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Research article

Commodity-linked bonds as an innovative financing instrument for African countries to build back better

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Abstract: Commodity-linked bond, a type of state contingent claims, presents an innovative tool for African countries to mobilize resources on the international capital markets. Given their colossal financing needs, which has been worsened by the COVID-19 pandemic, African countries need to put in place innovative financing mechanisms to support their development frameworks for building back better. The issuing of this type of bond could provide an opportunity for commodity-producing African countries to hedge against fluctuations in their export earnings. The results show that the value of a commodity-linked bond increases as the price of the commodity indexed to the bond rises, suggesting that African countries should issue debt contracts that are tied to their export commodities so that their debt declines with plummeting export prices (or export revenues). A simple portfolio rule derived suggests that countries should issue more commodity-linked bonds than conventional debt if the variance of the portfolio is greater than twice the spread between the expected total return of the conventional debt and the commodity-linked bond. This rule supports the view that African countries' debt-service payments, for debt issued in the form of commodity-linked bonds, would decline whenever the price of their export commodities decline thus lightening their debt load.

Keywords: innovative financing; debt; state contingent debt instruments; commodity linked bonds and option pricing; and building back better

JEL Codes: F30, F34, F49, G13, G11, O16

1. Introduction

In early January 2020, a call went out from Mr. Antonio Guterres, the UN Secretary General (UNSG) for a Decade of Action to all nations to increase their efforts in addressing the challenges impeding the achievement of the goals of the 2030 Agenda for Sustainable Development (see UNSG Report, 2020). Before the outbreak of the pandemic, the UNSG indicated that global hunger was on the accent as at least half of the world's population lacked essential health services. He also noted that ending extreme poverty by 2030 may not be achieved and violent conflicts and vulnerabilities to natural disasters were not easing. The world is currently experiencing rising sea levels; accelerating acidification of the ocean; global warming; land degradation; and one million plant and animal species are at risk of extinction; and land degradation. UNSG further observed that achieving quality education targets may have been jeopardized as more than half of the world's children do not meet standards in reading and mathematics. The world is also far from achieving gender equality as women in all parts of the world continue to face structural disadvantages and discrimination. Then comes COVID-19!

The coronavirus pandemic, which hit the world in late 2019 with the tentacles of the virus griping most countries around the globe, including those in Africa, has dealt a serious blow to the global economy. The lockdown, quarantine and restrictions on movement of people within many countries have slowed down the global economy as the number of those infected continues to rise. The threat of global recession is real and African economies have not been spurred. The gains made by many African countries in moving toward the achievement of the goals of 2030 Agenda on Sustainable Development stand to be reversed substantially as a result of the virus. COVID-19 is not discriminating with the spread of its venom as it is affecting both rich and poor countries, big and small businesses, rich and poor persons. The socioeconomic impact on Africans are direct and indirect. It ranges from a child who cannot be in school to families expressing their love to the old by staying away from them; from businesses who might have to close factories to employees, placing job security at risk. All sectors of the economy risk being "under water."

To address their economic challenges and make their economies resilient for the future, African countries need to Build Back Better (BBB). The Global Facility for Disaster Reduction and Recovery (GFDRR, 2018) defines BBB as an approach to post-crisis recovery that minimizes risks and vulnerability to future crisis while strengthening resilience to addressing physical, social, environmental, and economic vulnerabilities and shocks. As explained by GFDRR, BBB ensures that repaired or replaced assets are not only more resilient, but also that the recovery process is inclusive (particularly for the poorest and most vulnerable), shorter and more efficient. The pillars/strategies for BBB are therefore: rebuilding stronger, faster, and more inclusively. *Building Back Stronger* minimizes losses to the livelihoods of those impacted by ensuring that reconstructed assets are able to withstand/resist more intense future shocks.[1] In order to reduce the impact of a crisis it is important for

[1]GFDRR suggests that if all countries were to "build back stronger" in the next 20 years—ensuring that rebuilt assets can resist hazards with a 50-year return-period—then global well-being losses due to natural disasters would be reduced by 12 percent, a gain equivalent to US$65 billion annually.

countries to *Build Back Faster* through measures such as contingent reconstruction plans, pre-approved contracts, and financial arrangements.[2] BBB further requires *Building Back is Inclusive*.[3]

This means that countries should not return to the pre-COVID era of development practices that embark on environmentally destructive investment patterns and activities. The OECD (2020) advises that global environmental emergencies such as climate change and biodiversity loss could cause social and economic damages far larger than those caused by COVID-19. Hence countries need to do more than just resuscitating economies and livelihoods. BBB requires that countries fashion recovery policies that promote investment and behavioural changes to reduce the likelihood of future shocks and increase society's resilience to them when they do occur. These policies need to ensure that development is more inclusive, particularly for women, as well as resilient to climate impacts and deepening supply chains.

The question is what does Africa need to do to address its development challenges given the severe negative economic impact of COVID-19? The answer to this question is that Africa needs a much more ambitious intervention to achieve its development objectives. The challenges are very steep and may require unconventional methods. The drivers identified to accelerate the achievement of the sustainable development goals, within a framework of socio-economic transformation are: financing; resilience; sustainable and inclusive economies; more effective institutions; local action; better use of data; and harnessing science, technology and innovation with a greater focus on digital transformation. Africa will also need substantial financing needs to support all its development objectives. However, given the current global economic environment, it is extremely challenging for countries to raise funding. Africa would therefore need creative and innovative financing mechanisms to support the promotion of its development agenda.

How does Africa carry out development when it is faced with tight fiscal space? This question is being raised because some African countries have for years been faced with colossal debt. This debt, which is denominated in U.S. dollars at floating interest rates, became impaired in the 1970s and 1980s when interest rates were very high. Current lower interest rates have also encouraged many countries to pile on more debt. Moreover, unfavourable terms of trade, due to volatile prices of export commodities and falling export revenue, have hampered the ability of some African countries to retire and/or service their debts. Consequently, the debt "overhang" has limited many African countries access to new foreign capital, forcing them to adjust their domestic investment and consumption. Moreover, some African countries are still mired in debt crises, which is seriously stifling their economic growth. COVID-19 has further worsened the debt situation of some African countries as they had to borrow to acquire Personal Protective Equipment (PPEs) to fight COVID-19.

[2]Again GFDRR demonstrates that if the average reconstruction period is cut by two thirds (without compromising the quality of reconstruction), global well-being losses could be reduced by 14 percent, which is equivalent to increasing global consumption by over US$75 billion per year.

[3]This ensures that post-crisis remedial actions reach all sectors of the population, including the poor, vulnerable people and women. This is very important because, as indicated by GFDRR, if all governments were able to provide post-crisis support to the poorest people in developing countries, global well-being losses due to shocks/disaster could be reduced by 9 percent, an equivalent of about US$52 billion increase in annual global consumption. Such interventions particularly needed for countries with high inequality, and where poor people have little access to social protection and financial instruments.

The major contribution of this paper is to urge African countries to consider using state contingent debt instruments, such as commodity-linked bonds, as an innovative development-financial instrument to raise money on the international capital markets, rather than through standard forms of financing. Given the colossal financing needs, which have been worsened by the COVID-19 pandemic, African countries will have to put in place innovative financing mechanisms to mobilize resources in support of the development frameworks aligned to BBB. Commodity-linked bonds differ from conventional bonds in terms of their payoffs to the holder. The bearer of the conventional bond receives fixed coupon (interest) payments during the life of the bond, and face value (principal) at maturity. The principal of a commodity-linked bond, however, is paid in either the physical units of a reference commodity or its equivalent monetary value. Similarly, the coupon payments may or may not be in units of the commodity to which the bond is indexed. Therefore, the structural difference between the two bonds is that the nominal return of the conventional bond held to maturity is known with certainty, although the real return is unknown due to inflation uncertainty, whereas both the nominal and real returns of the commodity-linked bond are not known with certainty.

This paper is organized as follows. Section 2 discusses the state of Africa's debt and a search for innovative financing mechanism. Section 3 focuses on innovative financing mechanism and the potential role for commodity linked bonds in the mechanism. Section 4 talks about the various financing mechanisms to support development. Section 5 offers some conclusion remarks.

2. Africa's debt and a search for innovative financing

Prior to the COVID-19 pandemic, multilateral lending institutions were raising concerns about the high debt levels incurred by a number of African countries. As part of the measures to contain the virus, African countries had to borrow more to acquire Personal Protective Equipment (PPEs) for its citizens, contributing to worsening the debt situation for some African countries.

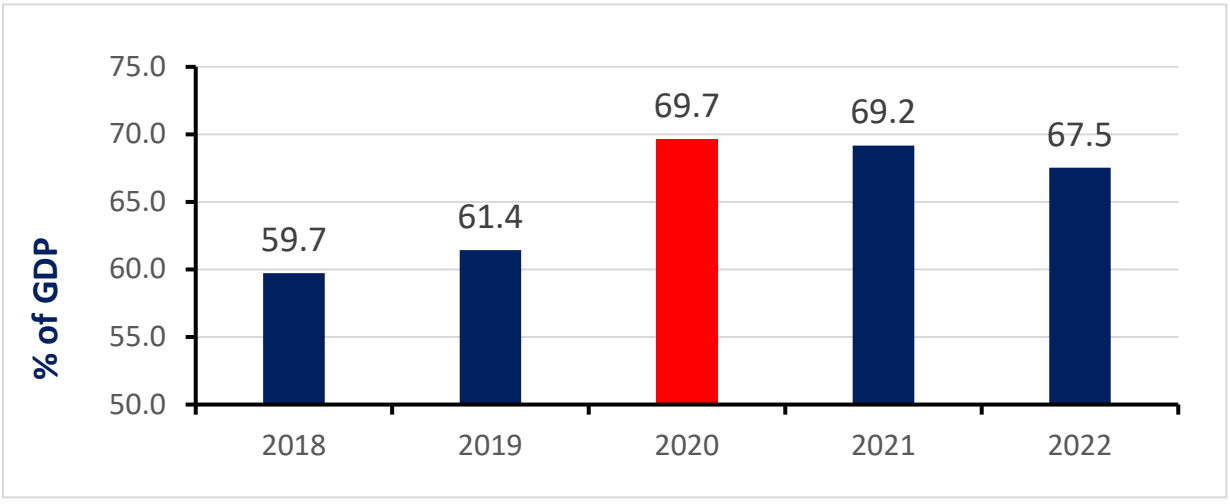

Figure 1. Africa's average Debt to GDP.
Source: Authors' calculations based on data from World Bank.

Available data shows that the percentage of Africa's debt-to-GDP is rising very fast. Figure 1 shows that Africa's average debt to GDP is expected to peak at 69.7 percent in 2020, which is higher than the 55 percent debt-to-GDP ratio suggested by the IMF and the 60 percent of GDP prescribed by the African Monetary Co-operation Programme (AMCP) but very close to the Economic Community of West African States' convergence requirement of 70 percent. Furthermore, Figure 2 indicates that Africa, on average, is spending more of its total GDP to service its external debts, peaking at 53 percent in 2020.

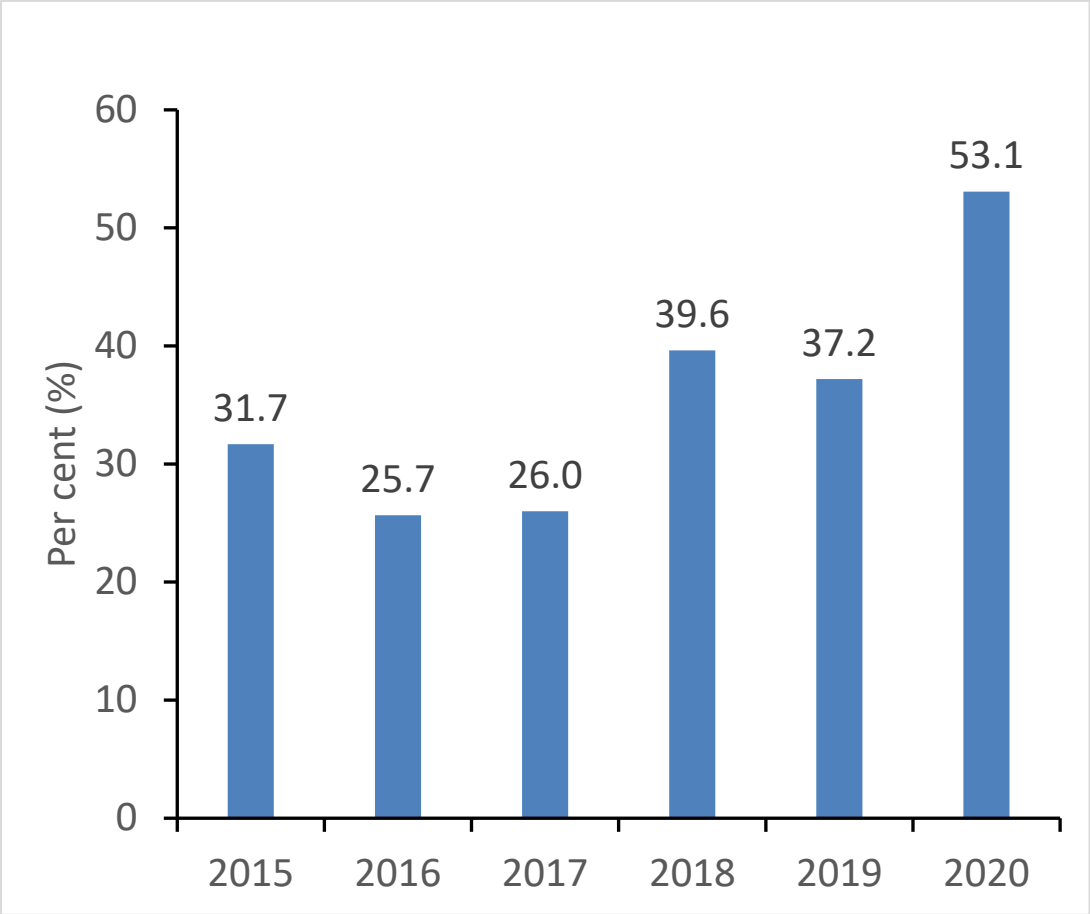

Figure 2. External debt service (% of GDP).
Source: Authors' calculations based on data from World Bank

Atta-Mensah and Ibrahim (2020) notes that Africa's Debt position is worsening because the growth in borrowing has not been matched with increases in tax revenues. They find that on average, Africa's debt is over four-folds of what it collects as tax revenues due to the worsening macroeconomic conditions and continuing oil and commodity price shocks. The dwindling tax revenues threaten the ability of countries to service their rising cost of debt. Furthermore, the changing structure of Africa's debt is also a challenge to measures needed to mitigate the worsening debt situation. As compared to the past, countries are tilting towards non-concessional and domestic debt with higher interest rates. In addition, African countries' access to and control over the domestic debt market is leading to excessive public debt accumulation and macroeconomic instability.

Cognisance of the Debt challenges of the low and lower middle-income countries, the G20, in collaboration with the Bretton Woods Institutions, agreed in April 2020 to the Debt Service Suspension Initiative (DSSI) for eligible countries to mitigate against the negative economic impact of the coronavirus pandemic. The DSSI, which came into effect on 1 May 2020, means that bilateral official creditors will, during a limited period, suspend debt service payments from the poorest countries (73 low- and lower middle-income countries) that request the suspension. The DSSI is expected to ease the financing constraints for these countries and free up scarce money that they could instead use to mitigate the human and economic impact of the COVID-19 crisis.[4] The DSSI is designed to support countries with their immediate liquidity needs but does not address debt vulnerabilities of countries.

The IMF has also come out with other financial support for low income countries including those in Africa. In March 2020 IMF announced the Extended Credit Facility (ECF) which provides financial assistance to countries with protracted balance of payments problems. The ECF was created under the Poverty Reduction and Growth Trust (PRGT) as part of a broader reform to make the Fund's financial support more flexible and better tailored to the diverse needs of low-income countries (LICs), including in times of crisis. The ECF is the Fund's main tool for providing medium-term support to LICs. The ECF supports countries' economic programmes aimed at moving toward stable and sustainable macroeconomic position consistent with strong and durable poverty reduction and growth. The ECF may also help catalyse additional foreign aid.

In April 2020, the IMF further announced the Rapid Credit Facility (RCF) which provides rapid concessional financial assistance with limited conditionality to LICs facing an urgent balance of payments need. The RCF was created under the Poverty Reduction and Growth Trust (PRGT) as part of a broader reform to make the Fund's financial support more flexible and better tailored to the diverse needs of LICs, including in times of crisis. The RCF places emphasis on a country's poverty reduction and growth objectives. In April 2020, the IMF announced the Rapid Financing Instrument (RFI) to provide rapid financial assistance to all of its member countries facing an urgent balance of payments need. The RFI was created as part of a broader reform to make the IMF's financial support more flexible to address the diverse needs of member countries. The RFI replaced the IMF's previous emergency assistance policy and can be used in a wide range of circumstances.

According to the IMF, the RCF and the RFI, which could amount up to $50 billion for low-income and emerging countries, can be disbursed very quickly to assist member countries to implement policies to address emergencies such as the coronavirus. Emergency lending, under the RCF, available to LICs is about $10 billion, and for emerging markets under the RFI it could amount to about $40 billion.

In addition to the above facilities, the IMF could modify, as needed, existing programmes through the Stand by Arrangements (SBA) in support of countries to accommodate urgent new needs arising from the COVID-19 pandemic. There is also the Catastrophe Containment and Relief Trust (CCRT) which allows the IMF to provide grants for debt relief to the poorest and most vulnerable countries with outstanding obligations to the IMF to help address disasters, including public health disasters.

[4]As of 30th April 2020 24 African Countries have taken up the DSSI. These are: Angola, Burkina Faso, Cameroon, Carbo Verde, Chad, Comoros, Cote d'Ivoire, Congo, Democratic Republic Congo, Djibouti, Ethiopia, Guinea, Lesotho, Madagascar, Mali, Mauritania, Mozambique, Niger, Sao Tome and Principe, Senegal, Sierra Leone, Tanzania, Togo and Zambia.

This facility was used to support Guinea, Liberia, and Sierra Leone during the 2014 Ebola outbreak. The CCRT is currently underfunded with just over $200 million available against possible needs of over $1 billion. All these facilities are available for some of the African countries to tap into to support their efforts in mitigating the ill-effects of the pandemic and to build back better as well as they return to the pathway of progress in achieving the SDGs.

Financing BBB would therefore require enormous resources. All the three BBB strategies—rebuilding stronger, faster, and more inclusively—could generate major benefits to all countries. Building back better is particularly important after the COVID-19 pandemic has exposed the vulnerabilities of many countries. African countries need to upgrade their healthcare facilities, infrastructure and other assets, including green technologies, so as to increase their productivity and sustainability. The benefits of generating increased economic benefits make the case for accelerated investments in better recovery and reconstruction processes. Such resilient and effective recovery and reconstruction can only come if the appropriate policies and tools are fashioned before the next crises.

It has to be emphasized that before the advent of the COVID-19 pandemic, most African countries were saddled with colossal levels of debt. One measure suggested by Atta-Mensah (1992) is that countries adopt innovative hedging strategies. Atta-Mensah (1992) as well as Fall (1986) recommends that countries shift the risk that their commodity prices face to the financial markets.

3. Commodity-linked bond as instrument for innovative financing scheme

African countries need adequate, predictable, sustainable and integrated financing mechanism to support its developmental agenda. This need is even more pronounced in a post-COVID-19 world. Africa therefore needs to devise innovative financing mechanism to support its development objectives. The development framework for Africa is the Agenda 2063 and 2030 Agenda for Sustainable Development. Achieving the goals of its development, in a post-COVID-19 world, means Africa would need billions of dollars. For example, Africa will need about 100 billion dollars annually to address its infrastructural deficits. Given that Africa cannot continue to depend on Overseas Development Assistance, it is imperative for African countries to devise new and innovative mechanisms for financing development projects on the continent. The question is how does Africa leverage innovative finances to support its development?

There are also discussions in the literature suggesting that many countries will have to restructure their debts following the COVID-19 pandemic (Cohen et al., 2020; Asonuma et al., 2019, 2020). The restructuring of debt is needed because following the pandemic, countries would be challenged in finding the path to economic recovery. The pandemic has elevated macroeconomic uncertainties as well as imposed risks to repayments of debts, and may force both debtors and creditors to enter into negotiations on the restructuring of debts. These negotiations are not going to be easy and straight forward and could therefore be prolonged. As argued by Cohen et al. (2020) the prolonging of the resolution of debt restructuring, could lead to suboptimal agreements, with long-lasting negative consequences for borrowers and creditors or lead to repeated restructuring of debts in a short space of time.

Furthermore, Asonuma et al. (2019, 2020) found that there are substantial costs to the restructuring of sovereign debts. The authors demonstrate that debt restructuring causes declines in GDP, investment, private sector credit and capital flows. The costs are however dependent on whether the restructuring takes place pre-emptively, without missing payments to creditors, or after a default has occurred. Their analysis found that post-default restructurings lead to larger declines in GDP, investment, private sector credit and capital inflows than pre-emptive restructurings.

Given the uncertainty and economic costs associated with debt restructuring, African countries could search for alternative resource mobilization strategies. Furthermore, most African countries are very vulnerable to major financial risks linked to commodity prices. Their exposure to these risks is because they are major exporters of raw commodities and also have limited capacities to effectively mitigate the risks when commodity prices plummet and there are sharp increases in interest rates. Such conditions lead countries carrying large debts to face the challenges of increased indebtedness and debt servicing difficulties. These difficulties also lead to the deterioration of the balance of payments of the countries as export revenues fall and consequently the depreciation of the value of their currencies. Forms of financing that have been in the financial markets of industrial countries and offer considerable potential for risk management for African countries are state-contingent debt instruments (SCDIs) which could facilitate speedier and less-costly debt restructuring as the payments of restructured debt contracts could be linked to future outcomes. SCDIs are contractual debt instruments where payments are linked to a predefined state variable such as GDP, exports, or commodity prices.[5] These instruments are also arranged so as to provide additional compensation to creditors in good times and/or provide some form of relief to debtors in bad times, such as the occurrence of a natural disasters or pandemics. Cohen et al. (2020) point out that by linking the debt service payments of restructured debt contracts to future outcomes, SCDIs may help avoid protracted disputes about current valuations and facilitate quicker agreements between creditors and debtors, thus allowing countries to restore debt sustainability and facilitating their return to market access.[6] Demertzis and Zenios (2019) suggest that SCDI could provide market-based insurance to protect the euro area from future debt crises.

An example of SCDI, which is the focus of the paper, is Commodity-linked bonds. These bonds present a significant vehicle that Africa could use to mobilize resources for the development of the continent given the vast mineral deposits on the continent. The most popular form of commodity-indexed bond is referenced to specified units of gold. A well-known example of gold bonds were issued in 1973 by the French government and accepted in the financial markets as the "Giscard." After the "Giscard," other types of commodity-linked securities were issued. In 1980, the Sunshine Mining Company, a large silver mine in the United States, issued US$25 million worth of silver-indexed bonds

[5]Cohen et al. (2020) indicate that a state variable is a measurable indicator that varies over time, and as used in a contract may trigger actions on the part of contract participants (such as an increase or decrease in payments).

[6]See Atta-Mensah (1992), Borensztein and Mauro (2004), Brooke et al. (2013) and Kim and Ostry (2018) and their references on earlier discussions in the literature on the potential usefulness of contingent debt instruments. Also see Cohen et al. (2020) for further discussions on why restructurings offer a unique opportunity for the introduction of SCDIs into a sovereign's debt portfolio. Cohen et al. (2020) further argues that restructuring allows SCDIs to be implemented across the entire renegotiated debt stock, eliminating the "first-mover" problem—namely, the differential treatment of legacy debt contracts that lack such clauses—which lowers their appeal in the context of new issuance.

to hedge against the fluctuations in the price of silver. Oil-backed bonds, or Petrobonds, appeared in the financial market during the late 1970s, issued by the Government of Mexico. Each 1,000-peso-bond was linked to 1.95354 barrels of oil. Other bonds have been indexed to other types of precious metals. As Privolos and Duncan (1991) report, Inco, which is one of the world's largest producers of nickel, copper, silver, cobalt, and platinum, in 1984, raised Can$90 million on the financial market through the issuance of bonds linked to the price of nickel or copper.[7]

In this paper, two approaches are taken to examine the potential benefits of African countries issuing commodity-linked bonds. First, the theory of option pricing is applied to determine the market price of a commodity-linked bond. An assessment is then made as to whether the value of the commodity-linked bond decreases with falling price of the underlying commodity. Second, the model of Myers and Thompson (1989) is extended to determine the optimal proportion of an African Country's total external debt that could be issued in the form of commodity-linked bonds. The relationship between the commodity price and the demand for the bond is also determined.

4. Model of optimal external debt allocation

In this section, the framework of portfolio theory is applied to derive simple rules African countries could follow in allocating debt instruments and their level of imports.[8] Besides the usual assumptions of no taxes, continuous trading, and zero transactions costs made in the financial literature, the following assumptions are made: the African country has a small open economy; all prices of assets are denominated in U.S. dollars; all external debt is issued by the government; there are no short sales, because a country cannot sell short its own debt; two sources of foreign finances are available to the government (the issue of conventional bonds and the issue of commodity-linked bonds); there is only one perishable and divisible imported good; and the rates of change in the price of the export commodity and the Libor rate follow a stochastic Brownian motion.

4.1. Conventional debt

The process followed by the price of the export commodity is postulated as:

$$\frac{dP}{P} = \alpha_p dt + \sigma_p dz_p \tag{1}$$

where is α_p the instantaneous average return of holding one unit of the export commodity, σ_p is the instantaneous standard deviation of the rate of change of the commodity price, and dz_p has a standard normal distribution with a mean of zero and a variance of *dt*. Note that α_p and σ_p may be functions of *P* and *t*. For the purpose of this exercise, however, they are assumed to be constants.

[7]See Krumm (1985) Adeniran et al. (2018), Mustapha and Prizzon (2019), Froot et al. (1989), Dornbusch (1988), Krugman (1989, 1999), Kenen (1990) and Sachs (1988), Caballero (2003), Williamson (2000), Meltzer (2000) and Fischer (2002) on the earlier academic research on Africa's debt and advocacy for debt relief under the Baker and Brady plans.

[8]See Merton (1971, 1973), Schwartz (1982), Privolos and Duncan (1991) and Atta-Mensah (1992) for a discussion on the methodology of the portfolio theory used for the derivations in this section.

The Libor rate is assumed to follow a mean-reversion stochastic process of the form:

$$dr = \kappa(\theta - r)dt + \sigma_r dz_r \tag{2}$$

The parameters are all constants. The Libor rate tends to be pulled towards the average target, θ. σ_r is the instantaneous standard deviation for the rate of change of the Libor rate, and dz_r is normally distributed with a mean of zero and a variance of *dt*. Also, dz_p and dz_r have an instantaneous correlation of $\rho_{prdt.}$.

Let the price of conventional debt, *Q(r, t)*, be dependent on the Libor rate. By applying Ito's Lemma, the rate of change of the price, *Q(r, t)*, is given as:

$$\frac{dQ}{Q} = \alpha_q dt + \sigma_q dz_r \tag{3}$$

where,

$$\alpha_q = \frac{\kappa(\theta - r)Q_r + 0.5\sigma_r^2 Q_{rr} - Q_t}{Q} \tag{4}$$

and

$$\sigma_q = \frac{\sigma_r}{Q} \tag{5}$$

Standard arbitrage arguments can be advanced to show that the partial differential equation that governs the pricing of the conventional debt with a coupon payment of c is given as:

$$[\kappa(\theta - r) - \sigma_r \lambda(r)]Q_r + 0.5\sigma_r^2 Q_{rr} - Q_t - rQ + c = 0 \tag{6}$$

where λ(r) is the market price of risk attached to all financial assets whose underlying state variable is the Libor rate. Also, *r* is the instantaneous riskless rate of interest. For a given boundary condition, a closed-form solution for Equation (6) cannot be determined. Assuming a face value of Q_o, no coupon payments, and a constant market price of interest rate risk (or λ(r) = λ), Atta-Mensah (1992) and Vasicek (1977) show that the price of the conventional debt satisfies:

$$Q(r, \tau) = Q_o \exp\left[\frac{1}{\kappa}(1 - e^{-\kappa\tau})(A - r) - \tau A - \frac{\sigma_r^2}{4\kappa^3}(1 - e^{-\kappa\tau})^2\right] \tag{7}$$

where τ is the time left to maturity, and

$$A = \theta + \frac{\lambda\sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} \tag{8}$$

Alternatively, if we assume a constant interest rate, then the market value of the conventional debt will be:

$$Q(r, \tau) = \frac{c}{r}(1 - e^{-r\tau}) + Q_o e^{-r\tau} \tag{9}$$

Based on either Equation (7) or (9), the main driver for the conventional debt is found to be the level of the interest rate.

4.2. Commodity-linked bond

Consider a commodity-linked bond, the value of which is solely a function of the Libor rate and the price of the export commodity. Let *H(r, P, t)* be the price of a commodity-linked bond. Applying Ito's lemma, the rate of change of the commodity-linked bond is obtained as:

$$\frac{dH}{H} = \alpha_h dt + \psi_r dz_r + \psi_p dz_p \tag{10}$$

where

$$\alpha_h = \frac{\{[\kappa(\theta - r) - \sigma_r \lambda(r)]H_r + \alpha_p P H_p + 0.5\sigma_p^2 P^2 H_{pp} + 0.5\sigma_r^2 H_{rr} + \rho_{pr}\sigma_p\sigma_r P H_{\rho r} - H_t\}}{H} \tag{11}$$

and

$$\psi_p = \frac{\sigma_p P H_p}{H} \tag{12}$$

$$\psi_r = \frac{\sigma_r H_r}{H} \tag{13}$$

The application of standard arbitrage arguments yields the partial differential equation that governs the valuation of the commodity-linked bond, which is of the form[9]:

$$[\kappa(\theta - r) - \sigma_r \lambda(r)]H_r + \alpha_p P H_p + 0.5\sigma_p^2 P^2 H_{pp} + 0.5\sigma_r^2 H_{rr} + \rho_{pr}\sigma_p\sigma_r P H_{\rho r} + rPH_p - H_t - rH + C^B = 0 \tag{14}$$

In Equation (14), C^B is the coupon rate of the commodity-linked bond. Furthermore, Equation (14) is restricted by the following conditions:

$$H(r, 0) = 0 \qquad \forall\, r \tag{15}$$

$$H(r, \infty) = Q(r; t) \qquad \forall\, r \tag{16}$$

$$H(\infty, P) = 0 \qquad \forall P \tag{17}$$

[9]See Schwartz (1982), Atta-Mensah (1992), or Miura and Yamauchi (1998) for expanded valuation models of commodity-linked bonds. Consiglio and Zenios (2018) show how GDP-linked bonds can be priced in incomplete markets which can be used for the commodities-indexed bonds. Blach (2020) also applies innovative financing to identify and prioritize the main types of barriers to the implementation of financial innovations by nonfinancial firms. Failler et al. (2020) have also used innovative financing to examine the feasibility of financial possibilities to support a Regional Marine Protected Area in Martinique.

4.2.1. The value of the commodity-linked bond

The price of the commodity-linked bond is shown by the solution of Equation (14) subject to a boundary condition. As stated earlier, a commodity-linked bond is indexed to an underlying commodity. Assume that the promised payment on the bond at maturity is set at the maximum of the face value of the bond (F) and the monetary value of a pre-specified unit of the referenced commodity. Let γ be the pre-specified unit of the commodity referenced to the bond, and $H^c(\cdot)$ the value of this particular bond; then the final payment of the bond is of the form:

$$H^c(P,r,0) = Max[F,\gamma P] \tag{18}$$

or,

$$H^c(P,r,0) = F + \gamma Max[0,P - F/\gamma], \tag{19}$$

Equation (19) implies that the promised payment of the bond is equivalent to the face value of a bond (F) for sure, plus γ amounts of a call option, which gives the bearer an option to buy the reference commodity bundle at a specified exercise price, F/γ.

On the other hand, to minimize default risk, the borrower could have an option to pay the minimum of the face value and the value of the reference amount of the commodity at the maturity date. In that case, the terminal value of the bond would be:

$$H^P(P,r,0) = Min[F,\gamma P] \tag{20}$$

or,

$$H^P(P,r,0) = F - \gamma Max[0,F/\gamma - P] \tag{21}$$

Equation (21) indicates that a commodity-linked bond that pays the minimum of the face value, F, and the monetary value of a pre-specified unit of a commodity is similar to a bond of face value, F, and a short position on γ amounts of a put option, which gives the bearer an option to sell the reference commodity bundle at a specified exercise price, F/γ.

A closed-form solution of Equation (14), subject to the boundary conditions of Equation (19) or Equation (21), is not a trivial exercise. Hence, for expositional reasons, consider a case in which the interest rate is constant. For simplicity and without loss of generality, also assume that γ is equal to unity. With these assumptions, the differential equation for pricing the commodity-linked bond subject to a boundary condition simplifies to:

$$\alpha_p P H_p + 0.5\sigma_p^2 P^2 H_{pp} + 0.5\sigma_r^2 H_{rr} + \rho_{pr}\sigma_p\sigma_r P H_{\rho r} + rPH_p - H_t - rH + C^B = 0 \tag{22}$$

and

$$H^c(P,0) = F + Max[0,P - F] \tag{23}$$

$$H^P(P,0) = F - Max[0,F - P] \tag{24}$$

The solution of Equation (22) subject to (23) is given as:

$$H^c(P,\tau) = \frac{C^c}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} + L(P,F,\tau) \tag{25}$$

where c^c is the coupon payment, *L(P, F, τ)* the Black-Scholes (1973) formula for valuing a call option on P with exercise price *F*, and τ the time left to maturity:

$$L(Q,F,\tau) = PN(d_1) - Fe^{-r\tau}N(d_2) \tag{26}$$

where

$$d_1 = \frac{\log\left(\frac{P}{F}\right)+(r+\frac{1}{2}\sigma_p^2)\tau}{\sigma_p\sqrt{\tau}} \tag{27}$$

$$d_2 = d_1 - \sigma_p\sqrt{\tau} \tag{28}$$

and *N(.)* is the cumulative normal distribution function.

On the other hand, the value of the bond could be the solution of Equation (22) subject to Equation (24):

$$H^P(P,\tau) = \frac{C^P}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} - Q(P,F,\tau) \tag{29}$$

where c^p is the coupon payment, *Q(P, F, τ)* the Black-Scholes (1973) formula for valuing a put option on *P* with exercise price *F*, and τ the time left to maturity:

$$Q(P,F,\tau) = Fe^{-r\tau}N(a_1) - Qe^{-r\tau}N(a_2) \tag{30}$$

where

$$a_1 = \frac{\log\left(\frac{F}{Q}\right)+\left(-r+\frac{1}{2}\sigma_P^2\right)\tau}{\sigma_P\sqrt{\tau}} \tag{31}$$

$$a_2 = a_1 - \sigma_P\sqrt{\tau} \tag{32}$$

and *N(.)* is the cumulative normal distribution function.

4.2.2. Commodity price and the value of the commodity-linked bond

The primary focus of this paper is to argue that African countries could, through the issuance of bonds linked to their main exports, hedge against fluctuations in their export earnings. Hence one expects the value of debt issued in the form of commodity-linked bonds to fall with the falling prices of (or revenues from) exports.

Proposition 1: *The value of the commodity-linked bond increases monotonically as the price of the commodity indexed to the bond increases.*

Proof:

Differentiating Equation (25) with respect to P:

$$\frac{\partial H^c}{\partial P} = N(d_1) + \frac{1}{\sigma_P\sqrt{\tau}}N'(d_1) - \frac{Fe^{-r\tau}}{P\sigma_P\sqrt{\tau}}N'(d_2) \tag{33}$$

but

$$N'(x) = \frac{1}{2\sqrt{\pi}} e^{-\left(\frac{1}{2}\right)x^2} \tag{34}$$

thus,

$$\frac{\partial H^C}{\partial P} = N(d_1) + \frac{1}{\sigma_P \sqrt{2\pi\tau}} \left[e^{-\left(\frac{1}{2}\right)d_1^2} - e^{-\log\left(\frac{P}{F}\right) + r\tau - \left(\frac{1}{2}\right)d_2^2} \right] \tag{35}$$

Substitute Equation (34) in the last part of Equation (35):

$$\frac{\partial H^C}{\partial P} = N(d_1) + \frac{1}{\sigma_P \sqrt{2\pi\tau}} \left[e^{-\left(\frac{1}{2}\right)d_1^2} - e^{d_1\sigma_P\sqrt{\tau} + \left(\frac{1}{2}\right)\sigma_P^2\tau - \left(\frac{1}{2}\right)\left(d_1 - \sigma_P\sqrt{\tau}\right)^2} \right] \tag{36}$$

which simplifies into:

$$\frac{\partial H^C}{\partial P} = N(d_1) + \frac{1}{\sigma_P \sqrt{2\pi\tau}} \left[e^{-\left(\frac{1}{2}\right)d_1^2} - e^{-\left(\frac{1}{2}\right)d_1^2} \right] \tag{37}$$

Hence:

$$\frac{\partial H^C}{\partial P} = N(d_1) \geq 0 \tag{38}$$

Alternatively, differentiating Equation (29) with respect to *P* also yields:

$$\frac{\partial H^P}{\partial P} = N(a_2) \geq 0 \tag{39}$$

Remarks: The first type of commodity-linked bond is equivalent to a portfolio that consists of a discount bond with a face value of *F* and a European call option on the commodity referenced to the bond with an exercise price of *F*. An explanation for Proposition 1 is that, as the commodity price increases, the probability that the call contained in the portfolio will end up "in the money" increases, which appreciates the value of the commodity-linked bond.

The second type of commodity-linked bond is equivalent to a portfolio that consists of a discount bond with a face value of *F* and a short position on a European put option on the commodity referenced to the bond with an exercise price of *F*. The value of the commodity-linked bond rises with the increase in the price of the referenced commodity, because of the value of the put option. The chances of the put option finishing out of the money also rises with the rise in the commodity price.

The two results clearly show that, if African countries had issued debt contracts that were tied to their main export commodities, then their debt load would have declined along with plummeting export prices (or export revenues). African countries could therefore have prevented their current potential debt crisis if they had issued commodity-linked bonds.

Proposition 2: An African country that has a volatile commodity price can minimize its debt burden by issuing bonds that pay holders, on maturity, the lesser of the face value of the bond and the monetary value of a pre-specified unit of a commodity, rather than the greater of these two.

Proof:

Differentiating Equation (25) with respect to σ_p and simplifying yields:

$$\frac{\partial H^C}{\partial \sigma_P} = \sqrt{\tau} P N'(d_1) \geq 0 \tag{40}$$

Differentiating Equation (29) with respect to σ_p and simplifying yields:

$$\frac{\partial H^P}{\partial \sigma_P} = -(\sqrt{\tau} P N'(a_2)) \leq 0 \tag{41}$$

Remarks: Equations (40) and (41) show that a commodity-linked bond that has an embedded put option falls in value when the volatility of the commodity price rises, whereas the opposite occurs with a bond that has an embedded call option. The increased volatility of the commodity price increases its value option attached to the bond, because a put call has no downside risk, since its value is zero irrespective of how far it finishes out of the money. Hence, an increase in the volatility of the commodity price increases the chances that the put option will expire in the money. Given that the commodity-linked bond of this type is equivalent to a regular bond and a short position on a put option, the value of the bond falls with a rise in the volatility of the commodity price. In other words, the heightened volatility of export commodity prices leads to an increase in the expected export revenue, and, with the debt burden falling with it, greatly reduces the chance of an African country defaulting on the bond.

Alternatively, if the African country was to issue a bond with an embedded call, then the rise in the volatility of the commodity price would increase the value of the bond. The value of the call rises with the volatility of the commodity price, because there is no downside risk to the call, since its value is zero irrespective of how far it finishes out of the money. An increase in σ_p, therefore, increases the chances that the call option will expire in the money. The implication is that an African country increases its debt burden when it issues commodity-linked bonds that are embedded with call options on a commodity price, because the value of the bond rises with the increase in the volatility of the commodity price.

Proposition 3: *In an environment where interest rates are not stochastic, the coupon rate for a conventional debt with an identical face value as a commodity-linked bond is generally less than the coupon rate for a commodity-linked bond that pays holders, on maturity, the minimum of the face value and the monetary value of a pre-specified unit of a commodity. The coupon rate for the conventional bond is, however, greater than its counterpart for a commodity-linked bond whose terminal payoff is the greater of the face value and the monetary value of a pre-specified unit of a commodity.*

Proof:

Given their identical face values, an investor on the margin would be indifferent between the two types of commodity-linked bonds and a conventional bond, which implies that the current market values of the two instruments must be the same. Using Equations (9) and (22), and setting Q_0 to F, we have:

$$\frac{c}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} = \frac{c^c}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} + L(P, F, \tau) = \frac{C^P}{r}(1 - e^{-r\tau}) + Fe^{-r\tau} - Q(P, F, \tau) \tag{42}$$

which implies that

$$\frac{c}{r}(1 - e^{-r\tau}) - \frac{c^c}{r}(1 - e^{-r\tau}) = L(P, F, \tau) \tag{43}$$

But $L(P, F, \tau) \geq 0$, because there is no downward risk for an option. It therefore follows that:

$$c - c^c \geq 0 \tag{44}$$

Similarly,

$$\frac{c^P}{r}(1 - e^{-r\tau}) - \frac{c^c}{r}(1 - e^{-r\tau}) = L(P, F, \tau) + Q(P, F, \tau) \geq 0 \tag{45}$$

Given that $L(P, F, \tau) \geq 0$ and $Q(P, F, \tau) \geq 0$,

$$c^P - c^c \geq 0 \tag{46}$$

Lastly

$$\frac{c^P}{r}(1 - e^{-r\tau}) - \frac{c}{r}(1 - e^{-r\tau}) = Q(P, F, \tau) \geq 0, \tag{47}$$

or

$$c^P - c \geq 0 \tag{48}$$

Putting Equations (44), (46), and (48) together, we have:

$$c^P \leq c \leq c^c \tag{49}$$

Remarks: Proposition 3 strengthens the economic rationale for the issue of a commodity-linked bond. It demonstrates that African countries or corporations in need of investment funds could share the appreciation of the market value of the underlying commodity with the bondholders, in return for a lower coupon rate. In this case, African countries would benefit by issuing commodity-linked bonds that pay, on maturity, the greater of the face value or the monetary value of a pre-specified unit of the underlying commodity. This supports Budd (1983), who argues that the issue of commodity- linked bonds offers an opportunity for commodity-producing issuers and international commodity organizations to borrow at below-market interest rates.

On the other hand, an African country could share the depreciation of the market value of its commodity price with bondholders in exchange for higher coupon rates. The African country would issue a commodity-linked bond whose final payoff is the lesser of the face value or the monetary value of a pre-specified unit of the underlying commodity. The issuance of such bonds would act as a hedge for an African country during times when the commodity price experiences a collapse (Caballero, 2003).

4.3. Net foreign debt

Without external financing, the value of imports must equal the value of exports, so that the current account is in balance each period. The assumption made in this paper, however, is that the government of the African country has access to two sources of external financing: one is to issue conventional debt and the other is to issue a commodity-linked bond.

Let $D(t) = \int_0^t D(t-1)d\tau$ be the quantity of conventional debt outstanding to the government of the African country.[10] The new quantity of debt issued in each period is, therefore, $\dot{D}(t) = dD/dt$. Similarly, the total quantity of commodity-linked bonds outstanding is $B(t) = \int_0^t B(t-1)d\tau$. The quantity of new commodity-linked bonds issued $\dot{B}(t) = dB/dt$. Furthermore, assume that both the conventional debt and the commodity-linked bond are of the console type. Also, the coupon payments to bearers of conventional debt and the commodity-linked bonds are, respectively, c and c^B. Hence, in each period, the contributions of the conventional debt and commodity-linked bond to the net foreign debt of the government are, respectively, $Q\dot{D}(t) - Dc$ and $H\dot{B}(t) - Bc^B$.

If x is the fixed rate of commodities exported and $m(t)$ is the rate of imports consumed, then, in every instant, imports must be financed by the sum of export revenue and the value of new total debt less the total coupon payments. In other words, the government's instantaneous import bill is constrained by the following function:

$$m(t) - Pxdt + QdD + HdB - Dcdt - Bc^B dt \tag{50}$$

Let W be the value of the total external debt of the government of the African country:

$$W = QD + HB \tag{51}$$

The change in W is, therefore,

$$dW = DdQ + BdH + QdD + HdB \tag{52}$$

But the import constraint of Equation (50) shows that:

$$QdD + HdB = m(t) - Pxdt + Dcdt + Bc^B dt \tag{53}$$

Substituting Equation (53) into Equation (52),

$$dW = DdQ + BdH + m(t) - Pxdt + Dcdt + Bc^B dt \tag{54}$$

Define ω_1 as the fraction of the total external debt held in conventional debt and ω_2 as the fraction of external debt held in commodity-linked bonds:

$$\omega_1 = QD/W \quad \text{and} \quad \omega_2 = HB/W.$$

Equation (54) then becomes:

$$W = \omega_1 W\frac{dQ}{Q} + \omega_2 W\frac{dH}{H} + m(t) - Pxdt + \frac{\omega_1 W}{Q}cdt + \frac{\omega_2 W}{H}c^B dt \tag{55}$$

Note that Q and H must satisfy Equations (8) or (9) and (25) or (29). Substitute Equations (3) and (10) into Equation (55) and note that $\omega_1 + \omega_2 = 1$. Since $\omega_2 = 1 - \omega_1$, the flow of the net external debt is:

[10] *D(t-1)* is a conventional debt that matures in *t-1* periods.

$$dW = [\omega_1 W(\alpha_q - \alpha_h + c/Q - c^B/H) + m - Px + W(\alpha_h + c^B/H)]dt + [\omega_1 W(\sigma_q - \psi_r) + W\psi_r]dz_r + (1 - \omega_1)W\psi_p dz_p \quad (56)$$

Equation (56) demonstrates that the value of the external debt of the African country changes with the market valuations of conventional bonds and commodity-linked bonds, the import bill, and export revenue. Shocks from interest rates and commodity prices, however, make the market valuation of the debt very uncertain.

4.4. The government's maximization problem

The government is faced with choosing in each period the level of imports, m, and the fractions of total external debt, ω_1 and ω_2, that must be held in conventional debt and commodity-linked bonds. The government embarks on this portfolio and imports rule in a manner that maximizes the expected value of a time-additive von Neumann-Morgenstern utility function[11]. The problem is formulated as:

$$\max_{m,\,\omega_1} E_0[\int_0^\infty e^{-\beta t} U(m(t), t)]dt \quad (57)$$

subject to Equation (56) and

$$W(0) = W_0 \quad (58)$$

Also, the utility function $U(\cdot)$ is restricted to be concave in m (i.e., $U_m > 0$ and $U_{mm} < 0$). E_0 is the expectations operator, conditional on $W(0) = W_0$, which is known.

Using dynamic programming techniques, a J function can be defined as:

$$J(W, P, r, t) \equiv \max_{m,\,\omega_1} E_0[\int_0^\infty e^{-\beta t} U(m(t), t)]dt \quad (59)$$

Equation (59) is also constrained by Equations (56) and (58). Equation (59) can therefore be rewritten as,

$$J(w(t_0), P, r, t_0) \equiv \max_{m,\,\omega_1} E_0[\int_{t_0}^{t_1} e^{-\beta t} U(m(t), t)]dt + J(w(t_1), P, r, t_1)] \quad (60)$$

As shown in the appendix, the optimization problem that faces the government is reduced to:

$$\max_{m,\,\omega_1} \Phi(\omega, m; W, P, r, t) = e^{-\beta t} U(m(t), t) + L(J) \quad (61)$$

where L, which is known as the Dynkin operator over the variables W, P, and r, is defined in the appendix. The first-order condition for a maximization problem is:

$$\Phi_m = e^{-\beta t} U_m + J_w = 0 \quad (62)$$

[11]Zenios et al. (2021) also proposes optimization strategies for sustainable financing in Eurozone.

$$\Phi_{\omega_1} = J_w W\left(\alpha_q - \alpha_h + \frac{c}{Q} - \frac{c^B}{H}\right) + J_{wp} WP\left(\rho_{pr}(\sigma_q - \psi_r) - \psi_p\right) + J_{wr} W\sigma_r\left((\sigma_q - \psi_r) - \rho_{pr}\psi_p\right) + 0.5 J_{ww} W^2\left(2\omega_1(\sigma_q - \psi_r)^2 + 2\psi_r(\sigma_q - \psi_r) + 2\rho_{pr}\psi_p(1 - 2\omega_1)(\sigma_q - \psi_r) - 2\rho_{pr}\psi_p\psi_r - 2(1 - \omega_1)\psi_p^2\right) = 0 \quad (63)$$

Before finding the optimum proportions of commodity-linked bonds and conventional debt that must be raised by the government externally, some comments on Equation (62) should be made.

Equation (62) implies that the marginal utility of external debt to the government of an African country is negative. The African country, therefore, chooses an optimum level of imported goods at the point where the sum of marginal utility derived from consuming imported goods and the marginal utility of external debt is zero. In other words, African countries will contract loans up to the point where the marginal disutility of total external debt is completely offset by the marginal utility derived from imported goods.

4.5. Optimal allocation of external debt

Equation (63) is used to obtain the optimum proportions of the total external debt that must be held in conventional debt and commodity-linked bonds. Thus, rearranging Equation (63) and simplifying, the optimum weight of conventional debt is expressed as:

$$\omega_1^* = \frac{J_w}{WJ_{ww}}\left[\frac{\alpha_q - \alpha_h + \frac{c}{Q} - \frac{c^B}{H}}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] - \frac{J_{wp}P}{WJ_{ww}}\left[\frac{\rho_{pr}\sigma_p(\sigma_q - \psi_r) - \sigma_q\psi_p}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] - \frac{J_{wr}}{WJ_{ww}}\left[\frac{\sigma_r(\sigma_q - \psi_r) - \sigma_r\rho_{pr}\psi_p}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] + \left[\frac{\psi_r(\sigma_q - \psi_r) - \psi_p(\rho_{pr}\psi_r - \psi_p)}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] \quad (64)$$

Without loss of generality, the last term of Equation (64) could be dropped as it can be assumed to be close to zero. The optimum proportion of external debt that is in the form of commodity-linked bonds is given as:

$$\omega_2^* = 1 - \omega_1^* \quad (65)$$

$$\omega_2^* = 1 + \frac{J_w}{WJ_{ww}}\left[\frac{\alpha_q - \alpha_h + \frac{c}{Q} - \frac{c^B}{H}}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] - \frac{J_{wp}P}{WJ_{ww}}\left[\frac{\rho_{pr}\sigma_p(\sigma_q - \psi_r) - \sigma_q\psi_p}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2} - \frac{J_{wr}}{WJ_{ww}}\left[\frac{\sigma_r(\sigma_q - \psi_r) - \sigma_r\rho_{pr}\psi_p}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right] + \left[\frac{\psi_r(\sigma_q - \psi_r) - \psi_p(\rho_{pr}\psi_r - \psi_p)}{(\sigma_q - \psi_r)^2 - 2\rho_{pr}\psi_p(\sigma_q - \psi_r) + \psi_p^2}\right]\right] \quad (66)$$

Assume that the government of the African country has a logarithmic utility function with a constant rate of time preference γ. Also, let the ratio of the government's instantaneous import bill to the external debt be λ. Thus, λ = m/W. With these equations, we have:

$$U(m,t) = e^{-\gamma t}\log(m) \tag{67}$$

$$m^*(W,P,r,t) = \lambda W \tag{68}$$

Equations (62), (66), and (67) can be used to obtain an expression for the $J(\cdot)$ value function:

$$J(W,P,r,t) = -\left(\frac{1}{\lambda}\right)e^{-\gamma t}\log(W) + \Gamma(P,r,t) \tag{69}$$

where $\Gamma(\cdot)$ is a function of the underlying state variables in the economy other than W.

Applying Equation (68), the optimum proportions of the total external debt in the form of conventional debt and commodity-linked bonds are expressed as:

$$\omega_1^* = \left[\frac{(\alpha_h-\alpha_q)-(\frac{c}{Q}-\frac{c^B}{H})}{(\sigma_q-\psi_r)^2-2\rho_{pr}\psi_p(\sigma_q-\psi_r)+\psi_p^2}\right] \tag{70}$$

and

$$\omega_2^* = 1-\left[\frac{(\alpha_h-\alpha_q)-(\frac{c}{Q}-\frac{c^B}{H})}{(\sigma_q-\psi_r)^2-2\rho_{pr}\psi_p(\sigma_q-\psi_r)+\psi_p^2}\right] \tag{71}$$

From Equations (69) and (70) it can be seen that the optimal proportions of the total external debt raised in commodity-linked bonds and conventional debt depend on the spread between the total returns (capital gains and coupon payments) of both bonds, adjusted by the riskiness of the portfolio.[12] The results accord with the literature on capital asset pricing. It can also be seen that the proportions respond positively to the debt's own total return and negatively to the return of the alternative debt instrument. Note that ω_1 and ω_2 would have to be non-negative, because a country cannot sell short its own debts. As in Atta-Mensah (1992) and Merton (1971), Equation (70) provides a rule of thumb that could be followed by an African country in its investment decisions. For example, the rule suggests that an African country should hold a larger share of commodity-linked bonds in its external debt portfolio whenever the variance of the portfolio (denominator of Equation (70)) is greater than twice the spread between the expected total return of the conventional debt and the commodity-linked bond (the numerator of Equation (70) with second term close to zero). This rule is obtained by setting $\omega_1^* < 0.5$ (Equation (70)).

The thrust of the paper is a call on African countries to consider issuing bonds linked to commodities their countries are endowed with instead of conventional or traditional bonds. There are a number of positive economic implications to Africa for issuance of these bonds per the results of the paper.

First, commodity linked bonds could potentially stabilize the debt of a country. Brooke et al. (2013) note that a country's debt-to-GDP ratio is impacted by two fundamental shocks: i) government spending shocks, emanating from shocks to the structural primary balance and interest payments; and ii) growth shocks, which comes from GDP growth. Given that commodity-linked bond compensate creditors with returns which varies with debtor country's nominal value of commodities, and by extension nominal GDP, the commodity-linked bonds can reduce the risks faced by a country from

[12]Riskiness is measured here as the correlation between the export price and the Libor rate, and the variances of the prices of the debt instruments.

growth shocks. Hence commodity-linked bonds provide a form of recession insurance to the issuer-country, and reduces the risk that growth shocks will push a sovereign into default.

Second, commodity-linked bonds, compared with conventional debt, can also increase a country's capacity to maintain higher debt levels without coming under market pressure. This is because as a country's probability of default increases with the rise in the level of debt or need for debt restructuring. Consequently, the yield demanded by creditors to hold sovereign debt rises. The size of this credit spread will depend on the size of potential shocks to the debt-to-GDP ratio. In other words, the spread will depend on the probability that the shock could push the country into default. As explained in Privolos and Duncan (1991) and Brooke et al. (2013), the debt-to-GDP ratio is much less volatile for countries with commodity-linked bonds than conventional debt. Therefore, at any given debt level, the probability that a sovereign will breach its debt limit is lower for commodity-linked bonds than that for conventional bonds. This implies a lower credit spread at any given debt level. Furthermore, and as noted above, the automatic stabilisation provided by commodity-linked bonds reduces the likelihood that a country would need to embark on debilitating fiscal consolidation in the depths of a crisis in order to control debt dynamics. Hence commodity-linked bonds has the effect of raising the debt ceiling or provide fiscal space.

Third, by issuing commodity-indexed bonds, African governments and corporations that need investment funds could share the appreciating market value of underlying commodities with bondholders in return for a lower coupon rate. Furthermore, the issuing of commodity-linked bonds offers an opportunity for African commodity-producing issuers and international commodity organizations to borrow at below-market interest rates (Budd, 1983). Through this process, African countries could place themselves in an advantageous position by being linked to the international markets, such as the U.S. commodity markets and Eurobond markets.

Fourth, African countries with a higher chance of defaulting on the final payment of a bond, because of serious balance-of-payment problems, could minimize the probability of default by asking for higher coupon payments during the life of the bond, in exchange for paying the minimum of the bond's face value and the monetary value of a pre-specified unit of the commodity indexed to the bond. The default probability is reduced because the contractual debt payments are reduced in precisely those circumstances when balance-of-payments problems occur. Also, under this arrangement the maximum the issuer would pay on the maturity date is the face value.

Fifth, the African countries could, through the issue of bonds linked to their main exports, hedge against fluctuations in their export earnings. Myers and Thompson (1989) note that the debt crisis faced by the African countries in the past were due to a fall in export revenues and a simultaneous rise in world interest rates and debt-service payments. Myers and Thompson (1989) argue that, if the African Countries' debt had been issued in the form of commodity-linked bonds, then the debt-service payment of the African countries would have declined along with export prices (or export revenues), thus lightening their debt load. Those who oppose African countries issuing commodity-linked bonds suggest that African countries should use the futures market to control for commodity price risk. Regulators of the futures markets, however, impose limits on the movements of the futures price in a single day. Thus, futures prices cannot move quickly to accommodate new information. Such limits are not in place for commodity options; therefore, commodity-linked bonds, which are a combination of straight bonds and commodity options, would react to new information to form the equilibrium price. Another advantage of commodity-linked

bonds over futures contracts is that futures contracts have a maturity of less than one year and exist for a limited number of commodities. By issuing commodity-linked bonds, African countries can have longer-term maturity and also index the bonds to any commodity of their choice.

Sixth, the issuance of commodity-linked bonds minimizes the default risk faced by financiers of African country loans. A way still must be found, however, to reach the necessary collateral arrangements between African countries and the developed nations that are major holders of the bond. One way is a legal contract between the African countries and investing nations such that holders of a commodity-linked bond are empowered to seize any proceeds from the African countries' exports in any of the signatory countries in the case of default. The drawback is that such a contract is not enforceable, and enormous transactions costs would have to be incurred to settle a dispute between an African country and a bearer of the bond. Kletzer and Wright (2000), however, demonstrate that, in the presence of credible punishment threats, sovereign borrowers would always choose to renegotiate an existing loan contract rather than default.

Seventh, the use of commodity-linked bonds for external financing would minimize the enormous transactions costs that would be incurred if African countries were to dynamically hedge their export revenues with futures contracts. In this paper, the model of Myers and Thompson (1989) is extended to determine the optimal proportion of total external debt that must be issued by an African country in the form of commodity-linked bonds.

Eighth, in a world of inflation, and given the general uncertainties in the markets, the availability of the commodity, indexed to the bonds, greatly reduces the default risk of the bonds. Hence, issuers of the bonds must maintain a threshold level of inventory similar to what banks hold as reserve requirements. Moreover, issuers of the bonds who do not have the commodity must back the bonds with a long position in the forward or futures contracts, whose maturity is timed with the redemption date of the bonds.

Ninth, the advantage of issuing commodities backed bonds is that the commodity acts as a hedge and therefore the probability of default risk reduces because debt service payments depends on price movement of the commodity. Furthermore, commodity-linked bonds could also be a hedge against inflation because commodity prices are leading indicator of inflation. This is due to the fact that commodity prices respond quickly to economic shocks. In addition, changes in the general price level correspond with systemic shocks such as natural disasters which disrupt the supply chain of agricultural products and subsequently increase supply costs, leading to inflation. The strongest case for commodity prices as a leading indicator of expected inflation is that commodities respond quickly to widespread economic shocks. Commodity price movement also affect the price of currencies because exports significantly contribute to economic growth, making exchange rates correlated with commodity prices. The relationships between trade, exchange rates and inflation make commodity linked bond a good hedging financial instrument.

Tenth, faced with commodity price uncertainty, one may suggest that countries holding conventional bonds could protect themselves against commodity risk by hedging in forward markets or other similar markets if they can have access to the markets or other markets such as futures, options, swaps. To begin with, most African countries do not have access to these markets. Beside the difficulties to access markets, countries financing their development through a combination of straight debt and forward sales, may not reduce sovereign risk as there are costs to rolling over hedging instruments, which have short-terms to maturity to be aligned with the longer terms to maturity of conventional debts. Hence

a strategy of a combination of conventional debt and a hedging instrument carries two credit risks, one associated with the debt and the other associated with hedging instruments. A commodity linked bond has only one credit risk as it combines a conventional debt and an option on the commodity at a strike price higher than the price of the commodity in the forward market.

The issuance of the commodity linked bonds by African countries could be fraught with some challenges. To allay the fears of investors, multilateral institutions could be called upon to provide risk insurance and credit enhancement guarantees at the smallest premium for any residual sovereign risk due the issuance the commodity-linked bond. Furthermore, to assist African countries with the issuance of the commodity linked bonds, development partners could support African countries in building capacities to enhance knowledge of African experts in the areas of financial engineering, state contingent debt instruments, risk management, bond brokerage, debt restructuring and other areas that will enhance the capacity of the countries to issue commodity backed instruments.

5. Conclusions

The issuing of commodity-linked bonds would provide an opportunity for commodity-producing African countries to tie their borrowing needs to an endowed resource. By issuing bonds indexed to their main export commodity, African countries could hedge against fluctuations in their export earnings and at the same time lessen the probability of defaulting on their external debt obligation.

Results indicate that the value of the commodity-linked bonds increases as the price of the commodity indexed to the bonds rises. This suggests that, if African countries had issued debt contracts that were tied to their main export commodities, then their debt loads would have declined along with plummeting export prices (or export revenues). This paper has also demonstrated that the coupon rate for a commodity-linked bond is less than its counterpart for a conventional debt instrument, if African countries share, on maturity, the appreciation in the commodity price with the bearer. The issuance of such bonds offers an opportunity for commodity-producing issuers and international commodity organizations to borrow at below-market interest rates.

On the other hand, African countries could issue a bond whose terminal payoff is the lesser of the face value and the monetary value of a pre-specified unit of a commodity. The coupon rate for this type of bond would have to be larger than that for a conventional bond, because investors would have to be compensated for accepting the prescribed terminal payoff. The importance of these types of bonds is that they act as a hedge for African countries against plummeting commodity prices.

Finally, using portfolio theory, a simple rule was derived for an African country to follow in its allocation of debt instruments and the level of imports. The rule suggests that an African country should hold a larger share of commodity-linked bonds in its external debt portfolio than that of a conventional debt whenever the variance of the portfolio is greater than twice the spread between the expected total return of the conventional debt and the commodity-linked bond.

Like most economic models, there are limitations to this model. The viability of a commodity-linked bond market cannot be guaranteed by simply letting risk-prone speculators issue these bonds to risk-averse hedgers. Hence, the commodity-linked bond market must be commercially guided and participants must be major market makers, such as corporations and governments. To reduce default risk, the issuers of the bonds must maintain a threshold level of inventory, similar to what banks hold as

reserves. Furthermore, issuers that do not have the commodity must back the bonds with a long position in the forward or futures contracts, whose maturity is timed with the redemption date of the bonds.

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Conflict of interest

The author declares no conflict of interest.

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