Quantitative Finance and Economics

https://www.aimspress.com/journal/QFE

QFE, 10(1): 130–161.
DOI: 10.3934/QFE.2026007
Received: 25 July 2025
Revised: 06 November 2025
Accepted: 03 December 2025
Published: 13 March 2026

Research article

Immunization of bond portfolios: A new general framework

Alberto Bueno-Guerrero[1], **Manuel Moreno**[2,*] **and Javier F. Navas**[3]

[1] Department of Economics, IES Francisco Ayala, Avda. Francisco Ayala, s/n, 18014 Granada, Spain

[2] Department of Economic Analysis and Finance, University of Castilla-La Mancha, Cobertizo San Pedro Mártir s/n, 45071 Toledo, Spain

[3] Department of Financial Economics and Accounting, Universidad Pablo de Olavide, Ctra. de Utrera, Km. 1, 41013 Sevilla, Spain

* **Correspondence:** Email: manuel.moreno@uclm.es; Tel: +34925268800.

Abstract: We present a new general setting for the classical immunization problem under which we recover and generalize many of the results in the literature related to immunization of bond portfolios. We also propose a new duration vector adapted to our framework that allows us to obtain immunized portfolios by duration matching. We introduce the concept of second-best portfolio that is the portfolio that produces the minimum loss in the worst-case scenario. We obtain explicit expressions for this class of portfolios when short positions are allowed or forbidden. We also show that, when immunization cannot be achieved, the second-best portfolio is the closest to the duration matching situation. Finally, we carry out a numerical illustration of our theoretical results, showing the effectiveness of the duration measures and the second-best portfolios.

Keywords: Immunization; duration; direction X; arbitrage; Gateaux differential; Kuhn-Tucker conditions

JEL Codes: C61, G11, G12.

1. Introduction

Duration is a key concept in the management of interest rate risk in bond portfolios. Macaulay (1938) defined the duration as a time-weighted average of a stream of known fixed payments, and Hicks (1939) interpreted it as a measure of interest rate risk. He showed that for a given (infinitesimal) change in interest rates, the percentage change in asset value is proportional to its Macaulay duration. This result was re-obtained by other authors (Samuelson (1945), Redington (1952)) but in all cases, the result is only valid if the entire yield curve undergoes a uniform additive displacement (Ingersoll et al. (1978)).

In subsequent papers, this idea has been extended to different types of duration measures associated with specific interest rate shifts (Bierwag (1977), Chambers et al. (1988), Diebold et al. (2006)). Duration can also be used to immunize a portfolio of bonds, that is, to find a portfolio that does not lose value at some predetermined time interval. Fisher and Weil (1971) were the first to relate immunization with duration: If all forward interest rates change by the same amount, then a portfolio is immunized if its duration is equal to the length of the holding period. Among the most recent works in this field, we can mention Jacoby (2003), which presented a new risk-adjusted duration for defaultable bonds, Ortobelli et al. (2018), which introduced a new duration-convexity risk measure that leads to the concept of average immunization, and Afik et al. (2022), which provided a review on duration analysis with special emphasis on applications.

The literature addressing the uses of duration mentioned before (measuring interest rate risk and immunizing bond portfolios) under general interest rate shifts is scarce. Balbás and Ibáñez (1998) and Balbás and Romera (2007) obtained important results working with Banach spaces, although they did not obtain concrete results on duration measures. In a more recent study, Lapshin (2019) developed a non-parametric approach to hedge a bond-like obligation with respect to changes in interest rates using a bond portfolio. Despite its non-parametric nature, this model is not completely general since the term structure of interest rates is estimated via a smoothing spline approach from the same set of bonds that is used for hedging.

On the other hand, an important line of research in this area has proposed the use of certain immunization risk (dispersion) measures and has obtained the immunized portfolios by minimizing these measures. Nawalkha and Chambers (1996) introduced the so-called M-Absolute measure and immunized bond portfolios by minimizing this measure through simple linear programming. Balbás and Ibáñez (1998) proposed the immunization strategy of selecting the portfolio that minimizes a dispersion measure ($\overline{N}$) from among the duration matching portfolios using the Macaulay duration. Minimization is carried out by evaluating the measure $\overline{N}$ for each portfolio. Balbás et al. (2002) extended the measures $\overline{N}$ and M^2 of Fong and Vasicek (1984) to a set of measures, M^n, with $n > 0$, empirically selecting the most appropriate values of n. These models do not impose restrictions on shifts in the term structure, but they lack certain desirable features of duration models, such as measuring portfolio sensitivity to specific interest rate shocks or achieving immunization based on duration matching. Nawalkha et al. (2003) extended this type of models to include polynomial-based vector measures, but the degree of these polynomials must be empirically determined.

Jointly with the previous papers, several studies have empirically analyzed different topics related to the management of interest rate risk. For instance, Bravo and Da Silva (2006) assessed the relative immunization performance of the M-vector model proposed in Nawalkha and Chambers (1997) using data for the Portuguese government debt market. Yan et al. (2009) empirically analyzed the effects of default risk, call risk, and the interactions between these risk factors on bond durations. Oliveira et al. (2014) compared the efficiency of traditional and stochastic interest rate risk measures under one-, two-, and three-factor term structure models, and for different immunization periods. Afik et al. (2018) presented a two-factor duration model that accounts for both foreign interest rate and exchange rate exposures. In a similar way, Fooladi et al. (2021) considered two duration measures (real and expected-inflation durations) that quantify the sensitivity of an asset value with respect to changes in, respectively, the real interest rate and expected inflation rate. All the studies discussed up to now have focused on duration as a measure of (first-order) interest rate risk and a hedging tool. An exception is

Zhu et al. (2018) which incorporated convexity in bond portfolio immunization using a multi-factor model and reformulated the corresponding problems as tractable semi-definite programs.

In recent years, infinite-dimensional models of the dynamics of the term structure of interest rates (TSIR) have been developed based on the idea that the forward curve can be affected by an infinite number of (orthogonal) shocks.[1] However, these infinite-dimensional models have not paid attention to the immunization of bond portfolios. To the best of our knowledge, the only paper addressing this issue under a Hilbert space framework is Balbás, Montagut, and Pérez-Fructuoso (2004), which has served as inspiration for the overall approach of the present paper.

The main objective of this paper is to propose a new infinite-dimensional framework for the immunization of bond portfolios. In this framework, we study immunization against an infinite number of term structure shifts, we generalize previously established properties of duration to a new (infinite-dimensional) duration vector, and we obtain important new results.

This paper is organized as follows. Section 2 presents our immunization framework. The Gateaux differentiability of the generalized portfolio value allows us to obtain the first general results on immunized and maxmin portfolios. We also obtain the direction of maximum decrease of the portfolio's value (the Direction X in Bowden (1997)). We introduce the new concept of factorial duration, which allows us to generalize the traditional duration and the result of Hicks (1939) to our infinite-dimensional framework. In the case of real-world portfolios, we show how factorial durations generalize many of the durations previously defined in the literature, and we state our main theorem on immunization. We also present a particular framework for immunization against a finite number of sources of risk.

Section 3 studies the case in which we cannot immunize against a finite number of sources of risk and we want to obtain a second-best portfolio, i.e., the one closest to the matched duration. We provide analytical expressions (when short positions are allowed or forbidden) for the second-best portfolio by solving a linear constrained optimization problem. Section 4 provides a numerical illustration of the theoretical results obtained in previous sections, showing the effectiveness of our factorial durations and second-best portfolios. Finally, Section 5 summarizes the main conclusions.

2. Immunized portfolios

This section obtains the classical immunization theorems (Fisher and Weil (1971), Bierwag and Kaufman (1977), and Khang (1979)) within the infinite-dimensional framework. The role played in these theorems by the different types of term structure shifts will now be assumed by elements of an orthonormal basis of a functional Hilbert space.

Assumption 2.1. *The forward curves in the Musiela (1993) parameterization belong to* $C^1(I) \cap \mathcal{H}(I)$*, where* I *is the set of times to maturity, with* $I = [0, \theta]$ *or* $I = \mathbb{R}^+$*,* $C^1(I)$ *is the set of continuously differentiable functions on* I*, and* $\mathcal{H}(I)$ *is some Hilbert space of functions defined on* I*.*

By this assumption, given an orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$ of $\mathcal{H}(I)$, with $\phi_k \in C^1(I)$, $k = 1, 2, \ldots$, we can write the instantaneous forward interest rate in the Musiela parameterization as

$$f(x; \boldsymbol{\mu}_t) = \sum_{k=1}^{\infty} \mu_k(t)\, \phi_k(x) \tag{1}$$

[1] See, for example, Kennedy (1994), Goldstein (2000), Santa-Clara and Sornette (2001), and Bueno-Guerrero et al. (2015).

and the price of a bond maturing at T as

$$P(t,T;\boldsymbol{\mu}_t) = \exp\left(-\sum_{k=1}^{\infty}\mu_k(t)\int_0^{T-t}\phi_k(x)\,dx\right) \tag{2}$$

where $\mu_k(t) = \langle\phi_k, f(\cdot;\boldsymbol{\mu}_t)\rangle_{\mathcal{H}(I)}$ and $\boldsymbol{\mu}_t \equiv \{\mu_k(t)\}_{k=1}^{\infty} \in l^2$. The forward curve (and therefore the bond prices) is completely determined, for each t, by $\boldsymbol{\mu}_t \in l^2$. However, not every $\boldsymbol{\gamma} \in l^2$ generates forward curves in $C^1(I) \cap \mathcal{H}(I)$. For this reason, we have to restrict our analysis to those admissible parameters in the sense of the next definition.

Definition 2.2. *A vector $\gamma \in l^2$ is* ***admissible*** *if $f(\cdot;\boldsymbol{\gamma})$ given by (1) belongs to $C^1(I) \cap \mathcal{H}(I)$. We set $\mathcal{A} = \left\{\boldsymbol{\gamma} \in l^2 : \boldsymbol{\gamma} \text{ is admissible}\right\}$.*

The next result, whose proof is straightforward, contains some fundamental properties of the set $\mathcal{A}$.

Proposition 2.3. *The following statements hold:*

i) $\mathbf{0} \in \mathcal{A}$.
ii) $\boldsymbol{e}_k \in \mathcal{A},\ k = 1, 2, \ldots$, *where $\boldsymbol{e}_k$ is the k-th element of the usual orthonormal basis of l^2.*
iii) If $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n \in \mathcal{A}$, then $\sum_{i=1}^{n}\alpha_i\boldsymbol{\mu}_i \in \mathcal{A}$ for $\alpha_i \in \mathbb{R},\ i = 1, \ldots, n$.
iv) $\mathcal{A}$ is a convex subset of l^2.

Remark 2.4. *In general, property iii) in this proposition is not true for infinite linear combinations because we cannot guarantee that the functions obtained are continuous.*

We consider a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual hypotheses.[2]

We also assume that $\mathcal{F} = \mathcal{F}_{\Upsilon}$, where Υ denotes the finite time horizon for trading zero-coupon bonds. Concrete details on the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le\Upsilon}$ will be specified later.

To maintain the analysis as general as possible, we adapt the definitions in Björk et al. (1997) for distribution-valued portfolios to our framework.

Definition 2.5. *A* ***bond portfolio*** *is given by a distribution-valued process $h(t,\cdot)$ that satisfies:*

i) For each ω and t, $h(\omega, t, \cdot)$ is a generalized function in (t, ∞).
ii) For each T, the process $h(t,T)$ is predictable.[3]

This definition is a generalization of the usual definition of bond portfolio that allows us to consider the existence of a continuum of bond maturities, a common feature in infinite-dimensional term structure models. The use of generalized functions allows us to recover the case of real-world portfolios by using Dirac deltas, as we will see later. The intuition behind this definition is that $h(t,T)\,dT$ is the number of bonds with maturities between T and $T + dT$ that are held in the portfolio at time t.

Definition 2.6. *The* ***value process****, V, of a bond portfolio h is defined as*

$$V_t = \int_t^{\infty} h(t,T)\,P(t,T;\boldsymbol{\mu}_t)\,dT.$$

[2]According to Protter (2005), a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le\Upsilon}, \mathbb{P})$ is said to satisfy the usual hypotheses if: a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and b) $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$, for all t with $0 \le t \le \Upsilon$.

[3]Note that, as usual, we are identifying distributions with generalized functions.

Note that a real-world portfolio composed of $q_i \in \mathbb{R}$ bonds with maturities T_i, $i = 1, \ldots, n$, can be obtained within this framework simply by taking $h(t,T) = \sum_{i=1}^{n} q_i \, \delta \, (T - T_i)$, where δ is the Dirac delta.

Following Prisman and Shores (1988), suppose that the term structure remains in the initial admissible state $\boldsymbol{\mu}_0$ from time 0 to time t, in which an investor spends C to buy the portfolio $h(t,T)$. The wealth constraint of the investor can be written as

$$\int_t^{\infty} h(t,T) \, P(t,T;\boldsymbol{\mu}_0) \, dT = C. \tag{3}$$

Extending the definition of Balbás and Ibáñez (1998) to our infinite-dimensional portfolios, we present the following definition.

Definition 2.7. *A portfolio $h(t,\cdot)$ is **feasible** if it satisfies condition (3). We will call* F *the set of feasible portfolios.*

We will assume that immediately after time t, an admissible shock $\boldsymbol{\Delta} = \{\Delta_k\}_{k=1}^{\infty}$ moves the forward curve to the (admissible) state $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\Delta}$ and this curve remains in $\boldsymbol{\mu}_t$ until a certain time H. Just after the shock, the investor would expect the value of the portfolio h at time H, as a function of the shift $\boldsymbol{\Delta}$, to be

$$L(h,\boldsymbol{\Delta}) = P^{-1}(t,H;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \int_t^{\infty} h(t,T) \, P(t,T;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \, dT. \tag{4}$$

From now on, we will work with the functional L_h defined for each portfolio h by

$$\begin{aligned} L_h &: \mathcal{A} \subset l^2 \longrightarrow \mathbb{R} \\ L_h(\boldsymbol{\Delta}) &= L(h,\boldsymbol{\Delta}). \end{aligned} \tag{5}$$

The next proposition states a fundamental property of L_h that will be used later.

Proposition 2.8. *L_h is twice Gateaux differentiable for each portfolio h, with Gateaux differentials given, for any $\boldsymbol{l}, \boldsymbol{m} \in \mathcal{A}$, by*

$$\delta L_h(\boldsymbol{\Delta};\boldsymbol{l}) = P^{-1}(t,H;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \int_t^{\infty} h(t,T) \left[\sum_{k=1}^{\infty} l_k \int_{x=T-t}^{H-t} \phi_k(x) \, dx \right] P(t,T;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \, dT \tag{6}$$

and

$$\begin{aligned} \delta^2 L_h(\boldsymbol{\Delta};\boldsymbol{l};\boldsymbol{m}) &= P^{-1}(t,H;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \int_t^{\infty} h(t,T) \left[\sum_{j=1}^{\infty} l_j \int_{x=T-t}^{H-t} \phi_j(x) \, dx \right] \\ &\quad \times \left[\sum_{k=1}^{\infty} m_k \int_{x=T-t}^{H-t} \phi_k(x) \, dx \right] P(t,T;\boldsymbol{\mu}_0 + \boldsymbol{\Delta}) \, dT. \end{aligned} \tag{7}$$

Proof. See the Appendix. ∎

As a first application of Proposition 2.8, we obtain the Direction X of Bowden (1997) in our framework.

Proposition 2.9. *The direction of maximum decrease of portfolio value from $L(h,\boldsymbol{0})$ is $\boldsymbol{X} = -\boldsymbol{m}/\|\boldsymbol{m}\|_{l^2}$, where*

$$m_k = \int_t^{\infty} h(t,T) \left[\int_{x=T-t}^{H-t} \phi_k(x) \, dx \right] \frac{P(t,T;\boldsymbol{\mu}_0)}{P(t,H;\boldsymbol{\mu}_0)} dT, \; k = 1, 2, \ldots. \tag{8}$$

Proof. See the Appendix. ■

Consider the vector $\alpha \boldsymbol{X}$ with $\alpha > 0$, i.e., a shift in the direction $\boldsymbol{X}$ of magnitude α. Then, the maximum loss in the portfolio value per unit of shift is given, to first order, by

$$\frac{L_h(\mathbf{0}) - L_h(\alpha \boldsymbol{X})}{\alpha} = \frac{|\delta L_h(\mathbf{0}; \alpha \boldsymbol{X})|}{\alpha} = \|\boldsymbol{m}\| . \tag{9}$$

Before stating the main results, we define the different types of immunization that we will consider.

Definition 2.10. *A portfolio h is*

- ***Locally immunized*** *if there is an open sphere $N \subset l^2$ containing $\mathbf{0}$ such that $L_h(\mathbf{0}) \leq L_h(\boldsymbol{\Delta})$ for all $\boldsymbol{\Delta} \in \mathcal{A} \cap N$.*
- ***Globally immunized*** *if $L_h(\mathbf{0}) \leq L_h(\boldsymbol{\Delta})$ for all $\boldsymbol{\Delta} \in \mathcal{A}$.*

The portfolio h is a ***maxmin portfolio*** *if it solves the program*

$$\max_{h \in \mathrm{F}} \min_{\boldsymbol{\Delta} \in \mathcal{A}} L(h; \boldsymbol{\Delta}) . \tag{10}$$

The definitions of locally and globally immunized portfolios correspond to local and global minima of functionals. A locally immunized portfolio is protected only against interest rate shifts near zero (in the l^2 sense) while globally immunized portfolios are protected against any kind of admissible term structure shifts. A portfolio is maxmin if it guarantees the maximum value of the portfolio at the horizon with the initial wealth constraint given the worst interest rate shift.

From the previous definition it is immediately clear that if h is a (locally or globally) immunized portfolio, then the portfolio αh with $\alpha \geq 0$ is also immunized. Thus, we have the following geometric interpretation.

Proposition 2.11. *The set of all immunized portfolios is a cone containing the origin.*

The following result was first obtained by Bierwag and Khang (1979) for parallel shifts and, later, by Balbás and Ibáñez (1998) in a very general framework. We adapt their proof to our model.

Proposition 2.12. *A feasible and globally immunized portfolio h^* is a maxmin portfolio.*

Proof. See the Appendix. ■

Proposition 2.13. *A necessary condition for a portfolio h to be locally immunized is*

$$\int_t^{\infty} h(t,T) \left(\int_{T-t}^{H-t} \phi_j(x)\, dx \right) P(t,T;\boldsymbol{\mu}_0)\, dT = 0, \;\; j = 1, 2, \ldots . \tag{11}$$

Proof. See the Appendix. ■

Note that, for feasible portfolios, a simple way for condition (11) to hold is to take $h(t,T) = C\, P^{-1}(t,H;\boldsymbol{\mu}_0)\, \delta(T-H)$, that is, to have only one bond with maturity H in such portfolios.

In an analogous way to Fisher and Weil (1971), Bierwag and Kaufman (1977), Khang (1979), or Prisman and Shores (1988), we can obtain a duration matching strategy equivalent to condition (11). To this end, we need to define a new concept of duration that generalizes some of the previous approaches.

Definition 2.14. *The* ***factorial duration*** $D^{(j)}(t)$*, associated to the* j*-th factor* ϕ_j*, is defined as*

$$D^{(j)}(t) = \frac{\displaystyle\int_t^{\infty} h(t,T)\left(\int_0^{T-t} \phi_j(x)\,dx\right) P(t,T;\boldsymbol{\mu}_0)\,dT}{\displaystyle\int_t^{\infty} h(t,T)\,P(t,T;\boldsymbol{\mu}_0)\,dT}. \tag{12}$$

From the previous definition and (11), it is trivial to obtain the following result.

Corollary 2.15. *A necessary condition for a portfolio to be locally immunized is*

$$D^{(j)}(t) = \int_0^{H-t} \phi_j(x)\,dx,\ \ j = 1, 2, \ldots. \tag{13}$$

Note that if a portfolio is locally immunized, then, by expression (13), we have $\boldsymbol{X} = 0$ and the portfolio does not lose value in any case.

The factorial durations just defined have two important properties:

1. These durations are not measured in time units except in a particular case that will be seen later. To show this, we must define the durations $d^{(j)}(t)$ by $\int_0^{d^{(j)}(t)} \phi_j(x)\,dx = D^{(j)}(t)$ with the immunization condition $d^{(j)}(t) = H - t,\ \ j = 1, \ldots$. The problem with this approach is that for $d^{(j)}(t)$ to be well defined, it is necessary that $\phi_j' \neq 0,\ \ j = 1, 2, \ldots$ (see Example 1 below). A similar problem appears with the stochastic duration measure defined in Munk (1999).
2. These durations measure the sensitivity of portfolio prices to changes in the respective factors. This is a property shared with other durations previously defined (Chambers et al. (1988), Diebold et al. (2006), Moreno (2007)).
   Using equations (4) to (6) and (12), we obtain the following result.

   Proposition 2.16. *It is satisfied that*

   $$\frac{\delta L_h\left(\mathbf{0};\boldsymbol{e}_j\right)}{L_h(\mathbf{0})} = \int_0^{H-t} \phi_j(x)\,dx - D^{(j)}(t),\ \ j = 1, 2, \ldots.$$

   The desired property is an immediate consequence of this proposition.

   Corollary 2.17. *The following relation holds:*

   $$D^{(j)}(t) = -\left.\frac{\delta L_h\left(\mathbf{0};\boldsymbol{e}_j\right)}{L_h(\mathbf{0})}\right|_{H=t},\ \ j = 1, 2, \ldots. \tag{14}$$

2.1. Finite-dimensional portfolios

In this subsection, we leave the general case and work with real-world portfolios. We assume that $h(t,T) = \sum_{i=1}^{n} q_i \delta(T - T_i)$ and we will refer to this finite-dimensional portfolio by $\boldsymbol{q} = (q_1, \ldots, q_n)$. We start by showing that, for finite-dimensional portfolios, our factorial durations generalize some of the most important duration measures in the literature.

Example 1. *If we take* $I = [0,\theta]$*,* $\mathcal{H}(I) = L^2([0,\theta])$*, and* $\phi_1(x) = 1,\ \forall x \in [0,\theta]$*, we have*

$$D_T^{(1)}(t) = \frac{\sum_{i=1}^{n} q_i (T_i - t) P(t, T_i; \boldsymbol{\mu}_0)}{\sum_{i=1}^{n} q_i P(t, T_i; \boldsymbol{\mu}_0)}$$

which is the traditional duration.

Note that in this particular case, $D_T^{(1)}(t) = d^{(1)}(t)$ *has dimensions of a time to maturity and the immunization condition for* $j = 1$ *is* $D_T^{(1)}(t) = H - t$*. This is the immunization result in Fisher and Weil (1971) and, under our framework, it is only the first-order condition.*

We also have that $\left\{\phi_j\right\}_{j=1}^{\infty}$ *can be obtained from simple polynomials* $\left\{1, x, x^2, \ldots\right\}$ *by the Gram-Schmidt orthonormalization procedure, and so, they will be linear combinations of this set of polynomials. In this case, the factorial durations* $D^{(j)}(t)$ *can be seen as a generalization of those defined by Chambers et al. (1988), Prisman and Shores (1988), and Soto (2001) for simple polynomials as factors. The main difference between the two approaches is that the polynomial factors in these three articles are not orthogonal in any sense, while in our case* $\left\{\phi_j\right\}_{j=1}^{\infty}$ *is an orthonormal set in* $L^2([0,\theta])$*.*

Example 2. *Taking* $I = \mathbb{R}^+$*,* $\mathcal{H}(I)$ *as the weighted Lebesgue space* $L^2_{p_\tau}(\mathbb{R}^+)$*, with weight* $p_\tau(x) = e^{(2\tau-1)x}$*,* $0 < \tau < \frac{1}{2}$ *(see Bueno-Guerrero et al. (2016)), and* $\phi_1(x) = e^{-\tau x},\ \forall x \in \mathbb{R}^+$*, we obtain*

$$D^{(1)}(t) = \frac{\sum_{i=1}^{n} q_i \left(1 - e^{-\tau(T_i - t)}\right) P(t, T_i; \boldsymbol{\mu}_0)}{\tau \sum_{i=1}^{n} q_i P(t, T_i; \boldsymbol{\mu}_0)}.$$

In this case, we can uniquely determine $d^{(1)}(t)$ *as*

$$d^{(1)}(t) = -\frac{1}{\tau} \ln \left(\frac{\sum_{i=1}^{n} q_i e^{-\tau(T_i - t)} P(t, T_i; \boldsymbol{\mu}_0)}{\sum_{i=1}^{n} q_i P(t, T_i; \boldsymbol{\mu}_0)} \right)$$

which is the stochastic duration obtained in Munk (1999) for the Hull and White (1990) (extended Vasicek) model.

2.2. Immunization and arbitrage opportunities

Ingersoll et al. (1978) were the first to study the relationship between the existence of an immunized portfolio and arbitrage opportunities. More concretely, with only long positions allowed and under parallel shifts in the term structure, the immunized portfolio is a dominant portfolio (Merton (1973)). Prisman and Shores (1988) solved this problem, showing that complete immunization can be achieved without arbitrage opportunities by a portfolio composed of just one zero-coupon bond maturing at time

H, when changes in the term structure are given by polynomials of degree not less than two. Apart from the restrictive condition on the composition of the portfolio, the proof of their result depends strongly on the choice of simple polynomials as factors and cannot be extended to a more general framework like the one we present here. Balbás and Ibáñez (1998) solved the problem by assuming that an immunized portfolio does not exist, which preserves the arbitrage-free character of the model.

In our case, and with real-world portfolios, we have the following result.

Theorem 2.18. *If only long positions are allowed, then the following statements are equivalent:*

i) $\boldsymbol{q}^*$ *is a globally immunized portfolio.*
ii) There exists a bond in $\boldsymbol{q}^*$ *with maturity* T_j *such that*

$$P\left(t, T_j; \boldsymbol{\Delta}\right) \geq P\left(t, H; \boldsymbol{\Delta}\right), \ \forall\, \boldsymbol{\Delta} \in \mathcal{A}. \tag{15}$$

iii) $\boldsymbol{q}^*$ *satisfies (11).*

If $\boldsymbol{q}^*$ *is also a feasible portfolio, then it is a maxmin immunized portfolio. In this case, there exist arbitrage opportunities.*

Proof. See the Appendix. ∎

The equivalence between i) and ii) in Theorem 2.18 was proved in Balbás and Ibáñez (1998) for the case in which $\boldsymbol{q}^*$ is a feasible portfolio. In this case condition ii) of the theorem is equivalent to the so-called *weak immunization condition*, i.e., the existence of a bond in the portfolio whose return is at least equal to the return of the bond maturing at H.

As an easy consequence of Theorem 2.18, we have the following.

Corollary 2.19. *With only long positions, any portfolio is globally immunized at the horizons given by the maturities of the bonds that compose it.*

2.3. Partial immunization

With only long positions and without arbitrage opportunities, feasible and globally immunized portfolios do not exist (see Theorem 2.18). Then we have to restrict our analysis to immunization against some of the infinite factors that we are working with. It is not difficult to show that if we match some durations in condition (13), then we have immunization against the matched factors. But at this point, we need more information on which are the most important factors to immunize against. To address this problem, we must review some results of Bueno-Guerrero et al. (2016, 2022).

For these authors, the source of randomness is the infinite-dimensional stochastic process (random field) $Z(t, x, \omega)$ (the *stochastic string process*), consisting of a continuum of adapted stochastic processes $Z(., x, \omega)$ indexed by the time to maturity. In more detail, we have

$$\begin{array}{rccc} Z: & \Delta^2 \times \Omega & \rightarrow & \mathbb{R} \\ & (t, x, \omega) & \mapsto & Z(t, x, \omega) \end{array}$$

where $\Delta^2 = \left\{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq \Upsilon,\ x \geq 0\right\}$. For simplicity, we drop the dependence on ω. It is assumed that the process Z satisfies the following properties:

a) For each $x \geq 0$, the stochastic process $Z(., x)$ is a continuous martingale.
b) For each $t \in [0, \Upsilon]$, the stochastic process $Z(t, .)$ is continuous and differentiable.
c) For each $x, y \geq 0$, we have

$$d[Z(., x), Z(., y)]_t = c(t, x, y)\, dt \tag{16}$$

where, for each t, $c(t, x, y)$ is a continuous, differentiable, and admissible[4] correlation function.

For each fixed time to maturity $x \geq 0$, the dynamics of the instantaneous forward interest rate $f(t, x)$ is given by

$$df(t, x) = \alpha(t, x)\, dt + \sigma(t, x)\, dZ(t, x)$$

where, for each t, $\alpha(t, .)$ and $\sigma(t, .) > 0$ are continuous and differentiable adapted stochastic processes and, for each x, $\alpha(., x)$ and $\sigma(., x) > 0$ are continuous adapted stochastic processes. The filtration $\mathcal{F}_t = \sigma\{Z(s, x) : 0 \leq s \leq t,\ x \geq 0\},\ t \leq \Upsilon$, reflects the available information.

In this setting, the no-arbitrage condition reads as follows:

$$\alpha(t, x) = \frac{\partial f(t, x)}{\partial x} + \int_0^x R_t(x, y)\, dyf$$

where $R_t(x, y)$ is the instantaneous conditional covariance between the shocks to the forward curve

$$R_t(x, y) = \frac{cov[df(t, x), df(t, y)]}{dt}.$$

Considering the weighted Lebesgue space $L^2_p(\mathbb{R}^+)$, for each $t \in [0, \Upsilon]$, they define the operator[5]

$$L_{R_t} : L^2_p(\mathbb{R}^+) \longrightarrow L^2_p(\mathbb{R}^+)$$
$$[L_{R_t} f](x) = \int_{\mathbb{R}^+} R_t(x, y) f(y) p(y)\, dy.$$

Assuming that R_t is a Hilbert-Schmidt kernel in $L^2_p(\mathbb{R}^+)$, it is shown that

$$R_t(x, y) = \sum_{i=1}^{\infty} \lambda_{t,i} g_{t,i}(x) g_{t,i}(y)$$

where $\{\lambda_{t,i}\}_{i=1}^{\infty}$ are the (positive) eigenvalues of L_{R_t}, with $\lambda_{t,i} > \lambda_{t,i+1}$, and $\{g_{t,i}\}_{i=0}^{\infty}$ are its respective eigenfunctions, orthonormal in $L^2_p(\mathbb{R}^+)$. Moreover, the following equalities are proven:

$$\begin{aligned} \sigma^2(t, x) &= \sum_{i=1}^{\infty} \lambda_{t,i} g^2_{t,i}(x) \\ \int_{x=0}^{\infty} p(x) \sigma^2(t, x)\, dx &= \sum_{i=1}^{\infty} \lambda_{t,i} < \infty. \end{aligned}$$

Therefore, each eigenvector $g_{t,i}$ explains a proportion $\lambda_{t,i}$ of the "total weighted variance" in the forward curve, $\sum_{i=1}^{\infty} \lambda_{t,i}$. Thus, we can interpret this analysis as an infinite-dimensional principal component analysis of the forward curve.

For the remainder of the subsection, we assume the following.

[4]It means that for each t, $c(t, \cdot, \cdot)$ is positive semi-definite, symmetric, and satisfies $|c(t, x, y)| \leq 1$ and $c(t, x, x) = 1,\ \forall x, y \geq 0$ (Santa-Clara and Sornette, 2001).

[5]As noted by Bueno-Guerrero et al. (2016), all the results in this framework can be established working on $L^2(\mathbb{R}^+)$. So, in the following, we can include in the analysis the case $p(x) = 1,\ \forall x \in \mathbb{R}^+$.

Assumption 2.20. *Our base Hilbert space is $\mathcal{H}(I) = L_p^2(\mathbb{R}^+)$ and the set $\{g_{t,i}\}_{i=1}^{\infty}$ of eigenfunctions of L_{R_t} verifies:*

i) $\{g_{t,i}\}_{i=0}^{\infty}$ *is independent of time, i.e.,* $g_{t,i} = g_i,\ t \in [0, \Upsilon],\ i \in \mathbb{N}$.
ii) $g_i \in C^1(\mathbb{R}^+),\ i = 1, 2, \ldots$.

Under these conditions, we can take $\phi_i \equiv g_i,\ i = 1, 2, \ldots, n$, and we are sure that if we immunize against the first n factors, we are immunized against the major sources of risk of the forward curve, because $\lambda_{t,i} > \lambda_{t,i+1} > 0$.

Definition 2.21. *We define $\mathcal{A}_n$ as the subset of $\mathcal{A}$ formed by linear combinations of $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$, i.e.,*

$$\mathcal{A}_n = \{\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots) \in \mathcal{A} : \mu_{n+1} = \mu_{n+2} = \cdots = 0\}.$$

Restricting our study to $\mathcal{A}_n$, we must rewrite Definition 2.10 in the following terms. Note that we do not transpose the definition of local immunization because, in this subsection, we are considering restrictions on short-selling.

Definition 2.22. *The portfolio $\boldsymbol{q}$ is **immunized against n factors** if $L_{\boldsymbol{q}}(\boldsymbol{0}) \leq L_{\boldsymbol{q}}(\boldsymbol{\Delta})$ for all $\boldsymbol{\Delta} \in \mathcal{A}_n$.*

Definition 2.23. *The portfolio $\boldsymbol{q}$ is a **maxmin portfolio with respect to n factors** if it solves the program*

$$\max_{\boldsymbol{q} \in \mathrm{F}} \min_{\boldsymbol{\Delta} \in \mathcal{A}_n} L(\boldsymbol{q}; \boldsymbol{\Delta}).$$

It is easy to check that all the results obtained previously can be translated into this n factor framework just by replacing $\mathcal{A}$ by $\mathcal{A}_n$ and $j = 1, 2, \ldots$ by $j = 1, \ldots, n$. The only exception is Theorem 2.18 in which the existence of a maxmin portfolio with respect to n factors does not imply the existence of arbitrage opportunities.

If we want to work with real-world portfolios, i.e., finite-dimensional ones, we have the following result. To the best of our knowledge, it is new in the literature.

Theorem 2.24. *With only long positions allowed, there exists a finite-dimensional portfolio, $\boldsymbol{q}^* = \left(q_1^*, \ldots, q_n^*\right)$, immunized against n factors, if and only if the determinant of the matrix with elements $\int_{T_j - t}^{T_{j+1} - t} \phi_i(x)\, dx,\ i, j = 1, \ldots, n,\ T_{n+1} = H$ becomes null.*

Proof. See the Appendix. ∎

This theorem tells us that immunization against some factors, if possible, can only be obtained at some horizons that depend on the set of available maturities $\{T_i\}_{i=1}^n$. Concretely, as in Corollary 2.19, we have the following result, which arises immediately from the theorem.

Corollary 2.25. *If any of the maturities of the bonds in the portfolio coincide with H, then the portfolio is immunized against any number of factors.*

In Prisman and Shores (1988), this is the only way in which a portfolio can be immunized, and only for portfolios with one bond and for polynomial factors with degree not less than two. As we have just seen, this is not true in our framework. The only problem that can arise is when we want to immunize against one factor, which is a function with no zeros.

Proposition 2.26. *If short positions are forbidden and $\phi_1(x) \neq 0,\ \forall x \in I$, then we can immunize a bond portfolio against one factor if and only if it includes just a bond maturing at H.*

Proof. See the Appendix. ∎

To put Theorem 2.24 in more operational terms, we take $\mathcal{H}(I) = L^2_{p_\tau}(\mathbb{R}^+)$ with $0 < \tau \leq \frac{1}{2}$ and select the orthonormal set $\left\{\overline{\phi}_j\right\}_{j=1}^n$ with $\overline{\phi}_j = e^{-\tau x} L_{j-1}(x)$, where L_j are Laguerre polynomials, and $L_m(x) = \frac{1}{m!} e^x \frac{d^n}{dx^n}\left(e^{-x} x^m\right)$. This choice, corresponding to Example 2, generalizes many models and verifies some important properties (see Bueno-Guerrero et al. (2016)). Unfortunately, the most we can obtain with this set of factors is something similar to the result of Prisman and Shores (1988) previously commented.[6]

Theorem 2.27. *With only long positions allowed, there exists a finite-dimensional portfolio immunized against the factors $\overline{\phi}_1, \ldots, \overline{\phi}_n$, if and only if the investor's horizon coincides with one of the maturities of the bonds in the portfolio.*

Proof. See the Appendix. ∎

3. Second-best portfolios

If immunization cannot be achieved, it would be nice to have a second-best strategy to manage possible losses in the portfolio. We will take a conservative approach and look for a portfolio that produces the minimum loss in the worst-case scenario. Within our framework, this corresponds to solving the program

$$\min_{h \in \mathrm{F}} \frac{|\delta L_h(\mathbf{0}; \alpha \boldsymbol{X})|}{\alpha}. \tag{17}$$

Proposition 3.1. *Program (17) is equivalent to*

$$\min_{h \in \mathrm{F}} \|\boldsymbol{D} - \boldsymbol{H}\| \tag{18}$$

where $\boldsymbol{D} = \left(D^{(1)}(t), D^{(2)}(t), \ldots\right)$ *and* $\boldsymbol{H} = \left(\int_0^{H-t} \phi_1(x)\, dx, \int_0^{H-t} \phi_2(x)\, dx, \ldots\right)$.

Proof. See the Appendix. ∎

Strategy (18) is an infinite-dimensional generalization of strategy C proposed in Prisman and Shores (1988). As these authors point out, the minimum is zero if and only if the portfolio is immunized. In any other case, the strategy (18) gives the minimum loss in portfolio value in the worst case. This reasoning gives sense to the following definition.

Definition 3.2. *The portfolio solution of (17) or (18) is defined as the* ***second-best portfolio****.*

We want now to search for second-best portfolios in the real-world case, thus, we consider feasible finite-dimensional portfolios and work in the partial immunization framework. We will obtain explicit expressions for the second-best portfolios in the cases where short positions are allowed or forbidden.

[6]These results provide a theoretical support to Balbás et al. (2002) which found that the portfolio with best empirical performance is the matching duration portfolio that minimizes the linear dispersion measure and that includes a maturity matching bond.

In both instances, we solve constrained optimization problems, with equality constraints in the first case and with equality and inequality constraints in the second.

To state the theorem when short positions are allowed, we need to consider the matrix Σ with elements

$$\Sigma_{ij} = \sum_{k=1}^{n} \left[\int_{T_i - t}^{H-t} \phi_k(x)\, dx \right] \left[\int_{T_j - t}^{H-t} \phi_k(x)\, dx \right], \ i, j = 1, \ldots, n. \tag{19}$$

Theorem 3.3. *Suppose that* $|\Sigma| \neq 0$ *and that there is no portfolio solution of*

$$\sum_{i=1}^{n} q_i P(t, T_i; \boldsymbol{\mu}_0) \int_{T_i - t}^{H-t} \phi_k(x) dx = 0, \ k = 1, \ldots, n. \tag{20}$$

Consider the program

$$\min_{\boldsymbol{q} \in \mathrm{F}} \left\| \boldsymbol{D}^{(n)} - \boldsymbol{H}^{(n)} \right\| \tag{21}$$

with $\boldsymbol{D}^{(n)} = \left(D^{(1)}(t), \ldots, D^{(n)}(t) \right)$ *and* $\boldsymbol{H}^{(n)} = \left(\int_0^{H-t} \phi_1(x)\, dx, \ldots, \int_0^{H-t} \phi_n(x)\, dx \right)$.

The solution of this program is the portfolio $\overline{\boldsymbol{q}} = (\overline{q}_1, \ldots, \overline{q}_n)$, *with*

$$\overline{q}_i = \frac{C\, |\Sigma_i|}{P(t, T_i; \boldsymbol{\mu}_0) \sum_{j=1}^{n} \left| \Sigma_j \right|}, \ i = 1, \ldots, n, \tag{22}$$

where Σ_i *is the matrix obtained replacing the i-th column of the matrix* Σ *by a column of ones.*

Proof. See the Appendix. ∎

If short positions are forbidden, we must use the matrices $\Phi = \left(\Phi_{ij} \right)$, $i, j = 1, \ldots, n-1$, and $\overline{\Phi} = \left(\Phi_{ij} \right)$, $\Omega = \left(\Omega_{ij} \right)$, $i, j = 1, \ldots, n$, where

$$\begin{aligned} \Phi_{ij} &= \int_{T_i - t}^{T_{i+1} - t} \phi_j(x)\, dx, \quad T_{n+1} = H, \\ \Omega_{ij} &= \int_{T j - t}^{H-t} \phi_i(x)\, dx. \end{aligned}$$

Theorem 3.4. *Consider the matrices* Φ^j, $j = 1, \ldots, n-1$, *obtained by replacing the j-th column of* Φ *by the column vector* $\left[\int_{T_1 - t}^{T_2 - t} \phi_n(x)\, dx, \ldots, \int_{T_{n-1} - t}^{T_n - t} \phi_n(x)\, dx \right]'$, $\Phi^n \equiv \Phi$, *and* $\overline{\Phi}^i$, *the matrix obtained by substituting the last row in* $\overline{\Phi}$ *by* $[\Delta_{1,i}, \ldots, \Delta_{n,i}]$, *where* $\Delta_{j,i}$ *is the cofactor of the element* (j, i) *of the matrix* Ω. *Consider the program*

$$\min_{\substack{\boldsymbol{q} \in \mathrm{F} \\ q_i \geq 0}} \left\| \boldsymbol{D}^{(n)} - \boldsymbol{H}^{(n)} \right\|. \tag{23}$$

Then, with only long positions allowed, if there is no portfolio solution of (20), the n-bond portfolio solution of this program exists if $\left| \overline{\Phi}^i \right| > 0, \ i = 1, \ldots, n$, *and is given by*

$$\overline{q}_i = \frac{C \left| \overline{\Phi}^i \right|}{P(t, T_i; \boldsymbol{\mu}_0) \sum_{j=1}^{n} |\Phi^j|^2}. \tag{24}$$

Proof. See the Appendix. ∎

4. Numerical implementation

In this section we numerically illustrate our results on factorial durations and second-best portfolios. To this end we consider the Hilbert space referred to in Example 2, namely, the weighted Lebesgue space $L^2_{p_\tau}(\mathbb{R}^+)$ with weight $p_\tau(x) = e^{(2\tau-1)x}$, $0 < \tau < \frac{1}{2}$, and the orthonormal basis $\left\{\overline{\phi}_j\right\}_{j=1}^{\infty}$, with $\overline{\phi}_j = e^{-\tau x} L_{j-1}(x)$, where L_j are the Laguerre polynomials. Note that $\left\{\overline{\phi}_j\right\}_{j=1}^{\infty}$ satisfies the conditions in Assumption 2.20. Moreover, the first two vectors $\overline{\phi}_1$ and $\overline{\phi}_2$ span a subset of curves of the Nelson and Siegel (1987) (NS) family, specifically, those curves without an independent term and with a constant coefficient in the exponent (Bueno-Guerrero et al. (2016)).[7] The NS family is widely used for term structure estimation and is able to produce different shapes. We will work with the first three vectors (factors) of $\left\{\overline{\phi}_j\right\}_{j=1}^{\infty}$, which are

$$\begin{aligned} \overline{\phi}_1(x) &= e^{-\tau x} \\ \overline{\phi}_2(x) &= e^{-\tau x}(-x+1) \\ \overline{\phi}_3(x) &= \frac{1}{2} e^{-\tau x}\left(x^2 - 4x + 2\right). \end{aligned}$$

Note that these three factors can be assimilated to the (exponentially smoothed) level, slope, and curvature factors of Litterman and Scheikman (1991), which according to these authors, explain 98.4% of the changes in the yield curve.

By (1) and (2), the instantaneous forward interest rates and the corresponding bond prices are given by

$$f(x; \boldsymbol{\mu}_t) = \mu_1(t)\overline{\phi}_1(x) + \mu_2(t)\overline{\phi}_2(x) + \mu_3(t)\overline{\phi}_3(x)$$

and

$$P(t, T; \boldsymbol{\mu}_t) = \exp\{-\mu_1(t) F_1(T-t) - \mu_2(t) F_2(T-t) - \mu_3(t) F_3(T-t)\} \tag{25}$$

where the functions F_i, $i = 1, 2, 3$, are given by

$$F_1(x) = \int_0^x e^{-\tau y} dy = -\frac{1}{\tau}\left(e^{-\tau x} - 1\right) \tag{26}$$

$$F_2(x) = \int_0^x e^{-\tau y}(-y+1) dy = \frac{1}{\tau} x e^{-\tau x} + \frac{1-\tau}{\tau^2}\left(e^{-\tau x} - 1\right) \tag{27}$$

$$\begin{aligned} F_3(x) &= \frac{1}{2}\int_0^x e^{-\tau y}\left(y^2 - 4y + 2\right) dy \\ &= \frac{1}{2}\left[-\frac{1}{\tau} x^2 e^{-\tau x} + \frac{4\tau - 2}{\tau^2} x e^{-\tau x} - \frac{2\tau^2 - 4\tau + 2}{\tau^3}\left(e^{-\tau x} - 1\right)\right]. \end{aligned} \tag{28}$$

We assume that at time $t = 0$, there are available zero-coupon bonds with maturities 1.5, 2.5, 4, and 6 years. Following Diebold and Li (2006), we also assume that $\tau = 0.0609$.[8]

[7] The absence of an independent term is not a problem here because for small values of τ, the first vector $\overline{\phi}_1 = e^{-\tau x}$ can be considered almost constant for short maturities. The constant value of τ is not a problem either, since it is usually considered fixed in works related to the NS family (see Bueno-Guerrero et al. (2016) and references therein).

[8] Diebold and Li (2006) obtained this value of τ maximizing the loading of the curvature factor.

4.1. Duration measures

To assess the effectiveness of factorial durations measuring the risk associated to different shocks to the term structure, we consider, at time $t = 0$, three different portfolios: equally weighted (EQ), bullet (BU), and barbell (BA), determined by the weights

$$\mathbf{q}_{EQ} = (1, 1, 1, 1)\,,\ \mathbf{q}_{BU} = (0, 0, 2, 2)\,,\ \mathbf{q}_{BA} = (2, 0, 0, 2)\,,$$

where the components indicate the number of bonds that mature in, respectively, 1.5, 2.5, 4, and 6 years.

We assume that, at time $t = 0$, the forward curve is determined by the vector $\boldsymbol{\mu}_0 = (0.05, 0, 0)$, corresponding to the initial forward curve $f_0(x; \boldsymbol{\mu}_0) = 0.05e^{-\tau x}$. We also assume that, at time $t = 1$, the initial forward curve suffers a shock $\boldsymbol{\Delta}$, remaining in the state $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\Delta}$ from that moment on. We consider three different scenarios for this shock: a parallel displacement (p), a change in the slope (s), and a change in the curvature (c). The corresponding shocks are determined by

$$\boldsymbol{\Delta}_p = (0.05, 0, 0)\,,\ \boldsymbol{\Delta}_s = (0, 0.005, 0)\,,\ \boldsymbol{\Delta}_c = (0, 0, 0.005)$$

with the associated forward curves

$$\begin{aligned} f_p(x; \boldsymbol{\mu}_1) &= 0.1e^{-\tau x} \\ f_s(x; \boldsymbol{\mu}_1) &= 0.05e^{-\tau x} + 0.005e^{-\tau x}(-x + 1) \\ f_c(x; \boldsymbol{\mu}_1) &= 0.05e^{-\tau x} + 0.0025e^{-\tau x}\left(x^2 - 4x + 2\right). \end{aligned}$$

Figure 1 presents the forward curve at time $t = 0$ and the three forward curves at time $t = 1$ considering the different scenarios for the shock. The continuous blue line indicates the initial forward curve while the curves at time $t = 1$ correspond, respectively, to the parallel displacement (orange, dashed line), change in the slope (green, dotted-dashed), and change in the curvature (red, dotted).

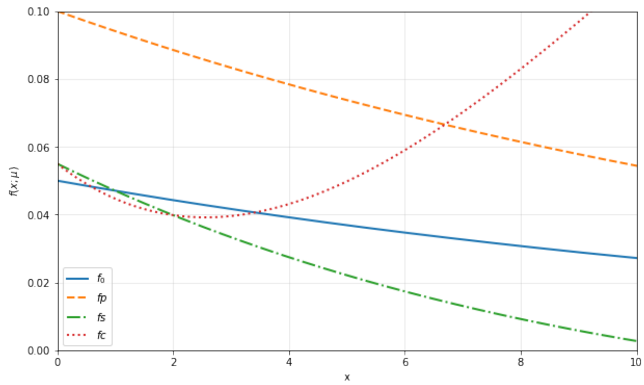

Figure 1. Forward curves at times $t = 0$ and $t = 1$.

From expression (25), we can obtain now the prices at time $t = 1$ (just before the shock) of the bonds maturing in 1.5, 2.5, 4, and 6 years as

$$P(1, 1.5; \boldsymbol{\mu}_0) = 0.9757,\ P(1, 2.5; \boldsymbol{\mu}_0) = 0.9308,\ P(1, 4; \boldsymbol{\mu}_0) = 0.8719,\ P(1, 6; \boldsymbol{\mu}_0) = 0.8061.$$

Hence, we can obtain the initial value of the different bond portfolios at time $t = 1$, before the shock:

$$V_{1,0}^{EQ} = 3.5845,\ V_{1,0}^{BU} = 3.356, V_{1,0}^{BA} = 3.5636.$$

For each of these portfolios, we calculate at $t = 1$, four duration measures: the traditional duration of Example 1, $D_T^{(1)}(t)$, and the three factorial durations $D^{(i)}(t)$, $i = 1, 2, 3$, associated to each of the factors $\overline{\phi}_1, \overline{\phi}_2$, and $\overline{\phi}_3$. The measure $D^{(1)}(t)$ is given explicitly in Example 2, and $D^{(2)}(t)$ and $D^{(3)}(t)$ are given by

$$\begin{aligned} D^{(2)}(t) &= \frac{\sum_{i=1}^{n} q_i F_2 (T_i - t) P(t, T_i; \mu_0)}{\sum_{i=1}^{n} q_i P(t, T_i; \mu_0)}, \\ D^{(3)}(t) &= \frac{\sum_{i=1}^{n} q_i F_3 (T_i - t) P(t, T_i; \mu_0)}{\sum_{i=1}^{n} q_i P(t, T_i; \mu_0)}. \end{aligned}$$

Table 1 shows the values at $t = 1$ of the four duration measures for the three portfolios.

Table 1. Values at $t = 1$ of the four duration measures for equally weighted (EQ), bullet (BU), and barbell (BA) portfolios.

Portfolio	$D_T^{(1)}(1)$	$D^{(1)}(1)$	$D^{(2)}(1)$	$D^{(3)}(1)$
EQ	2.3798	2.1426	−1.4363	−0.1830
BU	3.9608	3.4954	−3.4912	−0.4586
BA	2.5359	2.2198	−2.4756	0.3544

We assess now the performance of the different durations measuring the changes of portfolio values. To this aim, we compute the portfolio values, after the shocks, for the different scenarios. To focus on the effect of the shocks, we obtain the portfolio values at the same time, $t = 1$, but after the shock, with the bond prices $P(t, T_i; \boldsymbol{\mu}_1)$. Doing so, we do not need to include the discount factor $P^{-1}(t, H; \boldsymbol{\mu}_0 + \boldsymbol{\Delta})$ of expression (4).

Table 2 provides the bond prices under the three possible shocks: parallel displacement ($\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\Delta}_p$), changes in the slope ($\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\Delta}_s$), and changes in the curvature ($\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\Delta}_c$).

Table 2. Bond prices under the three possible shocks under analysis.

Bond price	Parallel displacement	Change in slope	Change in curvature
$P(1, 1.5; \boldsymbol{\mu}_1)$	0.9519	0.9739	0.9744
$P(1, 2.5; \boldsymbol{\mu}_1)$	0.8664	0.9291	0.9316
$P(1, 4; \boldsymbol{\mu}_1)$	0.7602	0.8773	0.8776
$P(1, 6; \boldsymbol{\mu}_1)$	0.6498	0.8303	0.8043

Table 3 reports the portfolio values at $t = 1$, with the new bond prices under the different scenarios.

Table 3. Portfolio values at $t = 1$ under the three possible shocks.

Portfolio	Parallel displacement	Change in slope	Change in curvature
EQ	3.2284	3.6106	3.5878
BU	2.8200	3.4154	3.3637
BA	3.2036	3.6084	3.5573

To see if the durations obtained capture the effect of the shocks, we need to rewrite equation (14) with finite increments. Taking into account the definition of the Gateaux differential, we have

$$\delta L_h\left(\mathbf{0};\mathbf{e}_j\right) = \lim_{\alpha\to 0}\frac{1}{\alpha}\left[L_h\left(\mathbf{0}+\alpha\mathbf{e}_j\right) - L_h\left(\mathbf{0}\right)\right] \simeq \frac{V_{1,f}^{(j)} - V_{1,0}}{\alpha^{(j)}}$$

where $j = p,\ s$, or c depending on the shock considered, and $\alpha^{(p)} = 0.05,\ \alpha^{(s)} = \alpha^{(c)} = 0.005$. Therefore, we can express equation (14) for our purposes as

$$D^{(j)}\left(t\right) \simeq -\frac{V_{1,f}^{(j)} - V_{1,0}}{\alpha^{(j)}V_{1,0}} \equiv \overline{R}^{(j)}$$

where $\overline{R}^{(j)}$ can be considered as the opposite of the instantaneous return of a portfolio, due to the shock j, and adjusted for finiteness.

Table 4 presents the values of $\overline{R}^{(j)}$, $j = p,\ s,\ c$, for the different portfolios and scenarios.

Table 4. Values of $\overline{R}^{(j)}$ for the different portfolios and alternative scenarios.

Portfolio	$\overline{R}^{(p)}$	$\overline{R}^{(s)}$	$\overline{R}^{(c)}$
EQ	1.9869	−1.4563	−0.1841
BU	3.1943	−3.5400	−0.4589
BA	2.0204	−2.5143	0.3536

Comparing Tables 1 and 4, we see that the instantaneous returns for parallel displacements are well captured by $D_T^{(1)}(1)$ and $D^{(1)}(1)$ for all the portfolios, although the results are better for the factorial duration $D^{(1)}(1)$. On the other hand, we can see that the returns associated to the changes in slope, $\overline{R}^{(s)}$, and curvature, $\overline{R}^{(c)}$, are also well captured by the factorial durations $D^{(2)}(1)$ and $D^{(3)}(1)$ for all the portfolios considered. Nevertheless, the first-order factorial durations $D_T^{(1)}(1)$ and $D^{(1)}(1)$ cannot provide information about the impact on the portfolio of shocks associated to higher-order factors (s or c). Therefore, the use of durations associated to first-order factors, such as $D_T^{(1)}$ and $D^{(1)}$, does not help to assess the risk derived from higher-order shocks, hence the importance of considering higher-order factorial durations in the risk management of bond portfolios.

4.2. *Second-best portfolios*

We use now the numerical results presented in the previous subsection to test the effectiveness of the second-best portfolios presented in Section 3 protecting the portfolio value under the worst scenarios. With this aim, we obtain the second-best portfolios from expressions (24) and (22) and three more portfolios, with different configurations, similar to the ones previously used. All the portfolios will satisfy the same wealth constraint (3). Then, we find the worst shock for each portfolio (the direction X) and identify the portfolio with the smallest loss in value under this shock at the final horizon H.

We assume that, at time $t = 0$, there exist three bonds with maturities 1.5, 4, and 6 years. We select a time horizon $H = 5$ years, different from the available maturities, and a wealth constraint $C = 3.5$. We consider the same factors $\overline{\phi}_i$, $i = 1, 2, 3$, of the previous subsection and assume the same initial forward curve

$$f_0(x; \mu_0) = 0.05 e^{-\tau x},$$

which provides the same prices at $t = 1$, before the shock:

$$P(1, 1.5; \boldsymbol{\mu}_0) = 0.9757, \; P(1, 4; \boldsymbol{\mu}_0) = 0.8719, \; P(1, 6; \boldsymbol{\mu}_0) = 0.8061.$$

The matrix Σ of equation (19) is given by

$$\begin{aligned} \Sigma_{ij} &= \sum_{k=1}^{3} \left[\int_{T_i - t}^{H-t} \overline{\phi}_k(x)\, dx \right] \left[\int_{T_j - t}^{H-t} \overline{\phi}_k(x)\, dx \right] \\ &= \sum_{k=1}^{3} \left[F_k(H-t) - F_k(T_i - t) \right] \left[F_k(H-t) - F_k\left(T_j - t\right) \right] \end{aligned}$$

with $T_1 = 1.5$, $T_2 = 4$, and $T_3 = 6$. Using expressions (26) to (28), we have, for $t = 1$,

$$\Sigma = \begin{bmatrix} 24.6336 & 9.6086 & -9.6083 \\ 9.6086 & 4.7352 & -6.1835 \\ -9.6083 & -6.1835 & 10.3239 \end{bmatrix}.$$

With this matrix and using (22), we can obtain the second-best portfolio with short positions allowed, SB1, as

$$\mathbf{q}_{SB1} = (-0.7370, 3.4905, 1.4586).$$

We will compare the performance of this portfolio with that of the following portfolios, which are the equivalent, with the wealth constraint $C = 3.5$, to the portfolios EQ, BU, and BA of subsection 4.1.

$$\begin{aligned} &\overline{\text{EQ}}: \quad \mathbf{q}_{\overline{EQ}} = (1.3189, 1.3189, 1.3189), \\ &\overline{\text{BU}}: \quad \mathbf{q}_{\overline{BU}} = (0, 2.0858, 2.0858), \\ &\overline{\text{BA}}: \quad \mathbf{q}_{\overline{BA}} = (1.9643, 0, 1.9643). \end{aligned}$$

Expression (8) can be written, with our finite-dimensional portfolios, as

$$\begin{aligned} m_k &= \sum_{i=1}^{3} q_i \left[\int_{T_i - t}^{H-t} \phi_k(x)\, dx \right] \frac{P(t, T_i; \boldsymbol{\mu}_0)}{P(t, H; \boldsymbol{\mu}_0)} \\ &= \frac{1}{P(t, H; \boldsymbol{\mu}_0)} \sum_{i=1}^{3} q_i \left[F_k(H-t) - F_k(T_i - t) \right] P(t, T_i; \boldsymbol{\mu}_0). \end{aligned}$$

Considering that the direction X is given by $\boldsymbol{X} = -\frac{\boldsymbol{m}}{\|\boldsymbol{m}\|_{l^2}}$, and that $P(t, H; \boldsymbol{\mu}_0) = P(1, 5; \boldsymbol{\mu}_0) = 0.8374$, we obtain

$$\begin{aligned} \boldsymbol{X}_{\overline{EQ}} &= (-0.5996, 0.6161, 0.5108), \\ \boldsymbol{X}_{\overline{BU}} &= (-0.0721, -0.3023, 0.9505), \end{aligned}$$

$$X_{\overline{BA}} = (-0.6111, 0.3613, 0.7043),$$
$$X_{SB1} = (0.7056, 0.4453, 0.5512).$$

Now we assume that each portfolio suffers, at $t = 1$, a shock in its direction X, with magnitude α. As second-best portfolios are obtained solving the program (17), we consider shocks of small magnitude, $\alpha = 0.05$. The introduction of these shocks leads to the bond prices $P(1, T_i; \boldsymbol{\mu}_1)$, with $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \alpha \boldsymbol{X}$, which we present in Table 5, for each of the portfolios considered.

Table 5. Values of bond prices for the different portfolios after the shocks in direction X.

Portfolio	$P(1, 1.5; \boldsymbol{\mu}_1)$	$P(1, 4; \boldsymbol{\mu}_1)$	$P(1, 6; \boldsymbol{\mu}_1)$
$\overline{\text{EQ}}$	0.9812	1.0700	1.1784
$\overline{\text{BU}}$	0.9705	0.9195	0.7325
$\overline{\text{BA}}$	0.9746	1.0153	1.0070
SB1	0.9774	1.0237	1.0573

To obtain the expected value of the portfolios at $H = 5$, we need the prices $P(1, 5; \boldsymbol{\mu}_1)$ for each of the shocks, which are

$$P(1, 5; \boldsymbol{\mu}_1^{\overline{EQ}}) = 1.1335,\ P(1, 5; \boldsymbol{\mu}_1^{\overline{BU}}) = 0.8538,\ P(1, 5; \boldsymbol{\mu}_1^{\overline{BA}}) = 1.0319,\ P(1, 5; \boldsymbol{\mu}_1^{SB1}) = 1.0543.$$

Now, we can obtain the expected values by means of the finite-dimensional version of expression (4):

$$L(h, \boldsymbol{\Delta}) = P^{-1}(t, H; \boldsymbol{\mu}_1) \sum_{i=1}^{3} q_i P(t, T_i; \boldsymbol{\mu}_1).$$

These expected values are the following:

$$L^{\overline{EQ}}\left(h, \alpha \boldsymbol{X}^{\overline{EQ}}\right) = 3.7579,\ L^{\overline{BU}}\left(h, \alpha \boldsymbol{X}^{\overline{BU}}\right) = 4.0356,\ L^{\overline{BA}}\left(h, \alpha \boldsymbol{X}^{\overline{BA}}\right) = 3.8680,\ L^{SB1}\left(h, \alpha \boldsymbol{X}^{SB1}\right) = 4.1687.$$

Taking into account that $P(t, H; \boldsymbol{\mu}_0) = P(1, 5; \boldsymbol{\mu}_0) = 0.8374$ and that the wealth constraint is $C = 3.5$, we have that the expected value under an absence of shocks for all the portfolios is given by

$$L(h, \mathbf{0}) = \frac{C}{P(1, 5; \boldsymbol{\mu}_0)} = 4.1798.$$

Table 6 shows the expected losses, $-\frac{L(h,\boldsymbol{\Delta}) - L(h,\mathbf{0})}{L(h,\mathbf{0})}$, for each portfolio. We can see that, as expected, the second-best portfolio generates the lowest expected losses in the worst-case scenario.

Table 6. Expected losses for the different portfolios after the shocks in direction X.

$\overline{\text{EQ}}$	$\overline{\text{BU}}$	$\overline{\text{BA}}$	SB1
0.1009	0.0345	0.0746	0.2700

Next we will address the effectiveness of second-best portfolios in the case of short positions forbidden, using expression (24). In this case, we will try to obtain the second-best portfolio, SB2, working with the same available bonds and horizon, and following the same procedure of the case with short positions allowed. We start by computing the matrices $\overline{\Phi}$, Φ, and Φ^j. For $i, j = 1, \ldots, n$, we have

$$\Phi_{ij} = \int_{T_i - t}^{H-t} \phi_j(x)\, dx = F_j(T_{i+1} - t) - F_j(T_i - t), \; T_{n+1} = H.$$

Applying again expressions (26) to (28), we obtain

$$\begin{aligned}
\overline{\Phi} &= \begin{bmatrix} 2.2494 & -1.6158 & -1.5750 \\ 1.5686 & -4.6739 & 1.7664 \\ -0.7604 & 2.6576 & -1.6379 \end{bmatrix}, \\
\Phi &= \begin{bmatrix} 2.2494 & -1.6158 \\ 1.5686 & -4.6739 \end{bmatrix}, \\
\Phi^1 &= \begin{bmatrix} -1.5750 & -1.6158 \\ 1.7664 & -4.6739 \end{bmatrix}, \\
\Phi^2 &= \begin{bmatrix} 2.2494 & -1.5750 \\ 1.5686 & 1.7664 \end{bmatrix}, \\
\Phi^3 &= \begin{bmatrix} 2.2494 & -1.6158 \\ 1.5686 & -4.6739 \end{bmatrix}.
\end{aligned}$$

Calculating the determinants of the matrices Φ^j, we obtain $\left|\Phi^3\right| < 0$ and then, by Theorem 3.4, it is impossible to obtain the second-best portfolio with short positions forbidden. We have analyzed different maturities and horizon configurations and, when short positions are forbidden, we were unable to obtain the second-best portfolio in any of the cases under consideration.

5. Conclusions

We have presented a general approach that covers from infinite-dimensional portfolios with infinitely many risk factors to real-world portfolios with a finite number of risk sources. In this setting, we have generalized classical concepts and important theoretical results on duration and immunization previously found in the academic literature to our new infinite-dimensional framework, and we have provided new relevant theoretical results.

Focusing on known results, we have obtained the concept analogous to Direction X of Bowden (1997) and we have reobtained different traditional, polynomial, and stochastic durations as particular cases of our general framework, with their associated immunization results. We have stated a theorem, valid for the case of only long positions allowed, extending several results in the literature, which clarify the relationship between locally and globally immunized portfolios, maxmin portfolios, and arbitrage opportunities. Moreover, based on some ideas taken from the stochastic string modeling of interest rates, we have introduced the so-called partial immunization framework, under which we have generalized the very-well-known result that, with only long positions allowed, a bond portfolio can be immunized if and only if the investor horizon coincides with one of the maturities of the bonds included in such a portfolio.

We have obtained new theoretical results. Among them, we can highlight a matching duration strategy for immunized portfolios based on the new factorial durations, a result that relates these durations to the sensitivity of portfolio value to shocks in the term structure of interest rates (measured by means of Gateaux differentials), and closed-form expressions for second-best portfolios, that is, portfolios that produce the minimum loss in the worst-case scenario.

Finally, we have numerically illustrated our new results with finite-dimensional portfolios, showing the performance of our duration measures in assessing the risk associated to different shocks and the effectiveness of the second-best portfolios in hedging the portfolio value in the worst-case scenario. In this way, we have shown that our results can be relevant for managing the interest rate risk in bond portfolios.

Two possible future directions for research that may arise from this paper are, on the one hand, to explore the negative result that we have obtained for the second-best portfolios when short positions are forbidden and, on the other hand, the applications of our results to the risk management of bond portfolios in a real-world environment, considering the available coupon bonds.

Author contributions

Alberto Bueno-Guerrero: Conceptualization, investigation, methodology, writing — original draft, writing — review and editing.

Manuel Moreno: Investigation, methodology, writing — review and editing, funding acquisition.

Javier F. Navas: Investigation, methodology, writing — review and editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are very grateful to the Editor in Chief and two anonymous referees for their truly helpful recommendations, comments, and suggestions. We gratefully acknowledge financial support from the grant PID2022-142143NB-100 financed by MCIN/AEI /10.13039/501100011033 / FEDER, UE Una manera de hacer Europa. The usual caveat applies.

Conflict of interest

All authors declare no conflicts of interest in this paper.

Data availability statement

This paper does not use any external data.

References

Afik Z, Fooladi I, Jacoby G, et al. (2022) Duration Concepts, Analysis, and Applications, In: Lee, CF., Lee, A.C. (eds.), *Encyclopedia of Finance. Springer*, Cham. https://doi.org/10.1007/978-3-030-91231-4_14

Afik Z, Jacoby G, Wiener Z (2018) Duration and Globalization. *J Fixed Income* 28: 31–43. https://doi.org/10.3905/jfi.2018.28.2.031

Balbás A, Ibáñez A(1998) When Can You Immunize a Bond Portfolio? *J Bank Financ* 22: 1571–1595. https://doi.org/10.1016/S0378-4266(98)00070-3

Balbás A, Ibáñez A, López S (2002) Dispersion Measures as Immunization Risk Measures. *J Bank Financ* 26: 1229–1244. https://doi.org/10.1016/S0378-4266(01)00168-6

Balbás A, Montagut EH, Pérez-Fructuoso MJ (2004) Hedging bond portfolios versus infinitely many ranked factors of risk. Available form: https://e-archivo.uc3m.es/rest/api/core/bitstreams/d94d5038-e94e-45a0-83ae-f1dcfd267d64/content.

Balbás A, Romera R (2007) Hedging Interest Rate Risk by Optimization in Banach Spaces. *J Optim Theory Appl* 132: 175–191. https://doi.org/10.1007/s10957-006-9124-6

Bierwag GO (1977) Immunization, Duration and the Term Structure of Interest Rates. *J Financ Quant Anal* 12: 725–742. https://doi.org/10.2307/2330253

Bierwag GO, Kaufman G (1977) Coping with the Risk of Interest Rate Fluctuations: A Note. *J Bus* 50: 364–370. http://www.jstor.org/stable/2352543

Bierwag GO, Khang C (1979) An Immunization Strategy is a Minimax Strategy. *J Financ* 34: 389–399. https://doi.org/10.2307/2326978

Björk T, Kabanov Y, Runggaldier W (1997) Bond Market Structure in the Presence of Marked Point Processes. *Math Financ* 7: 211–239. https://doi.org/10.1111/1467-9965.00031

Bowden RJ (1997) Generalising Interest Rate Duration with Directional Derivatives: Direction X and Applications. *Manage Sci* 43: 586–595. https://doi.org/10.1287/mnsc.43.5.586

Bravo JM, Da Silva CMP (2006) Immunization using a Stochastic-Process Independent Multi-Factor Model: The Portuguese experience. *J Bank Financ* 30: 133–156. https://doi.org/10.1016/j.jbankfin.2005.01.006

Bueno-Guerrero A, Moreno M, Navas JF (2015) Stochastic String Models with Continuous Semimartingales. *Physica A* 433: 229–246. http://dx.doi.org/10.1016/j.physa.2015.03.070

Bueno-Guerrero A, Moreno M, Navas JF (2016) The Stochastic String Model as a Unifying Theory of the Term Structure of Interest Rates. *Physica A* 461: 217–237. http://doi.org/10.1016/j.physa.2016.05.044

Bueno-Guerrero A, Moreno M, Navas JF (2022) Bond Market Completeness under Stochastic Strings with Distribution-Valued Strategies. *Quant Financ* 22: 197–211. https://doi.org/10.1080/14697688.2021.2018483

Chambers DR, Carleton WT, McEnally RW (1988) Immunizing Default-Free Bond Portfolios with a Duration Vector. *J Financ Quant Anal* 23: 89–104. https://doi.org/10.2307/2331026

Diebold FX, Ji L, Li C (2006) A Three Factor Yield Curve Model: Non-Affine Structure, Systematic Risk Sources, and Generalized Duration, In: L. R. Klein (ed.), *Long-Run Growth and Short-Run Stabilization: Essay in Memory of Albert Ando*, Cheltenham, U. K.: Edward Elgar, 240–274. https://doi.org/10.4337/9781781950500

Diebold FX, Li C (2006) Forecasting the Term Structure of Goverment Bond Yields. *J Econom* 130: 337–364. https://doi.org/10.1016/j.jeconom.2005.03.005

Fisher L, Weil R (1971) Coping with the Risk of Interest Rate Fluctuations and Return to Bondholders from Naive and Optimal Strategies. *J Bus* 44: 408–431. https://www.jstor.org/stable/2352056

Fong HG, Vasicek OA (1984) A Risk Minimizing Strategy for Portfolio Immunization. *J Finance* 39: 1541–1546. https://doi.org/10.2307/2327743

Fooladi IJ, Jacoby G, Jin L (2021) Real Duration and Inflation Duration: A Cross Country Perspective on a Multidimensional hedging Strategy. *J Int Financ Markets Inst Money* 70: 101265. https://doi.org/10.1016/j.intfin.2020.101265

Goldstein RS (2000) The Term Structure of Interest Rates as a Random Field. *Rev Financ Stud* 13: 365–384. https://www.jstor.org/stable/2646029

Hicks JR (1939) *Value and Capital*, Oxford: Clarendon Press.

Hull J, White A (1990) Pricing Interest-Rate Derivative Securities. *Rev Financ Stud* 3: 573–592. https://doi.org/10.1093/rfs/3.4.573

Ingersoll JE, Skelton J, Weil RL (1978) Duration Forty Years Later. *J Financ Quant Anal* 13: 627–650. https://doi.org/10.2307/2330468

Jacoby G (2003) A Duration Model for Defaultable Bonds. *J Financ Res* 26: 129–146. https://doi.org/10.1111/1475-6803.00049

Kennedy DP (1994) The Term Structure of Interest Rates as a Gaussian Random Field. *Math Financ* 4: 247–258. https://doi.org/10.1111/j.1467-9965.1994.tb00094.x

Khang C (1979) Bond Immunization when Short-Term Interest Rates Fluctuate more than Long-Term Rates. *J Financ Quant Anal* 14: 1085–1090. https://doi.org/10.2307/2330309

Lapshin V (2019) A Nonparametric Approach to Bond Portfolio Immunization. *Mathematics* 7: 1121. https://doi.org/10.3390/math7111121

Litterman RB, Scheikman J (1991) Common Factors Affecting Bond Returns. *J Fixed Income* 1: 54–61. https://doi.org/10.3905/jfi.1991.692347

Luenberger DG (1969) *Optimization by Vector Space Methods*, Wiley, New York.

Macaulay FR (1938) *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the U.S. since 1856*, Columbia University Press. https://doi.org/10.2307/2143533

Merton RC (1973) Theory of Rational Option Pricing. *Bell J Econ Manage Sci* 4: 141–183. https://doi.org/10.2307/3003143

Moreno M (2007) Managing Interest Rate Risk under Non-Parallel Changes: An Application of a Two-Factor Model, In: G. Gregoriou (ed.), *Advances in Risk Management*. Palgrave-MacMillan, London, United Kingdom, 69-85. https://doi.org/10.1057/9780230625846

Munk C (1999) Stochastic Duration and Fast Coupon Bond Option Pricing in Multi-Factor Models. *Rev Deriv Res* 3: 157–181. https://doi.org/10.1023/A:1009654427422

Musiela M (1993) Stochastic PDEs and Term Structure Models, Statistics Technical Reports, CSIR-002-93, SMS-017-93, University of South Wales.

Nawalkha SK, Chambers DR (1996) An Improved Immunization Strategy: M-Absolute. *Financ Anal J* 52: 69–76. https://doi.org/10.2469/faj.v52.n5.2026

Nawalkha SK, Chambers DR (1997) The M-squared Vector Model: Derivation and Testing of Extensions to M-Squared. *J Portf Manag* 23: 92–98. https://doi.org/10.3905/jpm.23.2.92

Nawalkha SK, Soto GM, Zhang J (2003) Generalized M-Vector Models for Hedging Interest Rate Risk. *J Bank Financ* 27: 1581–1604. https://doi.org/10.1016/S0378-4266(03)00089-X

Nelson CR, Siegel AF (1987) Parsimonious Modeling of Yield Curves. *J Bus* 60: 473–489. https://doi.org/10.1086/296409

Oliveira L, Nunes JPV, Malcato L (2014) The Performance of Deterministic and Stochastic Interest Rate Risk Measures: Another Question of Dimensions? *Port Econ J* 13: 141–165. https://doi.org/10.1007/s10258-014-0104-8

Ortobelli S, Vitali S, Cassader M, et al. (2018) Portfolio Selection Strategy for Fixed Income Markets with Immunization on Average. *Ann Oper Res* 260: 395–415. https://doi.org/10.1007/s10479-016-2182-8

Prisman EZ, Shores MR (1988) Duration Measures for Especific Term Structure Estimations and Applications to Bond Portfolio Immunization. *J Bank Financ* 12: 493–504. https://doi.org/10.1016/0378-4266(88)90011-8

Protter PE (2005) *Stochastic Integration and Differential Equations*, Second Edition, Springer. https://doi.org/10.1007/978-3-662-10061-5

Redington FM (1952) Review on the Principle of Life-Office Valuations. *J Inst Act* 78: 286–315.https://doi.org/10.1017/S0020268100052811

Samuelson PA (1945) The Effect of Interest Rates Increases on the Banking System. *Am Econ Rev* 35: 16–27.https://www.jstor.org/stable/1810106

Santa-Clara P, Sornette D (2001) The Dynamics of the Forward Interest Rate Curve with Stochastic String Shocks. *Rev Financ Stud* 14: 149–185. https://doi.org/10.1093/rfs/14.1.149

Soto GM (2001) Immunization Derived from a Polynomial Duration Vector in the Spanish Bond Market. *J Bank Financ* 25: 1037–1057. https://doi.org/10.1016/S0378-4266(00)00112-6

Yan AX, Liu S, Wu C, et al. (2009) The Effects of Default and Call Risk on Bond Duration. *J Bank Finan* 33: 1700–1708. https://doi.org/10.1016/j.jbankfin.2009.04.004

Zhu W, Zhang CH, Liu Q, et al. (2018) Incorporating Convexity in Bond Portfolio Immunization using Multifactor model: A Semidefinite Programming Approach. *J Oper Res Soc China* 6: 3–23. https://doi.org/10.1007/s40305-018-0196-4

Appendix of Proofs

Proof of Proposition 2.8

We will only prove the first differential, since the proof is essentially the same for the second one. Taking into account equations (4)–(5), the Gateaux differential of L_h at $\boldsymbol{\Delta}$ in the direction $\boldsymbol{l}$ is given by

$$\begin{aligned}\delta L_h(\boldsymbol{\Delta};\boldsymbol{l}) &= \lim_{\alpha\to 0}\frac{1}{\alpha}\left[L_h(\boldsymbol{\Delta}+\alpha\boldsymbol{l}) - L_h(\boldsymbol{\Delta})\right]\\ &= P^{-1}(t,H;\boldsymbol{\mu}_0+\boldsymbol{\Delta})\int_t^{\infty} h(t,T)\left[\sum_{k=1}^{\infty} l_k\int_{x=T-t}^{H-t}\phi_k(x)\,dx\right]P(t,T;\boldsymbol{\mu}_0+\boldsymbol{\Delta})\,dT\end{aligned}$$

where we have used the equalities $P(t,T;\boldsymbol{\mu}_0+\boldsymbol{\Delta}+\alpha\boldsymbol{l}) = P(t,T;\boldsymbol{\mu}_0+\boldsymbol{\Delta})P(t,T;\alpha\boldsymbol{l})$ and $P^{-1}(t,H;\alpha\boldsymbol{l})P(t,T;\alpha\boldsymbol{l}) = \exp\left\{\alpha\sum_{k=1}^{\infty} l_k\int_{T-t}^{H-t}\phi_k(x)\,dx\right\}$. ∎

Proof of Proposition 2.9

Rewriting (6) as

$$\delta L_h(\boldsymbol{0};\boldsymbol{l}) = \sum_{k=1}^{\infty} l_k\int_t^{\infty} h(t,T)\left[\int_{x=T-t}^{H-t}\phi_k(x)\,dx\right]\frac{P(t,T;\boldsymbol{\mu}_0)}{P(t,H;\boldsymbol{\mu}_0)}dT$$

and applying Hölder's inequality to the right-hand side, we obtain

$$\sum_{k=1}^{\infty} l_k m_k \le \|\boldsymbol{l}\|_{l^2}\|\boldsymbol{m}\|_{l^2}\,. \tag{29}$$

If we take $\boldsymbol{l} = \boldsymbol{m}/\|\boldsymbol{m}\|_{l^2}$, both sides in (29) become equal, providing the direction of the maximum increase in the portfolio value. Thus, $\boldsymbol{X} = -\boldsymbol{m}/\|\boldsymbol{m}\|_{l^2}$ gives the direction of the maximum decrease. ∎

Proof of Proposition 2.12

Condition (3) implies that

$$\min_{\boldsymbol{\Delta}\in\mathcal{A}} L(h,\boldsymbol{\Delta}) \le L(h,\boldsymbol{0}) = P^{-1}(t,H;\boldsymbol{\mu}_0)\,C,\ \forall h\in\mathcal{F}.$$

As h^* is globally immunized, it is also satisfied that

$$\min_{\boldsymbol{\Delta}\in\mathcal{A}} L(h^*,\boldsymbol{\Delta}) = L(h^*,\boldsymbol{0}) = P^{-1}(t,H;\boldsymbol{\mu}_0)\,C.$$

Thus, h^* solves program (10). ∎

Proof of Proposition 2.13

Portfolio h is locally immunized if the functional L_h has a local minimum at $\mathbf{0}$. A necessary condition for this is $\delta L_h(\mathbf{0};\boldsymbol{l}) = 0$ for any $\boldsymbol{l} \in \mathcal{A}$. Replacing this condition in (6), we obtain

$$\int_t^{\infty} h(t,T)\left(\sum_{j=1}^{\infty} l_j \int_{T-t}^{H-t} \phi_j(x)\,dx\right) P(t,T;\boldsymbol{\mu}_0)\,dT = 0.$$

If we take a fixed $j \in \mathbb{N}$ and make $l_k = \delta_{kj}$, we have $\boldsymbol{l} = \boldsymbol{e}_j$, the j-th vector of the usual basis of l^2 that is an admissible vector by statement ii) of Proposition 2.3. Repeating the process for every $j \in \mathbb{N}$, we obtain the desired result. ∎

Proof of Theorem 2.18

- i) ⇒ ii)
  By Definition 2.10 and expression (4), we have

$$\sum_{i=1}^{n} q_i^* \exp\left(\sum_{k=1}^{\infty} \mu_k(0) \int_{T_i-t}^{H-t} \phi_k(x)\,dx\right) \le \sum_{i=1}^{n} q_i^* \exp\left(\sum_{k=1}^{\infty} (\mu_k(0)+\Delta_k) \int_{T_i-t}^{H-t} \phi_k(x)\,dx\right)$$

  for every $\boldsymbol{\Delta} \in \mathcal{A}$. Since both sums involve non-negative terms, there exists at least one $j \in \{1,\ldots,n\}$ such that

$$\exp\left(\sum_{k=1}^{\infty} \mu_k(0) \int_{T_j-t}^{H-t} \phi_k(x)\,dx\right) \le \exp\left(\sum_{k=1}^{\infty} (\mu_k(0)+\Delta_k) \int_{T_j-t}^{H-t} \phi_k(x)\,dx\right).$$

  This inequality can be written in terms of bond prices as $P\left(t,T_j;\boldsymbol{\Delta}\right) \ge P(t,H;\boldsymbol{\Delta})$.
- ii) ⇒ i)
  It is not difficult to prove that condition (15) is equivalent to

$$V_j(\boldsymbol{\Delta}) \ge \frac{P\left(t,T_j;\boldsymbol{\mu}_0\right)}{P(t,H;\boldsymbol{\mu}_0)}\ \forall\ \boldsymbol{\Delta} \in \mathcal{A} \tag{30}$$

  with $V_j(\boldsymbol{\Delta}) \equiv \exp\left(\sum_{k=1}^{\infty} (\mu_k(0)+\Delta_k) \int_{T_j-t}^{H-t} \phi_k(x)\,dx\right)$. Consider the sets

$$A = \left\{\boldsymbol{\alpha} \in \mathbb{R}^n : \alpha_i \le \frac{P(t,T_i;\boldsymbol{\mu}_0)}{P(t,H;\boldsymbol{\mu}_0)},\ i = 1,\ldots,n\right\}$$

  and

$$B = \{\boldsymbol{\beta} \in \mathbb{R}^n : \exists\ \boldsymbol{\Delta} \in \mathcal{A} \text{ with } \beta_i \ge V_i(\boldsymbol{\Delta}),\ i = 1,\ldots,n\}.$$

  Obviously, set A is convex and, using the convexity of V_i, it is easy to show that B is also convex. Moreover, by the definitions of A and B, and by (30), it is immediate to obtain that $A^0 \cap B = \emptyset$, where A^0 is the interior of A. Under these conditions, by the Eidelheit separation theorem (Luenberger, 1969), there is a closed hyperplane separating A and B, and therefore, there exists $\boldsymbol{q}^*$ such that

$$\sum_{i=1}^{n} q_i^* \alpha_i \leq \sum_{i=1}^{n} q_i^* \beta_i$$

for any $\boldsymbol{\alpha} \in A, \boldsymbol{\beta} \in B$, and $\boldsymbol{q}^* \neq \mathbf{0}$ because $\mathbf{0} \in A^0$. If we take $\alpha_i = \dfrac{P(t, T_i; \boldsymbol{\mu}_0)}{P(t, H; \boldsymbol{\mu}_0)}$ and $\beta_i = V_i(\boldsymbol{\Delta}) + r_i$, with $\boldsymbol{\Delta} \in \mathcal{A}$ and $r_i \geq 0$, $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} q_i^* \frac{P(t, T_i; \boldsymbol{\mu}_0)}{P(t, H; \boldsymbol{\mu}_0)} \leq \sum_{i=1}^{n} q_i^* \left(V_i(\boldsymbol{\Delta}) + r_i\right). \tag{31}$$

Fixing $j \in \{1, \ldots, n\}$ and taking $r_j \longrightarrow \infty$, it follows that $q_j^* \geq 0$ because if we had $q_j^* < 0$, the right-hand side of (31) tends to $-\infty$ in contradiction with the inequality. Since j is arbitrary, we get $q_i^* \geq 0,\ i = 1, \ldots, n$. Finally, taking $r_i = 0,\ i = 1, \ldots, n$, in (31), we obtain

$$L_{\boldsymbol{q}^*}(\mathbf{0}) \leq L_{\boldsymbol{q}^*}(\boldsymbol{\Delta}),\ \forall\ \boldsymbol{\Delta} \in \mathcal{A}.$$

- i) $\Leftrightarrow$ iii)
  With only long positions, by expression (7) we have that $\delta^2 L_{\boldsymbol{q}}(\boldsymbol{\Delta}; \boldsymbol{l}, \boldsymbol{l}) \geq 0, \forall \boldsymbol{\Delta}, \boldsymbol{l} \in \mathcal{A}$, and therefore $L_{\boldsymbol{q}}$ is convex for each $\boldsymbol{q}$. Moreover, $L_{\boldsymbol{q}}$ is defined in a convex set by iv) of Proposition 2.3. Thus, the necessary condition (11) for local extremes becomes also sufficient for a global minimum, $\boldsymbol{q}^*$. If $\boldsymbol{q}^*$ is also feasible, then Proposition 2.12 guarantees that $\boldsymbol{q}^*$ is a maxmin immunized portfolio. For the arbitrage possibilities, following Prisman and Shores (1988), we have that $L(\boldsymbol{q}^*, \mathbf{0}) = CP^{-1}(t, H; \boldsymbol{\mu}_0)$. Thus, if no changes in interest rates occur, we will have C at time H. If any admissible shift $\boldsymbol{\Delta}$ in interest rates occurs, then $L(\boldsymbol{q}^*, \boldsymbol{\Delta}) \geq L(\boldsymbol{q}^*, \mathbf{0})$ and, in this case, we will have at least C at the horizon H. So, borrowing C now, investing it in the immunized portfolio, and returning it at time H is a risk-less arbitrage strategy. ■

Proof of Theorem 2.24

With only long positions, $\boldsymbol{q}^*$ is immunized against n factors if and only if the equation

$$\sum_{i=1}^{n} q_i \left(\int_{T_i - t}^{H-t} \phi_j(x)\, dx \right) P(t, T_i; \boldsymbol{\mu}_0) = 0, \quad j = 1, 2, \ldots, n,$$

has nonzero solutions. This occurs if and only if $\det \left(\int_{T_i - t}^{H-t} \phi_j(x)\, dx \right)_{i,j=1}^{n} = 0$, and the proof is complete. ■

Proof of Proposition 2.26

To immunize the portfolio $\boldsymbol{q}$ against ϕ_1, by (11), we have to impose

$$\sum_{i=1}^{n} q_i \left(\int_{T_i - t}^{H-t} \phi_1(x)\, dx \right) P(t, T_i; \boldsymbol{\mu}_0) = 0.$$

This equation is satisfied if and only if there exists $j \in \mathbb{N}$ such that $q_i = \delta_{ij} q_j$ and $T_j = H$. ■

Proof of Theorem 2.27

The necessary condition is immediate from Corollary 2.25. For sufficiency, consider the function $f : \mathbb{R}^{+} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{vmatrix} \int_{T_1-t}^{T_2-t} \overline{\phi}_1(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_1(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_1(u)\,du & \int_{T_n-t}^{x} \overline{\phi}_1(u)\,du \\ \int_{T_1-t}^{T_2-t} \overline{\phi}_2(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_2(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_2(u)\,du & \int_{T_n-t}^{x} \overline{\phi}_2(u)\,du \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_{T_1-t}^{T_2-t} \overline{\phi}_n(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_n(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_n(u)\,du & \int_{T_n-t}^{x} \overline{\phi}_n(u)\,du \end{vmatrix}. \tag{32}$$

Clearly, this function is continuous and differentiable in $\mathbb{R}^{+}$ and has n zeros given by $T_i-t, \ i = 1, 2, \ldots, n$. Differentiating (32), we obtain

$$f'(x) = \begin{vmatrix} \int_{T_1-t}^{T_2-t} \overline{\phi}_1(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_1(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_1(u)\,du & \overline{\phi}_1(x) \\ \int_{T_1-t}^{T_2-t} \overline{\phi}_2(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_2(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_2(u)\,du & \overline{\phi}_2(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_{T_1-t}^{T_2-t} \overline{\phi}_n(u)\,du & \int_{T_2-t}^{T_3-t} \overline{\phi}_n(u)\,du & \cdots & \int_{T_{n-1}-t}^{T_n-t} \overline{\phi}_n(u)\,du & \overline{\phi}_n(x) \end{vmatrix}.$$

As L_j is a polynomial of degree j, we have that f' has at most $n-1$ zeros, and thus f cannot have more than n zeros. ∎

Proof of Proposition 3.1

By expression (9), program (17) is equivalent to $\min_{h \in \mathrm{F}} \|\boldsymbol{m}\|$ and by (8), we have that

$$\begin{aligned} \|\boldsymbol{m}\|^2 &= P^{-2}(t, H; \boldsymbol{\mu}_0) \sum_{k=1}^{\infty} \left(\int_t^{\infty} h(t, T)\, P(t, H; \boldsymbol{\mu}_0) \left[\int_{T-t}^{H-t} \phi_k(x)\,dx \right] dT \right)^2 \\ &= \left[C P^{-1}(t, H; \boldsymbol{\mu}_0) \right]^2 \sum_{k=1}^{\infty} \left(D^{(k)}(t) - \int_0^{H-t} \phi_k(x)\,dx \right)^2. \end{aligned}$$

∎

Proof of Theorem 3.3

Program (21) can be written as

$$\min_{\boldsymbol{q}} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) \int_{T_j - t}^{H-t} \phi_k(x)\, dx \right)^2$$
$$s.t. \sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) = C$$

whose Lagrangian is

$$L(\boldsymbol{q}; \lambda) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) \int_{T_j - t}^{H-t} \phi_k(x)\, dx \right)^2 + \lambda \left[C - \sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) \right].$$

The first-order conditions are as follows:

$$2 \sum_{k=1}^{n} \left(\sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) \int_{T_j - t}^{H-t} \phi_k(x)\, dx \right) \left[\int_{T_i - t}^{H-t} \phi_k(x)\, dx \right] P\left(t, T_i; \boldsymbol{\mu}_0\right) - \lambda P\left(t, T_i; \boldsymbol{\mu}_0\right) = 0 \tag{33}$$

for $i = 1, \ldots, n$ and

$$\sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) = C. \tag{34}$$

From (33), we obtain the following:

$$\lambda' \equiv \frac{\lambda}{2} = \sum_{j=1}^{n} q_j P\left(t, T_j; \boldsymbol{\mu}_0\right) \left(\sum_{k=1}^{n} \int_{T_i - t}^{H-t} \phi_k(x)\, dx \int_{T_j - t}^{H-t} \phi_k(x)\, dx \right), \; i, j = 1, \ldots, n. \tag{35}$$

Condition (20) implies $\lambda' \neq 0$ and thus (35) is a non-homogeneous linear system with the unique solution

$$\overline{q}_i = \lambda' \frac{\prod_{j=1,\, j \neq i}^{n} P\left(t, T_j; \boldsymbol{\mu}_0\right) |\Sigma_i|}{\prod_{j=1}^{n} P\left(t, T_j; \boldsymbol{\mu}_0\right) |\Sigma|} = \lambda' \frac{|\Sigma_i|}{P\left(t, T_i; \boldsymbol{\mu}_0\right) |\Sigma|}, \quad i = 1, \ldots, n. \tag{36}$$

Replacing this solution in (34), we obtain the Lagrange multiplier

$$\lambda' = \frac{C\, |\Sigma|}{\sum_{j=1}^{n} \left|\Sigma_j\right|}$$

that, substituted back into (36), gives expression (22). ■

Proof of Theorem 3.4

For feasible $\boldsymbol{q}$, we have the following:

$$\left\|\boldsymbol{D}^{(n)}-\boldsymbol{H}^{(n)}\right\|^2=\sum_{k=1}^{n}\left[D^{(k)}(t)-\int_0^{H-t}\phi_k(x)\,dx\right]=\frac{1}{C^2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\int_{T_i-t}^{H-t}\phi_k(x)\,dx\right)^2.$$

Then, we can write the Lagrangian for program (23) as

$$L(\boldsymbol{q};\lambda;\boldsymbol{\beta})=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\int_{T_i-t}^{H-t}\phi_k(x)\,dx\right)^2+\lambda\left(C-\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\right)-\sum_{j=1}^{n}\mu_jq_j.$$

Then, for $j=1,\ldots,n$, the Kuhn-Tucker conditions read as follows:

$$2\sum_{k=1}^{n}\left(\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\int_{T_i-t}^{H-t}\phi_k(x)\,dx\right)\left[\int_{T_j-t}^{H-t}\phi_k(x)\,dx\right]P\left(t,T_j;\boldsymbol{\mu}_0\right)-\lambda P\left(t,T_j;\boldsymbol{\mu}_0\right)-\mu_j=0, \quad (37)$$

$$\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)=C, \quad (38)$$

$$\mu_jq_j=0, \quad (39)$$

where $q_j\geq 0,\ \mu_j\geq 0$. From (37) and (39), we get

$$2\sum_{k=1}^{n}\left(\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\int_{T_i-t}^{H-t}\phi_k(x)\,dx\right)\left[\int_{T_j-t}^{H-t}\phi_k(x)\,dx\right]q_jP\left(t,T_j;\boldsymbol{\mu}_0\right)=\lambda q_jP\left(t,T_j;\boldsymbol{\mu}_0\right). \quad (40)$$

Then, using (38), we obtain

$$\lambda=\frac{2}{C}\sum_{k=1}^{n}\left(\sum_{i=1}^{n}q_iP(t,T_i;\boldsymbol{\mu}_0)\int_{T_i-t}^{H-t}\phi_k(x)\,dx\right)^2\equiv\frac{2}{C}\sum_{k=1}^{n}\Theta_{k,H}^2. \quad (41)$$

Replacing (41) in (40) and assuming, without loss of generality, that $q_j>0,\ j=1,\ldots,n$, we have

$$\sum_{k=1}^{n}\Theta_{k,H}\int_{T_j-t}^{H-t}\phi_k(x)\,dx=\frac{1}{C}\sum_{k=1}^{n}\Theta_{k,H}^2,\quad j=1,\ldots,n.$$

Then, we obtain the following system:

$$\begin{aligned}\sum_{k=1}^{n}\Theta_{k,H}\int_{T_i-t}^{T_{i+1}-t}\phi_k(x)\,dx&=0,\ i=1,2,\cdots,n-1,\\ \sum_{k=1}^{n}\Theta_{k,H}\int_{T_n-t}^{H-t}\phi_k(x)\,dx&=\frac{1}{C}\sum_{k=1}^{n}\Theta_{k,H}^2.\end{aligned} \quad (42)$$

The first $n-1$ equations imply that

$$\sum_{k=1}^{n-1}\Theta_{k,H}\int_{T_i-t}^{T_{i+1}-t}\phi_k(x)\,dx=-\Theta_{n,H}\int_{T_i-t}^{T_{i+1}-t}\phi_n(x)\,dx,\ i=1,2,\cdots,n-1. \quad (43)$$

As (20) has no portfolio solution, then (43) is a Cramer system with the unique solution

$$\Theta_{i,H} = -\frac{\left|\Phi^{i}\right|}{|\Phi|}\Theta_{n,H},\ i = 1, 2, \ldots, n-1. \tag{44}$$

Replacing (44) in (42), we obtain that the left- and right-hand sides are given, respectively, by

$$\begin{aligned}\sum_{k=1}^{n}\Theta_{k,H}\int_{T_n-t}^{H-t}\phi_k(x)\,dx &= -\frac{\Theta_{n,H}}{|\Phi|}\sum_{k=1}^{n-1}\left|\Phi^{k}\right|\int_{T_n-t}^{H-t}\phi_k(x)\,dx + \Theta_{n,H}\int_{T_n-t}^{H-t}\phi_n(x)\,dx\\ &= \frac{\left|\overline{\Phi}\right|}{|\Phi|}\Theta_{n,H}\end{aligned} \tag{45}$$

and

$$\frac{1}{C}\sum_{k=1}^{n}\Theta_{k,H}^{2} = \frac{1}{C}\left(\frac{\Theta_{n,H}^{2}}{|\Phi|^{2}}\sum_{k=1}^{n-1}\left|\Phi^{k}\right|^{2} + \Theta_{n,H}^{2}\right) = \frac{1}{C}\frac{\Theta_{n,H}^{2}}{|\Phi|^{2}}\sum_{k=1}^{n}\left|\Phi^{k}\right|^{2}. \tag{46}$$

If $\Theta_{n,H} = 0$, expression (44) implies that $\Theta_{i,H} = 0,\ i = 1, \ldots, n$, which is in contradiction to our hypotheses. Thus, we have $\Theta_{n,H} \neq 0$, and equating (45) and (46) leads to

$$\Theta_{n,H} = \frac{C\left|\overline{\Phi}\right||\Phi|}{\sum_{k=1}^{n}\left|\Phi^{k}\right|^{2}},$$

and, by expression (44), we get

$$\Theta_{i,H} = -\frac{C\left|\overline{\Phi}\right|\left|\Phi^{i}\right|}{\sum_{k=1}^{n}\left|\Phi^{k}\right|^{2}},\quad i = 1, \ldots, n-1.$$

By the definition of $\Theta_{k,H}$ (see (41)), the portfolio $\overline{\boldsymbol{q}}$ solves the system

$$\sum_{j=1}^{n}\overline{q}_j\left[\int_{T_j-t}^{H-t}\phi_i(x)\,dx\right]P\left(t, T_j; \boldsymbol{\mu}_0\right) = \Theta_{i,H},\ i = 1, 2, \cdots, n. \tag{47}$$

Applying the same reasoning as in the proof of Theorem 3.3, it is straightforward to see that the determinant of the coefficient matrix is given by $\prod_{j=1}^{n}P\left(t, T_j; \boldsymbol{\mu}_0\right)\left|\overline{\Phi}\right|$, which is non-zero by hypothesis.

Thus, the unique solution to system (47) is given by

$$\begin{aligned}\overline{q}_i &= \frac{\prod\limits_{j=1,\ j\neq i}^{n}P\left(t, T_j; \boldsymbol{\mu}_0\right)}{\prod\limits_{j=1}^{n}P\left(t, T_j; \boldsymbol{\mu}_0\right)\left|\overline{\Phi}\right|}\begin{vmatrix}\int_{T_1-t}^{H-t}\phi_1(x)\,dx & \cdots & \overset{i)}{\Theta_{1,H}} & \cdots & \int_{T_n-t}^{H-t}\phi_1(x)\,dx\\ \vdots & & \vdots & & \vdots\\ \int_{T_1-t}^{H-t}\phi_n(x)\,dx & \cdots & \Theta_{n,H} & \cdots & \int_{T_n-t}^{H-t}\phi_n(x)\,dx\end{vmatrix}\\ &= \frac{1}{P\left(t, T_i; \boldsymbol{\mu}_0\right)\left|\overline{\Phi}\right|}\sum_{j=1}^{n}\Theta_{j,H}\Delta_{j,i}\\ &= \frac{C}{P\left(t, T_i; \boldsymbol{\mu}_0\right)\sum_{k=1}^{n}\left|\Phi^{k}\right|^{2}}\left(-\sum_{j=1}^{n-1}\left|\Phi^{j}\right|\Delta_{j,i} + |\Phi^{n}|\,\Delta_{n,i}\right)\end{aligned}$$

$$= \frac{C}{P(t, T_i; \mu_0)} \frac{\left|\overline{\Phi}^i\right|}{\sum_{k=1}^{n} \left|\Phi^k\right|^2}, \ i = 1, \ldots, n.$$

■

AIMS Press