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Research article

Nonlinear fractional boundary value problem with not instantaneous impulse

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Abstract: In this article, the main focus is to propose the solution for the nonlinear fractional boundary system with non-instantaneous impulse under some weak conditions. By applying well known classical fixed point theorems, we obtained the existence and uniqueness outcomes of the solution for the proposed problem. Moreover, an example is also discussed to explain the present work.

Keywords: Fractional order differential equation; fixed point theorems; impulsive conditions
Mathematics Subject Classification: 26A33, 34K05, 34A12, 34A37

1. Introduction

Fractional calculus is the most famous and useful branch of mathematics which provides a good framework for mathematical modeling of biological, engineering and physical phenomena etc. [1–5]. To get a few developments about the theory of fractional differential equations (FDEs), one can refer to [6–10] and the references therein.

Recently, the BVPs for nonlinear FDEs have been demonstrated by numerous authors. It has been seen that boundary conditions can be used to describe many physical systems and is therefore a popular part of mathematics. Interested readers may refer to [8–13] and the references therein for better understanding.

Most of the research papers deal with the existence of solutions for differential equations with instantaneous impulsive conditions see [8–10, 14, 15]. But many times it has seen that certain dynamics of evolution processes cannot describe by instantaneous impulses, For instance: Pharmacotherapy, high or low levels of glucose, this situation can be interpreted as an impulsive action which starts abruptly at certain point of time and continue with a finite time interval. Such type of systems are known as non-instantaneous impulsive systems which are more suitable to study the dynamics of evolution processes. This theory of a new class of impulsive differential equation (IDE) was developed by Hernndez et al. [14]. Afterwards, Pierri et al. [16] continued the work in this field and extend the theory of [14] in a $PC_\alpha-$ normed Banach space. The existence of solutions for non-instantaneous impulsive fractional differential equations (IFDEs) have also been studied [6, 7, 11, 16, 17].

Recently, Kumar et al. [15] investigated the FDE with not instantaneous impulse. By using the Banach fixed point theorem with condensing map the author's built up the presence and uniqueness results. Li et al. [6] considered the IFDEs where impulses are non-instantaneous. Yu [7] studied a new class of FDEs with non-instantaneous impulses and gave a suitable formula of piecewise continuous solutions. The author also established the concept of Ulam-Hyers stability on compact interval.

Wang et al. [11] investigated the following periodic BVP:

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \ q > 0, \\ u(t) = g_i(t, u(t)), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where ${}^cD^q$ represents the Caputo's fractional derivative. Authors also studied the same system for $q = 1$, and obtained the existence results by utilizing some classical fixed point theorems.

Recently, Z. Lin et al. [12] discussed the following system of order $q \in (0, 1)$:

$$\begin{cases} {}^cD^q u(t) + \lambda u(t) = f(t, u(t)), \ t \in (s_i, t_{i+1}), \ i = 0, 1, 2, \ldots, m, \\ u(t) = g_i(t, u(t_i^+)), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ u(0) = \sum_{i=1}^m \xi_i u(\tau_i), \ \tau_i \in (s_i, t_{i+1}), \ i = 1, 2, \ldots, m, \ \sum_{i=1}^m \xi_i = 1, \end{cases} \tag{1.2}$$

The two existence results are obtained by classical and generalized Mittag Leffer functions and fixed point theorems.

Inspired by the development in this field and the above mentioned work, we deliberate the following fractional BVP in case of non-instantaneous impulse:

$$\begin{cases} {}^cD^\alpha_{0,t} y(t) + f(t, y(t)) = 0, \ t \in (s_i, t_{i+1}] \subset [0, T], \ i = 0, 1, 2, \ldots, m, \ \alpha \in (0, 1], \\ y(t) = H_i(t, y(t)), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ y(0) = \mu \, {}^cD^q y(\psi), \ 0 < \psi < T, \ q \in (0, 1), \ \mu \in \mathbb{R}, \end{cases} \tag{1.3}$$

For the not instantaneous impulses, $0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = T$ are pre-fixed numbers. The functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $H : [t_i, s_i] \times \mathbb{R} \to \mathbb{R}$ are continuous.

In the literature, a limited work is reported on non-instantaneous impulsive effect together with FBVPs. To overcome this gap, we extend the work carried out in [11]. This paper is composed as follows. In Section 2 we demonstrate a couple of preliminaries, definitions, and lemmas which are to be employed to prove our essential outcomes. After that, the existence and uniqueness outcomes of solutions for the model (1.3) are analyzed under the Banach, Kransnoselskii's and Larey schauder's alternative fixed point theorems [18] in Section 3. In this article the Section 4 is introduced to demonstrate the validity and applicability of the techniques.

2. Preliminaries

Let $C([0, T], \mathbb{R})$ be the Banach space of all continuous functions with the sup norm $\|y\|_C := \sup\{|y(t)| : t \in [0, T]\}$ for $y \in C([0, T], \mathbb{R})$. From the associate literature we assume the space

$PC([0,T],\mathbb{R}) := \{y : [0,T] \to \mathbb{R} : y \in C((t_k,t_{k+1}],\mathbb{R}),\ \ k = 0,1,\dots,m$ and there exist $y(t_k^-)$ and $y(t_k^+),\ \ k = 0,1,\dots,m$, with $y(t_k^- = y(t_k))\}$ under the norm $\|y\|_{PC} := \sup\{|y(t)| : t \in [0,T]\}$. Set $PC^1([0,T],\mathbb{R}) := \{y \in PC([0,T],\mathbb{R}) : y' \in PC([0,T],\mathbb{R})\}$ endowed with the norm $\|y\|_{PC^1} := \max\{\|y\|_{PC},\ \|y'\|_{PC}\}$. Clearly, $PC^1([0,T],\mathbb{R})$ favoured with the norm $\|.\|_{PC^1}$ is a Banach space.

In this manuscript, we have used some fundamental definitions of fractional derivatives and integrals and preparatory facts of fractional calculus which are taken from the paper [19]. So we may relax all those notations and historical background.

Lemma 2.1. *[13] Let $\alpha > 0$, then ${}^cD^\alpha K(t) = 0$, has solutions $K(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{p-1}t^{p-1}$ and $I^{\alpha c}D^\alpha K(t) = K(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{p-1}t^{p-1}$ where $c_i \in \mathbb{R}, i = 0,1,\dots,p-1,\ p = [\alpha]+1$.*

In order to obtain the solution of the problem (1.3), we need the following lemma.

Lemma 2.2. *[11] Suppose $h : [0,T] \to \mathbb{R}$ and $G_i : [t_i,s_i] \to \mathbb{R}$ be continuous functions. A function $y \in PC^1([0,T],\mathbb{R})$ is a solution of the problem*

$$\begin{cases} y'(t) = h(t), & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m,\\ y(t) = G_i(t), & t \in (t_i,s_i],\ i = 1,2,\dots,m,\\ y(0) = y(T), \end{cases}$$

iff

$$y(t) = \begin{cases} G_m(s_m) + \int_{s_m}^T h(s)ds + \int_0^t h(s)ds, & t \in [0,t_1],\\ G_i(t), & t \in (t_i,s_i],\ i = 1,2,\dots,m,\\ G_i(s_i) + \int_{s_i}^t h(s)ds, & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m. \end{cases}$$

Theorem 2.3. *[18] Let C be a convex subset of a Banach space, E be a open subset of C with $0 \in E$. Then every completely continuous map $F : \overline{E} \to C$ has at least one of the two following properties*

1. *There exist an $e \in \overline{E}$ such that $Fe = e$.*
2. *There exists an $y \in \partial E$ and $\kappa \in (0,1)$, such that $y = \kappa Fy$.*

By using the concept of lemma 2.2, we can derive the following result.

Lemma 2.4. *A function $y \in PC^1([0,T],\mathbb{R})$ given by*

$$y(t) = \begin{cases} -\mu\int_0^\psi \frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\sigma(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, & t \in [0,t_1],\\ H_i(t), & t \in (t_i,s_i],\ i = 1,2,\dots,m,\\ H_i(s_i) + \int_0^{s_i} \frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m, \end{cases} \tag{2.1}$$

is a solution of the following system

$$\begin{cases} {}^cD^\alpha y(t) + \sigma(t) = 0, & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m,\ \alpha \in (0,1],\ t \in [0,T],\\ y(t) = H_i(t), & t \in (t_i,s_i],\ i = 1,2,\dots,m,\\ y(0) = \mu D^q y(\psi), & 0 < \psi < T,\ q > 0, \end{cases} \tag{2.2}$$

Proof. Suppose $y(t)$ satisfies (2.2). Then for $t \in [0, t_1]$, integrating the first equation of (2.2) from 0 to t, we have

$$y(t) = y(0) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds. \tag{2.3}$$

Moreover, if $t \in (s_i, t_{i+1}]$, and again integrating the first equation of (2.2) from s_i to t, we obtain

$$y(t) = y(s_i) - \int_{s_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds. \tag{2.4}$$

Now applying the impulsive condition $y(t) = H_i(t),\ t \in (t_i, s_i]$, we get

$$y(s_i) = H_i(s_i). \tag{2.5}$$

Consequently, from (2.4) and (2.5), we occur

$$y(t) = H_i(s_i) - \int_{s_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds, \tag{2.6}$$

and

$$y(t) = H_i(s_i) + \int_0^{s_i} \frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds. \tag{2.7}$$

Now using the boundary condition $y(0) = \mu D^q y(\psi),\ 0 < \psi < T$, we obtain

$$y(0) = \mu D^q y(\psi) = -\mu \int_0^{\psi} \frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\sigma(s)ds. \tag{2.8}$$

Hence, by the definitions of fractional derivatives, integrals, lemma 2.1 and proceeding the steps of lemma 4.1 [13], it is clear that (2.3), (2.7) and (2.8) imply (2.1). □

By lemma 2.4, we state the following definition.

Definition 2.5. *The continuously differentiable function $y : [0, T] \to \mathbb{R}$ such that $y \in PC^1([0, T], \mathbb{R})$ is said to be the solution of the system (1.3) if it satisfies the following integral equation*

$$y(t) = \begin{cases} -\mu \int_0^{\psi} \frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, y(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, & t \in [0, t_1], \\ H_i(t, y(t)), & t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ H_i(s_i, y(s_i)) + \int_0^{s_i} \frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m. \end{cases} \tag{2.9}$$

3. Main Results

In this section, we present the main outcomes and existence results of this article.

3.1. Existence and Uniqueness result via Banach fixed point theorem:

Theorem 3.1. *Consider the following condition hold*

(A_1) *There are positive constants* $L,\ L_{h_i},\ i = 1, 2, \ldots, m,$ *such that*

$$\begin{aligned} |f(t, y(t)) - f(t, v(t))| &\leq L|y - v|,\ \forall\ t \in [0, T],\ y, v \in \mathbb{R}, \\ |H_i(t, y_1) - H_i(t, y_2)| &\leq L_{h_i}|y_1 - y_2|,\ t \in [t_i, s_i],\ \forall\ y_1,\ y_2 \in \mathbb{R}. \end{aligned}$$

Then the system (1.3) has a unique solution on $[0, T]$ *provided* $\Delta < 1$, *such that*

$$\Delta := \max\Big\{\frac{\mu L \psi^{\alpha-q}}{\Gamma(\alpha - q + 1)} + \frac{LT^\alpha}{\Gamma(\alpha + 1)},\ \max_{i=1,2,\ldots,m} L_{h_i} + \frac{L(t_{i+1}^\alpha + s_i^\alpha)}{\Gamma(\alpha + 1)}\Big\}.$$

Proof. First, we transform the problem (1.3) into a fixed point problem. Define an operator $F : PC([0, T], \mathbb{R}) \to PC([0, T], \mathbb{R})$ by

$$(Fy)(t) = \begin{cases} -\mu \int_0^\psi \frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, y(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, & t \in [0, t_1], \\ H_i(t, y(t)), & t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \\ H_i(s_i, y(s_i)) + \int_0^{s_i} \frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, & t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m. \end{cases} \tag{3.1}$$

It also observe that F is well defined. Now, we show that F is a contraction mapping. Let $y_1,\ y_2 \in PC([0, T], \mathbb{R})$, we have

Case 1. For each $t \in [0, t_1]$, we get

$$\begin{aligned} |(Fy_1)(t) - (Fy_2)(t)| &\leq \mu \int_0^\psi \frac{(\psi - s)^{\alpha-q-1}}{\Gamma(\alpha - q)} |f(s, y_1(s)) - f(s, y_2(s))|ds \\ &+ \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_1(s)) - f(s, y_2(s))|ds \\ &\leq L\Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha - q + 1)} + \frac{T^\alpha}{\Gamma(\alpha + 1)}\Big]\|y_1 - y_2\|_{PC}. \end{aligned}$$

Case 2. For each $t \in (t_i, s_i]$, we find

$$|(Fy_1)(t) - (Fy_2)(t)| \leq |H_i(t, y_1(t)) - H_i(t, y_2(t))| \leq L_{h_i}\|y_1 - y_2\|_{PC}.$$

Case 3. For each $t \in (s_i, t_{i+1}]$, we obtain

$$\begin{aligned} |(Fy_1)(t) - (Fy_2)(t)| &\leq |H_i(s_i, y_1(s_i)) - H_i(s_i, y_2(s_i))| + \int_0^{s_i} \frac{(s_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_1(s)) - f(s, y_2(s))|ds \\ &+ \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_1(s)) - f(s, y_2(s))|ds \\ &\leq \Big[L_{h_i} + \frac{L}{\Gamma(\alpha + 1)}(t_{i+1}^\alpha + s_i^\alpha)\Big]\|y_1 - y_2\|_{PC}. \end{aligned}$$

From the above simulation, we conclude that $\|(Fy_1) - (Fy_2)\|_{PC} \leq \Delta\|y_1 - y_2\|_{PC}$, which implies that F is a contraction and $\exists$ a unique solution $y \in PC([0, T], \mathbb{R})$ on $[0, T]$. □

3.2. Existence result via Krasnoselskiis fixed point theorem:

Theorem 3.2. *Suppose that A_1 satisfied and the following axioms hold*

(A_2) *There is a constant $L_f > 0$, such that*

$$|f(t,y)| \leq L_f(1+|y|),\ t \in [s_i, t_{i+1}],\ \forall\ y \in \mathbb{R}.$$

(A_3) *There exist a function $\eta_i(t),\ i = 1,2,\dots,m$, such that*

$$|H_i(t,y)| \leq \eta_i(t),\ t \in [t_i, s_i],\ \forall\ y \in \mathbb{R}.$$

For the convenience, we also assume that $M_i := sup_{t\in[t_i,s_i]}\eta_i(t) < \infty$, and $K := \max\{L_{h_i}\} < 1$, such that for all $i = 1,2,\dots,m$. Then the system (1.3) has at least one solution.

Proof. For each $r > 0$, let us consider two operators P and Q on $B_{\alpha,r} = \{y \in PC([0,T],\mathbb{R}) : \|y\|_{PC} \leq r\}$ such as

$$(Py)(t) = \begin{cases} 0, & t \in [0,t_1], \\ H_i(t,y(t)), & t \in (t_i,s_i],\ i = 1,2,\dots,m, \\ H_i(s_i,y(s_i)), & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m. \end{cases}$$

and

$$(Qy)(t) = \begin{cases} -\mu\int_0^{\psi}\frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}f(s,y(s))ds - \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds, & t \in [0,t_1], \\ 0, & t \in (t_i,s_i],\ i = 1,2,\dots,m, \\ \int_0^{s_i}\frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds - \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds, & t \in (s_i,t_{i+1}],\ i = 0,1,2,\dots,m. \end{cases}$$

For better readability, we break the proof into a sequence of following steps.

Step 1. Under the assumption (A_2), we prove that $Py + Qy \in B_{\alpha,r}$.

Case 1. For $t \in [0,t_1]$, we get

$$\begin{aligned} |(Py+Qy)(t)| &\leq \mu\int_0^{\psi}\frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}|f(s,y(s))|ds + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds \\ &\leq \Big[\frac{\mu L_f\psi^{\alpha-q}}{\Gamma(\alpha-q+1)} + \frac{L_f T^{\alpha}}{\Gamma(\alpha+1)}\Big](1+r) \leq r. \end{aligned}$$

Case 2. For each $t \in (t_i,s_i]$, we have

$$|(Py+Qy)(t)| \leq |H_i(t,y(t))| \leq M_i \leq r.$$

Case 3. For each $t \in (s_i,t_{i+1}]$, we obtain

$$\begin{aligned} |(Py+Qy)(t)| &\leq |H_i(s_i,y(s_i))| + \int_0^{s_i}\frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds \\ &+ \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds \end{aligned}$$

$$\leq M_i + \Big[\frac{L_f(s_i^\alpha + t_{i+1}^\alpha)}{\Gamma(\alpha+1)}\Big](1+r) \leq r.$$

So, we infer that $Py + Qy \in B_{\alpha,r}$.

Step 2. By using the condition (A_1), we show that P is contraction on $B_{\alpha,r}$. Let $y_1,\ y_2 \in B_{\alpha,r}$, we have

Case 1. For $t \in [0, t_1]$, we occur

$$|Py_1(t) - Py_2(t)| \leq 0. \tag{3.2}$$

Case 2. For $t \in (t_i, s_i]$, we find

$$|Py_1(t) - Py_2(t)| \leq |H_i(t, y_1(t)) - H_i(t, y_2(t))| \leq L_{h_i}\|y_1 - y_2\|_{PC}. \tag{3.3}$$

Case 3. For $t \in (s_i, t_{i+1}]$, we obtain

$$|Py_1(t) - Py_2(t)| \leq |H_i(s_i, y_1(s_i)) - H_i(s_i, y_2(s_i))| \leq L_{h_i}\|y_1 - y_2\|_{PC}. \tag{3.4}$$

From the above inequalities (3.2), (3.3) and (3.4), we find

$$\|Py_1(t) - Py_2(t)\|_{PC} \leq K\|y_1 - y_2\|_{PC}.$$

Hence P is a contraction on $B_{\alpha,r}$.

Step 3. In this step, we show that Q is continuous. Let y_n be any convergent sequence such that $y_n \to y$ in $PC([0,T],\mathbb{R})$.

Case 1. For each $t \in [0, t_1]$, we have

$$\begin{aligned} |Qy_n(t) - Qy(t)| &\leq \mu\int_0^\psi \frac{(\psi-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}|f(s,y_n(s)) - f(s,y(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s)) - f(s,y(s))|ds \\ &\leq \Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha-q+1)} - \frac{T^\alpha}{\Gamma(\alpha+1)}\Big]\|f(.,y_n(.)) - f(.,y(.))\|_{PC}. \end{aligned}$$

Case 2. For each $t \in (t_i, s_i]$, we obtain

$$|Qy_n(t) - Qy(t)| \leq 0.$$

Case 3. For each $t \in (s_i, t_{i+1}]$, we occur

$$\begin{aligned} |Qy_n(t) - Qy(t)| &\leq \int_0^{s_i} \frac{(s_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s)) - f(s,y(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s)) - f(s,y(s))|ds \\ &\leq \frac{s_i^\alpha + t_{i+1}^\alpha}{\Gamma(\alpha+1)}\|f(.,y_n(.)) - f(.,y(.))\|_{PC}. \end{aligned}$$

Thus, we conclude from the above cases that $\|Qy_n - Qy\|_{PC} \to 0$ as $n \to \infty$.

Step 4. Finally by using the assumption (A_2), we show that Q is compact. Since $\|Qy\|_{PC} \leq \frac{L_f T}{\Gamma(\alpha+1)}(1+r) < r$, so that we can say Q is uniformly bounded on $B_{\alpha,r}$. Therefore, it remains to prove that Q maps bounded set into equicontinuous set of $B_{\alpha,r}$.

Case 1. For each $t \in [0, t_1],\ 0 \le E_1 \le E_2 \le t_1,\ y \in B_{\alpha,r}$, we obtain

$$|Qy(E_2) - Qy(E_1)| \le \int_0^{E_1} \frac{(E_1-s)^{\alpha-1}-(E_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, y(s))|ds - \int_{E_1}^{E_2} \frac{(E_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, y(s))|ds,$$
$$\le \begin{cases} \frac{L_f(1+r)}{\Gamma(1+\alpha)}\{(E_2^{\alpha} - E_1^{\alpha}) + 2(E_1 - E_2)^{\alpha}\}, & 0 < \alpha < 1, \\ \frac{L_f(1+r)}{\Gamma(\alpha+1)}(E_1^{\alpha} - E_2^{\alpha}), & \alpha \ge 1. \end{cases}$$

Case 2. For each $t \in (t_i, s_i],\ t_i < E_1 < E_2 \le s_i,\ y \in B_{\alpha,r}$, we have

$$|Qy(E_2) - Qy(E_1)| = 0.$$

Case 3. For each $t \in (s_i, t_{i+1}],\ s_i < E_1 < E_2 \le t_{i+1},\ y \in B_{\alpha,r}$, we establish

$$|Qy(E_2) - Qy(E_1)| \le \begin{cases} \frac{L_f(1+r)}{\Gamma(1+\alpha)}\{(E_2^{\alpha} - E_1^{\alpha}) + 2(E_1 - E_2)^{\alpha}\}, & 0 < \alpha < 1, \\ \frac{L_f(1+r)}{\Gamma(\alpha+1)}(E_1^{\alpha} - E_2^{\alpha}), & \alpha \ge 1. \end{cases}$$

From the above estimation, we observe that $|Qy(E_2) - Qy(E_1)| \to 0$ as $E_2 \to E_1$. So the operator Q is equicontinuous.

Hence, we can say that $Q : B_{\alpha,r} \to B_{\alpha,r}$ is continuous and completely continuous. With the statement of Krasnoselskii's fixed point theorem [13], we explore that $F = P + Q$ has atleast a fixed point on $[0, T]$. The proof is complete. □

3.3. Existence result via Nonlinear Alternative of Leray-Schauder type fixed point theorem:

Theorem 3.3. *Assume that*

(A_4) *There exist* $S_f \in L^1([0, T], \mathbb{R}^+),\ S_{h_i} \in L^1([t_i, s_i], \mathbb{R}^+)$ *and the continuous nondecreasing functions* $\omega,\ \omega^* : [0, \infty) \to [0, \infty)$ *such that*

$$|f(t, y)| \le S_f(t)\omega(|y|),\ (t, y) \in [0, T] \times \mathbb{R},$$

and

$$|h_i(t, y)| \le S_{h_i}(t)\omega^*(|y|),\ (t, y) \in (t_i, s_i] \times \mathbb{R}.$$

(A_5) *There exist a number* $\mathcal{M} > 0$ *such that*

$$\frac{\mathcal{M}}{\omega^*(\mathcal{M})\|S_{h_i}\|_{L^1} + \omega(\mathcal{M})\|S_f\|_{L^1} \max\left(\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha-q+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)},\ \max_{i=1,2,\dots,m} \frac{(s_i{}^{\alpha} + t_{i+1}^{\alpha})}{\Gamma(\alpha+1)}\right)} > 1,$$

satisfy.

Then the system (1.3) has at least one solution on $[0, T]$.

Proof. For the proof of this result first we consider an operator F defined as in (3.1) and rest of the part is divided in the following steps.

Step 1. In starting, we show that F maps bounded sets (balls) into bounded sets in $PC([0, T], \mathbb{R})$ sothat we consider a ball $B_{\alpha,r}$ defined in Theorem 3.2 then by the axiom (A_4), we have

Case 1. For each $t \in [0, t_1]$, we find

$$\begin{aligned} |(Fy)(t)| &\leq \mu \int_0^{\psi} \frac{(\psi - s)^{\alpha-q-1}}{\Gamma(\alpha - q)} |f(s, y(s))| ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds \\ &\leq S_f(s)\omega(\|y\|_{PC})\Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha - q + 1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big] \\ &\leq \omega(r)\|S_f\|_{L^1}\Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha - q + 1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big]. \end{aligned}$$

Case 2. For $t \in (t_i, s_i]$, we get

$$|(Fy)(t)| = |H_i(t, y(t))| \leq S_{h_i}(t)\omega^*(\|y\|_{PC}) \leq \omega^*(r)\|S_{h_i}\|_{L^1}.$$

Case 3. For $t \in (s_i, t_{i+1}]$, we obtain

$$\begin{aligned} |(Fy)(t)| &\leq |H_i(s_i, y(s_i))| + \int_0^{s_i} \frac{(s_i - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds \\ &\leq S_{h_i}(s_i)\omega^*(\|y\|_{PC}) + S_f(s)\omega(\|y\|_{PC})\frac{s_i^{\alpha}}{\Gamma(\alpha+1)} + S_f(s)\omega(\|y\|_{PC})\frac{t_{i+1}^{\alpha}}{\Gamma(\alpha+1)} \\ &\leq \omega^*(r)\|S_{h_i}\|_{L^1} + \omega(r)\|S_f\|_{L^1}\frac{(s_i^{\alpha} + t_{i+1}^{\alpha})}{\Gamma(\alpha+1)}. \end{aligned}$$

Step 2. Now, we shall show that F maps bounded sets into equicontinuous sets of $PC([0, T], \mathbb{R})$. Let $k_1,\ k_2 \in [0, T]$ with $k_1 < k_2$ and $y \in B_{\alpha,r}$ then under the condition (A_4), we need to discuss the following cases

Case 1. For $t \in [0, t_1]$, we have

$$\begin{aligned} |(Fy)(k_2) - (Fy)(k_1)| &\leq \int_0^{k_2} \frac{(k_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + \int_0^{k_1} \frac{(k_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds \\ &\leq \int_0^{k_1} \frac{(k_2 - s)^{\alpha-1} + (k_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds + \int_{k_1}^{k_2} \frac{(k_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s))| ds \\ &\leq \frac{\omega(r)}{\Gamma(\alpha)}\Big[\int_0^{k_1} (k_2 - s)^{\alpha-1} + (k_1 - s)^{\alpha-1} S_f(s) ds + \int_{k_1}^{k_2} (k_2 - s)^{\alpha-1} S_f(s) ds\Big]. \end{aligned}$$

Case 2. For $t \in (t_i, s_i]$, we get

$$|(Fy)(k_2) - (Fy)(k_1)| = |H_i(k_2, y(k_2)) - H_i(k_1, y(k_1))|.$$

Case 3. For $t \in (s_i, t_{i+1}]$, we obtain

$$|(Fy)(k_2) - (Fy)(k_1)| \leq \frac{\omega(r)}{\Gamma(\alpha)}\Big[\int_0^{k_1} (k_2 - s)^{\alpha-1} + (k_1 - s)^{\alpha-1} S_f(s) ds + \int_{k_1}^{k_2} (k_2 - s)^{\alpha-1} S_f(s) ds\Big].$$

As $k_1 \to k_2$, the right-hand side of above inequalities in Step 2 tends to zero independently of $y \in B_{\alpha,r}$. Hence, by the Arzel-Ascoli theorem the operator $F : PC([0, T], \mathbb{R}) \to PC([0, T], \mathbb{R})$ is completely continuous.

Step 3. In this last step of the proof, we show that F has a fixed point in $\overline{E}$. Let $y = \kappa F y$ for some $\kappa \in (0, 1)$ then from the conditions (A_4) and (A_5), we conclude that

Case 1. For $t \in [0, t_1]$, we have

$$|y| = |\kappa F y| \leq \omega(\|y\|_{PC})\|S_f\|_{L^1}\Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha-q+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big],$$

which implies that

$$\frac{\|y\|_{PC}}{\omega(\|y\|_{PC})\|S_f\|_{L^1}\Big[\frac{\mu\psi^{\alpha-q}}{\Gamma(\alpha-q+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}\Big]} \leq 1.$$

Case 2. For $t \in (t_i, s_i]$, we get

$$|y| = |\kappa F y| \leq \omega^*(\|y\|_{PC})\|S_{h_i}\|_{L^1},$$

which implies that

$$\frac{\|y\|_{PC}}{\omega^*(\|y\|_{PC})\|S_{h_i}\|_{L^1}} \leq 1.$$

Case 3. For $t \in (s_i, t_{i+1}]$, we obtain

$$|y| = |\kappa F y| \leq \omega^*(\|y\|_{PC})\|S_{h_i}\|_{L^1} + \omega(\|y\|_{PC})\|S_f\|_{L^1}\frac{({s_i}^\alpha + t_{i+1}^\alpha)}{\Gamma(\alpha+1)},$$

which implies that

$$\frac{\|y\|_{PC}}{\omega^*(\|y\|_{PC})\|S_{h_i}\|_{L^1} + \omega(\|y\|_{PC})\|S_f\|_{L^1}\frac{({s_i}^\alpha + t_{i+1}^\alpha)}{\Gamma(\alpha+1)}} \leq 1.$$

According to the assumed condition (A_5), we know that there exist $\mathcal{M}$ such that $\mathcal{M} \neq \|y\|_{PC}$ and let

$$E = \{y \in PC([0, T], \mathbb{R}) : \|y\|_{PC} < \mathcal{M}\}. \tag{3.5}$$

Thus, we observe that the operator $F : \overline{E} \to PC([0, T], \mathbb{R})$ is continuous and completely continuous. There is no $y \in \partial E$ such that $y = \kappa F y$ for some $\kappa \in (0, 1)$ with the choice of E. Therefore, by the nonlinear alternative of Leray-Schauder type Theorem 2.3, we deduce that F has a fixed point $y \in \overline{E}$ which is a solution of (1.3). The proof is complete. □

4. Example

We consider the following example to verify the uniqueness result:

$$\begin{cases} D^{\frac{1}{4}}u(t) = \frac{|u(t)|}{(1+e^t)},\ t \in (0, 1] \cup (2, 3], \\ u(t) = \frac{|u(t)|}{2(1+|u(t)|)},\ t \in (1, 2], \\ u(0) = 3D^{\frac{1}{2}}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

Here, we have $\alpha = \frac{1}{4},\ t \in [0,3],\ 0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3,\ f(t,u(t)) = \frac{|u(t)|}{(1+e^t)}$ and $h_1(t,u(t)) = \frac{|u(t)|}{2(1+|u(t)|)}$. Let $u_1,\ u_2 \in \mathbb{R}$ and $t \in (0,1] \cup (2,3]$, we obtain $|f(t,u_1) - f(t,u_2)| \leq \frac{1}{2}|u_1 - u_2|$. Let $u_1,\ u_2 \in \mathbb{R}$ and $t \in (1,2]$, we get $|h_1(t,u_1) - h_1(t,u_2)| \leq \frac{1}{2}|u_1 - u_2|$. Moreover, we have $L = \frac{1}{2},\ \mu = .25,\ \psi = \frac{1}{2},\ q = \frac{1}{2}$. We determine that

$$\frac{\mu L \psi^{\alpha-q}}{\Gamma(\alpha - q + 1)} + \frac{LT^\alpha}{\Gamma(\alpha + 1)} < 1.$$

or

$$\max_{i=1,2,\ldots,m} L_{h_i} + \frac{L(t_{i+1}^\alpha + s_i^\alpha)}{\Gamma(\alpha + 1)} < 1.$$

Since all the assumptions of Theorem 3.1 are satisfied so that the system 4.1 has a unique solution on $[0,3]$.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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