Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 2(2): 215-229
DOI:10.3934/Math.2017.2.215
Received: 19 March 2017
Accepted: 28 March 2017
Published: 6 April 2017

Research article

On higher-order anisotropic conservative Caginalp phase-field type models

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Abstract: Our aim in this paper is to study the well-posedness of higher-order (in space) anisotropic conservative phase-field systems. More precisely, we prove the existence and uniqueness of solutions.

Keywords: Conserved phase-field systems; Higher-order systems; Anisotropy Well-posedness

1. Introduction

G. Caginalp proposed in [3] and [4] two phase-field system, namely,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T, \tag{1.1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.2}$$

called nonconserved system, and

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta T, \tag{1.3}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.4}$$

called concerved system (in the sense that, when endowed with Neumann boundary conditions, the spacial average of u is conserved). In this context, u is the order parameter, T is the relative temperature (defined as $T = \tilde{T} - T_E$, where $\tilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature) and f is the derivative of a double-well potential F (a typical choice is $F(s) = \frac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, we have set all physical parameters equal to one. These systems have been introduced to model phase transition phenomena, such as melting-solidication phenomena, and have been much studied from a mathematical point of view. We refer the reader to, e.g., [3, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 21, 22, 23, 25].

Both systems are based on the (total Ginzburg-Landau) free energy

$$\Psi_{GL} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \tag{1.5}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^3$, with boundary Γ), and the enthalpy

$$H = u + T. \tag{1.6}$$

As far as the evolution equations for the order parameter are concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial u} = -\frac{D\Psi_{GL}}{Du}, \tag{1.7}$$

for the nonconserved model, and

$$\frac{\partial u}{\partial u} = \Delta\frac{D\Psi_{GL}}{Du}, \tag{1.8}$$

for the conserved one, where $\frac{D}{Du}$ denotes a variational derivative with respect to u, which yields (1.1) and (1.3), respectively. Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\mathrm{divq}, \tag{1.9}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla T, \tag{1.10}$$

we obtain (1.2).

In (1.5), the term $|\nabla u|^2$ models short-ranged interactions. It is however interesting to note that such a term is obtained by truncation of higher-order ones; it can also be seen as a first- order approximation of a nonlocal term accounting for long-ranged interactions [11].

G. Caginalp and Esenturk recently proposed in [6] (see also [20]) higher-order phase-field models in order to account for anisotropic interfaces (see also [7] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these autors proposed the following modified (total) free energy

$$\Psi_{HOGL} = \int_\Omega (\frac{1}{2}\sum_{i=1}^{k}\sum_{|\beta|=i} a_\beta|\mathcal{D}^\beta u|^2 + F(u) - uT - \frac{1}{2}T^2)\mathrm{d}x, \quad k \in \mathbb{N}, \tag{1.11}$$

where, for $\beta = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$,

$$|\beta| = k_1 + k_2 + k_3$$

and, for $\beta \neq (0,0,0)$,

$$\mathcal{D}^{\beta} = \frac{\partial^{|\beta|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)} v = v$).

A. Miranville studied in [17] the corresponding nonconserved higher-order phase-field system.

As far as the conserved case is concerned, the above generalized free energy yields, procceding as above, the following evolution equation for the order parameter u:

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\beta|=i} a_{\beta} \mathcal{D}^{2\beta} u - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right), \tag{1.12}$$

In particular, for $k = 1$ (anisotropic conserved Caginalp phase-field), we have an equation of the form

$$\frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)$$

and, for $k = 2$ (fourth-order anisotropic conserved Caginalp phase-field system), we have an equation of the form

$$\frac{\partial u}{\partial t} - \Delta \sum_{i,j=1}^{3} a_{ij} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} + \Delta \sum_{i=1}^{3} b_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right).$$

L. Cherfils A. Miranville and S. Peng have studied in [8] the corresponding higher-order isotropic equation (without the coupling with the temperature), namely, the equation

$$\frac{\partial u}{\partial t} - \Delta P(-\Delta) u - \Delta f(u) = 0,$$

where

$$P(s) = \sum_{i=1}^{k} a_i s^i, \quad a_k > 0, \quad k \geqslant 1,$$

endowed with the Dirichlet/Navier boundary conditions

$$u = \Delta u = ... = \Delta^k u = 0 \quad on \quad \Gamma.$$

Our aim in this paper is to study the model consisting of the higher-order anisotropic equation (1.12) and the temperature equation

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}. \tag{1.13}$$

In particular, we obtain the existence and uniqueness of solutions.

2. Setting of the problem

We consider the following initial and boundary value problem, for $k \in \mathbb{N}$, $k \geqslant 2$ (the case $k = 1$ can be treated as in the original conserved system; see, e.g., [23]):

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\beta|=i} a_\beta \mathcal{D}^{2\beta} u - \Delta f(u) = -\Delta(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}), \tag{2.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.2}$$

$$\mathcal{D}^\beta u = \alpha = 0 \quad on \quad \Gamma, \quad |\beta| \leqslant k, \tag{2.3}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We assume that

$$a_\beta > 0, \quad |\beta| = k, \tag{2.5}$$

and we introduce the elliptic operator A_k defined by

$$\langle A_k v, w \rangle_{H^{-k}(\Omega), H_0^k(\Omega)} = \sum_{|\beta|=k} a_\beta ((\mathcal{D}^\beta v, \mathcal{D}^\beta w)), \tag{2.6}$$

where $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$. Furthermore, $((.,.))$ denotes the usual L^2-scalar product, with associated norm $\|.\|$. More generally, we denote by $\|.\|_X$ the norm on the Banach space X; we also set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Dirichlet boudary conditions. We can note that

$$(v, w) \in H_0^k(\Omega)^2 \mapsto \sum_{|\beta|=k} a_\beta ((\mathcal{D}^\beta v, \mathcal{D}^\beta w))$$

is bilinear, symmetric, continuous and coercive, so that

$$A_k : H_0^k(\Omega) \to H^{-k}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2k$ (see [1] and [2]) that A_k is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A_k) = H^{2k}(\Omega) \cap H_0^k(\Omega),$$

where, for $v \in D(A_k)$,

$$A_k v = (-1)^k \sum_{|\beta|=k} a_\beta \mathcal{D}^{2\beta} v.$$

We further note that $D(A_k^{\frac{1}{2}}) = H_0^k(\Omega)$ and, for $(v, w) \in D(A_k^{\frac{1}{2}})^2$,

$$((A_k^{\frac{1}{2}} v, A_k^{\frac{1}{2}} w)) = \sum_{|\beta|=k} a_\beta ((\mathcal{D}^\beta v, \mathcal{D}^\beta w)).$$

We finally note that (see, e.g., [24]) $\|A_k.\|$ (resp., $\|A_k^{\frac{1}{2}}.\|$) is equivalent to the usual H^{2k}-norm (resp., H^k-norm) on $D(A_k)$ (resp., $D(A_k^{\frac{1}{2}})$).

Similarly, we can define the linear operator $\bar{A}_k = -\Delta A_k$

$$\bar{A}_k : H_0^{k+1}(\Omega) \to H^{-k-1}(\Omega)$$

which is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(\bar{A}_k) = H^{2k+2}(\Omega) \cap H_0^{k+1}(\Omega),$$

where, for $v \in D(\bar{A}_k)$,

$$\bar{A}_k v = (-1)^{k+1} \Delta \sum_{|\beta|=k} a_\beta \mathcal{D}^{2\beta} v.$$

Furthermore, $D(\bar{A}_k^{\frac{1}{2}}) = H_0^{k+1}(\Omega)$ and, for $(v, w) \in D(\bar{A}_k^{\frac{1}{2}})$,

$$((\bar{A}_k^{\frac{1}{2}} v, \bar{A}_k^{\frac{1}{2}} w)) = \sum_{|\beta|=k} a_\beta ((\nabla \mathcal{D}^\beta v, \nabla \mathcal{D}^\beta w)).$$

Besides $\|\bar{A}_k.\|$ (resp., $\|\bar{A}_k^{\frac{1}{2}}.\|$) is equivalent to the usual H^{2k+2}-norm (resp., H^{k+1}-norm) on $D(\bar{A}_k)$ (resp., $D(\bar{A}_k^{\frac{1}{2}})$).

We finally consider the operator $\tilde{A}_k = (-\Delta)^{-1} A_k$, where

$$\tilde{A}_k : H_0^{k-1}(\Omega) \to H^{-k+1}(\Omega);$$

note that, as $-\Delta$ and A_k commute, then the same holds for $(-\Delta)^{-1}$ and A_k, so that $\tilde{A}_k = A_k(-\Delta)^{-1}$.

We have the (see [17])

Lemme 2.1. *The operator $\tilde{A}_k$ is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain*

$$D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega),$$

where, for $v \in D(\tilde{A}_k)$

$$\tilde{A}_k v = (-1)^k \sum_{|\beta|=k} a_\beta \mathcal{D}^{2\beta} (-\Delta)^{-1} v.$$

Furthermore, $D(\tilde{A}_k^{\frac{1}{2}}) = H_0^{k-1}(\Omega)$ and, for $(v, w) \in D(\tilde{A}_k^{\frac{1}{2}})$,

$$((\tilde{A}_k^{\frac{1}{2}} v, \tilde{A}_k^{\frac{1}{2}} w)) = \sum_{|\beta|=k} a_\beta ((\mathcal{D}^\beta (-\Delta)^{-\frac{1}{2}} v, \mathcal{D}^\beta (-\Delta)^{-\frac{1}{2}} w)).$$

Besides $\|\tilde{A}_k.\|$ (resp., $\|\tilde{A}_k^{\frac{1}{2}}.\|$) is equivalent to the usual H^{2k-2}-norm (resp., H^{k-1}-norm) on $D(\tilde{A}_k)$ (resp., $D(\tilde{A}_k^{\frac{1}{2}})$).

Proof. We first note that $\tilde{A}_k$ clearly is linear and unbounded. Then, since $(-\Delta)^{-1}$ and A_k commute, it easily follows that $\tilde{A}_k$ is selfadjoint.

Next, the domain of $\tilde{A}_k$ is defined by

$$D(\tilde{A}_k) = \{v \in H_0^{k-1}(\Omega), \tilde{A}_k v \in L^2(\Omega)\}.$$

Noting that $\tilde{A}_k v = f, f \in L^2(\Omega), v \in D(\tilde{A}_k)$, is equivalent to $A_k v = -\Delta f$, where $-\Delta f \in H^2(\Omega)'$, it follows from the elliptic regularity results of [1] and [2] that $v \in H^{2k-2}(\Omega)$, so that $D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega)$.

Noting then that $\tilde{A}_k^{-1}$ maps $L^2(\Omega)$ onto $H^{2k-2}(\Omega)$ and recalling that $k \geqslant 2$, we deduce that $\tilde{A}_k$ has compact inverse.

We now note that, considering the spectral properties of $-\Delta$ and A_k (see, e.g., [24]) and recalling that these two operators commute, $-\Delta$ and A_k have a spectral basis formed of common eigenvectors. This yields that, $\forall s_1, s_2 \in \mathbb{R}$, $(-\Delta)^{s_1}$ and $A_k^{s_2}$ commute.

Having this, we see that $\tilde{A}_k^{\frac{1}{2}} = (-\Delta)^{-\frac{1}{2}} A_k^{\frac{1}{2}}$, so that $D(\tilde{A}_k^{\frac{1}{2}}) = H_0^{k-1}(\Omega)$, and for $(v, w) \in D(\tilde{A}_k^{\frac{1}{2}})^2$,

$$((\tilde{A}_k^{\frac{1}{2}} v, \tilde{A}_k^{\frac{1}{2}} w)) = \sum_{|\beta|=k} a_\beta((\mathcal{D}^\beta(-\Delta)^{-\frac{1}{2}} v, \mathcal{D}^\beta(-\Delta)^{-\frac{1}{2}} w)).$$

Finally, as far as the equivalences of norms are concerned, we can note that, for instance, the norm $\|\tilde{A}_k^{\frac{1}{2}}.\|$ is equivalent to the norm $\|(-\Delta)^{-\frac{1}{2}}.\|_{H^k(\Omega)}$ and, thus, to the norm $\|(-\Delta)^{\frac{k-1}{2}}.\|$.

□

Having this, we rewrite (2.1) as

$$\frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta f(u) = -\Delta(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}), \tag{2.7}$$

where

$$B_k v = \sum_{i=1}^{k-1} (-1)^i \sum_{|\beta|=i} a_\beta \mathcal{D}^{2\beta} v.$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^2(\mathbb{R}), \quad f(0) = 0, \tag{2.8}$$

$$f' \geqslant -c_0, \quad c_0 \geqslant 0, \tag{2.9}$$

$$f(s)s \geqslant c_1 F(s) - c_2 \geqslant -c_3, \quad c_1 > 0, \quad c_2, \quad c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.10}$$

$$F(s) \geqslant c_4 s^4 - c_5, \quad c_4 > 0, \quad c_5 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.11}$$

where $F(s) = \int_0^s f(\tau) d\tau$. In particular, the usual cubic nonlinear term $f(s) = s^3 - s$ satisfies these assumptions.

Throughout the paper, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line. Similary, the same letter Q denotes (positive) monotone increasing (with respect to each argument) and continuous functions which may vary from line to line.

3. A priori estimates

We multiply (2.7) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (2.2) by $\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}$, sum the two resulting equalities and integrate over Ω and by parts. This gives

$$\frac{d}{dt}(\|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2)$$
$$+ 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 = 0 \tag{3.1}$$

(note indeed that $\|\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 = \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2$), where

$$B_k^{\frac{1}{2}}[u] = \sum_{i=1}^{k-1}\sum_{|\beta|=i} a_\beta\|\mathcal{D}^\beta u\|^2 \tag{3.2}$$

(note that $B_k^{\frac{1}{2}}[u]$ is not necessarily nonnegative). We can note that, owing to the interpolation inequality

$$\|(-\Delta)^{\frac{i}{2}}v\| \leqslant c(i)\|(-\Delta)^{\frac{m}{2}}v\|^{\frac{i}{m}}\|v\|^{1-\frac{i}{m}}, \tag{3.3}$$

$$v \in H^m(\Omega), \quad i \in \{1, ..., m-1\}, \quad m \in \mathbb{N}, \quad m \geqslant 2,$$

there holds

$$|B_k^{\frac{1}{2}}[u]| \leqslant \frac{1}{2}\|A_k^{\frac{1}{2}}u\|^2 + c\|u\|^2. \tag{3.4}$$

This yields, employing (2.11),

$$\|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x \geqslant \frac{1}{2}\|A_k^{\frac{1}{2}}u\|^2 + \int_\Omega F(u)\mathrm{d}x + c\|u\|_{L^4(\Omega)}^4 - c'\|u\|^2 - c'',$$

whence

$$\|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x \geqslant c(\|u\|_{H^k(\Omega)}^2 + \int_\Omega F(u)\mathrm{d}x) - c', \quad c > 0, \tag{3.5}$$

nothing that, owing to Young's inequality,

$$\|u\|^2 \leqslant \epsilon\|u\|_{L^4(\Omega)}^4 + c(\epsilon), \quad \forall\epsilon > 0. \tag{3.6}$$

We then multiply (2.7) by $(-\Delta)^{-1}u$ and have, owing to (2.10) and the interpolation inequality (3.3),

$$\frac{d}{dt}\|u\|_{-1}^2 + c(\|u\|_{H^k(\Omega)}^2 + \int_\Omega F(u)\mathrm{d}x) \leqslant c'(\|u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2) + c'',$$

hence, proceeding as above and employing, in particular, (2.11)

$$\frac{d}{dt}\|u\|_{-1}^2 + c(\|u\|_{H^k(\Omega)}^2 + \int_\Omega F(u)\mathrm{d}x) \leqslant c'(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2) + c'', \quad c > 0. \tag{3.7}$$

Summing (3.1) and δ_1 times (3.7), where $\delta_1 > 0$ is small enough, we obtain a differential inegality of the form

$$\frac{d}{dt}E_1 + c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\frac{\partial u}{\partial t}\|^2_{-1} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) \leqslant c', \quad c > 0, \tag{3.8}$$

where

$$E_1 = \|A_k^{\frac{1}{2}}u\|^2 + B_k^{\frac{1}{2}}[u] + 2\int_\Omega F(u)\mathrm{d}x + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 + \delta_1\|u\|^2_{-1}$$

satisfies, owing to (3.5)

$$E_1 \geqslant c(\|u\|^2_{H^k(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0. \tag{3.9}$$

Multiplying (2.2) by $-\Delta\alpha$, we then obtain

$$\frac{d}{dt}(\|\Delta\alpha\|^2 - 2((\frac{\partial \alpha}{\partial t}, \Delta\alpha)) + 2((\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha))) + \|\Delta\alpha\|^2 \leqslant \|\frac{\partial u}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2,$$

which yields, employing the interpolation inequality

$$\|v\|^2 \leqslant c\|v\|_{-1}\|v\|_{H^1(\Omega)}, \quad v \in H^1_0(\Omega), \tag{3.10}$$

the differential inequality, with $0 < \epsilon << 1$ is small enough

$$\begin{aligned} &\frac{d}{dt}(\|\Delta\alpha\|^2 - 2((\frac{\partial \alpha}{\partial t}, \Delta\alpha)) + 2((\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha))) + c\|\alpha\|^2_{H^2(\Omega)} \\ &\leqslant c'(\|\frac{\partial u}{\partial t}\|^2_{-1} + \epsilon\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}), \quad c > 0. \end{aligned} \tag{3.11}$$

We now differentiate (2.7) with respect to time to find, owing to (2.2),

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta A_k\frac{\partial u}{\partial t} - \Delta B_k\frac{\partial u}{\partial t} - \Delta(f'(u)\frac{\partial u}{\partial t}) = -\Delta(\Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha - \frac{\partial u}{\partial t}), \tag{3.12}$$

together with the boundary condition

$$\mathcal{D}^\beta\frac{\partial u}{\partial t} = 0 \quad on \quad \Gamma, \quad |\beta| \leqslant k. \tag{3.13}$$

We multiply (3.11) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and obtain, owing to (2.9) and the interpolation inequality (3.3),

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2_{-1} + c\|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)} \leqslant c'(\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\alpha\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2), \quad c > 0,$$

hence,owing to (3.10), the differential inequality

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2_{-1} + c\|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)} \leqslant c'(\|\frac{\partial u}{\partial t}\|^2_{-1} + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}), \quad c > 0. \tag{3.14}$$

Summing finally (3.8), δ_2 times (3.11) and δ_3 times (3.14), where $\delta_2, \delta_3 > 0$ are small enough, we find a differential inequality of the form

$$\frac{dE_2}{dt} + c(E_2 + \|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)}) \leqslant c', \quad c > 0, \tag{3.15}$$

where

$$E_2 = E_1 + \delta_2(\|\Delta\alpha\|^2 - 2((\frac{\partial \alpha}{\partial t}, \Delta\alpha)) + 2((\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha))) + \delta_3\|\frac{\partial u}{\partial t}\|^2_{-1}.$$

Owing to the continuous embedding $H^{2k+1}(\Omega) \subset C(\bar{\Omega})$, we deduce that

$$|\int_\Omega F(u_0)\mathrm{d}x| \leqslant Q(\|u_0\|_{H^{2k+1}(\Omega)})$$

and since

$$(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}(0) = -(-\Delta)^{\frac{1}{2}}A_k u_0 - (-\Delta)^{\frac{1}{2}}B_k u_0 - (-\Delta)^{\frac{1}{2}}f(u_0) + (-\Delta)^{\frac{1}{2}}(\alpha_1 - \Delta\alpha_1),$$

we see that $(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\|\frac{\partial u}{\partial t}(0)\|_{-1} \leqslant Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}). \tag{3.16}$$

Furthermore E_2 satisfies

$$E_2 \geqslant c(\|u\|^2_{H^k(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0. \tag{3.17}$$

It thus follows from (3.15), (3.16), (3.17) and Growall's lemma that

$$\begin{aligned}&\|u(t)\|^2_{H^k(\Omega)} + \|\frac{\partial u}{\partial t}(t)\|^2_{-1} + \|\alpha(t)\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|^2_{H^2(\Omega)}\\ &\leqslant e^{-ct}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0,\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\int_t^{t+r} \|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)}\mathrm{d}s\\ &\leqslant e^{-ct}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'(r), \quad c > 0, \quad t \geqslant 0,\end{aligned} \tag{3.19}$$

$r > 0$ given.

Multiplying next (2.7) by $\tilde{A}_k u$, we find, owing to the interpolation inequality (3.3),

$$\frac{d}{dt}\|\tilde{A}_k^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^{2k}(\Omega)} \leqslant c'(\|u\|^2 + \|f(u)\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2), \quad c > 0,$$

hence, since f and F are continuous and owing to (3.18),

$$\frac{d}{dt}\|\tilde{A}_k^{\frac{1}{2}}u\|^2 + c\|u\|^2_{H^{2k}(\Omega)}$$

$$\leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c, c' > 0, \quad t \geqslant 0. \tag{3.20}$$

Summing (3.15) and (3.22), we have a differential inequality of the form

$$\begin{aligned}&\frac{dE_3}{dt} + c(E_3 + \|u\|^2_{H^{2k}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)})\\ &\leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c, c' > 0, \quad t \geqslant 0,\end{aligned} \tag{3.21}$$

where

$$E_3 = E_2 + \|\tilde{A}_k^{\frac{1}{2}} u\|^2$$

satisfies

$$E_3 \geqslant c(\|u\|^2_{H^k(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}) - c', \quad c > 0. \tag{3.22}$$

In particular, it follows from (3.21) − (3.22) that

$$\int_t^{t+r} \|u\|^2_{H^{2k}(\Omega)}\mathrm{d}s \leqslant e^{-ct}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'(r), \quad c > 0, \quad t \geqslant 0, \tag{3.23}$$

$r > 0$ given.

We now multiply (2.7) by u and obtain, employing (2.9) and the interpolation inequality (3.3)

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^{k+1}(\Omega)} \leqslant c'(\|u\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2), \quad c > 0,$$

whence, proceeding as above,

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^{k+1}(\Omega)} \leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c, c' > 0. \tag{3.24}$$

We also multiply (2.7) by $\frac{\partial u}{\partial t}$ and find

$$\frac{d}{dt}(\|\bar{A}_k^{\frac{1}{2}} u\|^2 + \bar{B}_k^{\frac{1}{2}}[u]) + c\|\frac{\partial u}{\partial t}\|^2 \leqslant c'\|\Delta f(u)\|^2 - 2((\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t})),$$

where

$$\bar{B}_k^{\frac{1}{2}}[u] = \sum_{i=1}^{k-1}\sum_{|\beta|=i} a_\beta \|\nabla \mathcal{D}^\beta u\|^2.$$

Since f is of class C^2, it follows from the continuous embedding $H^2(\Omega) \subset C(\bar{\Omega})$ that

$$\|\Delta f(u)\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}),$$

hence, owing to (3.18),

$$\frac{d}{dt}(\|\bar{A}_k^{\frac{1}{2}} u\|^2 + \bar{B}_k^{\frac{1}{2}}[u]) + c\|\frac{\partial u}{\partial t}\|^2$$

$$\leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) - 2((\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t})) + c'', \quad c, c' > 0. \tag{3.25}$$

Multiply next (2.2) by $-\Delta(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t})$, we have

$$\begin{aligned}&\frac{d}{dt}(\|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t} - \nabla\Delta\frac{\partial \alpha}{\partial t}\|^2) + c(\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2)\\ &\leqslant 2((\Delta\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t})), \quad c > 0\end{aligned} \tag{3.26}$$

(note indeed that $\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2 = \|\nabla\frac{\partial \alpha}{\partial t} - \nabla\Delta\frac{\partial \alpha}{\partial t}\|^2$).

Summing (3.25) and (3.26), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\bar{A}_k^{\frac{1}{2}}u\|^2 + \bar{B}_k^{\frac{1}{2}}[u] + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t} - \nabla\Delta\frac{\partial \alpha}{\partial t}\|^2) + c(\|\frac{\partial u}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\Delta\frac{\partial \alpha}{\partial t}\|^2)\\ &\leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c, c' > 0.\end{aligned} \tag{3.27}$$

Summing finally (3.21), (3.24) and (3.27), we find a differential inegality of the form

$$\begin{aligned}&\frac{dE_4}{dt} + c(E_3 + \|u\|^2_{H^{k+1}(\Omega)} + \|u\|^2_{H^{2k}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^k(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^3(\Omega)})\\ &\leqslant e^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c, c' > 0, \quad t \geqslant 0\end{aligned} \tag{3.28}$$

where

$$E_4 = E_3 + \|u\|^2 + \|\bar{A}_k^{\frac{1}{2}}u\|^2 + \bar{B}_k^{\frac{1}{2}}[u] + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \|\nabla\frac{\partial \alpha}{\partial t} - \nabla\Delta\frac{\partial \alpha}{\partial t}\|^2$$

satisfies, owing to (2.11) and the interpolation inegality (3.3)

$$E_4 \geqslant c(\|u\|^2_{H^{k+1}(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{-1} + \int_\Omega F(u)\mathrm{d}x + \|\alpha\|^2_{H^3(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^3(\Omega)}) - c', \quad c > 0. \tag{3.29}$$

In particular, it follows from (3.28) – (3.29) that

$$\begin{aligned}&\|u(t)\|_{H^{k+1}(\Omega)} + \|\alpha(t)\|_{H^3(\Omega)} + \|\frac{\partial \alpha}{\partial t}(t)\|_{H^3(\Omega)}\\ &\leqslant e^{-ct}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c', \quad c > 0, \quad t \geqslant 0,\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}&\int_t^{t+r}(\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^3(\Omega)})\mathrm{d}s\\ &\leqslant e^{-ct}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'(r), \quad c > 0, \quad t \geqslant 0,\end{aligned} \tag{3.31}$$

r given.

We finally rewrite (2.7) as an elliptic equation, for $t > 0$ fixed,

$$A_k u = -(-\Delta)^{-1}\frac{\partial u}{\partial t} - B_k u - f(u) + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}, \quad \mathcal{D}^\beta u = 0 \quad on \quad \Gamma, \quad |\beta| \leqslant k - 1. \tag{3.32}$$

Multiplying (3.32) by $A_k u$, we obtain, owing to the interpolation inequality (3.3),

$$\|A_k u\|^2 \leqslant c(\|u\|^2 + \|f(u)\|^2 + \|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2),$$

hence, since f is continuous and owing to (3.18)

$$\|u(t)\|_{H^{2k}(\Omega)}^2 \leqslant ce^{-c't}Q(\|u_0\|_{H^{2k+1}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, \|\alpha_1\|_{H^3(\Omega)}) + c'', \quad c' > 0 \quad t \geqslant 0. \tag{3.33}$$

4. Existence and uniqueness of solutions

We first have the following theorem.

Theorem 4.1. *(i) We assume that* $(u_0, \alpha_0, \alpha_1) \in H_0^k(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$*, with* $\int_\Omega F(u_0)\mathrm{d}x < +\infty$*. Then,* (2.1) − (2.4) *possesses at last one solution* $(u, \alpha, \frac{\partial \alpha}{\partial t})$ *such that,* $\forall T > 0$*,* $u(0) = u_0$*,* $\alpha(0) = \alpha_0$*,* $\frac{\partial \alpha}{\partial t}(0) = \alpha_1$*,*

$$u \in L^\infty(\mathbb{R}^+; H_0^k(\Omega)) \cap L^2(0, T; H^{2k}(\Omega) \cap H_0^k(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; H^{-1}(\Omega)) \cap L^2(0, T; H_0^k(\Omega)),$$

$$\alpha, \frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega))$$

and

$$\frac{d}{dt}((-\Delta)^{-1}u, v)) + \sum_{i=1}^{k}\sum_{|\beta|=i} a_i((\mathcal{D}^\beta u, \mathcal{D}^\beta v)) + ((f(u), v)) = \frac{d}{dt}(((u, v)) + ((\nabla u, \nabla v))), \forall v \in C_c^\infty(\Omega),$$

$$\frac{d}{dt}(((\frac{\partial \alpha}{\partial t}, w)) + ((\nabla \frac{\partial \alpha}{\partial t}, \nabla w)) + ((\nabla \alpha, \nabla w))) + ((\nabla \alpha, \nabla w)) = -\frac{d}{dt}((u, w)), \forall w \in C_c^\infty(\Omega),$$

in the sense of distributions.

(ii) If we futher assume that $(u_0, \alpha_0, \alpha_1) \in (H^{k+1}(\Omega) \cap H_0^k(\Omega)) \times (H^3(\Omega) \cap H_0^1(\Omega)) \times (H^3(\Omega) \cap H_0^1(\Omega))$*, then,* $\forall T > 0$*,*

$$u \in L^\infty(\mathbb{R}^+; H^{k+1}(\Omega) \cap H_0^k(\Omega)) \cap L^2(\mathbb{R}^+; H^{k+1}(\Omega) \cap H_0^k(\Omega))$$

$$\frac{\partial u}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega)),$$

$$\alpha \in L^\infty(\mathbb{R}^+; H^3(\Omega) \cap H_0^1(\Omega))$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H^3(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega))$$

The proofs of existence and regularity in (i) and (ii) follow from the a priori estimates derived in the previous section and, e.g., a standard Galerkin scheme.

We then have the following theorem.

Theorem 4.2. *The system* (1.1) − (1.4) *possesses a unique solution with the above regularity.*

Proof. Let $(u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t})$ and $(u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$ be two solutions to (2.1) − (2.3) with initial data $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively. We set

$$(u, \alpha, \frac{\partial \alpha}{\partial t}) = (u^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}) - (u^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t})$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

Then, (u, α) satisfies

$$\frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta(f(u^{(1)}) - f(u^{(2)})) = -\Delta(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}), \tag{4.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}, \tag{4.2}$$

$$\mathcal{D}^\beta u = \alpha = 0 \quad on \quad \Gamma, \quad |\beta| \leqslant k, \tag{4.3}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \tag{4.4}$$

Multiplying (4.1) by $(-\Delta)^{-1}u$ and integrating over Ω, we obtain

$$\frac{d}{dt}\|u\|_{-1}^2 + c\|u\|_{H^k(\Omega)}^2 \leqslant c'(\|u\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2) - 2((f(u^{(1)}) - f(u^{(2)}, u)).$$

We note that

$$f(u^{(1)}) - f(u^{(2)}) = l(t)u,$$

with l defined as

$$l(t) = \int_0^1 f'(su^{(1)}(t) + (1-s)u^{(2)}(t))\mathrm{d}s.$$

Owing to (2.9), we have

$$\begin{aligned} -2((f(u^{(1)}) - f(u^{(2)}, u)) &\leqslant 2c_0\|u\|^2, \\ &\leqslant c\|u\|^2 \end{aligned}$$

and we obtain owing to the intepolation inequalities (3.3) and (3.10),

$$\frac{d}{dt}\|u\|_{-1}^2 + c\|u\|_{H^k(\Omega)}^2 \leqslant c'(\|u\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2), \quad c > 0. \tag{4.5}$$

Next, multiplying (4.2) by $(-\Delta)^{-1}(u + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t})$, we find

$$\begin{aligned} &\frac{d}{dt}(\|\alpha\|^2 + \|\nabla\alpha\|^2 + \|u + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|_{-1}^2) + c(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1(\Omega)}^2) \\ &\leqslant c'(\|u\|^2 + \|\alpha\|^2). \end{aligned} \tag{4.6}$$

Summing then δ_4 times (4.5) and (4.6), where $\delta_4 > 0$ is small enough, we have, employing once more the interpolation inequality (3.10), a differential inequality of the form

$$\frac{dE_5}{dt} \leqslant cE_5, \tag{4.7}$$

where

$$E_5 = \delta_4\|u\|_{-1}^2 + \|\alpha\|^2 + \|\nabla\alpha\|^2 + \|u + \frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|_{-1}^2$$

satisfies

$$E_5 \geqslant c(\|u\|_{-1}^2 + \|\alpha\|_{H^1(\Omega)}^2 + \|\frac{\partial\alpha}{\partial t} - \Delta\frac{\partial\alpha}{\partial t}\|^2), c > 0. \tag{4.8}$$

It follows from (4.7) $-$ (4.8) and Gronwall's lemma that

$$\|u(t)\|_{-1}^2 + \|\alpha(t)\|_{H^1(\Omega)}^2 + \|\frac{\partial\alpha}{\partial t}(t)\|_{H^1(\Omega)}^2 \leqslant ce^{c't}(\|u_0\|_{-1}^2 + \|\alpha_0\|_{H^1(\Omega)}^2 + \|\alpha_1\|_{H^1(\Omega)}^2), t \geqslant 0, \tag{4.9}$$

hence the uniquess, as well as the continuous dependence with respect to the initial data in $H^{-1} \times H^1 \times H^1$-norm. □

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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