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Research article

The mathematics teacher's specialized and interdisciplinary knowledge when implementing a STEAM-based activity on the logistic function

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Abstract: This paper identifies and characterizes the specialized and interdisciplinary knowledge of two mathematics teachers, with different academic profiles, when solving an activity with a STEAM approach on different topics in the area of calculus, such as the study of functions and applications of the derivative. For this purpose, the Mathematics Teacher's Specialized Knowledge model was used, which allowed us to analyze the knowledge of mathematics teachers when faced with activities based on real problems. We opted for a purely qualitative study of an interpretative and instrumental nature, using classroom observations, field notes, and semi-structured interviews as instruments for the collection of information. The research results show the potential of the model when it comes to deepening the understanding and specificity of mathematical content and its didactics. However, it would be convenient to reinforce some of its categories when the objective is to use mathematics to explain the behavior of an event in which other disciplines are involved, especially aspects of contextualization and the structure and teaching of mathematics. From the integrated and interdisciplinary point of view, the didactic knowledge of mathematical content is quite complex to detect in teaching practice, and that is why it is so necessary to operationalize it. Finally, the reflections obtained through this research show the importance of identifying both specialized and interdisciplinary knowledge of the mathematics teacher when dealing with class activities considered within the STEAM framework.

Keywords: MTSK, mathematics teacher knowledge, STEAM education, interdisciplinary approach, teaching and learning of mathematics, problem-based learning, logistic function

1. Introduction

In recent decades, there has been an increasing interest in studying and exploring in depth the elements that should be part of mathematics teachers' knowledge. In fact, different analytical and methodological approaches have been proposed with which to analyze this knowledge, such as those developed by Shulman [1,2], Ponte [3], Ball, Thames and Phelps [4], Tatto et al. [5], Godino [6], Baumbert and Kunter [7], or Carrillo-Yáñez et al. [8]. These models not only highlight the factors involved in mathematics teaching and learning but also help organize, classify, and describe the knowledge used in teaching practice [9]. However, while many studies have focused on characterizing and analyzing teachers' mathematical knowledge in traditional settings, fewer have explored how this knowledge is mobilized in interdisciplinary frameworks such as STEAM. This study seeks to address that gap.

Shulman [1] was the first to try to understand the process of didactic transposition that teachers use to transform their knowledge into understandable knowledge for students. For this reason, he organized the teacher's knowledge into three basic domains: disciplinary knowledge, pedagogical content knowledge, and curricular knowledge, thus recognizing the different natures of knowledge that a teacher must use in practice. From Shulman's work, other theoretical models were proposed with the purpose of systematizing and specifying the elements of mathematics teacher knowledge, in particular. Although each has a different way of characterizing knowledge, they all share the influence of Shulman's pedagogical knowledge content. In fact, it is this construct that has motivated these models to interrelate various knowledge related to the teaching and learning of mathematics [10].

One of the most relevant contributions to the progress of this research comes from Ball et al. [4], who, in order to conceptualize the mathematics teacher's knowledge, proposed the Mathematical Knowledge for Teaching (MKT) model. The most relevant aspect of this model lies in the inclusion of a subdomain of specialized knowledge as part of the domain of subject-matter knowledge: specialized content knowledge. This knowledge will only be useful to the mathematics teacher as opposed to the common mathematical knowledge that any person with a certain level of mathematics may have.

However, for Carrillo et al. [11], this specialization should not refer only to mathematical knowledge itself. It should also encompass professional knowledge more broadly, including aspects related to the teaching and learning of mathematics [12], such as the relationships between mathematical concepts, mathematical language, learning characteristics and theories, or mathematical reasoning, among others. For this reason, Carrillo et al. [11] designed a new knowledge model, called the Mathematics Teacher's Specialized Knowledge (MTSK), with which they intended to provide a solution to the specialization of both mathematical and didactic knowledge.

This model has been used in recent years as a methodological tool to analyze different mathematics teacher practices, such as teaching schedules, peer discussion, or activity design [13]. This study offers a novel contribution by applying the MTSK model within a STEAM framework, an approach that, to our knowledge, has not yet been explored in the literature. Unlike prior studies that examined mathematics teachers' specialized knowledge in conventional classroom environments, our research investigates how this model functions in an interdisciplinary context that integrates mathematical content with scientific, social, and technological elements. By doing so, it not only provides new insights into teacher knowledge in STEAM settings but also opens the discussion about the need to expand current theoretical models to meet emerging educational demands.

In recent years, STEAM education has emerged as an essential framework for preparing students to understand and respond to complex, real-world problems through interdisciplinary learning [14]. In the specific case of mathematics education, this interdisciplinary framework challenges teachers to move beyond the traditional transmission of content and toward the design of learning experiences that connect mathematical concepts with technological tools, scientific inquiry, and socially relevant phenomena [15]. However, despite the momentum of STEAM-based pedagogy, there is still a lack of theoretical models that support mathematics teachers in identifying and organizing the specialized knowledge required to implement this kind of teaching.

In this regard, models like MTSK, which aim to conceptualize the professional knowledge specific to mathematics teachers, must be examined not only in terms of content-specific knowledge but also in their capacity to incorporate dimensions such as technological integration, interdisciplinary reasoning, and context-based understanding. Our study contributes to this field by exploring how the MTSK model performs when applied to a STEAM learning scenario and by identifying whether its current structure adequately reflects the complex knowledge demands placed on teachers in STEAM-oriented classrooms.

Specifically, the mathematical concepts selected for this study (mathematical functions and derivatives) were chosen as they are very useful in other disciplines to express the dependence between two magnitudes and to study the behavior of a phenomenon. From an educational perspective, a function is associated with a graph in the Cartesian plane that, by mere observation, allows us to extract certain relevant information about the variables to be analyzed. Likewise, functions, apart from providing a structured way of analyzing and manipulating data, also allow us to describe and predict behaviors in nature and society, such as population growth, the spread of diseases, or climatic variations, among others.

On the other hand, derivatives allow us to know how a function changes with respect to one of its variables and are useful for analyzing social and scientific phenomena, modeling and predicting behavior in science and engineering, or for promoting analytical reasoning skills, which are valuable in STEAM disciplines. Therefore, it is essential that the mathematics teacher, apart from having all the mathematical and didactic knowledge about these concepts, also knows how to approach them from other fields of knowledge and knows how to use technological resources that help to better understand the concept.

In particular, the proposal we put forward here deals with the logistic function. This type of function is used, for example, in artificial intelligence as applications of neural networks for spam detection, image classification, financial fraud detection, or audio signal processing, and also in probability modeling for predicting rainfall or crop yield and soil salinity in agriculture or even to detect heart diseases or measure the time evolution of an epidemic [17]. In this case, the objective of the activity is to study, with the help of the educational software GeoGebra, the logistic function defined by

$$f(t) = \frac{ab}{a + (b-a)e^{-ct}}, \qquad \text{with} \quad a > 0, \quad b > a \quad \text{and} \quad c > 0,$$

which indicates the evolution over time of the total number of cases generated by an epidemic. The use of the logistic function to describe epidemic behavior exemplifies how mathematical modeling can serve as a bridge between mathematics and the broader STEAM framework, promoting deeper conceptual understanding and authentic problem-solving.

Although the primary focus of this study is on mathematics and science, we intentionally adopt the STEAM acronym to reflect the role of creativity and divergent thinking in mathematical modeling. In this context, the “A” in STEAM is understood not merely as artistic expression, but as a cognitive and intellectual disposition toward creative problem solving, flexibility in reasoning, and innovative approaches to exploring complex phenomena [16]. The use of GeoGebra, the analysis of epidemic data, and the graphical exploration of functions also require representational and design-based thinking, which are central to the artistic and communicative dimension of STEAM. Therefore, this approach invites a more holistic and contextualized interpretation of mathematical modeling beyond traditional STEM boundaries.

In this regard, the study pursues a dual objective. First, it aims to analyze the specialized knowledge that two mathematics teachers mobilize when implementing a STEAM-based learning activity, using the MTSK model as an analytical lens. Second, it seeks to assess the extent to which the MTSK model is adequate for capturing the complex, interdisciplinary knowledge demands involved in STEAM-oriented instruction. Through this analysis, we aim to contribute both to a deeper understanding of teacher knowledge in integrated learning environments and to a critical reflection on the potential need to adapt existing theoretical models to better support interdisciplinary mathematics education.

In sum, this study contributes to the field of mathematics education by bridging a well-established theoretical model—MTSK—with the emerging demands of STEAM instruction. Through the concrete case of a learning activity based on the logistic function and supported by GeoGebra, we aim to illuminate how mathematics teachers activate specialized knowledge when faced with interdisciplinary challenges. This context not only tests the robustness of the MTSK model but also reveals potential areas for its refinement. By analyzing teachers’ practices in this setting, we offer insights that are both theoretically significant and practically relevant for the design of integrated mathematics learning experiences.

2. Theoretical framework

The Mathematics Teacher's Specialized Knowledge (MTSK) is a theoretical model developed by Carrillo-Yáñez et al. [11] that arises from the work of Shulman [1] and Ball et al. [4] in order to identify, characterize, and organize the specific and specialized knowledge of mathematics teachers that differentiates them from other professionals in the sector [8,18]. In particular, this model allows mathematics teachers to have a greater understanding of the nature of such specialized knowledge and its relationship with the teaching-learning of mathematics [19].

The former sought to differentiate subject-matter knowledge (SMK) from pedagogical content knowledge (PCK). It emphasized how teachers manage and plan the different thematic units of the subject, how they formulate questions, or how they assess students’ understanding of disciplinary concepts [1]. For their part, Ball et al. [4], assuming this differentiation of knowledge proposed by Shulman, created a specific analytical model for the mathematics teacher in order to “operationalize the knowledge of the mathematics teacher” [18] (p. 592). They called this model the Mathematical Knowledge for Teaching (MKT).

While pedagogical content knowledge was the most relevant aspect of Shulman's model, horizon content knowledge (HCK) and specialized content knowledge (SCK) are the most relevant aspects of the MKT model. HCK refers to the interrelation of content throughout the different educational stages (from less to more complexity) and to value creativity in students' mathematical reasoning, while SCK is that "mathematical knowledge that will only make sense to the mathematics teacher" [18] (p. 593). However, for Carillo, Contreras, and Flores [20] this model contains some shortcomings as far as the exclusivity of knowledge is concerned: how can we differentiate that exclusive knowledge of the mathematics teacher from the knowledge that other professionals in the subject who are far from teaching, or any other person minimally educated in mathematics, might have? In this regard, Carrillo-Yáñez et al. [11] refined the MKT model by incorporating elements that enable mathematics teachers to organize their knowledge in a comprehensive way. This approach considers both the specialization and the specificity of mathematical and didactic content [12].

Thus, the MTSK model is composed of three main domains: mathematical knowledge, pedagogical content knowledge, and beliefs. In turn, each of these domains is composed of a set of subdomains related to the mathematical content (concepts and results, structure, and language), to the didactic content (characteristics of teaching and learning and curricular content), and to the beliefs and conceptions of the mathematics teacher [8] (Figure 1).

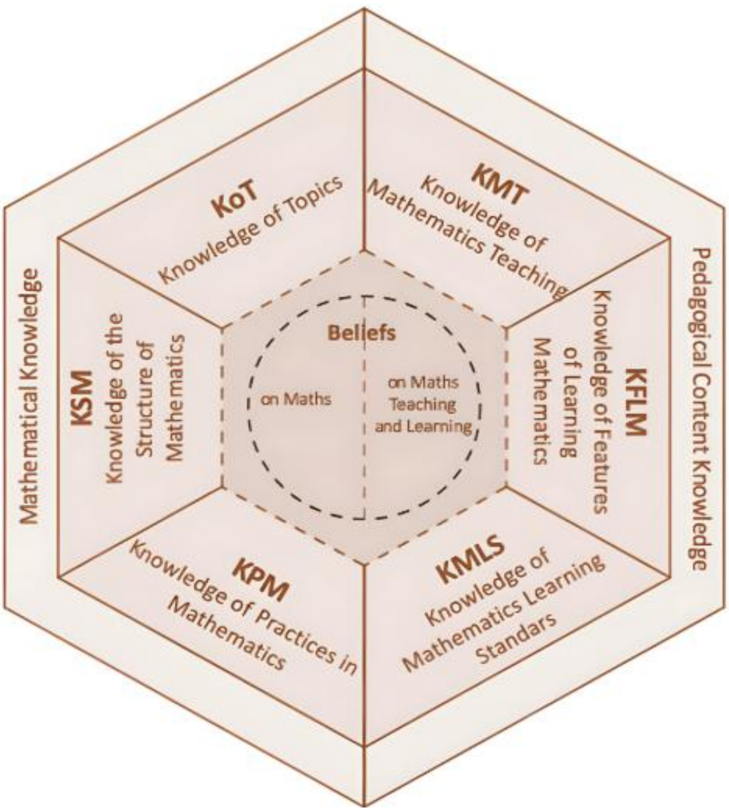

Figure 1. Schematic of the MTSK model [9].

2.1. Mathematical knowledge

The domain of mathematical knowledge refers to the knowledge that the teacher should have about the mathematics taught in school. This knowledge is grounded in the training received and the experience gained in mathematics teaching [9]. It integrates the concepts from all didactic units and their interrelations within each educational stage, as well as "the ways of proceeding, creating, and producing mathematics" [12] (p. 57). This domain consists of three specific interrelated subdomains: knowledge of topics (KoT), knowledge of structure mathematics (KSM), and knowledge of practices in mathematics (KPM).

KoT refers to the in-depth knowledge of the contents of each of the mathematical blocks taught in school [21]. It includes definitions and properties, procedures, representation registers, and/or applications [10], i.e., it integrates all those formal aspects of mathematics that we want the student to understand, but with a higher level of depth [12]. For example, the knowledge that a teacher has about the concept of exponential function is part of this subdomain, since it implies knowing its properties, its representation, and its possible applications.

For its part, KSM refers to the knowledge of mathematics from an integral and relational approach. The mathematics teacher must know how mathematical objects are related, either within the same academic course or throughout his or her training [22]. In this sense, the teacher must be able to connect the most advanced concepts with those more elementary ones that are the basis that support the mathematical construction process [8]. In reference to the example proposed in the KoT on exponential function, it is clear that properly understanding this concept requires prior knowledge of what a function is, its classification, the elements that define it, and how it is represented. It is also necessary to understand its main properties, as well as the concepts of domain and image. In addition, knowledge about sets is required. Likewise, to find some of the properties, such as the cut points with the coordinate axes, it is necessary to know how an exponential equation is solved; for this, in turn, we need to know its relationship with logarithms, etc. As can be seen, there is a structured learning sequence whose knowledge is exclusive to the mathematics teacher.

Finally, KPM refers to the different ways in which the mathematics teacher can approach the concept through practice, as well as the ways of arguing and reasoning. This knowledge encompasses the mathematical language used to explain the concept, the formal aspects on which to rely to go deeper into it, such as theorems, propositions, axioms, etc., and even the use of demonstrations and logical-deductive schemes. In other words, the mathematics teacher must know how to transfer their knowledge in an adequate and understandable way [23]. Continuing with the example of the exponential function, the teacher must know, among other things, how to formally write a function, how to argue the properties of the exponential function, which theorems to rely on, whether it is necessary to study the proof of any of them beforehand, etc.

2.2. Pedagogical content knowledge

The teacher's domain of didactic knowledge of mathematics refers to knowledge related to mathematical content as an object of teaching and learning. This domain is the one that has been most investigated due to its pedagogical nature, since it is not enough to have the knowledge of the subject; indeed, it is also necessary to know how and when to transmit it and why it is important to possess it. Consequently, to adequately develop the knowledge of this domain, previous training in didactics is necessary, which will be nurtured through teaching experience in the field of mathematics. This domain consists of three other subdomains: knowledge of mathematics teaching (KMT), knowledge of features of learning mathematics (KFLM), and knowledge of mathematics learning standards (KMLS) [8].

KMT refers to a teacher's ways of teaching mathematics and the resources they use to explain the content. KMT includes the teacher's knowledge of teaching theories, which can be derived from research in mathematics education or personal experience. It also encompasses the use of active methodologies, activities, examples, techniques, and strategies. Additionally, KMT considers the potential and limitations of both material and technological resources. Excluded from this knowledge are elements of general pedagogical knowledge [24]. For example, to explain the properties of an exponential function, some real examples can be presented, in which it is used to explain a social or scientific phenomenon, such as the evolution of an epidemic or compound interest. GeoGebra software can also be used as a technological resource to study the properties of the function in each case.

For its part, KFLM refers to the teacher's knowledge of "how mathematical content is learned and thought about, as well as the ways in which students interact with each content" [18] (pp. 597,598). This includes the teacher's knowledge about the strengths, difficulties, or errors associated with the content, the language that students tend to use to refer to that content, the creative and intuitive ideas used by students when faced with a problem, or even aspects related to both mathematics and the mathematical attitudes of students. In short, all the knowledge that involves the way students think when they put into practice the mathematical concepts learned in the classroom [25].

Finally, KMLS refers to the teacher's knowledge of institutional curricular regulations in mathematics. This may also include references outside the curriculum that provide extra information for the teacher's professional development, such as research results, contributions from professional associations, or even the informed opinions of expert teachers [26].

2.3. Beliefs

The domain of beliefs refers to the statements that the teacher expresses or the interests that the teacher has about the mathematical content that is taught and how it is taught. In this domain, a distinction is made between beliefs about the nature of mathematics and beliefs about the teaching and learning of mathematics [27]. In the context of the MTSK model, the former are related to mathematical knowledge, while the latter are related to didactic knowledge of mathematical content. In this regard, it is admitted that "beliefs permeate knowledge and that, if both constructs are intrinsic to the teacher, their joint study enables us to have a more complete picture of the object to be analyzed" [28] (p. 111).

3. Methodology

3.1. Approach

The approach adopted in this research is purely qualitative, as it seeks to analyze and interpret the specialized knowledge mobilized by mathematics teachers when implementing a STEAM-based learning activity. Achieving this analysis requires understanding and interpreting the nature of this knowledge; as such, the research is framed within an interpretative paradigm [29,30]. According to Lincoln and Cuba [31], this interpretative approach is characterized by the nature of the phenomenon, the relationship between the researcher and reality, and the possibility of generalization and enquiry (ideographic interpretation).

On the other hand, an instrumental case study is carried out because, as Stake [32] points out, it will allow us to deepen our understanding of a specific topic or to reformulate a theoretical model. This design enables a detailed exploration of how the MTSK model functions in STEAM contexts and allows us to reflect on the potential refinement or extension of the model based on empirical evidence. It is worth mentioning that all participants gave informed consent, and the study complied with institutional ethical guidelines. Data were anonymized, and teachers were given pseudonyms.

3.2. Context and participants

Two teachers, pseudonymously named Sophie and Georg, were purposefully selected based on their contrasting profiles—academic background, school context, and years of experience—as these differences were expected to provide rich, comparative insights. Sophie is a graduate mathematics teacher with seven years of experience teaching math at secondary school levels, from 7^{th} to 12^{th} grade, in public schools in Spain. She is currently teaching in a school located in an urban area with a diverse student population, characterized by low-to-medium socioeconomic backgrounds and a wide range of cognitive development levels. Georg is a teacher with a degree in industrial engineering who has twelve years of experience teaching math, technology, physics, and chemistry at secondary school levels, from 7^{th} to 12^{th} grade, in private or charter schools in Spain. He is currently teaching in a school located in a suburban area, serving students from predominantly medium-to-high socioeconomic environments, with generally lower cognitive demands and fewer behavioral challenges reported by staff. The diversity in teacher profiles was considered essential to explore how different professional backgrounds shape the activation of specialized knowledge in STEAM-oriented instruction.

3.3. Research phases and activity design

The research was developed in three phases: (1) design and validation of the didactic proposal, supported by two university experts in mathematics education; (2) implementation of the activity in both classrooms; and (3) analysis and interpretation of the data collected. Each phase was strategically designed to align with the research objectives: the first to ensure the didactic and theoretical rigor of the intervention, the second to provide authentic empirical data, and the third to allow for an in-depth analysis of teachers' specialized knowledge from a grounded perspective.

The design process followed a theory-informed and practice-oriented approach, incorporating both current research in mathematics education and real classroom constraints. The process began with a preliminary analysis of the literature on STEAM education, with special attention to methodological strategies that facilitate interdisciplinary learning and active student participation. Among these, problem-based learning (PBL) was selected as the overarching framework due to its compatibility with STEAM principles and its emphasis on student-led inquiry, contextual problem-solving, and the integration of multiple domains of knowledge [33].

Within the PBL structure, the mathematical object selected—the logistic function—was chosen for its dual value: it provides meaningful opportunities to work with complex mathematical reasoning (functions, derivatives, modeling) and has wide applicability in real-world phenomena, especially in epidemiology and data science. This alignment was key to ensuring that the task would genuinely require the teacher to draw upon various elements of specialized and interdisciplinary knowledge.

To ensure pedagogical coherence and content validity, the proposal was reviewed by two university professors, both experts in mathematics education and teacher training. Their role extended beyond mere content review; they engaged in a collaborative validation process, which involved:

- Assessing the mathematical and didactic rigor of the task.
- Verifying that the proposed activities appropriately triggered the dimensions of MTSK.
- Ensuring alignment with typical curricular expectations at the upper secondary level.
- Identifying potential cognitive or procedural difficulties that students might face and suggesting modifications accordingly.

This validation process occurred in two iterative rounds. After the first review, modifications were made to clarify task instructions, reinforce the logical flow of the activity, and better scaffold the use of GeoGebra. For example, the experts recommended simplifying the initial task statement to avoid overwhelming students, reorganizing the order of questions to follow a more natural progression from basic to advanced properties, and including an initial visual example in GeoGebra to reduce cognitive load. In the second round, experts confirmed that the structure and objectives of the activity were appropriate for implementation.

Furthermore, to simulate classroom conditions, the proposal was subjected to pilot testing in a university mathematics education course, where pre-service teachers analyzed and discussed its structure. Their feedback helped refine the clarity of explanations, the pacing of conceptual progression, and the usability of GeoGebra in each step. This two-stage validation process contributed to the didactic robustness of the proposal and enhanced its reliability as a data-generating tool. By ensuring content coherence and anticipating student difficulties, we created an activity capable of eliciting observable and analyzable knowledge elements.

The final version of the activity was structured into three clearly differentiated parts, each designed to progressively elicit different categories of specialized knowledge: (a) the introduction, in which the teacher introduces the activity and explains the concepts that will be involved during its development; (b) the core exploration, with a theoretical-practical approach, in which students will have to answer a series of questions of purely mathematical content about the logistic function; and (c) the epilogue, with a totally practical approach, in which one of the applications of this function is introduced. Table 1 summarizes the sequence of tasks.

Although the activity is grounded in mathematical modeling, it also incorporates scientific reasoning (through the epidemiological interpretation of the logistic function), technological tools (via GeoGebra), and societal relevance (through real-world epidemic modeling). Students are not only asked to compute or analyze mathematical properties but to interpret and contextualize results from a scientific and social perspective, using representational technology to explore implications and derive conclusions.

In this regard, Sophie and Georg were guided to cultivate mathematical creativity and flexibility throughout the activity. Students were encouraged to explore different ways of interpreting the logistic function, to justify their reasoning using multiple representational registers, and to reflect on the assumptions and limitations of their models. In this way, the activity sought to engage students not only in procedural calculations but also in the creative dimensions of mathematical thinking, such as hypothesizing, re-framing the problem context, and generating meaningful interpretations. This integration of disciplinary insights reflects the core objectives of STEAM education.

Table 1. Sequence of exercises comprising the activity.

A. Introduction	1. Presentation of the activity.	a.	Define the objectives and context.
		b.	Cite some applications of the logistic function in real
B. Core exploration	2. Definition of the logistic function: $$f(t) = \frac{ab}{a + (b - a)e^{-ct}},$$ being t the time, in days, of the evolution and a the number of cases registered on day zero.	a.	Represent the function with GeoGebra, knowing that on day zero of the epidemic, only one person was infected.
		b.	Calculate the domain, image, asymptotes, characteristic points, monotonicity, and curvature of the function based on the parameters b and c.
C. Epilogue	3. Definition of daily incidence (DI): $$DI(t) = \frac{df}{dt}.$$	a.	Represent the function $DI(t)$ with GeoGebra. What is its shape?
		b.	Taking $b = 800$ and $c = 0{,}2$, calculate the exact time (day and hour) when the peak of the DI was reached and the number of cases (per hundred thousand inhabitants) that occurred at that time.
	4. Reasoning question about $f(t)$ (cumulative incidence).	c.	Explain the meaning of the parameters b and c.
		d.	If during the epidemic the population was 47,398,695 people, how many people had (or were suffering from) the disease from day zero until the time when the maximum of DI is reached?

This rigorous design and validation process enhanced the didactic robustness and research reliability of the study, ensuring that the activity would both function as an authentic classroom experience and generate the conditions necessary to observe and analyze the MTSK-related knowledge elements under investigation.

3.4. Data collection and evaluation instruments

The proposal was applied by Sophie and Georg in their first-year upper secondary education classes over two 45-minute sessions. Both sessions were conducted in a computer lab where each student had access to a computer to complete the exercises. The introduction and core exploration were carried out during the first session, while the epilogue and final reflection took place in the second. These sessions were scheduled after the teachers had completed the unit on derivatives and had begun covering their applications. It is assumed that students had already studied functions and limits in previous lessons, so the foundational concepts required for the activity were considered to be already understood. A total of fifteen students participated voluntarily in Sophie's group, and twelve in Georg's.

With regard to evaluation instruments, the main sources of evidence were classroom observations, field notes, and semi-structured interviews, as proposed by Kagan [34]. The combination of these three assessment instruments allowed for the triangulation of methods, strengthening the credibility and reliability of the findings. Specifically:

1. Classroom observations were conducted using the non-participant direct observation method

[35], with researchers remaining in the background and not interacting with the teacher's work, except when invited to clarify aspects of the activity. This minimized disruption to the usual classroom environment. The method allowed for the collection of reliable data, useful for answering research questions and identifying new theoretical insights. It also enabled us to observe how teachers and students behaved during the activity, and to analyze the learning sequences, recognizing patterns related to elements of specialized knowledge. Additionally, we were able to formulate new questions and track changes that provided insight into how these knowledge elements affected the teaching and learning of mathematics.

2. Field notes were taken both during and after the sessions, following Lofland and Lofland's recommendations [36]. We used conventional descriptive and procedural formats, including transcriptions, immediate descriptions, and checklists. After the intervention, these notes were expanded based on direct classroom observations and enriched with the researchers' personal reflections and viewpoints.
3. A semi-structured interview was also conducted with each of the teachers in order to generate discussion around the proposal designed and to clarify some aspects derived from direct observation in the classroom. In this way, we were able to combine the questions already defined previously with the flexibility to explore additional topics depending on the conversation [37].

Although the classroom observations and the notes taken during the intervention allowed us to organize and evaluate the different elements of knowledge in both case studies, the interviews conducted with Sophie and Georg helped us to reinforce some of them and to include those subdomains that could not be identified through observation, especially the constituents of the KMLS subdomain of the PCK. These interviews also helped us extract new relevant information about the foundations of the knowledge model and to learn about Sophie and Georg's personal reflection on the methodology used or their interest in the content (elements included within the beliefs domain of the MTSK model). The triangulation of these three instruments not only strengthens the validity of our findings but also enables a multi-dimensional reconstruction of teachers' knowledge in action. This holistic approach was essential to map out the complexity of the MTSK model in dynamic classroom environments.

As for the analysis of the activity, we carried out a content analysis as suggested by Bardin [38] or Krippendorff [39], taking into account the different categories of the subdomains of the MTSK model and the indicators of specialized knowledge specific to the activity. To ensure the reliability of the analysis, two researchers independently coded the data. Discrepancies were resolved through collaborative discussion until a full agreement was reached, reinforcing inter-coder consistency. This process followed qualitative validity criteria such as credibility, dependability, and confirmability [38]. In addition, emerging codes were cross-checked against the initial framework to ensure that no relevant aspects were overlooked, supporting the theoretical and interpretative soundness of the findings.

Table 2 lists the aspects considered to evaluate the knowledge elements detected during Sophie and Georg's interventions based on the MTSK model. After the interventions and after all the data had been collected, we were able to compare both case studies, obtaining relevant information about the categories of each of the MTSK subdomains. This helped us differentiate the types of knowledge identified in each subdomain and delve into some interdisciplinary aspects that would deserve to be included (or redefine the already known ones) as possible indicators of knowledge within the subdomains of the model.

3.5. Ethical considerations

This study complied with established ethical research standards for studies involving human participants. All participants were informed of the objectives and procedures of the research and gave their voluntary, informed consent prior to participating. Participation was entirely optional, and participants were informed that they could withdraw at any time without any consequence. To ensure data confidentiality and anonymity, all identifying information was removed from transcripts and observation notes. The names used in the manuscript (Sophie and Georg) are pseudonyms selected by the researchers to protect participants' identities. The data collected were stored securely and used solely for the purposes of this research.

Table 2. Elements of knowledge to assess.

Subdomain	Elements of knowledge
KoT	1. Context. 2. Concepts involved: definitions, properties, and procedures. 3. Representational registers: numerical, graphic, verbal, analytical, etc.
KSM	1. Relationship between concepts of the same subject. 2. Relationship between concepts from different themes: complexification and simplification.
KPM	1. Definitions, propositions, theorems, examples, or counterexamples that complement or help to understand a concept or to refute an idea. 2. Syntax used. 3. Structure and organization. 4. Rigor, analysis, deduction, and reasoning skills.
KMT	1. Resources used. 2. Teaching theories, techniques, or strategies. 3. Mathematical flexibility: different ways of proceeding.
KFLM	1. Strengths and weaknesses of the students: identification of capacities, mistakes, difficulties, and limitations. 2. Provoke students' interest in the content and engage them in interaction.
KMLS	1. Activity specific to the level. 2. Sequencing of the concepts and issues involved.
Beliefs on maths (teaching-learning)	1. Teacher's interest in content. 2. Personal reflection. 3. Personal experiences.

4. Results

To gain a comprehensive understanding of how the didactic proposal was implemented in the classroom and the role teachers play in STEAM-oriented activities, a data triangulation strategy was adopted. First, through direct classroom observation, it was possible to identify how the intervention unfolded in practice and which elements of knowledge emerged organically during teacher-student interaction. This was followed by the use of a checklist, grounded in the theoretical framework, allowing for a more systematic recording of the presence or absence of specific knowledge components. Finally, semi-structured interviews provided deeper insight into less observable aspects, such as teachers' beliefs, their planning processes, and their views on the value of interdisciplinary teaching in mathematics. The integration of these three approaches not only validated the relevance and coherence of the proposed activity but also offered a richer understanding of teaching practices, including their limitations and potential within a STEAM framework. What follows is a detailed breakdown of the results gathered through each of the evaluation instruments.

4.1. Results obtained through observation and field notes

First, we will analyze the results obtained from classroom observations and field notes (transcriptions and immediate descriptions) collected in both case studies. In particular, we will analyze the elements of knowledge that were detected throughout the sessions, aligned with those specified in Table 2, and indicating in brackets the subdomain from which it comes. Thus, for example, when [KPM-2] is written, the knowledge element identified is “syntax used”, which comes from the KPM subdomain.

The first indicator of knowledge that emerges directly from exercises is the need to know all the characteristic properties involved in the study of a function. Of course, knowledge is also needed on how to graph functions with GeoGebra using parameter sliders, as well as to calculate limits and derivatives and solve equations with said software. Likewise, since the activity deals with a real-life application of the logistic function, mathematics teachers must know the object of application and, of course, know how to transmit said knowledge to their students.

The technique used by both Sophie and Georg was to start by explaining an alternative example with a more elementary function [KPM-1; KMT-1; KMT-2]. In this example, they showed the students the code they had to use to enter points, graph a function with parameters (using sliders), calculate its first and second derivative, calculate limits, and solve simple equations, i.e., the concepts that they would have to use to complete the exercise [KoT-3; KSM-1]. Once the students had internalized all these basic notions, they read and carried out the exercise at the same time as the students, leaving a reasonable amount of time between sections before projecting the solution [KFLM-1; KFML-2].

Next, the sequence followed by both teachers in each of the sections of the activity was described. In the first section, the function for cumulative incidence (per one hundred thousand inhabitants) had to be represented with respect to time, knowing that on day 0 of the epidemic, only one infected person was recorded (Figure 2).

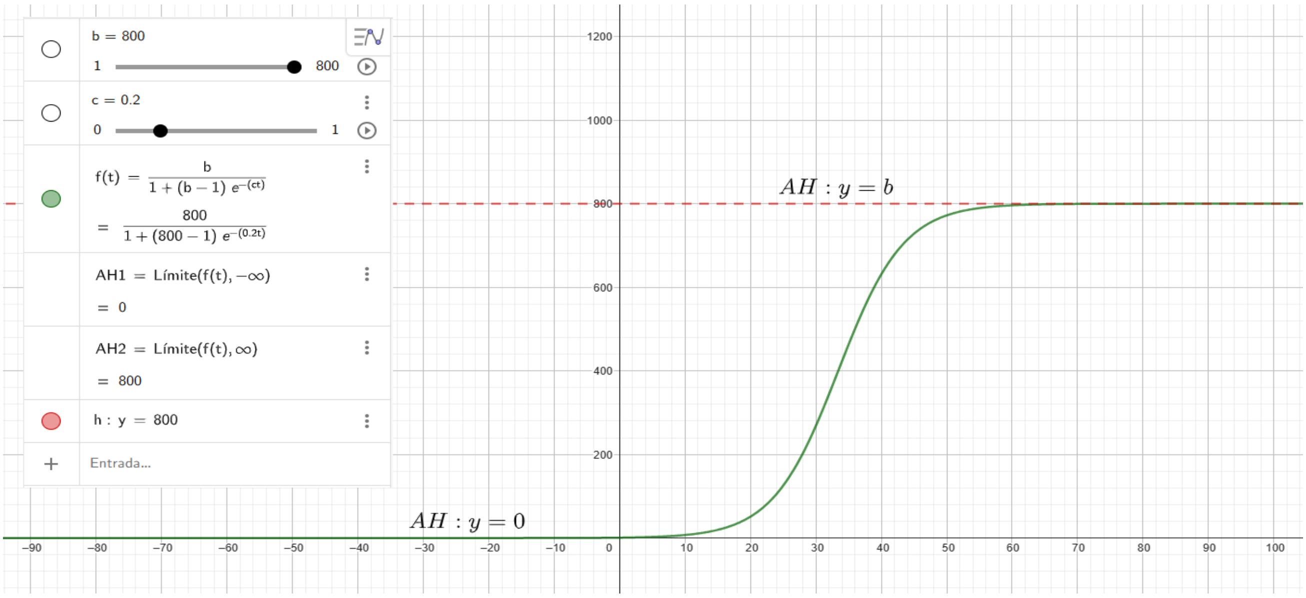

Figure 2. Representation of the cumulative incidence function $f(t)$ with GeoGebra.

Sophie: *Notice that we have a function that depends on the variable* t*, which is time, and three unknown parameters* a*,* b *and* c*. We know that a is the number of cases recorded on day 0 of the epidemic. Therefore,* $a = 1$ *because it is given to us as data. We have to write a code in GeoGebra to plot a function whose independent variable is time, and which depends on two unknown parameters* b *and* c*, for which we will use two sliders that you have to define according to the data of the problem. We know that* b *must be greater than* a *and that* c *must be greater than 0. Therefore, you can define the sliders with the limits that you think are most appropriate.*

From this introduction, we can detect how Sophie introduces the exercise in a structured and organized way [KPM-3], asking students to define the sliders so that they can see how the function varies [KFLM-2]. She uses a technical verbal register in her dialogue [KoT-3] and demonstrates that she has sufficient knowledge to cope with the exercise [KoT-2].

Sophie: *Once you have the graph of the function, we can identify the domain, the image and the asymptotes very easily, but what if we had not used GeoGebra to represent it? This would not have been so easy* [KFLM-1; KFLM-2]. *To study the domain, we would have to have previously checked if there are values of* t *for which the function* $f(t)$ *is not defined. Since it is a rational function, we would do this by equating the denominator to zero, right* [KoT-2]*?*.

Sophie writes on the board [KSM-2]:

$$1 + (b-1)e^{-ct} = 0 \longrightarrow (b-1)e^{-ct} = -1 \longrightarrow e^{-ct} = \frac{-1}{b-1} \longrightarrow -ct = Ln\left(\frac{-1}{b-1}\right).$$

Sophie: *Remember that a logarithm cannot be negative* [KSM-2]. *Therefore, in this case, there will be no value of* t *that cancels the denominator. This means that the domain of the function is the set of real numbers* [KoT-2]. *However, calculating the image analytically is somewhat more complex because we would have to calculate the domain of the inverse of the function* [KoT-2; KSM-2]. *As for the asymptotes, we can study how the function behaves when* t *tends to minus infinity and to infinity.*

Sophie writes on the board [KSM-1]:

$$\lim_{t\to\infty}\frac{b}{1+(b-1)e^{-ct}}=\frac{b}{1+(b-1)\underbrace{e^{-\infty}}_{0}}=\frac{b}{1}=b,$$

$$\lim_{t\to-\infty}\frac{b}{1+(b-1)e^{-ct}}=\frac{b}{1+(b-1)\underbrace{e^{\infty}}_{\infty}}=\frac{b}{\infty}=0.$$

As can be seen, Sophie demonstrates knowledge of the analytical calculus of the function domains, solving exponential equations, and calculating limits at infinity [KoT-2]. In addition, she is rigorous and organized in her reasoning [KPM-3; KPM-4] and in the way of transmitting knowledge [KoT-3; KPM-2]. Furthermore, it shows that the results obtained with the software do indeed coincide with those obtained analytically [KPM-1; KFLM-2]. On the other hand, she is aware of the limitations that students would have in the analytical calculus of the function image and opts to calculate it by visual inspection [KFLM-1]. However, it could have been used as a resource to give simple values to the parameters to reduce the complexity of the function and thus be able to calculate the inverse (reinforce KMT and KSM). Regarding the association of the parameters b and c with the context of the exercise, we can think that Sophie presents some deficiencies, since she only refers to the asymptotic meaning of b and the change in slope of the function when c varies. At this point, it is very important to relate the behavior of the function to the study phenomenon.

Georg, on the other hand, does not reveal as many details as Sophie and lets the students graph the function themselves and study its properties based on the known data. However, it is more technical when explaining the meaning of parameters b and c.

Georg: *If we set a value for b and move the slider of c, notice how as c increases, the slope of the curve becomes steeper and steeper. This means that as* c *gets larger, the number of infections will increase for a smaller and smaller interval of* t. *This is what is known, in scientific terms, as the growth rate. Therefore, the growth rate will tell us how the cumulative incidence changes between two different points in time of* t. *For example, if we set* $b=800$ *and we want to know how much the cumulative incidence has grown in the first ten days of the epidemic, we must do* $f(10)$ *and, by moving the slider of* c*, you can see how the number of cases grows as* c *increases. For* $c=0.1$*, we have* 2.71*, and if we take* $c=0.2$*, the value of the cumulative incidence increases to* 7.33*. If we keep increasing the value of* c*, taking now* $c=0.3$*, the cumulative incidence increases to* 19.62 *cases per hundred thousand inhabitants in the first 10 days, and for* $c=1$*, it increases to 772. You can see how the cumulative incidence grows very fast*.

Georg: *If we now set the slider of* c *and move the slider of* b*, you can see that what changes is the upper horizontal asymptote. Looking at the function, we can see how the function has a horizontal asymptote at* $y=b$*. Therefore, the value of* b *is the maximum number of individuals affected by the epidemic that can be reached in a population. There will come a time when an equilibrium situation, or stationary situation, is reached, where the accumulated incidence no longer grows and remains constant, or may even decrease. In an epidemic, this occurs when the number of individuals left uninfected is less than the minimum necessary for further spread of infection. In scientific terms, this parameter is known as saturation*".

From Georg's explanation, we can detect that he has outstanding scientific knowledge [KoT-2], he knows how to correctly explain the meaning of parameters b and c, providing examples that help understand the meaning and using technical verbal registers and a vocabulary typical of the level [KoT-3; KPM-1]. In this case, he focuses on explaining the behavior of the function and not so much on its mathematical properties, as Sophie did, thus demonstrating great capacity for analysis and deduction [KPM-4].

However, there is one thing that the two teachers overlook: pointing out that it does not make sense to evaluate the function for negative time values. It is clear that the domain of the function $f(t)$ is the set of real numbers, but in the context of the exercise, the epidemic begins on day 0 and, therefore, the function would be restricted to values of t greater than 0. A negative time, in this case, indicates the number of people who were infected before the epidemic started, which does not make sense. Consequently, we identify in both Sophie and Georg an absence of knowledge precisely in this part of the study of the phenomenon [reinforce KoT-1].

Regarding the representation of the daily incidence with respect to time (Figure 3), Sophie chooses to dwell on its shape, explaining to the students that it is shaped like a Gaussian bell, which in terms of statistics and probability is known as a normal distribution. She also explains that it is very useful for describing natural and scientific phenomena, such as measurement errors, financial returns, and epidemiological studies, among others. In this way, we can detect that Sophie knows how to relate the form of the function to other mathematical concepts [KSM-2].

However, Georg focuses on explaining why the daily incidence is the derivative of the cumulative incidence, thus enhancing the KoT and KPM subdomains. Georg points out that this is because, in the context of functions, it tells us the instantaneous rate of change. He adds that the derivative of a function at a specific point gives us information on how that function changes at that point. This gives us the rate of change of cumulative incidence at each specific point in time, i.e., how many new cases are being added each day. As we can see, Georg demonstrates knowledge of the concept of derivatives applied to a social phenomenon [KoT-1; KoT-2] using an analytical representation register [KoT-3] and providing definitions and theorems implicit in its argument [KPM-1], again demonstrating his analytical and deductive ability [KPM-4].

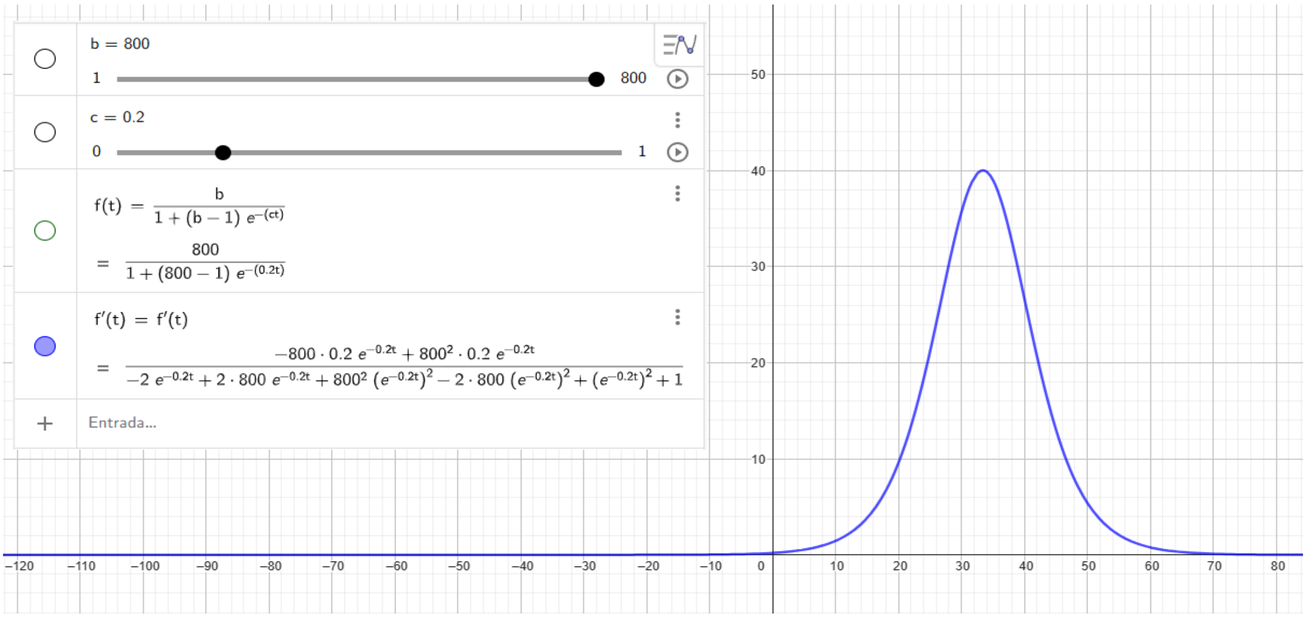

Figure 3. Representation of the daily incidence function $DI(t)$ with GeoGebra.

When it comes to calculating the instant in which the maximum peak of the daily incidence was reached and the number of cases (per one hundred thousand inhabitants) that were registered at that moment (Figure 4), both Sophie and Georg show that they know the procedure [KoT-2] and the codes to be entered in GeoGebra to calculate them [KMT-1]. In this case, as the exact time (day and hour) must be calculated, a conversion factor must be made to express 33.42 days in "days, hours, minutes, and seconds". Both teachers demonstrate their knowledge to explain the procedure [KoT-2; KSM-2] and use the same solving strategy. Sophie and Georg write on the board [KMT-2]:

$$0.42\,d \cdot \frac{24\,h}{1\,d} = \underbrace{10.08\,h}_{10\,horas} \longrightarrow 0{,}08\,h \cdot \frac{60\,min}{1\,h} = \underbrace{4.8\,min}_{4\,minutos} \longrightarrow 0.8\,min \cdot \frac{60\,s}{1\,min} = \underbrace{48\,s}_{48\,segundos}.$$

At this point, both Georg and Sophie comment that the value of t obtained coincides with the inflection point of the cumulative incidence graph. However, while Sophie chooses to explain its mathematical meaning by referring to the change in curvature of the function, Georg focuses his attention on relating this change in curvature to the growth rate of the epidemic.

Georg: *Notice that just on day 33 at 10 hours, 4 minutes, and 48 seconds, there is a change in the growth of the number of registered cases. Also, notice how after that instant, the growth rate starts to slow down until the population stabilizes. This indicates that at* $t = 33.42$ *there is a turning point in the number of total cases registered, which, if it did not exist, the epidemic would have continued growing at the same rate without the possibility of stopping or stabilizing*".

Once again, Georg demonstrates that he knows how to connect a mathematical term with its scientific explanation using the usual terminology [KoT-1; KoT-3].

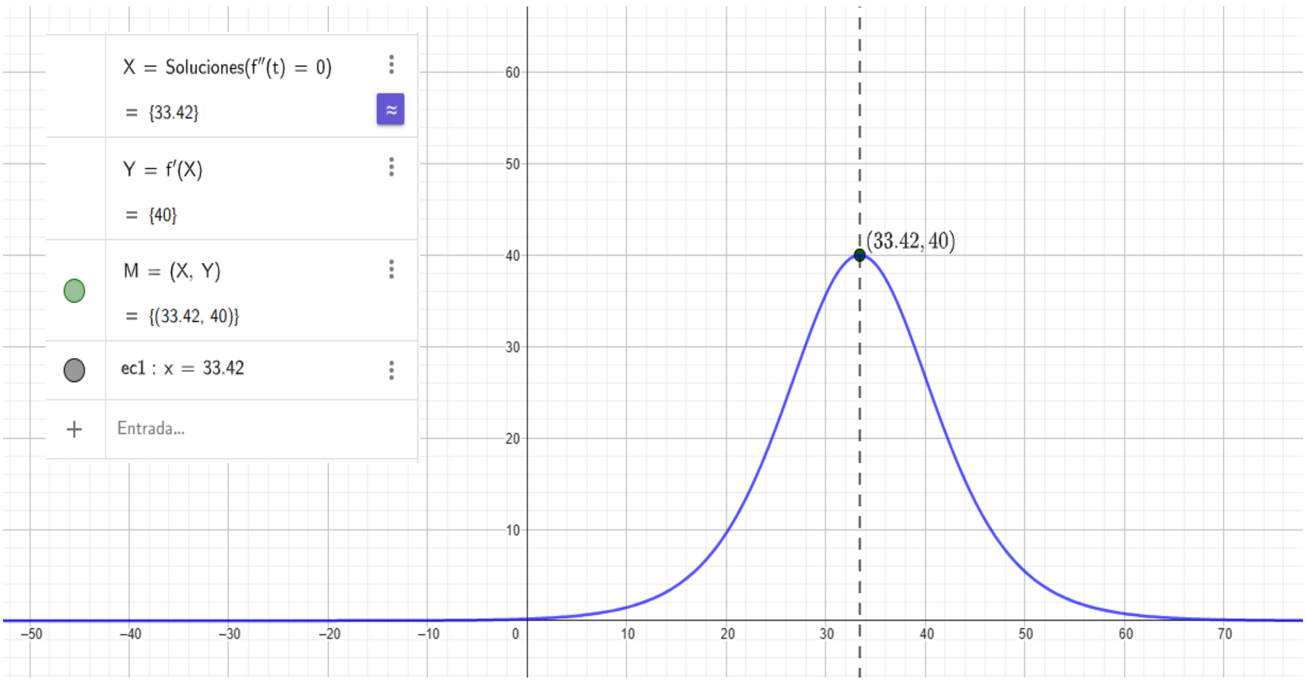

Figure 4. Representation of the maximum peak of the daily incidence with GeoGebra.

Finally, with regard to the reasoning question asking to calculate the number of people who suffered from the disease from day 0 to the time when the daily maximum is reached, once again Sophie and Georg demonstrate the mathematical and didactic knowledge from the exercise [KoT-2] and know how to relate it to the topic of proportionality [KSM2]. However, here the students have some comprehension problems because they are not able to relate the size of the population with the value of the cumulative incidence at the instant $t = 33.42$ days. Teachers, aware of this limitation, explain the process using the rule of three as a teaching strategy [KPM-1; KMT-2; KFLM-1]. While Sophie uses fraction equivalence, Georg chooses to apply the traditional rule [KMT-3]. Table 3 shows the syntax used by each teacher.

Table 3. Syntax used by Sophie and Georg to solve the exercise.

Sophie's technique: Proportionality	Georg's technique: Rule of three
$\frac{f(33.42)}{100{,}000} = \frac{x}{47{,}398{,}695}$	$f(33.42) \longrightarrow 100{,}000$ $x \longrightarrow 47{,}398{,}695$

Note how the syntax used by Sophie is the most correct, in mathematical terms, since it uses concepts of direct proportionality and equivalence of fractions [KSM-2; KPM-2]. However, Georg opts for a syntax that, although not entirely rigorous, is more familiar to students [KMT-2].

To finalize the results obtained from the classroom observations, a checklist was used as a field note that served as a guide to monitor the teachers' adherence to the expected standards throughout the implementation process, providing a structured way to assess how well the teachers integrated and applied the knowledge required for the activity.

4.2. Checklist

Table 4 shows the data from the checklist used during the intervention to identify the interdisciplinary knowledge of both teachers. This instrument was developed based on key knowledge elements that were considered essential for a teacher to successfully carry out the proposed activity. Each item in the checklist reflects a specific aspect of teacher knowledge, ranging from content understanding to pedagogical strategies and interdisciplinary connections.

During the classroom intervention, this checklist was used as an observation tool to systematically record whether or not each item was demonstrated by the participating teachers. This allowed for a structured and objective way to assess the presence of these knowledge components in real teaching contexts, complementing the more qualitative insights obtained through observation and interviews. Specifically, 10 items, aligned with the knowledge elements in Table 2, were analyzed. We indicate with the check symbol if the item was identified and with a cross if it was not.

In summary, the checklist enabled us to identify, in a concise and structured manner, which interdisciplinary knowledge elements were made explicit in each teacher's practice. However, certain aspects of teacher knowledge, particularly those related to beliefs, motivations, and pedagogical intentions, cannot always be directly observed in the classroom. For this reason, the following section analyzes the results of the interviews, which serve not only to complement and triangulate the findings from the intervention but also to highlight elements that remained implicit during the observed sessions.

Table 4. Checklist for detecting teachers' interdisciplinary knowledge.

Subdomains	Item to be evaluated	Sophie	Georg
KoT-1	1. Define the objectives of the activity correctly.	✓	✓
	2. Provide various examples of the applicability of the logistic function in real life.	✗	✓
	3. Adequately introduce the cumulative incidence function by explaining the meaning of the concepts.	✓	✓
KoT-2 KPM-4 KMT-2	4. Explain the meaning of the parameters b and c of the logistic function, relating them to the study phenomenon.	✗	✓
	5. Use scientific terminology to explain the meaning of the parameters b and c of the logistic function.	✗	✓
KPM-1 KMT-1	6. Explain why the derivative of the cumulative incidence is the daily incidence in scientific terms.	✗	✓
KoT-2	7. Know and can convey the meaning of the relative maximum of the function $DI(t)$ within the context of the study phenomenon.	✓	✓
KoT-2 KMT-2	8. Use conversion factors to express the exact day and time at which the maximum daily incidence is reached.	✓	✓
KMT-1	9. Use GeoGebra correctly to represent and study a function (use of sliders, first derivative, limits, equations, …).	✓	✓
KoT-3 KPM-4	10. Divide the cumulative incidence into a piecewise function to indicate that the domain of the function within the context of the study phenomenon is $(0, \infty)$ and not $\mathbb{R}$.	✗	✗

4.3. Results of the teacher interviews

As anticipated, the semi-structured interviews were essential for deepening our understanding of both Sophie's and Georg's knowledge, particularly regarding elements that were difficult to observe directly in the classroom. These interviews not only allowed us to reinforce and validate findings identified through the observations but also offered insights into the teachers' pedagogical beliefs, perceptions of the activity's value, and views on the interdisciplinary and STEAM-oriented approach. The responses are analyzed below, aligned with the subdomains from Table 2 and marked accordingly.

4.3.1. Interview with Sophie

Here is an analysis of the questions asked to Sophie (we use R to refer to the researcher and S to Sophie):

1. R: *What motivated you to implement the activity in your classroom?*

   S: I think it is quite a complete activity; it develops concepts from different topics in the area of mathematical analysis [KMLS-2] and, above all, it serves to enhance the value of mathematics and make students aware of its usefulness in real life [KoT-1; Beliefs-1]. They think that mathematics is finished and that there is no more outside the school environment, and this activity is a clear example that mathematics is everywhere and that it is used to explain natural and scientific phenomena in real life [Beliefs-2].

2. R: *Do you think the activity is appropriate for the level?*

   S: Yes, without a doubt. According to my experience, I am used to always following the same pattern in my classes: explanation-exercises-doubts. Students often make the mistake of studying by memorization, and this is a big mistake in mathematics [Beliefs-3]. For example, in the topic of applications of derivatives, we have worked with the typical polynomial and rational functions. I think it is essential that they get out of this comfort zone and face more complex functions which are not so common for them [Beliefs-2]. Of course, the concepts worked on during the activity form part of the minimum knowledge that must be taught at this level, and it is essential that they master them, as in future courses they may probably take them for granted [KFLM-1].

3. R: *Do you think more of these activities should be implemented in the classroom and fewer model exercises?*

   S: I think a mix of the two would be ideal. You must work on exercises of different levels of complexity, from the most basic ones, where they only must apply definitions and properties, to some more complex ones, where they have to demonstrate their reasoning skills [KMT-2; KMLS-2]. On the other hand, activities such as these can be implemented at the end of a topic with the purpose of consolidating knowledge, identifying their skills and difficulties regarding the concepts studied [KFLM-1] and, as I said before, enhancing the value of mathematics in society.

4. R: *In terms of the design of the proposal, do you think that the structure is appropriate considering the objective of the activity? Do you think the exercises have the correct syntax? Is there coherence between the questions? Would you change anything about the structure?*

   S: The structure is very well developed, and so is the sequencing of the exercises. I think that approaching the problem first from an entirely mathematical approach will help to understand how another function with the same form as the one studied in the approach will behave, but from an interdisciplinary approach [KMT-2; Beliefs-1]. The syntax used is also appropriate; there is no abuse of complex mathematical notation, which sometimes has a negative impact on students. In general, the fact that the statements of the exercises are not clear, or that technical terms are abused, is usually a cause of rejection among students, and they directly say that they do not know how to do it without even having tried [KFLM-1; Beliefs-3]. However, in this case, this does not happen because the sequencing of the exercises is very well structured, and it is perfectly understood what is asked in each case. That said, I would not change anything in the way the activity is presented. However, during the development of the sessions, some interesting questions occurred to me that could have been included [Beliefs-1].

5. R: *What are those questions?*

   S: For example, in the approach part of the PBL, they could have been asked about other properties of the function, such as symmetry [KSM-1; KMLS-2]. This function is precisely ideal for this because it has a shape that can confuse students into thinking that it may have odd symmetry when passing through the origin. This is a very common mistake among students [KFLM-1; Beliefs-3]. It would have been good if they had done it analytically.

6. R: *When would be the best time for you to make this proposal?*

   S: You only have one possibility, and that is to do it after the topic on applications of derivatives has been taught, or during its teaching as an example of application in real life. It is understood that, at this point, students already know all the previous concepts needed to do the activity [KSM-1, KSM-2; KMLS-2].

7. R: *During your intervention, you show rigor in your explanations, and the vocabulary and syntax you use are quite technical. Do you think this is good for students?*

   S: Depending on the course, I am more or less rigorous in my explanations. At this level, I think that students must already demonstrate their mathematical skills in both reasoning and writing. Being organized and methodical is fundamental to solve a mathematical problem [Beliefs-2].

8. R: W*hich of the proposed exercises do you think the students have had the most difficulties with, and why do you think this is?*

   S: Assuming that it is not a function that usually appears in textbooks and that it probably makes an impression on them at first, I think that they have had the greatest difficulties when it comes to reasoning about the meaning of the parameters, because they are not used to reasoning, but to doing the exercises mechanically. So, in line with what I mentioned before, if they see an exercise in which they are asked about something that is not "the usual", they directly switch off and say that they do not know how to do it [KFLM-1; Beliefs-3]. For example, in the last proportionality exercise, they were totally thrown off by the fact that the function was evaluated per hundred thousand inhabitants. Most of them thought that the result they obtained when evaluating the value of t in the function was the number of cases registered in relation to the entire population, instead of per hundred thousand inhabitants [KFLM-1]. This happens because they lack reading comprehension or because they are not used to doing exercises in which they have to reason.

9. R: *Apart from GeoGebra, what other resources do you think could have been applied??*

   S: I cannot think of any resources beyond the usual ones (schemes, notes, similar exercises, …) [reinforce KMT-1]. Perhaps some other symbolic or numerical calculation software, such as WolframAlpha or Wxmaxima, could have been used to calculate the analytical expressions of cumulative incidence and daily incidence [KMT-1]. Although these programs are not familiar to students and may have a negative impact on learning [KFLM-1].

10. R: *If you were to apply the proposal again, what didactic aspects would you change?*

    S: I would possibly spend some more time during the development of the epilogue to explain and contextualize the study phenomenon. [Beliefs-1; KoT-1].

11. R: *Finally, regarding STEAM practice, do you usually implement examples or activities with this approach in class? Do you think it is necessary in mathematics? If so, what knowledge do you think a mathematics teacher should have to deal with them?*

    S: No, the truth is that, in general, my classes are very traditional. The examples and exercises we carry out are purely mathematical and have little relation to other disciplines, except for some contextualized problems that appear in the textbook [reinforce KMT]. I consider that in the subject of mathematics it is difficult to look for activities from this perspective in some topics, such as algebra or arithmetic, but it is true that, in many other areas, especially in geometry, calculus, or statistics, it is an approach with a lot of potential. However, I consider that we, as mathematics teachers, are the ones who must train students with a solid mathematical foundation that they can develop in other disciplines. In terms of the knowledge that a mathematics teacher should have to develop this approach, I do not think we need to know more than what we have studied in our degree. If we decide to implement STEAM activities in mathematics class, we simply need to prepare well for the topic, do some research, and ask for help from the teachers in the disciplines involved if necessary [Beliefs-2].

In general terms, Sophie shows interest in implementing activities with this approach, although she recognizes that she does not usually prepare them. Likewise, she demonstrates curricular knowledge by knowing how the areas of study are organized and recognizes that the activity contains exercises specific to the educational level. She also expresses her opinion on the proposal according to her personal experience as a teacher and suggests alternatives to improve the proposal based on the students' limitations, thus reinforcing the KFLM subdomain. Finally, she acknowledges not doing problems that connect different areas of knowledge and is also aware of a lack of context in her argument (reinforce KoT-1), but she offsets this limitation through her solid analytical, deductive, and reasoning skills.

4.3.2. Interview with Georg

We now turn to the interview with Georg (we use R to refer to the researcher and G to Georg):

1. R: *What motivated you to implement the activity in your classroom?*

   G: Firstly, because it is an activity with an interdisciplinary approach. I think that with the great advances that have been made in education in recent years, more emphasis should be placed on implementing more activities of this type and with this methodology [Beliefs-2]. And, secondly, because I think it is important for students to know and be aware of the great applicability of mathematics outside the classroom [KoT-1; Beliefs-1]].

2. R: *Do you think the activity is appropriate for the level?*

   G: Yes, the first pre-university level is a level where students are generally already familiar with basic concepts of functions, graphs, and elementary calculus. Here we include a new concept for them, which is the derivative, and it is essential that they understand its meaning and its applicability [KMLS-1; KMLS-2; Beiefs-1]. However, the analysis of functions such as, in this case, the logistic function, is certainly a challenge for them, although we have already seen that it is not impossible [KFLM-1]. Perhaps, in a higher grade it would have been more accessible to them as they would already have the concept of derivative more internalized, but, in general terms, it is a very valid activity that will help students develop critical skills in the handling of functions and graphic analysis and, at the same time, it provides them with a good foundation in the area of calculus [KMLS-1; Beliefs-2].

3. R: *Do you think more of these activities should be implemented in the classroom and fewer model exercises?*

   G: Of course, activities that require a deeper and more comprehensive analysis of mathematical concepts can be much more enriching for students, as they encourage critical thinking and problem-solving skills. Moreover, such activities can also arouse students’ curiosity, as they tend to be more engaging than repetitive and mechanical exercises. But there is a balance to be struck [Beliefs-1, Beliefs-2]. Model exercises are also important because they help students practice and consolidate specific techniques and methods, such as the study of a function [KMT-2].

4. R: *When would be the best time for you to make this proposal?*

   G: It must be applied after having seen the topic of derivatives and having done many exercises on complete studies of functions. But if such activities are to be applied, it is important to plan them in advance and include them in the course programming [KMLS-2].

5. R: *In terms of the design of the proposal, do you think that the structure is appropriate considering the objective of the activity? Do you think the exercises have the correct syntax? Is there coherence between the questions? Would you change anything about the structure?*

   G: Yes, I think the idea of using problem-based learning as an active methodology is a good choice in this case. The questions are well formulated, and it is easy to understand what you are asked to calculate. Perhaps they are too argumentative, although it is necessary to contextualize the problem [Beliefs-2]. When students see a lot of text in an exercise, they already associate it with it being difficult or it simply makes them unmotivated [KFLM-1]. In my experience, it is better to give them shorter and more direct exercises [KFLM-2; Beliefs-3].

6. R: *Which of the proposed exercises do you think the students have had the most difficulties with, and why do you think this is?*

   G: Possibly in the reasoning exercises. In my experience, students tend to mechanize mathematical procedures instead of reasoning them out, and this is something that needs to be changed [Beliefs-2; Beliefs-3]. That is why I found this activity extraordinary in that respect [Beliefs-1]. On the other hand, I have noticed that they have had difficulties with the last exercise, which, at first, was the easiest. This is because they are not used to reasoning [KFLM-1; Beliefs-2].

7. R: *Apart from GeoGebra, what other resources do you think could have been applied??*

   G: I cannot think of any now [reinforce KMT-1].

8. R: *If you were to apply the proposal again, what didactic aspects would you change?*

   G: I would proceed in the same way, perhaps focusing more on those points where I think students may have the most difficulties [KFLM-1].

9. R: *During the intervention, you demonstrate that you have knowledge of very specific scientific terms such as, for example, growth rate, saturation, equilibrium or stationary situation. You also provide very technical definitions on the application of the exponential function and the logistic function. In general terms, do you think it is important for a mathematics teacher to internalize concepts like these when facing an activity with a STEAM focus?*

   G: Yes, of course it is important, but always from a mathematical point of view. You do not need to know the exact terms, but you do need to know how to reason their meaning, which is what we ask the students to do [KPM-4]. In any case, if you are going to implement activities of this type in class, it is essential that you research and inform yourself before applying them [Beliefs-1].

10. R: *Finally, regarding STEAM practice, do you usually implement examples or activities with this approach in class? Do you think it is necessary in mathematics? If so, what knowledge do you think a mathematics teacher should have to deal with them?*

    G: Yes, I like to connect mathematics with other disciplines whenever I can. Above all, I like to relate it a lot to physical and historical concepts [Beliefs-1]. Or, for example, I really like sports, and I often give them exercises applied to a sport. It tends to motivate them [KFLM-2; Beliefs-3]. Of course, integrating STEAM into the teaching and learning of mathematics is necessary and also beneficial and enriching for both the student and the teacher as it can transform the educational experience and better prepare students for the future. Undoubtedly, teachers who implement this approach in their classes must be familiar with the application of mathematics in other subject areas and know interdisciplinary teaching strategies. In addition, they must be competent with the use of technological tools and educational software that complement the learning of mathematical concepts and be skilled in posing problems that require critical thinking [Beliefs-2].

As can be seen, Georg, like Sophie, demonstrates knowledge of curricular aspects in terms of level and content. In addition, he conveys his interest in carrying out activities from a STEAM perspective, emphasizing the importance of interdisciplinarity in the teaching/learning of mathematics. He also demonstrates that he is aware of students' limitations in terms of understanding and reasoning, and based on his experience, he recognizes that he knows how to present the exercises to the students so that they do not become demotivated. Due to his engineering background, he shows that he has knowledge of more technical aspects, but insists that what is important is knowing how to transmit to students the meaning of the concepts they are studying and their connection with other disciplines.

4.4. Final reflection

In conclusion, the triangulation of data from classroom observations, the checklist of interdisciplinary knowledge elements, and the interviews provided a comprehensive view of how both teachers approach the implementation of interdisciplinary STEAM activities in mathematics education. Both Sophie and Georg demonstrated a solid grasp of curricular content and acknowledged the importance of connecting mathematics to real-world contexts, although their approaches and teaching styles differ. Sophie's responses emphasized a more structured and analytical mindset, rooted in rigor and traditional classroom practices, while Georg appeared more inclined toward interdisciplinarity and contextualization, possibly due to his engineering background. Despite these differences, both recognized the value of incorporating STEAM-based activities to enrich students' mathematical understanding and foster critical thinking skills.

These findings not only validate the coherence and appropriateness of the designed activity but also highlight the need for more professional development opportunities focused on interdisciplinary approaches, especially in equipping teachers with strategies to help students move beyond procedural learning and toward meaningful reasoning.

5. Discussion

From the proposal made, we were able to identify the knowledge indicators that teachers used as a basis to address the proposed STEAM activity, finding some shortcomings in terms of the interdisciplinary nature of their knowledge. These findings are consistent with prior concerns in the literature that disciplinary knowledge alone is insufficient for STEAM instruction. As other scholars have noted, the integration of real-world phenomena into mathematics classrooms requires both pedagogical flexibility and a rethinking of content boundaries. In the following, each of the subdomains of the MTSK model is analyzed on the basis of the results obtained with the purpose of answering the research questions posed in the introduction.

Regarding the KoT, the mathematical knowledge of the topics involved in the activity was identified in both teachers. Both know the properties that make the mathematical objects involved definable, use standard procedures to deal with the content and justify and evaluate them appropriately, and use the usual representation registers in this type of exercise. However, the key issue in this activity was to use the proposed functions to explain a real-life social and scientific phenomenon. While Georg demonstrates interdisciplinary knowledge on the subject matter by using technical terms from scientific disciplines and exemplifications that help to understand the study phenomenon, Sophie is more involved with the mathematical content and not so much with the transversal nature of the activity, thus deviating from the main objective. This gap reflects a broader challenge in mathematics education: the need for teachers not only to possess conceptual fluency but also to mobilize it in socially and scientifically meaningful ways. This aligns with the concerns raised by authors such as Stillman et al. [40], Capone and Faggiano [41], and Huincahue and Gaete-Peralta [42], who stress that real-world contextualization is essential for students to engage meaningfully with mathematical content in interdisciplinary environments.

Thus, assuming the epistemological aspects linked to mathematics and the nature of the contents included in the KoT [11], we believe it is essential for teachers to know how to use this knowledge. In particular, they should be able to apply it to explain social, natural, scientific, or experimental events. In the same way, they must also be able to identify the mathematics on which any phenomenon with these characteristics is based [43]. For this reason, it is useful to include, as knowledge indicators within the KoT, those referring to the applicability of mathematical content in real-world contexts. These may include connections between areas and other STEAM disciplines, the historical development of concepts, and their interpretation to understand certain phenomena. In this regard, we consider that it is essential for the mathematics teacher to know how, why, and for what purpose the different concepts that are studied at school have arisen from a holistic point of view [44,45].

As far as KSM is concerned, Sophie and Georg show that they know the relationships between the mathematical contents of the level. In some cases, they refer to some other content from lower levels to explain a more complex concept or justify a procedure. However, although we do appreciate the connections of complexification and simplification referring to mathematical content, the lack of interconceptual connection of transversal concepts is obvious [46–48], especially in the case study of Sophie, who, despite having a solid mathematical knowledge, did not necessarily lead to an effective interdisciplinary application. This aligns with previous findings in interdisciplinary education research [40–42], which highlight how even experienced teachers often struggle to make connections across disciplinary lines.

Although the KSM does contemplate the relationship of mathematical content with concepts from other disciplines [8,18], it is advisable to reinforce it by including not only connections within mathematics but also structured links to scientific and technological knowledge, especially in contexts where phenomena require multi-variable reasoning. For the mathematics teacher to be a competent figure in the framework of STEAM education, it is convenient to expand the transversal connections that Carrillo-Yáñez et al. [8] defined in their model as the interrelationship between concepts with common characteristics. This implies having the ability to understand how the theoretical concepts that we have already internalized are transversally related to introduce other new concepts in practice or to prove or refute a hypothesis [43].

We finish with the subdomains of mathematical knowledge, analyzing the KPM subdomain. Both teachers' ways of creating and writing mathematics are appropriate for the level. While Sophie chooses to use more rigorous mathematics, Georg chooses to use more elementary and less complex structures. In any case, both demonstrate knowledge of the formalist and practical parts of mathematics. As for the presentation of the content, they also follow a logical sequence as the problem progresses, planning and hierarchizing the relationship of the content [12,49]. In this sense, we do not consider it necessary to redefine the knowledge elements of the subdomain beyond logical argumentation and the use of logical, formal, and reasoning systems [43,50] specific to the disciplines involved in the activity. This is because the way mathematics is produced in a STEAM activity is the same as that used to address any mathematical problem.

Regarding the KMT and KFLM subdomains of didactic knowledge, we were able to identify in both teachers the main characteristics that a mathematics teacher should have and know to develop effective and efficient teaching-learning of mathematics. Both demonstrate knowledge of how students think and interact when facing a task. They are aware of their limitations and support their understanding by offering guidelines and learning strategies, strengthening their mathematical skills, and effectively guiding them through the mathematical procedure. As a result, they help reinforce students' mathematical attitude. However, as far as the teaching-learning of the study phenomenon is concerned, many deficiencies are observed, especially in Sophie, in whom we observed a slight disengagement with the subject.

For this reason, we consider that for the mathematics teacher to be competent in the STEAM framework, they must know how to organize teaching around the central problem, considering two basic elements: contextualization and emotional and cognitive impact [51]. While the MTSK model acknowledges knowledge of students' misconceptions and learning processes, our findings suggest the need to strengthen the emotional and cognitive engagement dimension within interdisciplinary contexts. As suggested by Tytler and Self [52], STEAM education should not only be cognitively rigorous but also personally meaningful for students, which requires emotional sensitivity from the teacher and a reframing of failure as part of the inquiry process.

Contextualization, unlike what was previously defined in the KoT, where the focus was on understanding the context of the problem and how to introduce it properly, should, in the teaching sense, be understood differently. From a STEAM perspective, it refers to the intellectual and emotional preparation of students for problem-solving [53]. For this reason, teachers must be able to identify the problem, analyze it, and know how to convey to their students the importance of its application beyond formal mathematics. In this sense, for teachers to perform a good interdisciplinary training with their students, it is necessary that they prepare adequately by doing prior research work.

With regard to the emotional and cognitive part, although it is true that the KFLM contemplates knowledge about the students' perception of different mathematical contents, it is important that teachers also know their intellectual and affective needs, especially when applying an activity with an interdisciplinary nature. In this way, it is possible to avoid frustration among students if they do not manage to solve the exercise correctly, conveying to them the idea that what is important in activities with a STEAM approach is not so much the result but the learning they obtained throughout the process [54]. Likewise, in line with the characteristics defined in the KMT subdomain, the teacher, when incorporating STEAM activities into their programming, should know how to guide discussions during the intervention, encourage communication, and provide feedback on the progress achieved [49].

Finally, from the interviews conducted, we found that the KMLS subdomain, apart from contemplating the learning standards related to the discipline of mathematics, should also include the existing curricular specifications on STEAM education when the teachers' objective is to develop the mathematical ability of their students from an interdisciplinary perspective. In this sense, teachers should have reference to existing curricular models for STEAM education, such as those designed by Tytler and Self [52] or Aguilera et al. [49], and also be aware of the sequencing of the topics of the disciplines involved to know the best moment to implement the activity.

This suggests that teacher education programs must go beyond disciplinary preparation to include sustained interdisciplinary experiences, curriculum planning across STEAM fields, and training in using digital tools for modeling real phenomena. Therefore, we argue that the MTSK model could evolve into a more comprehensive framework that explicitly integrates curriculum awareness, epistemological diversity, and sociocultural responsiveness in mathematics teaching within STEAM education.

6. Conclusions

We agree with Shulman [1] in that it is essential for teachers to have a deep knowledge of the content they teach and why they teach it, and also, of course, how and when to apply it. Regarding mathematics teachers, characterizing their knowledge is one of the most studied topics in recent years [55–58], especially regarding the structure of professional knowledge. However, the educational community is engaged in a continuous process of reflection, and mathematics teachers are increasingly required to have an interdisciplinary profile [59]. As we already know, Carrillo et al. [11] designed the MTSK model to be used as a teaching framework to integrate mathematical content with pedagogical practices from the point of view of the specificity and specialization of the mathematics teacher's knowledge. But, in view of the results obtained in this research, we consider the possibility that for this model to be suitable for guiding mathematics teachers toward interdisciplinary knowledge, it is advisable to adapt the focus of some of the subdomains defined therein, evidently assuming as valid all the elements of knowledge that arise from the original model.

In this sense, it is not only necessary to focus on what makes the mathematics teacher's knowledge specialized [60] but also to analyze what makes it integrated and interdisciplinary. That is, mathematics teachers must prepare their students to be able to solve problems in real contexts in which mathematical concepts or procedures are needed to solve them [52]. In this regard, we consider that the elements that form part of the specialized knowledge of the mathematics teacher within the MTSK model are not entirely sufficient to address methodological proposals put forward from the STEAM perspective. As discussed before, it is necessary to include knowledge indicators that enhance the mathematical content and its didactics from a holistic point of view and that also adequately prepare the way toward professional learning of this practice before implementing it.

Research shows that having a great knowledge of mathematical content does not always translate into greater competence in teaching it when it comes to linking it to other disciplines or areas of knowledge. We agree that implementing activities that require in-depth analysis and a comprehensive understanding of mathematical concepts can be much more enriching for students. However, this presupposes a deeper and more systemic knowledge of the topics on the part of mathematics teachers, as well as contemplating a more cross-disciplinary view of content and highlighting the importance of knowledge of mathematical practice within the STEAM framework. It is also essential to broaden the curricular references with a view to a good adaptation of the content to the curricular process, where the intellectual, affective, and cognitive demands and needs of the students are considered.

Building on the findings of this study, we offer recommendations to enhance the implementation of interdisciplinary approaches in mathematics education within the STEAM framework:

1. Professional development programs for both pre-service and in-service mathematics teachers should explicitly incorporate training in the design and delivery of STEAM-integrated lessons. These programs should prioritize methodologies such as PBL, which facilitate the connection between mathematical concepts and their applications in real-world scientific and technological contexts.
2. It is essential that curricular guidelines and teacher education frameworks reflect the interdisciplinary nature of STEAM education. Policymakers are encouraged to revise current standards to include explicit references to cross-disciplinary competencies and the integration of mathematical knowledge with other STEM fields. This alignment will not only support the development of coherent instructional strategies but also ensure consistency across educational stages.
3. We highlight the need to create learning environments that address not only the cognitive demands of interdisciplinary tasks but also the emotional and motivational dimensions of student learning. Teachers should be equipped with strategies to promote student engagement, such as encouraging a growth mindset, valuing process over product, and providing tailored scaffolding to mitigate potential frustration in complex, multi-disciplinary activities.

By embedding these recommendations into both teacher training and curricular policy, we believe that mathematics educators will be better prepared to meet the demands of STEAM education, and students will be more effectively supported in developing the integrated knowledge and skills necessary for success in today’s STEM-oriented society.

7. Limitations and future lines of research

While this study offers valuable insights into the specialized knowledge mobilized by mathematics teachers in STEAM-oriented instruction, certain methodological limitations should be acknowledged. First, the qualitative nature of the research, based on two instrumental case studies, does not aim for generalization but rather in-depth understanding of specific teaching contexts. The findings reflect the perspectives and practices of two particular teachers within a limited temporal and curricular frame. Consequently, the results may not be directly transferable to other educational contexts with different institutional structures, student profiles, or cultural backgrounds.

Second, the design of the activity, although validated by experts and piloted with future teachers, may have introduced some bias in the type of knowledge mobilized, particularly favoring mathematical and didactic components of the MTSK model. It is possible that other types of knowledge or subdomains, such as those more related to beliefs or long-term curricular planning, were less visible in the context of a single intervention.

Finally, the analysis relies heavily on classroom observations and interview data. Although triangulation and intercoder reliability measures were applied to enhance credibility and ensure interpretive validity, the inclusion of additional data sources could enrich future investigations by offering a broader view of teacher knowledge in action; for example, student artifacts (not to assess student learning, but to gain indirect evidence of how teacher knowledge is enacted in classroom practice) or teacher reflection journals, which may provide a richer insight into decision-making processes.

These limitations point to promising avenues for continued exploration of interdisciplinary teaching practices and teachers' knowledge in STEAM contexts. In fact, we are working on designing a new theoretical model that, assuming the subdomains of the MTSK model, allows us to characterize the specialized and interdisciplinary knowledge of the mathematics teacher when faced with this type of activity. To do this, we are basing ourselves on research carried out in recent years, in addition to our own, such as the one presented here, in which the meaning and interpretation of the different subdomains of the MTSK model are analyzed. This allows us to refine the model and highlight those characteristics that we think give meaning to this interdisciplinary approach.

Author contributions

Daniel Martín-Cudero: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Visualization, Writing – original draft, Writing – review & editing; Rocío Guede-Cid and Ana Isabel Cid-Cid: Conceptualization, Project administration, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

Ethics declaration

This work has received the approval of the research ethics committee of the Rey Juan Carlos University with internal registration number 290220241352024.

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