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Research article

Logarithmically improved regularity criteria for the Boussinesq equations

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Abstract: In this paper, logarithmically improved regularity criteria for the Boussinesq equations are established under the framework of Besov space $\dot{B}^{-r}_{\infty,\infty}$. We prove the solution (u,θ) is smooth up to time $T > 0$ provided that

$$\int_0^T \frac{\|u(\cdot,t)\|^{\frac{2}{1-r}}_{\dot{B}^{-r}_{\infty,\infty}}}{\log(e+\|u(t,.)\|_{\dot{B}^{-r}_{\infty,\infty}})}dt < \infty$$

for some $0 \leq r < 1$ or

$$\|u(\cdot,t)\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} << 1.$$

This result improves some previous works.

Keywords: Regularity criterion; Boussinesq equations; A priori estimates
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1. Introduction and main result

This paper is concerned with the regularity criterion of the 3D Boussinesq equations with the incompressibility condition :

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi = \theta e_3, \\ \partial_t \theta + u \cdot \nabla \theta - \Delta \theta = 0, \\ \nabla \cdot u = 0, \\ (u,\theta)(x,0) = (u_0,\theta_0)(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity vector field and the scalar function temperature, while u_0, θ_0 with $\nabla \cdot u_0 = 0$ in the sense of distribution are given initial data. $e_3 = (0,0,1)^T$. $\pi = \pi(x,t)$ the pressure of fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. The Boussinesq equation is one of important subjects for researches in nonlinear sciences [14]. There are a huge literatures on the incompressible Boussinesq equations such as [1–4, 6, 8–10, 17, 19–22] and the references therein.

When $\theta = 0$, (1.1) reduces to the well-known incompressible Navier-Stokes equations and many results are available. Besides their physical applications, the Navier-Stokes equations are also mathematically significant. From that time on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations.

However, similar to the classic Navier-Stokes equations, the question of global regularity of the weak solutions of the 3D Boussinesq equations still remains a big open problem and the system (1.1) has received many studies. Based on some analysis technique, some regularity criteria via the velocity of weak solutions in the Lebesgue spaces, multiplier spaces and Besov spaces have been obtained in [5, 17, 19, 20, 22, 23].

More recently, the authors of the present paper [7] showed that the weak solution becomes regular if

$$\int_0^T \frac{\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} + \|\theta(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{1+\log(e+\|u(\cdot,t)\|_{H^s}+\|\theta(\cdot,t)\|_{H^s})}dt < \infty \text{ for some } 0 \le r < 1 \quad \text{and} \quad s \ge \frac{1}{2}, \tag{1.2}$$

where $\dot{B}^{-r}_{\infty,\infty}$ denotes the homogeneous Besov space. Definitions and basic properties of the Sobolev spaces and the Besov spaces can be find in [18]. For concision, we omit them here.

The purpose of this paper is to improve the regularity criterion (1.2) in the following form.

Theorem 1.1. *Let (u, θ) be a smooth solution to (1.1) in $[0, T)$ with the initial data $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. Suppose that the solution (u, θ) satisfies*

$$\int_0^T \frac{\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{\log(e+\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}})}dt < \infty \quad \textit{for some } r \textit{ with } 0 \le r < 1. \tag{1.3}$$

Then it holds

$$\sup_{0\le t\le T}\left(\|u(\cdot,t)\|_{H^3}^2 + \|\theta(\cdot,t)\|_{H^3}^2\right) < \infty.$$

That is, the solution (u, θ) can be smoothly extended after time $t = T$. In other word, if T_* is the maximal time existence of the solution, then

$$\int_0^{T_*} \frac{\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{\log(e+\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}})}dt < \infty.$$

Then the solution can be smoothly extended after $t = T$.

Remark 1.1. *The condition (1.3) can be regarded as a logarithmically improved version of the assumption*

$$\int_0^T \|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} dt < \infty \quad \textit{for some } r \textit{ with } 0 \le r < 1.$$

For the case $r = 1$, we have the following result.

Theorem 1.2. *Let* (u,θ) *be a smooth solution to (1.1) in* $[0,T)$ *with the initial data* $(u_0,\theta_0)\in H^3(\mathbb{R}^3)\times H^3(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *in* $\mathbb{R}^3$. *Suppose that there exists a small positive constant* η *such that*

$$\|u(\cdot,t)\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \le \eta, \tag{1.4}$$

then solution (u,θ) *can be smoothly extended after time* $t = T$.

Remark 1.2. *Theorem 1.2 can be regarded as improvements and limiting cases of those in [7]. It is worth to point out all conditions are valid for the usual Navier-Stokes equations. We refer to a recent work [7] and references therein.*

Remark 1.3. *For the case* $r = 0$*, see [23].*

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by the standard energy method.

Proof. Let $T > 0$ be a given fixed time. The existence and uniqueness of local smooth solutions can be obtained as in the case of the Navier-Stokes equations. Hence, for all $T > 0$ we assume that (u,θ) is a smooth solution to (1.1) on $[0,T)$ and we will establish a priori bounds that will allow us to extend (u,θ) beyond time T under the condition (1.3).

Owing to (1.3) holds, one can deduce that for any small $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^{T} \frac{\|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{\log(e + \|u(\cdot,t)\|_{\dot{B}^{-r}_{\infty,\infty}})}\, dt \le \epsilon << 1. \tag{2.1}$$

Thanks to the divergence-free condition $\nabla\cdot u = 0$, from $(1.1)_2$, we get immediately the global a priori bound for θ in any Lebesgue space

$$\|\theta(\cdot,t)\|_{L^q} \le C\,\|\theta_0\|_{L^q} \quad \text{for all } \ q\in[2,\infty] \ \text{ and all } \ t\in[0,T].$$

Now, multiplying $(1.1)_2$ by θ and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 = 0.$$

Hence, we obtain

$$\theta \in L^\infty\left(0,T; L^2(\mathbb{R}^3)\right) \cap L^2\left(0,T; H^1(\mathbb{R}^3)\right). \tag{2.2}$$

Next, multiplying $(1.1)_1$ by u, we have after integration by part,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_{\mathbb{R}^3} (\theta e_3)\cdot u dx \le \|\theta\|_{L^2}\|u\|_{L^2} \le C\,\|u\|_{L^2},$$

which yields

$$u \in L^\infty\left(0,T; L^2(\mathbb{R}^3)\right) \cap L^2\left(0,T; H^1(\mathbb{R}^3)\right), \tag{2.3}$$

where we used (2.2) and

$$\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot u dx = \frac{1}{2} \int_{\mathbb{R}^3} (u \cdot \nabla) u^2 dx = -\frac{1}{2} \int_{\mathbb{R}^3} (\nabla \cdot u) u^2 dx = 0$$

by incompressibility of u, that is, $\nabla \cdot u = 0$.

Now, apply ∇ operator to the equation of $(1.1)_1$ and $(1.1)_2$, then taking the inner product with ∇u and $\nabla\theta$, respectively and using integration by parts, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\ = & -\int_{\mathbb{R}^3} \nabla (u \cdot \nabla) u \cdot \nabla u dx + \int_{\mathbb{R}^3} \nabla(\theta e_3) \cdot \nabla u dx - \int_{\mathbb{R}^3} \nabla (u \cdot \nabla) \theta \cdot \nabla\theta dx \\ = & \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{2.4}$$

Employing the Hölder and Young inequalities, we derive the estimation of the first term $\mathcal{I}_1$ as

$$\begin{aligned} \mathcal{I}_1 &= \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx \le \|\nabla \cdot (u \otimes u)\|_{L^2} \|\Delta u\|_{L^2} \\ &\le C \|u\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla u\|_{\dot{H}^r} \|\Delta u\|_{L^2} \\ &\le C \|u\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla u\|_{L^2}^{1-r} \|\Delta u\|_{L^2}^{1+r} \\ &\le \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} \|\nabla u\|_{L^2}^2 \\ &\le \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2), \end{aligned}$$

where we have used the inequality due to [16] :

$$\|u \otimes u\|_{\dot{H}^1} \le C \|u\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla u\|_{\dot{H}^r}$$

and the interpolation inequality

$$\|w\|_{\dot{H}^s} = \left\| |\xi|^s \widehat{w} \right\|_{L^2} \le \|w\|_{L^2}^{1-s} \|\nabla w\|_{L^2}^s \quad \text{for all } 0 \le s \le 1.$$

The term $\mathcal{I}_3$ can be estimated as

$$\begin{aligned} \mathcal{I}_3 &\le C \|\nabla u\|_{L^2} \|\nabla\theta\|_{L^4}^2 \\ &\le C \|\nabla u\|_{L^2} \|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}} \|\Delta\theta\|_{L^2} \\ &\le C \|\nabla u\|_{L^2} \|\theta\|_{\dot{B}^{0}_{\infty,\infty}} \|\Delta\theta\|_{L^2} \\ &\le \frac{1}{2} \|\Delta\theta\|_{L^2}^2 + C \|\theta\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \\ &\le \frac{1}{2} \|\Delta\theta\|_{L^2}^2 + C \|\theta\|_{L^\infty}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2), \end{aligned}$$

where we have used

$$\|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}} \le C \|\theta\|_{\dot{B}^{0}_{\infty,\infty}} \le C \|\theta\|_{L^\infty} .$$

The term $\mathcal{I}_2$ can be estimated as

$$\mathcal{I}_2 \leq \|\nabla u\|_{L^2} \|\nabla\theta\|_{L^2} \leq \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).$$

Plugging all the estimates into (2.4) yields that

$$\begin{aligned}
& \frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
\leq \;& C(\frac{1}{2} + \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} + \|\theta\|_{L^\infty}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 \\
\leq \;& C\left[\frac{\frac{1}{2} + \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} + \|\theta\|_{L^\infty}^2}{\log(e + \|u\|_{\dot{B}^{-r}_{\infty,\infty}})}\right](\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\log(e + \|u\|_{\dot{B}^{-r}_{\infty,\infty}}) \\
\leq \;& C\left[\frac{\frac{1}{2} + \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} + \|\theta\|_{L^\infty}^2}{\log(e + \|u\|_{\dot{B}^{-r}_{\infty,\infty}})}\right](\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\log(e + \|u\|_{H^3} + \|\theta\|_{H^3}) \\
\leq \;& C\left[\frac{\frac{1}{2} + \|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}} + \|\theta\|_{L^\infty}^2}{\log(e + \|u\|_{\dot{B}^{-r}_{\infty,\infty}})}\right](\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\log(e + \kappa(t))
\end{aligned}$$

where $\kappa(t)$ is defined by

$$\kappa(t) = \sup_{T_0 \leq \tau \leq t}(\|u(\cdot,\tau)\|_{H^3} + \|\theta(\cdot,\tau)\|_{H^3}) \quad \text{for all } \; T_0 < t < T.$$

It should be noted that the function $\kappa(t)$ is nondecreasing. Moreover, we have used the following fact :

$$\|u\|_{\dot{B}^{-r}_{\infty,\infty}} \leq C\|u\|_{H^3}.$$

Integrating the above inequality over $[T_0, t]$ and applying Gronwall's inequality, we have

$$\begin{aligned}
& \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla\theta(\cdot,t)\|_{L^2}^2 + \int_{T_*}^t \|\Delta u(\cdot,\tau)\|_{L^2}^2 + \|\Delta\theta(\cdot,\tau)\|_{L^2}^2\, d\tau \\
\leq \;& (\|\nabla u(\cdot,T_0)\|_{L^2}^2 + \|\nabla\theta(\cdot,T_0)\|_{L^2}^2) \\
& \times\exp\left(C\int_{T_0}^t \frac{\|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{\log(e + \|u(\cdot,\tau)\|_{\dot{B}^{-r}_{\infty,\infty}})}\log(e + \kappa(\tau))\, d\tau\right) \\
\leq \;& (\|\nabla u(\cdot,T_0)\|_{L^2}^2 + \|\nabla\theta(\cdot,T_0)\|_{L^2}^2)
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left(C \log\left(e+\kappa(t)\right) \int_{T_0}^{t} \frac{\|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}}{\log(e+\|u(\cdot,\tau)\|_{\dot{B}^{-r}_{\infty,\infty}})} d\tau \right) \\
\leq\ & \widetilde{C} \exp\left(C\epsilon \log\left(e+\kappa(t)\right)\right) = \widetilde{C}\left(e+\kappa(t)\right)^{C\epsilon} \qquad (2.5)
\end{aligned}
$$

where $\widetilde{C}$ is a positive constant depending on $\|\nabla u(\cdot,T_0)\|_{L^2}^2$, $\|\nabla\theta(\cdot,T_0)\|_{L^2}^2$, T_0, T and θ_0.

H^3**−norm.** Next, we start to obtain the H^3−estimates under the above estimate (2.5). Applying $\Lambda^3 = (-\Delta)^{\frac{3}{2}}$ to $(1.1)_1$, then taking L^2 inner product of the resulting equation with $\Lambda^3 u$, and using integration by parts, we obtain

$$
\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^4 u(\cdot,t)\right\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^3\left(u\cdot\nabla u\right)\cdot\Lambda^3 u dx + \int_{\mathbb{R}^3}\Lambda^3(\theta e_3)\cdot\Lambda^3 u dx \qquad (2.6)
$$

Similarly, applying $\Lambda^3 = (-\Delta)^{\frac{3}{2}}$ to $(1.1)_2$, then taking L^2 inner product of the resulting equation with $\Lambda^3\theta$, and using integration by parts, we obtain

$$
\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 \theta(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^4 \theta(\cdot,t)\right\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^3\left(u\cdot\nabla \theta\right)\cdot\Lambda^3 \theta dx, \qquad (2.7)
$$

Using $\nabla\cdot u = 0$, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\left\|\Lambda^3 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^3 \theta(\cdot,t)\right\|_{L^2}^2) + \left\|\Lambda^4 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^4 \theta(\cdot,t)\right\|_{L^2}^2 \\
=\ & -\int_{\mathbb{R}^3}\left[\Lambda^3(u\cdot\nabla u) - u\cdot\Lambda^3\nabla u\right]\cdot\Lambda^3 u dx + \int_{\mathbb{R}^3}\Lambda^3(\theta e_3)\cdot\Lambda^3 u dx \\
& -\int_{\mathbb{R}^3}^{3}\left[\Lambda^3(u\cdot\nabla \theta) - u\cdot\Lambda^3\nabla \theta\right]\cdot\Lambda^3 \theta dx \\
=\ & \Pi_1 + \Pi_2 + \Pi_3. \qquad (2.8)
\end{aligned}
$$

To bound Π_1, we recall the following commutator estimate due to [12]:

$$
\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leq C\left(\left\|\Lambda^{\alpha-1}g\right\|_{L^{q_1}}\|\nabla f\|_{L^{p_1}} + \|\Lambda^\alpha f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \qquad (2.9)
$$

for $\alpha > 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Hence Π_1 can be estimated as

$$
\begin{aligned}
\Pi_1 &\leq C\|\nabla u\|_{L^3}\|\Lambda^3 u\|_{L^3}^2 \\
&\leq C\|\nabla u\|_{L^2}^{\frac{3}{4}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{1}{3}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{3}} \\
&\leq \frac{1}{6}\|\Lambda^4 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{13}{2}}\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}}, \qquad (2.10)
\end{aligned}
$$

where we used (2.9) with $\alpha = 3, p = \frac{3}{2}, p_1 = q_1 = p_2 = q_2 = 3$, and the following Gagliardo-Nirenberg inequalities

$$
\begin{cases}
\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{3}{4}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}}, \\
\|\Lambda^3 u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{1}{6}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{6}}.
\end{cases} \qquad (2.11)
$$

If we use the existing estimate (2.1) for $T_0 \leq t < T$, (2.10) reduces to

$$\Pi_1 \leq \frac{1}{2}\|\Lambda^4 u\|_{L^2}^2 + \widetilde{C}(e+\kappa(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon}. \tag{2.12}$$

Using (2.11) again, we get

$$\begin{aligned}
\Pi_3 &\leq C(\|\nabla u\|_{L^3}\|\Lambda^3\theta\|_{L^3} + \|\nabla\theta\|_{L^3}\|\Lambda^3 u\|_{L^3})\left\|\Lambda^3\theta\right\|_{L^3} \\
&\leq C(\|\nabla u\|_{L^3} + \|\nabla\theta\|_{L^3})(\left\|\Lambda^3 u\right\|_{L^3}^2 + \left\|\Lambda^3\theta\right\|_{L^3}^2) \\
&\leq \frac{1}{6}(\left\|\Lambda^4 u\right\|_{L^2}^2 + \left\|\Lambda^4\theta\right\|_{L^2}^2) + \widetilde{C}(e+\kappa(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon}.
\end{aligned}$$

For Π_2, we have

$$\Pi_2 \leq \frac{1}{2}(\left\|\Lambda^3 u\right\|_{L^2}^2 + \left\|\Lambda^3\theta\right\|_{L^2}^2) \leq \widetilde{C}(e+\kappa(t))^2.$$

Inserting all the inequalities into (2.8) and absorbing the dissipative terms, one finds

$$\frac{d}{dt}(\left\|\Lambda^3 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^3\theta(\cdot,t)\right\|_{L^2}^2) \leq \widetilde{C}(e+\kappa(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon} + \widetilde{C}(e+\kappa(t))^2, \tag{2.13}$$

with together with the basic energy (2.2)-(2.3) yields

$$\frac{d}{dt}(\|u(\cdot,t)\|_{H^3}^2 + \|\theta(\cdot,t)\|_{H^3}^2) \leq \widetilde{C}(e+\kappa(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon} + \widetilde{C}(e+\kappa(t))^2, \tag{2.14}$$

Choosing ϵ sufficiently small provided that $\frac{13}{2}C\epsilon < \frac{1}{2}$ and applying the Gronwall inequality to (2.14), we derive that

$$\sup_{T_0\leq\tau\leq t}\left(\|u(\cdot,\tau)\|_{H^3}^2 + \|\theta(\cdot,\tau)\|_{H^3}^2\right) \leq \widetilde{C} < \infty, \tag{2.15}$$

where $\widetilde{C}$ depends on $\|\nabla u(\cdot,T_0)\|_{L^2}^2$ and $\|\nabla\theta(\cdot,T_0)\|_{L^2}^2$.

Noting that the right-hand side of (2.15) is independent of t for $T_0 \leq t < T$, we know that $(u(\cdot,T),\theta(\cdot,T)) \in H^3(\mathbb{R}^3)\times H^3(\mathbb{R}^3)$. Consequently, (u,θ) can be extended smoothly beyond $t = T$. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first recall the following local existence theorem of the three-dimensional Boussinesq equations.

Lemma 3.1. *Suppose* $(u,\theta) \in L^\alpha\left(\mathbb{R}^3\right)$, *for some* $\alpha \geq 3$ *and* $\nabla\cdot u = 0$. *Then, there exists* $T_0 > 0$ *and a unique solution of (1.1) on* $[0,T_0)$ *such that*

$$(u,\theta) \in BC\left([0,T_0);L^\alpha\left(\mathbb{R}^3\right)\right)\cap L^s\left([0,T_0);L^r\left(\mathbb{R}^3\right)\right),\quad t^{\frac{1}{s}}u \in BC\left([0,T_0);L^\alpha\left(\mathbb{R}^3\right)\right) \tag{3.1}$$

Moreover, let $(0,T^*)$ *be the maximal interval such that* (u,θ) *solves (1.1) in* $C\left((0,T^*);L^\alpha\left(\mathbb{R}^3\right)\right)$, $\alpha > 3$. *Then for any* $t \in (0,T^*)$

$$\|u(\cdot,t)\|_{L^\alpha} \geq \frac{C}{(T^*-t)^{\frac{\alpha-3}{2\alpha}}} \quad and \quad \|\theta(\cdot,t)\|_{L^\alpha} \geq \frac{C}{(T^*-t)^{\frac{\alpha-3}{2\alpha}}},$$

with the constant C independent of T^* *and* α.

Let (u,θ) be a strong solution satisfying

$$(u,\theta)\in L^{\alpha}\left((0,T);L^{\beta}\left(\mathbb{R}^3\right)\right) \quad \text{for} \quad \frac{2}{\alpha}+\frac{3}{\beta}=1 \quad \text{and} \quad \beta>3.$$

Then (u,θ) belongs to $C^{\infty}\left(\mathbb{R}^3\times(0,T)\right)$.

Proof. For all $T>0$, we assume that (u,θ) is a smooth solution to (1.1) on $[0,T)$ and we will establish a priori bounds that will allow us to extend (u,θ) beyond time T under the condition (1.4).

Similar to the proof of Theorem 1.1, we can show that

$$(u,\theta)\in L^{\infty}\left(0,T;L^2(\mathbb{R}^3)\right)\cap L^2\left(0,T;H^1(\mathbb{R}^3)\right). \tag{3.2}$$

The proof of Theorem 1.2 is divided into steps.

Step I. H^1−estimation. In order to get the H^1−estimates, we apply ∇ operator to the equation of $(1.1)_1$ and $(1.1)_2$, multiply by ∇u and $\nabla\theta$, respectively to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2)+\|\Delta u(\cdot,t)\|_{L^2}^2+\|\Delta\theta(\cdot,t)\|_{L^2}^2\\
=\ & -\int_{\mathbb{R}^3}\nabla\left(u\cdot\nabla\right)u\cdot\nabla u dx+\int_{\mathbb{R}^3}\nabla(\theta e_3)\cdot\nabla u dx-\int_{\mathbb{R}^3}\nabla\left(u\cdot\nabla\right)\theta\cdot\nabla\theta dx\\
=\ & \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.
\end{aligned} \tag{3.3}$$

Next we estimate $\mathcal{I}_1,\mathcal{I}_2$ and $\mathcal{I}_3$ in another way. Hence,

$$\begin{aligned}
\mathcal{I}_1 &\le \|\nabla u\|_{L^3}^3\le C\|\nabla u\|_{\dot{B}^{-2}_{\infty,\infty}}\|\Delta u\|_{L^2}^2\\
&\le C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta u\|_{L^2}^2,
\end{aligned}$$

where we have used the following interpolation inequality due to [16] :

$$\|w\|_{L^3}\le C\|\nabla w\|_{L^2}^{\frac{2}{3}}\|w\|_{\dot{B}^{-2}_{\infty,\infty}}^{\frac{1}{3}}.$$

By means of the Hölder and Young inequalities, the term $\mathcal{I}_3$ can be estimated as

$$\begin{aligned}
\mathcal{I}_3 &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta\theta\|_{L^2}\\
&\le C\|\theta\|_{\dot{B}^{0}_{\infty,\infty}}^2\|\Delta\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\\
&\le C\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}^2,
\end{aligned}$$

where we have used the following interpolation inequality due to [16] :

$$\|\nabla\theta\|_{L^4}^2\le C\|\nabla\theta\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta\theta\|_{L^2}.$$

The term $\mathcal{I}_2$ can be estimated as

$$\mathcal{I}_2\le\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\le\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).$$

Plugging all the estimates into (3.3) yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2)+\|\Delta u(\cdot,t)\|_{L^2}^2+\|\Delta\theta(\cdot,t)\|_{L^2}^2\\
\leq\ & C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta u\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2+C(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Under the assumption (1.4), we choose η small enough so that

$$C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\leq\frac{1}{2}\,.$$

Hence, we find that

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\leq C(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).$$

Integrating in time and applying the Gronwall inequality, we infer that

$$\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\theta(\cdot,t)\|_{L^2}^2+\int_0^T(\|\Delta u(\cdot,\tau)\|_{L^2}^2+\|\Delta\theta(\cdot,\tau)\|_{L^2}^2)d\tau\leq C. \tag{3.4}$$

Step II. H^2-**estimation.** Next, we start to obtain the H^2-estimates under the above estimate (3.4). Applying Δ to $(1.1)_1$, then taking L^2 inner product of the resulting equation with Δu, and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u(\cdot,t)\|_{L^2}^2+\left\|\Lambda^3u(\cdot,t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\Delta\left(u\cdot\nabla u\right)\cdot\Delta u dx+\int_{\mathbb{R}^3}\Delta(\theta e_3)\cdot\Delta u dx \tag{3.5}$$

Similarly, applying Δ to $(1.1)_2$, then taking L^2 inner product of the resulting equation with $\Delta\theta$, and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\theta(\cdot,t)\|_{L^2}^2+\left\|\Lambda^3\theta(\cdot,t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\Delta\left(u\cdot\nabla\theta\right)\cdot\Delta\theta dx. \tag{3.6}$$

Adding (3.5) and (3.6), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta u(\cdot,t)\|_{L^2}^2+\|\Delta\theta(\cdot,t)\|_{L^2}^2)+\left\|\Lambda^3u(\cdot,t)\right\|_{L^2}^2+\left\|\Lambda^3\theta(\cdot,t)\right\|_{L^2}^2\\
=\ & -\int_{\mathbb{R}^3}\Delta\left(u\cdot\nabla u\right)\cdot\Delta u dx+\int_{\mathbb{R}^3}\Delta(\theta e_3)\cdot\Delta u dx-\int_{\mathbb{R}^3}\Delta\left(u\cdot\nabla\theta\right)\cdot\Delta\theta dx\\
=\ & \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3.
\end{aligned}\tag{3.7}$$

Using Hölder's inequality and Young's inequality, $\mathcal{K}_1$ can be estimated as

$$\begin{aligned}
\mathcal{K}_1 &= \int_{\mathbb{R}^3}\Delta\left(u\otimes u\right)\cdot\Delta\nabla u dx\leq\|\Delta\left(u\otimes u\right)\|_{L^2}\|\Delta\nabla u\|_{L^2}\\
&\leq C\|u\|_{L^\infty}\|\Delta u\|_{L^2}\left\|\Lambda^3u\right\|_{L^2}\\
&\leq \frac{1}{2}\left\|\Lambda^3u\right\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\Delta u\|_{L^2}^2\,.
\end{aligned}$$

Here we have used the bilinear estimates due to Kato-Ponce [12] and Kenig-Ponce-Vega [13] :

$$\|\Lambda^{\alpha}(fg)\|_{L^p} \leq C\left(\|\Lambda^{\alpha} g\|_{L^{q_1}} \|f\|_{L^{p_1}} + \|\Lambda^{\alpha} f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right),$$

for $\alpha > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

From the incompressibility condition, Hölder's inequality and Young's inequality, one has

$$\begin{aligned}
\mathcal{K}_3 &= \int_{\mathbb{R}^3} \Delta(u\theta) \cdot \Delta\nabla\theta dx \leq \|\Delta(u\theta)\|_{L^2} \|\Delta\nabla\theta\|_{L^2} \\
&\leq C(\|u\|_{L^\infty} \|\Delta\theta\|_{L^2} + \|\theta\|_{L^\infty} \|\Delta u\|_{L^2}) \left\|\Lambda^3\theta\right\|_{L^2} \\
&\leq \frac{1}{2} \left\|\Lambda^3\theta\right\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2)(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).
\end{aligned}$$

For $\mathcal{K}_2$, we have

$$\mathcal{K}_2 \leq \frac{1}{2}(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2).$$

Inserting all the inequalities into (3.7) and absorbing the dissipative terms, one finds

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta u(\cdot,t)\|_{L^2}^2 + \|\Delta\theta(\cdot,t)\|_{L^2}^2) + \left\|\Lambda^3 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^3\theta(\cdot,t)\right\|_{L^2}^2 \\
\leq\; & C(\|u\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2)(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2). \qquad (3.8)
\end{aligned}$$

Using the following interpolation inequality

$$\|f\|_{L^\infty} \leq C\, \|f\|_{L^2}^{\frac{1}{4}} \|\Delta f\|_{L^2}^{\frac{3}{4}},$$

together with the key estimate (3.4) yield that

$$\int_0^T (\|u(\cdot,\tau)\|_{L^\infty}^2 + \|\theta(\cdot,\tau)\|_{L^\infty}^2) d\tau \leq C < \infty.$$

Applying the Gronwall inequality to (3.8), we derive that

$$\|\Delta u(\cdot,t)\|_{L^2}^2 + \|\Delta\theta(\cdot,t)\|_{L^2}^2 + \int_0^T (\left\|\Lambda^3 u(\cdot,t)\right\|_{L^2}^2 + \left\|\Lambda^3\theta(\cdot,t)\right\|_{L^2}^2) dt \leq C. \qquad (3.9)$$

By estimates (3.4) and (3.9) as well as the following Gagliardo-Nirenberg's inequality

$$\|f\|_{L^6} \leq C\, \|f\|_{L^2}^{\frac{1}{2}} \|\Delta f\|_{L^2}^{\frac{1}{2}},$$

it is easy to see that

$$(u, \theta) \in L^4(0, T; L^6(\mathbb{R}^3)),$$

from which and Lemma 3.1 the smoothness of (u, θ) follows immediately. This completes the proof of Theorem 1.2. □

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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