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Tutorial note

Unification of the common methods for solving the first-order linear ordinary differential equations

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Abstract: A good understanding of the mathematical processes of solving the first-order linear ordinary differential equations (ODEs) is the foundation for undergraduate students in science and engineering programs to progress smoothly to advanced ODEs and/or partial differential equations (PDEs) later. However, different methods for solving the first-order linear ODEs are presented in various textbooks and resources, which often confuses students in their choice of the method for solving the ODEs. This special tutorial note presents the practices the author used to address such confusions in solving the first-order linear ODEs for students engaged in the bachelorette engineering studies at a regional university in Australia in recent years. The derivation processes of the four commonly adopted methods for solving the first-order linear ODEs, including three explicit methods and one implicit method presented in many textbooks, are presented first, followed by the logical interconnections that unify these four methods to clarify student's confusions on different presentations of the procedures and the solutions in different sources. Comparisons among these methods are also made.

Keywords: ordinary differential equations (ODEs), implicit solution, explicit solution, advanced mathematics

1. Introduction

Solving ordinary differential equations (ODEs) is an important part in advanced applied mathematics in undergraduate programs involving science and engineering. The fundamentals of different types of ODEs and the techniques to solve common ODEs are usually introduced immediately after the completion of elementary calculus in many Australian institutions. Solving ODEs begins with the first-order differential equations, and then gradually progresses to special ODEs, such as the Bernoulli equation, and the higher-order ODEs. Hence, a comprehensive understanding of the mathematical rationales and processes of solving the first-order linear ODEs sets up a solid foundation for students to deal with other types of ODEs later with confidence.

However, in most tertiary institutions in the world, one textbook, or one set of lecture notes, is usually prescribed for an advanced mathematics course in the science or engineering program according to the curriculum requirements. Teaching and learning would largely follow the context of the prescribed textbook or the provided lecture notes although students can still search own references from different sources. This raises a common problem with student's learning as different textbooks or sources usually provide students with different approaches for solving the same ODEs, particularly in solving the first-order linear ODEs in textbooks used in universities in different regions of the world [1-9]. Such diversity would benefit a few talented students but may confuse many other students who always face the challenge in dealing with advanced mathematics. This has been experienced by the author in teaching applied mathematics for the past decade. The confusion may come from the mathematical process of a method that requires some particular prerequisite knowledge of which students are lack, or from different presentations of a procedure in different sources.

To address such confusions in solving the first-order linear ODEs for students engaged in the bachelorette engineering studies at a regional university in Australia in recent years, the author prepared this special tutorial note on unifying the common methods for solving the ODEs from some popular textbooks. Since the full discourse of the unification of multiple methods would take more than two teaching hours and some parts require knowledge of multivariable calculus of which most students were lack, this special tutorial note was offered as an optional activity for any keen student, outside the normal lecture and tutorial sessions. Although this was an optional activity, it was found that most students were significantly benefited from studying this special note. This paper intends to share the success of this special tutorial note on unifying the common methods for solving the first-order linear ODEs.

2. The common methods for solving the first-order linear ODEs

The first-order linear ODEs have different forms, in which equation (1) below is commonly known as the standard form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

There are various approaches to solve this ODE in different sources. The four commonly adopted methods in different textbooks are summarized in the following subsections.

2.1. Method of variation of parameters

Variation of parameters, also called variation of constants, is a general method to solve the inhomogeneous linear ODE (1) [1, 2]. It is extended from the solution to the homogenous linear ODE of the corresponding inhomogeneous ODE (1). For the standard ODE (1), if $Q(x) = 0$, it becomes a homogenous linear ODE,

$$\frac{dy}{dx} + P(x)y = 0\,. \tag{2}$$

This homogenous ODE can be solved using separation of variables as follows:

$$\frac{dy}{dx} = -P(x)y \longrightarrow \frac{dy}{y} = -P(x)dx$$

$$\int \frac{dy}{y} = \int -P(x)dx \longrightarrow \ln y = \int -P(x)dx + c_1.$$

Hence,

$$y = e^{\int -P(x)dx + c_1} = e^{c_1} e^{\int -P(x)dx} = ce^{\int -P(x)dx}\,. \tag{3}$$

This is the general solution to the homogenous ODE (2). It already contains one unknown constant c so there is no need to add any new unknown constant from integral $\int -P(x)dx$.

To find the general solution to the inhomogeneous ODE (1), we replace the constant c in the general solution (3) to the homogenous ODE (2) by an unknown function $u(x)$, i.e., assuming

$$y = u(x)e^{\int -P(x)dx} \tag{4}$$

to be the solution to the inhomogeneous ODE (1). Once we find $u(x)$, the general solution is then obtained by the formula (4). Apply the product rule of differentiation to the formula (4)

$$\frac{dy}{dx} = \frac{d[u(x)]}{dx} e^{\int -P(x)dx} + u(x)\frac{d[e^{\int -P(x)dx}]}{dx} = u'(x)e^{\int -P(x)dx} + u(x)e^{\int -P(x)dx}\,\frac{d[\int -P(x)dx]}{dx}$$

$$= u'(x)e^{\int -P(x)dx} + u(x)e^{\int -P(x)dx}[-P(x)]$$

or

$$\frac{dy}{dx} = u'(x)e^{\int -P(x)dx} - P(x)u(x)e^{\int -P(x)dx}\,. \tag{5}$$

Substitute the formulae (4) and (5) into the ODE (1)

$$u'(x)e^{\int -P(x)dx} - P(x)u(x)e^{\int -P(x)dx} + P(x)u(x)e^{\int -P(x)dx} = Q(x)$$

$$u'(x)e^{\int -P(x)dx} = Q(x) \longrightarrow u'(x) = Q(x)e^{\int P(x)dx},$$

then integrate both sides

$$u(x) = \int Q(x)e^{\int P(x)dx}dx + c\,. \tag{6}$$

Substitute the formula (6) into the formula (4) to obtain the general solution to the inhomogeneous ODE (1)

$$y = u(x)e^{\int -P(x)dx} = e^{\int -P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + c\right]. \tag{7}$$

This is a unified explicit solution to the ODE (1). The advantage of variation of parameters is that students are able to fully understand the derivation process as the process only requires an understanding of the basic concepts of elementary calculus. The other good fact about this method is that it provides students with an explicit solution, by which one can directly find the solution $y = f(x)$ to the ODE (1) if it comprises of relatively simple $P(x)$ and $Q(x)$.

2.2. Integrating factor by the quotient rule of elementary calculus

The standard ODE (1) can be rewritten to its differential form (8),

$$dy + P(x)ydx = Q(x)dx \tag{8}$$

Divide both sides of the above equation by a common function $\bar{\mu}(x)$

$$\frac{1}{\bar{\mu}(x)}dy + \frac{P(x)}{\bar{\mu}(x)}ydx = \frac{Q(x)}{\bar{\mu}(x)}dx \tag{9}$$

so that the left side becomes the differential of a quotient, i.e.,

$$d[\frac{y}{\bar{\mu}(x)}] = \frac{Q(x)}{\bar{\mu}(x)}dx. \tag{10}$$

Expand the equation (10) to the following form

$$\frac{\bar{\mu}(x)dy - \bar{\mu}'(x)ydx}{[\bar{\mu}(x)]^2} = \frac{Q(x)}{\bar{\mu}(x)}dx$$

or

$$\frac{1}{\bar{\mu}(x)}dy - \frac{\bar{\mu}'(x)}{[\bar{\mu}(x)]^2}ydx = \frac{Q(x)}{\bar{\mu}(x)}dx. \tag{11}$$

By comparing both sides of the equations (9) and (11), both equations will be equal if the following condition is met:

$$\frac{P(x)}{\bar{\mu}(x)} = -\frac{\bar{\mu}'(x)}{[\bar{\mu}(x)]^2} \quad \text{or} \quad \frac{\bar{\mu}'(x)}{\bar{\mu}(x)} = -P(x). \tag{12}$$

The equation (12) is equivalent to

$$d[\ln \bar{\mu}(x)] = -P(x) \longrightarrow \ln \bar{\mu}(x) = -\int P(x)dx.$$

This can be explicitly expressed as

$$\bar{\mu}(x) = e^{-\int P(x)dx}. \tag{13}$$

Therefore, if the common function, or the integrating factor $\bar{\mu}(x)$, is determined by the equation (13), the standard ODE (1) can be transferred to the equation (10). Integrate both sides of the equation (10)

$$\frac{y}{\bar{\mu}(x)} = \int \frac{Q(x)}{\bar{\mu}(x)} dx + c$$

or

$$y = \bar{\mu}(x)\left[\int \frac{Q(x)}{\bar{\mu}(x)} dx + c\right] \longleftarrow \bar{\mu}(x) = e^{-\int P(x)dx}. \tag{14}$$

Hence, the solution defined by the formula (14) produces a split explicit solution to the standard ODE (1), by which the whole process is divided into two separate steps for a better control, particularly for complicated integrations associated with $P(x)$ and $Q(x)$. This split explicit solution (14) is presented in textbook [2].

2.3. Integrating factor by the product rule of elementary calculus

Multiply both sides of the differential equation (8) by a common function $\mu(x)$

$$\mu(x)dy + \mu(x)P(x)ydx = \mu(x)Q(x)dx \tag{15}$$

so that the left side becomes the differential of a product, i.e.,

$$d[\mu(x)y] = \mu(x)Q(x)dx. \tag{16}$$

Expand the equation (16) to the following form

$$\mu(x)dy + \mu'(x)ydx = \mu(x)Q(x)dx$$

or

$$dy + \frac{\mu'(x)}{\mu(x)} ydx = Q(x)dx. \tag{17}$$

By comparing both sides of the equations (8) and (17), both equations will be equivalent if the following condition is met

$$P(x) = \frac{\mu'(x)}{\mu(x)}. \tag{18}$$

The equation (18) is equivalent to

$$\frac{d[\mu(x)]}{\mu(x)} = P(x)dx\,.$$

Integrate both sides

$$\ln[\mu(x)] = \int P(x)dx\,, \quad \text{or} \quad \mu(x) = e^{\int P(x)dx}\,.$$

Therefore, if the common function, or the integrating factor $\mu(x)$, is determined by the above formula, by integrating both sides of the transferred ODE (16), a general solution to the standard ODE (1) can be obtained from

$$\mu(x)y = \int \mu(x)Q(x)dx + c \longleftarrow \mu(x) = e^{\int P(x)dx}\,. \tag{19}$$

This split implicit solution appeared in many textbooks [3-9].

2.4. Integrating factor by the exact differential equation of multivariable calculus

In the popular textbook for advanced engineering mathematics [10], if the integrating factor $F(x)$ that can make the first-order ODE in the differential form (20) below

$$\overline{P}(x,y)dx + \overline{Q}(x,y)dy = 0 \tag{20}$$

to the exact differential equation (21)

$$F(x)\overline{P}(x,y)dx + F(x)\overline{Q}(x,y)dy = 0 \tag{21}$$

so that

$$\frac{\partial(F\overline{P})}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial(F\overline{Q})}{\partial x}\,, \tag{22}$$

the solution to the differential equation (20) becomes

$$u(x,y) = c = \int F\overline{P}dx + g(y) \text{ or } u(x,y) = c = \int F\overline{Q}dy + f(x)\,, \tag{23}$$

where c is a constant. Such integrating factor is determined by

$$F(x) = e^{\int R(x)dx} \longleftarrow R = \frac{1}{\overline{Q}}\left(\frac{\partial \overline{P}}{\partial y} - \frac{\partial \overline{Q}}{\partial x}\right). \tag{24}$$

The differential form (8) of the standard ODE (1) can be rewritten as

$$[P(x)y - Q(x)]dx + dy = 0\,. \tag{25}$$

Compared with the equation (20),

$$\overline{P}(x,y)=P(x)y-Q(x),\quad \overline{Q}(x,y)=1\,. \tag{26}$$

Substitute the formula (26) into the formula (24)

$$R=\frac{1}{\overline{Q}}\left(\frac{\partial\overline{P}}{\partial y}-\frac{\partial\overline{Q}}{\partial x}\right)=P(x)\longrightarrow F(x)=e^{\int R(x)dx}=e^{\int P(x)dx}\,. \tag{27}$$

Through $F(x)$, a split explicit solution to the ODE (1) can be obtained as

$$y=e^{-h}\left[\int e^{h}Q(x)dx+c\right]\longleftarrow h=\int P(x)dx\,. \tag{28}$$

Example 1: Find the general solution to the ODE $y'+y\cos x=e^{-\sin x}$ by the formulae (7), (14), (19), and (28), respectively.

This ODE is in the standard form with $P(x)=\cos x;\ Q(x)=e^{-\sin x}$. By the formula (7), the general solution can be obtained as

$$\begin{aligned}y&=e^{\int -P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx+c\right]=e^{\int -\cos xdx}\left[\int e^{-\sin x}e^{\int \cos xdx}dx+c\right]\\&=e^{-\sin x}\left[\int e^{-\sin x}e^{\sin x}dx+c\right]=e^{-\sin x}\left[\int dx+c\right]=e^{-\sin x}(x+c).\end{aligned}$$

By the formula (14), the general solution can be obtained as

$$\begin{aligned}&\overline{\mu}(x)=e^{-\int P(x)dx}=e^{-\int \cos xdx}=e^{-\sin x}\\&y=\overline{\mu}(x)\left[\int\frac{Q(x)}{\overline{\mu}(x)}dx+c\right]=e^{-\sin x}\left[\int\frac{e^{-\sin x}}{e^{-\sin x}}dx+c\right]=e^{-\sin x}\left[\int dx+c\right]=e^{-\sin x}(x+c).\end{aligned}$$

By the formula (19), the general solution can be obtained as

$$\begin{aligned}&\mu(x)=e^{\int P(x)dx}=e^{\int \cos xdx}=e^{\sin x}\\&e^{\sin x}y=\int\mu(x)Q(x)dx+c=\int e^{\sin x}e^{-\sin x}dx+c=\int dx+c=x+c\\&\therefore e^{\sin x}y=x+c.\end{aligned}$$

Note this is not an explicit solution to the ODE.

By formula (28), the general solution can be obtained as

$$\begin{aligned}&h=\int P(x)dx=\int\cos xdx=\sin x\\&y=e^{-h}\left[\int e^{h}Q(x)dx+c\right]=e^{-\sin x}\left[\int e^{\sin x}e^{-\sin x}dx+c\right]=e^{-\sin x}\left[\int dx+c\right]=e^{-\sin x}(x+c).\end{aligned}$$

If the process is correct, all explicit methods will result in the same solution $y = f(x)$. The implicit solution (19) requires an extra step to obtain the explicit solution.

Example 2: Find the general solution to the ODE $\frac{dy}{dx}+\frac{1}{x^2-x-2}y = 28\sqrt[3]{(x+1)^4}$ by the formulae (7), (14), (19), and (28), respectively.

This ODE is in the standard form with

$$P(x)=\frac{1}{x^2-x-2}=\frac{1}{(x-2)(x+1)}=\frac{1}{3}\left(\frac{1}{x-2}-\frac{1}{x+1}\right),\ \text{and}\ Q(x)=28\sqrt[3]{(x+1)^4}=28(x+1)^{\frac{4}{3}}.$$

By the formula (7), the general solution can be obtained as

$$
\begin{aligned}
y &= e^{\int -P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx+c\right]=e^{\int -\frac{1}{x^2-x-2}dx}\left[\int 28\sqrt[3]{(x+1)^4}e^{\int \frac{1}{x^2-x-2}dx}dx+c\right]\\
&= e^{\int -\frac{1}{3}(\frac{1}{x-2}-\frac{1}{x+1})dx}\left[\int 28(x+1)^{\frac{4}{3}}e^{\int \frac{1}{3}(\frac{1}{x-2}-\frac{1}{x+1})dx}dx+c\right]=e^{-\frac{1}{3}\ln\frac{x-2}{x+1}}\left[\int 28(x+1)^{\frac{4}{3}}e^{\frac{1}{3}\ln\frac{x-2}{x+1}}dx+c\right]\\
&= e^{\frac{1}{3}\ln\frac{x+1}{x-2}}\left[\int 28(x+1)^{\frac{4}{3}}e^{\frac{1}{3}\ln\frac{x-2}{x+1}}dx+c\right]=\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left[\int 28(x+1)^{\frac{4}{3}}\left(\frac{x-2}{x+1}\right)^{\frac{1}{3}}dx+c\right]\\
&= \left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left[\int 28(x-2)^{\frac{1}{3}}(x+1)^{\frac{4}{3}-\frac{1}{3}}dx+c\right]=\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left[\int 28(x-2)^{\frac{1}{3}}(x+1)dx+c\right]\\
&= \left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left\{28(x+1)\left[\left(\frac{3}{4}(x-2)^{\frac{4}{3}}\right]-\int 21(x-2)^{\frac{4}{3}}dx+c\right\}\\
&= \left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left\{21(x+1)(x-2)^{\frac{4}{3}}-\left[21\times\frac{3}{7}(x-2)^{\frac{7}{3}}\right]+c\right\}\\
&= \left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left[21(x+1)(x-2)^{\frac{4}{3}}-9(x-2)^{\frac{7}{3}}+c\right]=\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\left\{3(x-2)^{\frac{4}{3}}[7(x+1)-3(x-2)]+c\right\}\\
&= \left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}[3(x-2)^{\frac{4}{3}}(7x+7-3x+6)+c]=3(x+1)^{\frac{1}{3}}(x-2)^{\frac{4}{3}-\frac{1}{3}}(4x+13)+c\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\\
&= 3(x+1)^{\frac{1}{3}}(x-2)(4x+13)+c\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}=3(x+1)^{\frac{1}{3}}(4x^2+5x-26)+c\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}\\
&= (x+1)^{\frac{1}{3}}\left[3(4x^2+5x-26)+c\left(\frac{1}{x-2}\right)^{\frac{1}{3}}\right]=\sqrt[3]{x+1}\left(12x^2+15x-78+\frac{c}{\sqrt[3]{x-2}}\right)
\end{aligned}
$$

By the formula (14), the general solution can be obtained as

$$\overline{\mu}(x)=e^{-\int P(x)dx}=e^{\int -\frac{1}{x^2-x-2}dx}=e^{\int -\frac{1}{3}(\frac{1}{x-2}-\frac{1}{x+1})dx}=e^{-\frac{1}{3}\ln\frac{x-2}{x+1}}=e^{\frac{1}{3}\ln\frac{x+1}{x-2}}=\left(\frac{x+1}{x-2}\right)^{\frac{1}{3}}$$

$$y=\bar{\mu}(x)\left[\int\frac{Q(x)}{\bar{\mu}(x)}dx+c\right]=(\frac{x+1}{x-2})^{\frac{1}{3}}\left[\int\frac{28(x+1)^{\frac{4}{3}}}{(\frac{x+1}{x-2})^{\frac{1}{3}}}dx+c\right]$$

$$=(\frac{x+1}{x-2})^{\frac{1}{3}}\left[\int 28(x+1)(x-2)^{\frac{1}{3}}dx+c\right]=...=\sqrt[3]{x+1}(12x^2+15x-78+\frac{c}{\sqrt[3]{x-2}})$$

By the formula (19), the general solution can be obtained as

$$\mu(x)=e^{\int P(x)dx}=e^{\int\frac{1}{x^2-x-2}dx}=e^{\int\frac{1}{3}(\frac{1}{x-2}-\frac{1}{x+1})dx}=e^{\frac{1}{3}\ln\frac{x-2}{x+1}}=(\frac{x-2}{x+1})^{\frac{1}{3}}$$

$$\mu(x)y=\int\mu(x)Q(x)dx+c\longrightarrow(\frac{x-2}{x+1})^{\frac{1}{3}}y=\int(\frac{x-2}{x+1})^{\frac{1}{3}}28(x+1)^{\frac{4}{3}}dx+c$$

$$=\int 28(x-2)^{\frac{1}{3}}(x+1)dx+c=...=(x-2)^{\frac{4}{3}}(12x+39)+c$$

$$\therefore(\frac{x-2}{x+1})^{\frac{1}{3}}y=(x-2)^{\frac{4}{3}}(12x+39)+c.$$

Note that this is only an intermediate solution to the ODE.

By the formula (28), the general solution can be obtained as

$$h=\int P(x)dx=\int\frac{1}{x^2-x-2}dx=\int\frac{1}{3}(\frac{1}{x-2}-\frac{1}{x+1})dx=\frac{1}{3}\ln\frac{x-2}{x+1}=\ln(\frac{x-2}{x+1})^{\frac{1}{3}}$$

$$y=e^{-h}\left[\int e^hQ(x)dx+c\right]=e^{-\ln(\frac{x-2}{x+1})^{\frac{1}{3}}}\left[\int e^{\ln(\frac{x-2}{x+1})^{\frac{1}{3}}}28(x+1)^{\frac{4}{3}}dx+c\right]$$

$$=(\frac{x+1}{x-2})^{\frac{1}{3}}\left[\int 28(x+1)^{\frac{4}{3}}(\frac{x-2}{x+1})^{\frac{1}{3}}dx+c\right]=...=\sqrt[3]{x+1}(12x^2+15x-78+\frac{c}{\sqrt[3]{x-2}}).$$

Note that the process of the formula (28) becomes the same as that of the formula (7) once substituting h into the solution (28).

In this example, all integrations involved are complicated. If the process is correct, all explicit methods will result in the same solution $y = f(x)$ directly. However, the solution from the implicit method (19) requires extra steps to obtain the explicit solution. In this case, removing $\mu(x)$ from $\mu(x)y$ is not straightforward and will involve more manipulations to reach the same solution produced by other explicit methods. Often some students would make mistakes during the final manipulations for the explicit solution.

3. Unification of the common methods

As students could choose any of these common methods to solve the first-order linear ODEs, the different presentations often cause confusions to some students particularly when the derivation process for a method is not well presented or is fully understood by the students. The effort on unifying these common methods is intended to clarify any confession to the different forms of solution on which students may have.

3.1. Unification among the explicit methods (7), (14) and (28)

This unification is intended to answer student's question "*Are the explicit methods (7), (14) and (28) resulted from different derivation processes actually equal to each other?*" or alike.

These three explicit methods can be mutually converted to each other by some simple substitutions.

To convert the split method (14) to the unified method (7), we substitute

$$\bar{\mu}(x)=e^{\int -P(x)dx} \text{ and } \frac{1}{\bar{\mu}(x)}=\frac{1}{e^{\int -P(x)dx}}=e^{\int P(x)dx}$$

into the split form (14), i.e.,

$$y=\bar{\mu}(x)\left[\int\frac{Q(x)}{\bar{\mu}(x)}dx+c\right]=e^{-\int P(x)dx}\left[\int\frac{Q(x)}{e^{-\int P(x)dx}}dx+c\right]=e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx+c\right],$$

which is the same as the unified explicit method (7).

To convert the method (7) to the method (14), let $e^{\int -P(x)dx}$ in the method (7) be

$$e^{\int -P(x)dx}=\bar{\mu}(x). \text{ Then } e^{\int P(x)dx}=e^{-\int -P(x)dx}=\frac{1}{e^{\int -P(x)dx}}=\frac{1}{\bar{\mu}(x)}.$$

The method (7) becomes

$$y=e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx+c\right]=\bar{\mu}(x)\left[\int\frac{Q(x)}{\bar{\mu}(x)}dx+c\right],$$

which is the same as the split explicit solution (14).

The split explicit solution (28) resulted from a process involving multivariable calculus presented in textbook [10] is also convertible to the unified explicit solution (7) by substituting $h=\int P(x)dx$ into the solution (28), i.e.,

$$y=e^{-h}\left[\int e^{h}Q(x)dx+c\right]=e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx+c\right].$$

This is the same as the unified explicit solution (7).

3.2. Unification between the split explicit method (14) and the split implicit method (19)

This unification is intended to answer student's question "*Why in many textbooks the integrating factor is associated with P(x) but in other textbooks is associated with –P(x)?*" or alike.

These two methods can be mutually converted to each other. In the implicit method (19), divide both sides by $\mu(x)$

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + c\right] = \frac{1}{e^{\int P(x)dx}}\left[\int e^{\int P(x)dx}Q(x)dx + c\right].$$

Considering

$$e^{\int P(x)dx} = e^{-\int -P(x)dx} = \frac{1}{e^{\int -P(x)dx}} = \frac{1}{\bar{\mu}(x)} \text{ and } \frac{1}{e^{\int P(x)dx}} = \frac{1}{e^{-\int -P(x)dx}} = e^{\int -P(x)dx} = \bar{\mu}(x),$$

$$y = \frac{1}{e^{\int P(x)dx}}\left[\int e^{\int P(x)dx}Q(x)dx + c\right] = \bar{\mu}(x)\left[\int \frac{Q(x)}{\bar{\mu}(x)}dx + c\right].$$

This is the same as the explicit method (14). Hence, both methods are convertible.

By using the integrating factor determined with $-P(x)$, i.e., $\bar{\mu}(x) = e^{\int -P(x)dx}$, the solution $y = f(x)$ to a first-order linear ODE can be obtained directly by resolving all integrations involved. This means that we will have the final solution to the ODE by using $\bar{\mu}(x) = e^{\int -P(x)dx}$.

By using the integrating factor determined with $P(x)$, i.e., $\mu(x) = e^{\int P(x)dx}$, the solution to a first-order linear ODE obtained is only an intermediate solution $\mu(x)y$. One has to work on this intermediate solution further to find the final solution $y = f(x)$.

In a few textbooks [6, 9, 10], the split implicit form was converted to an explicit form similar to

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx + c\right],$$

which is the same as the unified explicit method (7). However, such converted form is presented as "an auxiliary form" of the implicit method (19) in these books.

3.3. Integrating factor by the multivariable total differential

In the textbook [10], an attempt on using the multivariable total differential to derive a method of integrating factor to solve a general first-order ODE was made but not applied to the standard first-order linear ODE (1). Some students were curious about whether using the multivariable total differential could lead to the same solution as any of the solutions (7), (14), (19), and (28). The following derivation is to address such curiosity a few students had.

The standard ODE (1) can be reorganised as a differential form as

$$dy + P(x)ydx - Q(x)dx = 0 \longrightarrow [P(x)y - Q(x)]dx + dy = 0.$$

Multiply both sides of the above equation by a common function $\mu(x)$

$$\mu(x)[P(x)y - Q(x)]dx + \mu(x)dy = 0. \qquad (29)$$

so that the left side becomes the total differential of a function $f(x, y)$, i.e.,

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0\,. \tag{30}$$

By integrating both sides of the equation (30), such function *f*(*x*, *y*) as the solution of the transferred ODE (29) from the ODE (1) can be obtained as

$$f(x,y) = c\,. \tag{31}$$

where *c* is an unknown constant. The equation (29) can be expressed as

$$M(x,y)dx + N(x,y)dy = 0 \tag{32}$$

where *M*(*x*, *y*) and *N*(*x*, *y*) are defined by

$$\begin{cases} M(x,y) = \mu(x)[P(x)y - Q(x)] \\ N(x,y) = \mu(x) \end{cases}. \tag{33}$$

By comparing the equations (30) and (32), the following correlations can be obtained

$$\begin{cases} \dfrac{\partial f}{\partial x} = M(x,y) = \mu(x)[P(x)y - Q(x)] \\ \dfrac{\partial f}{\partial y} = N(x,y) = \mu(x) \end{cases}. \tag{34}$$

If *f*(*x*, *y*) has continuous second-order partial derivatives, its two mixed second-order partial derivatives will be the same, i.e.,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}\,. \tag{35}$$

Since

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} = \mu(x)P(x) \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} = \mu'(x)\,,$$

The equation (35) becomes

$$\mu(x)P(x) = \mu'(x) \longrightarrow \frac{\mu'(x)}{\mu(x)} = P(x)\,. \tag{36}$$

This is the same as the equation (18), which produces the same integrating factor $\mu(x) = e^{\int P(x)dx}$. By the second equation in the formula (34), we obtain

$$f(x,y) = \int N(x,y)dy = \int \mu(x)dy = \mu(x)y + A(x)\,, \tag{37}$$

where $A(x)$ is an unknown function only involving x. Apply partial derivative with respect to x to the equation (37)

$$\frac{\partial f}{\partial x} = \mu'(x)y + A'(x)\,. \tag{38}$$

Consider the first equation in the formula (34) and the equation (38),

$$\mu'(x)y + A'(x) = \mu(x)P(x)y - \mu(x)Q(x)\,,$$

By the equation (36), $\mu(x)P(x) = \mu'(x)$; hence

$$\mu'(x)y + A'(x) = \mu'(x)y - \mu(x)Q(x) \longrightarrow A'(x) = -\mu(x)Q(x)\,.$$

Thus,

$$A(x) = -\int \mu(x)Q(x)dx\,. \tag{39}$$

Substitute the formula (39) into the equation (37) and consider the equation (31) $f(x, y) = c$,

$$c = \mu(x)y - \int \mu(x)Q(x)dx \quad \text{or} \quad \mu(x)y = \int \mu(x)Q(x)dx + c \longleftarrow \mu(x) = e^{\int P(x)dx}\,.$$

This is exactly the same as the split implicit solution (19).

4. Concluding remarks

This special tutorial note firstly presents the derivation processes of the four commonly adopted methods for solving the first-order linear ODEs, including three explicit methods and one implicit method that is arguably the most widely presented method in many textbooks and learning resources. Among them, the derivation of the split explicit solution (14) may be the first time derived using the quotient rule of differentiation. All these processes presented in the same tutorial note together, unlike many textbooks that only present one or two methods in a book, assist in a comprehensive understanding of the mathematical rationales and processes for solving the first-order linear ODEs with any of the four methods.

Secondly, the unifications among the four methods presented in this note provide the logical interconnections between any two of them, which clarifies student's confusions on different presentations of the procedures and the solutions in different sources. In the light of the unifications, students can choose any method to solve a given ODE without worrying how and why other students may use different methods to solve the same problem.

Thirdly, this note also demonstrates that the methods derived from elementary calculus can also be realized using multivariable calculus, which is helpful for those students who are lack of knowledge of multivariable calculus but still curious about the interconnections between the processes.

Finally, this special tutorial note explains the difference between the solutions resulted from the explicit methods and the implicit method, that is, all the explicit methods lead to the final solution $y = f(x)$ to the ODE but the implicit method leads to an intermediate solution that needs further manipulations towards the final solution $y = f(x)$ to the ODE.

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