\mathbb{R}$. *Since*

$$\begin{aligned} V^t_{-\frac{\pi}{2}}(h \circ f; \mathbb{R}) &= \int_{-\frac{\pi}{2}}^{t} |1 - \sin u| du \\ &= t + \cos t + \frac{\pi}{2}, \end{aligned}$$

then $g_1(t) = t + \cos t + \frac{\pi}{2}$ *and* $g_2(t) = \frac{\pi}{2}$. *Hence*

$$f_1(t) = \frac{1}{2}\left(t + \cos t + \frac{\pi}{2}, t + \cos t + \frac{\pi}{2}\right)$$

and

$$f_2(t) = \frac{1}{2}\left(\frac{\pi}{2}, \frac{\pi}{2}\right),$$

which are increasing functions with respect to $\mathcal{H}_{\overline{x}_0+}$. *We note that indeed:*

$$\langle f(t) - [f_1(t) - f_2(t)], (1, 1)\rangle = \langle(-\frac{t}{2} + \frac{\cos t}{2}, \frac{t}{2} - \frac{\cos t}{2}), (1, 1)\rangle = 0,$$

whereby

$$(\cos t, t) \sim_{(1,1)} [f_1(t) - f_2(t)].$$

If we take into account the cone $\mathcal{H}_+ = \{(x_1, x_2) \in \mathbb{R} : x_1, x_2 \leq 0\}$, *it is easy to see that* f_1 *is not increasing with respect to this cone.*

Lemma 2.16. *Let* $(\mathcal{H}, \mathcal{H}_{x_0+})$ *be an ordered Hilbert space with a cone* $\mathcal{H}_{x_0+}$ *defined in (3), with* $x_0 \neq 0$, *and let* $h_0(x) = \langle x, x_0\rangle$. *If* f *is an increasing function in* $(\mathcal{H}, \mathcal{H}_{x_0+})$, *then* $h_0 \circ f : [a, b] \to \mathbb{R}$ *is increasing and satisfies that* $f \sim_{x_0} h_0 \circ f(t)x_0$.

Proof. Because f is increasing in $(\mathcal{H}, \mathcal{H}_{x_0+})$, then, by proposition 2.10 and theorem 1.6, f is of bounded variation. Hence there exist f_{x_01} and f_{x_02} increasing in $(\mathcal{H}, \mathcal{H}_{x_0+})$ such that $f \sim_{x_0} [f_{x_01} - f_{x_02}]$. Because we can choose

$$f_{x_01}(t) = V^t_a(h_0 \circ f, \mathbb{R})x_0$$

and

$$f_{x_02}(t) = \left[V^t_a(h_0 \circ f, \mathbb{R}) - h_0 \circ f(t)\right] x_0,$$

we have

$$f \sim_{x_0} h_0 \circ f(t)\, x_0.$$

If $t_1 < t_2$, then: $h_0 \circ f(t_2) - h_0 \circ f(t_1) = \langle f(t_2) - f(t_1), x_0\rangle \geq \alpha \|f(t_2) - f(t_1)\| \geq 0$, therefore $h_0 \circ f$ is increasing. □

Using lemma 2.16 in the following proposition we show that if we have an ordered Hilbert space $(\mathcal{H}, \mathcal{H}_+)$, where $\mathcal{H}_+$ is a given extensible cone and $f : I \to \mathcal{H}$ is a bounded variation function, then the Jordan decomposition cannot be possible.

Proposition 2.17. *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{H}_{x_0+}$, with $x_0 \neq 0$, the extensible cone defined in (3). There exists $f : I \to \mathcal{H}$ of bounded variation such that the only possibility of satisfying $f \sim_{x_0} (f_1 - f_2)$, where f_1 and f_2 are increasing, is that $f_1 \sim_{x_0} f_2$.*

Proof. Let $x_1 \in \mathcal{H}$ with $x_1 \neq 0$ such that $\langle x_1, x_0\rangle = 0$. Let $\lambda : [a,b] \to \mathbb{R}$ an increasing function and $f(t) = \lambda(t)x_1$. Then f is a bounded variation function in $\mathcal{H}$. Suppose that there exist f_1, f_2 increasing functions in $(\mathcal{H}, \mathcal{H}_{x_0+})$ such that $f \sim_{x_0} [f_1 - f_2]$. By lemma 2.16, we have

$$f_1(t) \sim_{x_0} h_0 \circ f_1(t)x_0 \quad \text{and} \quad f_2(t) \sim_{x_0} h_0 \circ f_2(t)x_0, \tag{8}$$

which is equivalent to

$$\langle f_i(t), x_0\rangle = h_0 \circ f_i(t)\|x_0\|^2, \quad i = 1, 2.$$

Thus:

$$\begin{aligned} 0 &= \langle f(t) - (f_1(t) - f_2(t)), x_0\rangle \\ &= \langle \lambda(t)x_1 - [h_0 \circ f_1(t) - h_0 \circ f_2(t)]\, x_0, x_0\rangle \\ &= [h_0 \circ f_1(t) - h_0 \circ f_2(t)]\, \|x_0\|^2. \end{aligned}$$

Therefore: $h_0 \circ f_1(t) = h_0 \circ f_2(t)$, for all $t \in [a,b]$. By (8), we conclude that

$$f_1 \sim_{x_0} f_2.$$

□

As consequence of the previous proposition, we can find bounded variation functions with values in a Hilbert space (therefore normed) for which the Jordan decomposition is satisfied only if they are related with the zero function. The difference of the previous proposition with proposition 2.10 is that, at this last, the extensible cone is predetermined, while in proposition 2.10 the cone depends on x_0.

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Conflict of Interest

All authors declare no conflicts of interest in this paper

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