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*Research article*

# An application of Regular Vine copula in portfolio risk forecasting: evidence from Istanbul stock exchange

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**Abstract:** In times of financial turbulence, it is a well-documented fact that the co-movement of financial returns tends to increase leading to unexpected portfolio losses. The magnitude of the losses can severely be underestimated when the characteristics of univariate return series and the dependence structure between the returns are not represented well by a risk forecasting model. From a growing literature on the available multivariate modelling tools, this paper aims to investigate daily portfolio Value at Risk and Expected Shortfall forecasting performance of elliptical as well as Regular Vine copulas. For this purpose, return series of twelve stocks that are listed in Istanbul Stock Exchange are obtained for the period of June 2010 to December 2018. The series are modelled with univariate Generalized Auto-Regressive Conditional Heteroskedasticity models with Normal and Student's t innovations. Equally weighted portfolio returns are forecasted depending on the univariate GARCH marginals and their multivariate dependence structure. Estimated daily portfolio Value at Risk and Expected Shortfall values with varying levels are compared with the traditional Variance-Covariance and Dynamic Conditional Correlation Multivariate GARCH model estimates. While the models performed well at the Value at Risk backtests, according to the applied ES backtests R-vine copula GARCH found better at yielding more accurate Expected Shortfall forecasts.

**Keywords:** copula theory; Regular Vine copula; value at risk; expected shortfall; risk management

**JEL Codes:** C13, C15, C52, G17

## 1. Introduction

Due to growing financial integration between international markets and severe financial crises witnessed on the last century, a greater emphasis is placed on financial risk management, especially on risk management tools that are able to correctly specify the magnitude of portfolio losses. One of the standard tools applied to measure market risk of a financial asset or portfolio is Value at Risk (VaR).

VaR is defined as the maximum loss amount of a portfolio in a certain time and at a given confidence level. The identification of an accurate VaR estimation method is a challenging research area since the standardized portfolio returns are not normally distributed with extreme values that cannot be modelled with linear dependence measures. When VaR is estimated for a portfolio, parametric estimation based on the assumptions of multivariate normal or Student's t-distributed returns is a common application leading to not correctly specified VaR measures if the assumptions do not hold.

Furthermore, it is shown that the correlation between financial returns tends to increase in times of financial turbulence compared to the other times, indicating a stronger lower tail dependence between the returns, see Ang and Chen (2002). Correlation coefficient is a measure of linear dependence and has the multivariate normal or elliptic distribution assumptions. Additionally, it does not have the property of being invariant under non-linear strictly increasing transformations. These assumptions and properties of the correlation coefficient are heavily criticized (Embrechts et al., 2002).

Additional to the issues related to multivariate normality and linear dependence in estimating portfolio VaR, it is also criticized for not being a coherent risk measure (Artzner et al., 1999). VaR violates the diversification principle of portfolio construction by not being sub-additive. Acerbi and Tasche (2002) proposed a risk measure as an extension of VaR named as *Expected Shortfall* (ES). ES is defined as the expected value or the average of portfolio losses exceeding VaR in a certain time and at a given confidence level. ES is sub-additive and takes into account losses below a specified quantile.

On the other hand, *Copulas* (Sklar, 1959), that are the functions linking multivariate distributions to their univariate marginals, are suggested for defining nonlinear and tail dependent multivariate dependence structures. Copula functions are able to derive multivariate joint distributions with the given fixed marginals even if the marginals belong to varying distributional families. Moreover, copula functions are invariant under non-linear strictly increasing transformations and are able to represent more sophisticated dependence structures (Nelsen, 2006). Nevertheless, most of the copula families are bivariate restricting their application on portfolios with many assets. Mainly, Normal and Student-t copulas from elliptical copulas family and copulas from the Archimedean family are used to model the dependence structure between the assets of a portfolio in dimensions higher than two. At the same time, the application of Normal copula in inferring tail related risks is found insufficient since Normal copula has a zero-tail dependence coefficient. Student-t copula is criticized for having only one degrees of freedom parameter and symmetric tail dependence. Moreover, Archimedean copulas are also found insufficient by trying to model complicated dependence structures with only one parameter. Originally proposed by Joe (1997), Nested Archimedean copulas (NAC) made an improvement over symmetric and exchangeable Archimedean copulas by allowing asymmetry in high dimensional dependency modelling. On the other hand, NAC is able to model a $d$-dimensional dependence with only $d$-$1$ number of bivariate Archimedean copulas including strong restrictions on copula parameters. As a result, NAC is also criticized for not being flexible enough in modelling varying types of multivariate dependencies (Aas and Berg, 2009).

One of the another recently developed approaches, first proposed by Joe (1996), later developed by Bedford and Cooke (2001, 2002), Kurowicka and Cooke (2006) and Aas et al. (2009), *Regular Vine* (R-vine) copulas are very flexible multivariate dependency modelling tools that are constructed by using only bivariate copulas and can model highly complicated dependence structures even in high dimensions. As a result, in this paper R-vine copulas are applied to model the dependence structure between the assets of a 12-dimensional portfolio additional to Normal and Student-t copulas. Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) processes (Engle, 1982; Bollerslev, 1986) are employed to model time-varying volatility, volatility clustering and fat tail properties of univariate stock returns and combined with copula functions to capture the dependence between twelve stocks of an equally weighted portfolio to generate portfolio returns without a prior multivariate joint distribution assumption. From the forecasted daily portfolio returns, daily portfolio VaR and ES values are estimated for varying levels. In order to evaluate the performance of daily portfolio VaR and ES forecasts, VaR backtests of Kupiec (1995) and Christoffersen (1998) and ES backtests of McNeil and Frey (2000) and Embrechts et al. (2005) are applied.

The contribution of this paper is threefold. First, compared to the univariate modelling, there are fewer papers assessing portfolio VaR and ES forecasting performance of various methods other than traditional Variance-Covariance and Historical Simulation. Second, portfolio VaR and ES forecasts of R-vine copulas are generally compared with C- and/or D-vines that are already a special type of R-vine. This paper employs four types of statistical multivariate modelling tools; Dynamic Conditional Correlation Multivariate GARCH model (DCC-GARCH), Normal copula, Student-t copula and R-vine copula to compare their portfolio VaR and ES forecasting performance. Third, to the best of our knowledge, this is the first paper comparing daily portfolio VaR and ES forecasting performance of R-vine copulas with other multivariate copula and DCC-GARCH models for the stocks listed in Istanbul Stock Exchange.

The rest of the paper is organized as follows. Section 2 gives an overview of the studies on R-vines. Section 3 introduces the data and the methodology employed including the marginal models, copula functions, multivariate DCC-GARCH model, R-vine copulas and the estimations of VaR and ES and their backtesting procedures. Section 4 presents the empirical procedures and the results. The paper is concluded in Section 5.

## 2. Literature review

In empirical finance, copulas have been mainly used in modelling dependence between asset returns, risk management, asset and derivative pricing since mid-1990s, forty years later from the introduction by Sklar (1959). For example, Li (2000) employed copula functions to model default risks of financial instruments in a portfolio. Patton (2001) used copulas to define time-varying joint distributions of two exchange rate returns. Rockinger and Jondeau (2001) applied copula functions to model conditional dependence of five major international stock indices. Cherubini and Luciano (2001) used copula functions (Archimedean copulas) to evaluate tail probabilities, market risk (VaR) trade-offs and capital allocation of two stock market indices. For an overview on the applications of copulas on economic forecasting see also Patton (2012). Messaoud and Aloui (2015) estimated portfolio Value at Risk and Conditional Value at Risk by combining GJR-GARCH model with Extreme Value Theory and varying copula functions. Hung (2019) investigated the dependence structure between crude oil and three US dollar exchange rates of China, India and South Korea with copula-based GARCH models.

Even though they are recently developed statistical tools, that are first introduced by Joe (1996), later developed by Bedford and Cooke (2001, 2002) and Kurowicka and Cooke (2006), there is a growing interest in the theory and application of R-vine copulas. Following the work of Aas et al. (2009), their application can be found in varying scientific disciplines, such as finance, hydrology and biology. For example, Vaz de Melo Mendes et al. (2010) provided an application of a six-dimensional D-vine copula in selecting optimal portfolios, constructing efficient frontiers and estimating portfolio value-at-risk. Nikoloulopoulos et al. (2012) used C- and D-vine copulas on a data composed of five European stock indices. Vine copulas are constructed solely from the families of BB1, BB7 and Student-t copulas for all tree levels of the vines in order to assess whether the fit of the C- and D-vines obtained from bivariate Student-t copulas are improved. Authors compared goodness-of-fit of the Vine copulas not by only their log-likelihood and AIC values, but by also their extreme quantile (daily VaR) forecasts. Righi and Ceretta (2012) applied D-vine copulas to estimate the dependence structure of three different markets (developed, Latin emerging and Asia-pacific emerging) and obtained their daily portfolio VaR forecasts.

Until the work of Dißmann et al. (2013), the use of R-vines in finance were mainly focused on their two special types; Canonical Vines (C-vines) and Drawable Vines (D-vines) due to the difficulty of selecting and estimating a regular vine in high dimensions. Dißmann et al. (2013) introduced an algorithm searching for an optimal regular vine tree structure depending on the pairwise dependence of variables in terms of absolute kendall's tau and sequentially selects the ones with the maximum absolute dependence. The authors applied the suggested algorithm on a 16-dimensional data composed of international financial indices and showed the more flexible dependency modelling ability of R-vines compared to C- and D-vines.

Brechmann et al. (2013) applied C-vine copulas to construct conditional dependence structure of credit default swap spreads of 38 financial institutions with a systemic risk stress testing exercise. Zhang et al. (2014) employed Canonical (C-vine), Drawable (D-vine) and Regular Vine (R-vine) copulas to model the dependence structure and forecast daily portfolio VaR and ES values of ten international stock indices. Weiß and Scheffer (2015) suggested to use mixture pair-copula constructions in a Vine model. Authors compared the suggested model in terms of portfolio VaR forecasting accuracy with a benchmark Vine model constructed from parametric pair copulas that are selected according to AIC criteria. One can also consult to Aas (2016) for a literature review on the applications of Vine copulas in finance.

Furthermore, Allen et al. (2017) used R-vine and C-vine copulas in a static approach to model the co-dependencies of eleven European indices composed of ten major European market indices and one composite European index for three different periods. Authors also constructed an equally weighted portfolio in order to assess the portfolio VaR forecasting performance of R- and C-vine copulas compared to the forecasts obtained from the simulations of univariate GARCH(1,1) processes. Müller and Righi (2018) compared portfolio risk forecasting performance of Historical Simulation, DCC-GARCH of Engle (2002) and multivariate copula models (Regular, Vine and Nested Archimedean copulas) using a data simulated from AR(1)-GARCH(1,1) process with multivariate Normal and Student's t marginal distribution assumptions. From the more recent papers, Allevi et al. (2019) employed C-vine, D-vine and R-vine copulas in the optimal asset allocation of seven underlying assets of a long-term natural gas contract with respect to five different risk measures. Acar et al. (2019) suggested dynamic vine copula models with a local likelihood estimator based on kernel smoothing in order to be to able capture highly nonlinear time-varying dependence patterns. Tofoli et al. (2019) used dynamic D-vine copula to model the dependence structure between five international market indices for the crises and non-crises periods and compared VaR forecasting performance of dynamic and static D-vine copulas. Boako et al. (2019) modelled dependence structure of six cryptocurrencies (Bitcoin and five other altcoins) with C- and R-vine copulas and estimated daily equally weighted portfolio VaR using a rolling-windows approach. Nagler et al. (2019a) introduced a pair-copula selection method called "modified Bayesian Information Criterion for Vines (mBICV)" that is specific to sparse Vine copulas. The authors tested the proposed method in a high dimensional portfolio VaR forecasting task. According to the results, it is argued that mBICV selection criteria leads to more efficient sparse models with valid portfolio VaR forecasts.

## 3. Data and methodology

This paper combines copula functions with symmetric univariate GARCH processes in order to forecast daily portfolio VaR and ES risk measures and applies risk measure specific back-testing procedures. A brief introduction to the data and applied models is given on the following subsections.

### *3.1. Data*

This research uses a data of 2160 daily closing prices of twelve stocks listed in BIST30 Index of Istanbul Stock Exchange between the period of June 2010 and December 2018. BIST30 index lists the top thirty stocks of Istanbul Stock Exchange with the biggest market value and the highest transaction volume in each quarter. The index is updated four times in a year and its composition changes depending on the stocks' market value and transaction volume. As a result, only the stocks that are continuously listed in BIST30 index during the June 2010 and December 2018 research period and that can be a good proxy of the index performance are considered for this research. The data is obtained from Istanbul Stock Exchange and the stock tickers with their industry specific company information is summarized in Table 1. Daily stock returns are estimated as:

$$r_t = \ln(P_t/P_{t-1}) * 100 \tag{1}$$

where $r_t$ is the daily stock return, $P_t$ is the daily closing price of a stock at time t. Descriptive statistics for the in-sample period of June 2010 – January 2012 are given in Table 2. Mean values of the daily logarithmic returns are not significantly different from zero at 95% significance level. Returns are skewed with excess kurtosis suggesting fatter tails than standard normal distribution. According to Jarque-Bera tests for normality, return series are not normally distributed. Applied ADF (Dickey and Fuller, 1979) unit root test confirmed the stationarity property of the series.

**Table 1.** Stock ticker, company and industry classification.

| Stock Ticker | Company | Sector | Stock Ticker | Company | Sector |
|---|---|---|---|---|---|
| AKBNK | Akbank | Financial | SAHOL | Sabancı Holding | Holding |
| ARCLK | Arçelik | Industrial | SISE | Şişe Cam | Industrial |
| GARAN | Garanti Bankası | Financial | TUPRS | Tüpraş | Energy |
| ISCTR | İş Bankası | Financial | TTKOM | Türk Telekom | Telecom |
| KCHOL | Koç Holding | Holding | VAKBN | Vakıflar Bankası | Financial |
| KRDMD | Kardemir | Industrial | YKBNK | Yapi ve Kredi Bankası | Financial |

### *3.2. Univariate GARCH Marginals*

Today, it is a well-known fact that financial return series exhibit auto-correlation and volatility clustering but back time, it was Engle (1982) who first introduced finance literature with the Auto-Regressive Conditional Heteroskedasticity (ARCH) Model. Even though, the first application of the

**Table 2.** Descriptive statistics of scaled [100x] daily logarithmic stock returns / June 2010 – January 2012.

| Stock | Mean | Median | Min | Max | SD | Skew. | Kurt. | JB | ADF stat. | LB(10) | LM(7) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| AKBNK | −0.05 | 0.00 | −9.53 | 7.00 | 2.26 | −0.36 | 4.20 | 0.00 | −8.12 ** | 0.55 | 0.00 |
| ARCLK | −0.01 | 0.00 | −8.73 | 5.70 | 1.98 | −0.40 | 4.19 | 0.00 | −7.37 ** | 0.09 | 0.03 |
| GARAN | −0.02 | 0.00 | −7.76 | 7.43 | 2.27 | −0.28 | 3.33 | 0.02 | −8.21 ** | 0.30 | 0.01 |
| ISCTR | −0.08 | 0.00 | −8.36 | 6.90 | 2.14 | −0.33 | 4.33 | 0.00 | −7.81 ** | 0.97 | 0.00 |
| KCHOL | 0.02 | 0.00 | −7.82 | 8.59 | 2.11 | −0.18 | 3.97 | 0.00 | −8.97 ** | 0.06 | 0.03 |
| KRDMD | 0.06 | 0.00 | −8.59 | 7.10 | 2.06 | −0.04 | 4.38 | 0.00 | −7.50 ** | 0.48 | 0.05 |
| SAHOL | −0.03 | 0.00 | −8.73 | 10.46 | 2.17 | 0.26 | 5.85 | 0.00 | −8.51 ** | 0.40 | 0.00 |
| SISE | 0.12 | 0.12 | −11.47 | 6.96 | 2.36 | −0.61 | 5.21 | 0.00 | −6.51 ** | 0.60 | 0.02 |
| TUPRS | 0.09 | 0.00 | −8.19 | 9.31 | 2.21 | −0.03 | 4.25 | 0.00 | −8.41 ** | 0.14 | 0.00 |
| TTKOM | 0.10 | 0.00 | −12.28 | 9.62 | 2.04 | −0.29 | 8.83 | 0.00 | −8.20 ** | 0.51 | 0.00 |
| VAKBN | −0.07 | 0.00 | −8.90 | 6.67 | 2.21 | −0.35 | 3.82 | 0.00 | −8.07 ** | 0.57 | 0.00 |
| YKBNK | −0.09 | 0.00 | −10.40 | 7.47 | 2.40 | −0.53 | 4.78 | 0.00 | −8.02 ** | 0.48 | 0.00 |

Notes: ** Indicates rejection of the null hypothesis at 5%. p values of the estimated Jarque-Bera (JB), Ljung-Box (LB) (Ljung and Box, 1978) and ARCH LM (LM) (Engle, 1984) statistics are reported.

model was on macroeconomic data, it became extremely popular in Finance. Engle (1982) used lagged values of squared error terms to model variance, as a result variance for the first time was conditioned on past. Later, Bollerslev (1986) extended the model by including lagged values of variance itself into the equation allowing for a more flexible modelling structure. Since then, ARCH and Generalized ARCH (GARCH) models are the tools commonly applied in finance to capture volatility clustering and fat tail properties of financial series.

ARCH(1) model of Engle (1982) is defined as:

$$\varepsilon_t = z_t\sigma_t, \quad \varepsilon_t|\Omega_{t-1} \sim D(0, \sigma_t^2) \tag{2}$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 \tag{3}$$

where $\varepsilon_t$ is the error term, $\Omega_{t-1}$ is the information set (the $\sigma$-field) generated by $\varepsilon_t$ and D is the distribution. $z_t$ is an independent and identically distributed (i.i.d.) process with zero mean and unit variance. $\sigma_t^2$ is the variance conditioned on the lagged value of the squared error term. The extension of the ARCH(1) model to higher orders is straightforward. ARCH(q) model with parameter positivity constraints of $\omega > 0,\ \alpha \geq 0$ is defined as:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i\varepsilon_{t-i}^2 \tag{4}$$

The initial GARCH(p,q) model as proposed by Bollerslev (1986) is:

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \beta_j\sigma_{t-j}^2 + \sum_{i=1}^{q} \alpha_i\varepsilon_{t-i}^2 \tag{5}$$

GARCH(p,q) model has the parameter constraints of $\omega > 0 \ \wedge \ \alpha_i \geq 0 \ \wedge \ \beta_j \geq 0$ for $i = 1,\ldots,q$ and $j = 1,\ldots,p$ so that the conditional variance is strictly positive. The stationary condition for GARCH(1,1) process is $\alpha_1 + \beta_1 < 1$.

In this paper, univariate marginals are assumed to follow either the first window fitted GARCH(p,q) (with zero mean) process introduced in the previous paragraphs or the following ARMA(1,1)-GARCH(p,q) specification:

$$r_{d,t} = \mu_d + \phi_d(r_{d,t-1} - \mu_d) + \theta_d \varepsilon_{d,t-1} + \varepsilon_{d,t} \tag{6}$$

$$\varepsilon_{d,t} = z_{d,t}\sigma_{d,t} \;\wedge\; \sigma^2_{d,t} = \omega_d + \sum_{j=1}^{p} \beta_{d,j}\sigma^2_{d,t-j} + \sum_{i=1}^{q} \alpha_{d,i}\varepsilon^2_{d,t-i} \tag{7}$$

where $r_{d,t}$ is the return of stock $d$ at time $t$ for $d = 1, 2, ..., 12$. $\mu_d$ is the mean of return $d$ and assumed to be zero since the mean of the stock returns are not significantly different than zero in 95% significance level. $\phi_d$ and $\theta_d$ are the stock specific parameters of the conditional mean processes.

### 3.3. Dynamic Conditional Correlation Multivariate GARCH Model

Dynamic Conditional Correlation Multivariate GARCH model is defined by Engle (2002) as a two-stage model with a purpose to estimate large time-varying covariance matrices. On the first stage of the multivariate DCC-GARCH model, univariate GARCH processes are used to obtain conditional variances and on the second stage multivariate conditional covariance matrices are estimated. If $r_t$ is multivariate normally distributed return series stemming from $d$ number of assets:

$$r_t|\Omega_{t-1} \sim \mathrm{N}(0, H_t) \tag{8}$$

$$H_t = D_t R_t D_t, \quad D_t = diag\left\{\sqrt{\sigma_{d,t}}\right\} \tag{9}$$

where $H_t$ is the conditional covariance matrix, $R_t$ is the time varying correlation matrix and $D_t$ is the $d \times d$ diagonal matrix of time varying standard deviations obtained from the univariate GARCH models. The time varying correlation matrix $R_t$ is defined as:

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha\epsilon_{t-1}\epsilon'_{t-1} + \beta Q_{t-1} \tag{10}$$

$$R_t = diag\{Q_t\}^{-1} Q_t \, diag\{Q_t\}^{-1} \tag{11}$$

where $\alpha + \beta < 1$, $\bar{Q}$ is the unconditional covariance matrix of standardized residuals obtained from the first stage univariate GARCH models and $Q_t$ is the conditional covariance matrix.

### 3.4. Copula functions

A *copula* is defined as a multivariate cumulative distribution function (cdf) with uniform univariate marginal distributions (Nelsen, 2006). Formal definition of copula functions is due to Sklar's Theorem (Sklar, 1959): Let $X = (X_1, X_2, ..., X_d)$ be a $d$-dimensional random vector and $F(x) = \mathbb{P}(X_1 \leq x_1, ..., X_d \leq x_d)$, $x \in \mathbb{R}^d$ is its distribution function. Every $d$-dimensional cumulative distribution function $F$ can be decomposed into a $d$-dimensional C copula and univariate marginal distribution functions $F_1, F_2, ..., F_d$.

$$F(x) = C(F_1(x_1), ...., F_d(x_d)), \quad x \in \mathbb{R}^d \tag{12}$$

Thus, a copula function C maps the univariate marginals $F_1, F_2, ..., F_d$ to the joint distribution F and contains the information about the dependence structure of the marginals. On the other hand, when a $d$-dimensional C copula and univariate marginal distribution functions $F_1, F_2, ..., F_d$ are given, a $d$-dimensional multivariate distribution function F can be defined as in Equation 12. If the univariate margins $F_1, F_2, ..., F_d$ are continuous, then the defined copula C is unique.

A $d$-dimensional C copula: $[0,1]^d \to [0,1]$ for the univariate distribution functions $F_1, F_2, ..., F_d$ : $\mathbb{R} \to [0,1]$ is defined as:

$$C(u_1, u_2, ..., u_d) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), ...., F_d^{-1}(u_d)) \tag{13}$$

where $F_i(x_i) = u_i$, $i = 1, 2, ..., d$ and $F_1^{-1}$, $F_2^{-1}$, ..., $F_d^{-1}$ are the inverse probability transformations of $F_1$, $F_2, ..., F_d$.

### 3.4.1. Normal and Student-t copulas

In this paper, twelve dimensional Normal (Gaussian) and Student-t copulas are used to model the dependence structure between the return series. Student-t and Normal copulas with multivariate elliptical distributions belong to the family of *Elliptical* copulas.

Gaussian or Normal copulas are obtained from multivariate normal distributions according to the Sklar's Theorem. A $d$-dimensional Normal copula $C_{\rho,d}^{Gauss}$ with a correlation matrix $\rho$ is defined as:

$$C_{\rho,d}^{Gauss}(u_1, u_2, ..., u_d) = \phi_{\rho,d}(\phi_1^{-1}(u_1), ..., \phi_d^{-1}(u_d)) \tag{14}$$

where $\phi$ is the cumulative standard normal distribution function, $\phi^{-1}$ is the inverse of $\phi$ and $\phi_{\rho,d}$ is a $d$-dimensional multivariate standard normal distribution with a correlation matrix $\rho$. Normal copula is a radially symmetric copula with $\rho$ ranging in $[-1,1]$.

Student-t copulas, as can be deduced from the name, are obtained from multivariate Student's t distributions with $\upsilon$ degrees of freedom and are defined by:

$$C_{\rho,\upsilon,d}^{t}(u_1, u_2, ..., u_d) = T_{\rho,\upsilon,d}(T_\upsilon^{-1}(u_1), T_\upsilon^{-1}(u_2), ..., T_\upsilon^{-1}(u_d)) \tag{15}$$

where $T_\upsilon$ is a standard univariate Student's t distribution with $\upsilon$ degrees of freedom, $T_\upsilon^{-1}$ is the inverse probability transform of $T_\upsilon$ and $T_{\rho,\upsilon,d}$ is a $d$-dimensional multivariate Student's t distribution with $\upsilon$ degrees of freedom and correlation matrix $\rho$. Similar to the Normal copulas, bivariate Student-t copulas are also radially symmetric and exchangeable. However, unlike Normal copulas, they exhibit tail dependence with positive dependence coefficients.

### 3.4.2. Regular Vine copulas

Vine copulas or *pair copula constructions* (PCC) are first defined by Joe (1996) as multivariate distribution functions constructed only from pair or bivariate copulas. Bedford and Cooke (2001, 2002) contributed to the development of these flexible dependency models by providing a general framework for the bivariate copula constructions and their density. In multivariate distributions with more than two margins, the identification of unconditional bivariate and their conditioning copulas is enabled by graphical models consisting a sequence of trees. These trees are called *Regular Vines* (R-vines). According to Bedford and Cooke (2001, 2002) and Kurowicka and Cooke (2006), a regular vine on $d$ elements is a linked sequence of trees with a set of $V = \{T_1, T_2, ..., T_{d-1}\}$, nodes $N_i$ and edges $E_i$ $(1 \leq i \leq d-1)$ that must hold the following properties:

1. $T_1$ is a tree with nodes $N_1 = \{1, ..., d\}$ and with edges $E_1 = \{1, ..., d-1\}$.
2. For $i = 2, ..., d-1$, $T_i$ is a tree with nodes $N_i = E_{i-1}$ and edge set $E_i$.

3. For $i = 2, ..., d-1$, $\{a, b\} \in E_i$ with $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$, the proximity condition of $\#(a \cap b) = 1$ must hold, where $\#(a \cap b)$ is the cardinality of the set $(a \cap b)$.

A $d$-dimensional R-vine copula is constructed with a product of $d(d-1)/2$ pair copulas by sequentially associating pair copulas on each tree level with an edge, and can be expressed with $d-1$ number of trees (T). In fact, an R-vine copula is a regular vine distribution with uniform marginals and its density can be written as (Kurowicka and Cooke, 2006; Czado, 2010):

$$c_{1,...,d}(u) = \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)}(C_{j(e)|D(e)}\left(u_{j(e)}|u_{D(e)}\right),$$
$$C_{k(e)|D(e)}\left(u_{k(e)}|u_{D(e)}\right)) \tag{16}$$

where $F_i(x_i) = u_i$ for $i = 1, 2, ..., d$. $j(e)$ and $k(e)$ are the *conditioned* and $D(e)$ is the *conditioning set*. The edges $e = j(e), k(e)|D(e)$ are in $E_i$. $\mathrm{B} = \left\{c_{j(e),k(e)|D(e)} \middle| e \in E_i\right\}$ is the set of bivariate copula densities with $1 \le i \le d-1$.

Depending on their graphical representation, two specific types of R-vines are defined by Bedford and Cooke (2001, 2002). *Drawable Vine* (D-vine) has a tree structure in which each node is connected with at most 2 edges, while *Canonical Vine* (C-vine) has a star like tree structure with a root node connected to all other nodes in each tree $T_i$. Aas et al. (2009) enabled the application of C- and D-vines in finance by providing their statistical inference algorithms.

The use of C-vine might be optimal when the dependency structure between the variables has a root or key variable with the highest correlation to all other variables and is able to govern the interactions of them. D-vine has a line or path structure and the dependency between the variables is modelled by the path and the proximity condition. On the other hand, R-vine without a specific tree structure assumption can flexibly model the dependency between the variables by only depending on the statistical feature of the multivariate distribution.

### *3.5. Value at risk and Expected Shortfall*

Value at risk (VaR) is a concept developed to measure maximum loss of a portfolio for a specific time period and confidence level. More formally, let $R_t$ be the return of a portfolio at time $t$ and the confidence level $1-\alpha$ with $\alpha \in (0, 1)$. VaR of a portfolio at level $\alpha$ and time $t$ is:

$$VaR_t^\alpha = F_R^{-1}(\alpha) = inf\{r \in \mathbb{R} : F_R(r) \ge \alpha\} \tag{17}$$

where $F_R$ is the distribution function of the portfolio return at time $t$ and $F^{-1}$ is the standardized quantile of $F_R$.

On the other hand, Expected Shortfall (ES) is proposed by Acerbi and Tasche (2002) in order to overcome some of the shortcomings of VaR as a risk measure, such as not being a coherent measure. For a specific time $t$ and confidence level $1-\alpha$, ES is the average value of portfolio return smaller than portfolio $VaR_t^\alpha$ value. ES of a portfolio at level $\alpha$ can be defined as:

$$ES_t^\alpha = \frac{1}{\alpha} \int_0^\alpha F_R^{-1}(r) dr \tag{18}$$

3.5.1. Estimation methods

Daily portfolio VaR and ES values are first estimated with the classical Variance Covariance (Var-Covar) method. Portfolio returns are assumed to follow a normal distribution and one day ahead portfolio VaR and ES values are calculated as (Ruppert and Matteson, 2015; Jorion, 2000):

$$VaR_t^{\alpha} = \mu + \sigma q_{\alpha} \tag{19}$$

$$ES_t^{\alpha} = \mu - \sigma\left(\frac{d(q_{\alpha})}{\alpha}\right) \tag{20}$$

where $\mu$ is the mean and $\sigma$ is the standard deviation of the portfolio returns, $q_{\alpha}$ is the $\alpha$ quantile and $d$ is the standard normal density. One day ahead portfolio VaR and ES forecasts of DCC-GARCH model are obtained by:

$$VaR_t^{\alpha} = \mu_p + q_{\alpha}\sqrt{\omega H \omega^T} \tag{21}$$

$$ES_t^{\alpha} = \mu_p - \sqrt{\omega H \omega^T}\left(\frac{d(q_{\alpha})}{\alpha}\right) \tag{22}$$

where $\mu_p$ is the portfolio conditional mean, $\omega$ represents the vector of portfolio weights, $H$ is the one step ahead forecast of the covariance matrix, $q_{\alpha}$ is the $\alpha$ quantile and $d$ is the standard normal density.

Following the one day ahead portfolio VaR and ES estimates of Var-Covar and multivariate DCC-GARCH models, the steps given below are applied for the copula based GARCH models to obtain forecasts of the one day ahead portfolio VaR and ES:

1. $n$ = 10000 samples of pseudo-uniform variables ($u_{d,t}$) are simulated from the fitted copulas, where $d$=1,2...,12 is the stock number and $t : T + 1$ is the time of 1-day ahead VaR forecast horizon, $T$:is the last day of the model fit window.
2. Standardized residual series are obtained by converting the simulated pseudo-uniform variables to $z_{d,t,n}$ through the inverse probability transformations of the marginal distributions that can be represented as: $z_{d,t,n} = F_d^{-1}(u_{d,t,n})$, where $F_d$ is the univariate marginal distribution function of stock $d$ and $F_d^{-1}$ is quantile function associated with $F_d$. By applying probability transformations, continuous random variables are transformed to standard uniform random variables. On the other hand, applying the inverse probability or quantile transformations do the converse.
3. Once the standardized residuals are obtained, in conjunction with the conditional mean and volatility specifications, 10000 1-day ahead return forecasts of the univariate return series are obtained through: $r_{d,t,n} = \mu_{d,t,n} + \sigma_{d,t,n} z_{d,t,n}$ .
4. Based on the simulated stock returns, equally weighted 10000 portfolio returns are calculated as: $R_{t,n} = 12^{-1}\sum_{d=1}^{12} r_{d,t,n}$ .
5. From the daily distributions of the portfolio returns $\alpha$th quantile daily portfolio VaR and ES values are estimated: $VaR_t^{\alpha}(R_{t)} = F_{R_t}^{-1}(\alpha)$ where $F_{R_t}$: is the distribution function of portfolio return at time $t$, $ES_t^{\alpha} = mean\ (R_{t,n} < VaR_t^{\alpha})$.

For all of the above five estimation methods (Var-Covar, DCC-GARCH, Normal, Student-t and R-vine copula-GARCH), portfolio VaR and ES forecasts for the remaining days are obtained by one day rolling windows approach. Following the first one day ahead portfolio risk forecasts, the in-sample data is rolled forward by one day including a day from the out of sample period and leaving the earliest (first) observation out. Using the new rolled data, univariate GARCH, multivariate DCC-GARCH and copula models are re-estimated leading to a dynamic approach incorporating new information from marginals to the dependence structure. This approach is applied until all VaR and ES forecasts are obtained for the out of sample period.

3.5.2. Backtesting

In this paper, in order to evaluate the portfolio VaR forecasting performance of the univariate copula-based GARCH models, *Unconditional Coverage* (UC) test of Kupiec (1995) and *Conditional Coverage* (CC) test of Christoffersen (1998) are applied.

Kupiec (1995) suggested to use a test that compares total number of realized VaR breaches with the expected number for a chosen probability level $\alpha$. A VaR breach or exceedance is observed when a realized return is smaller than the forecasted VaR ($R_t < VaR_t^{\alpha}$) value at a given $\alpha$ level and time horizon. According to Kupiec (1995), the percent of total number of realized VaR breaches should be consistent with the chosen $\alpha$ level. The test, also known as the Proportion of Failures (POF) test and has the null hypothesis of:

$$H_0: \ p = \hat{p} = \frac{n_1}{T} \tag{23}$$

where $n_1$ is the total number of realized VaR breaches and is assumed to follow a binomial distribution. $T$ is the total number of out-of-sample observations and $p$ is the chosen $\alpha$ level. UC tests the null hypothesis that the proportion of the realized VaR breaches ($\hat{p}$) is consistent with $p$. The test statistic ($LR_{UC}$) is $\chi^2(1)$ distributed and given by:

$$LR_{UC} = -2\ln\left[(1-p)^{n_0} p^{n_1}\right] + 2\ln\left[(1-\hat{p})^{n_0}\hat{p}^{n_1}\right] \tag{24}$$

where $n_0 = T - n_1$.

Christoffersen (1998) suggested a Conditional Coverage (CC) test that takes into account both the correct coverage and the independence (probable time-dependent clusters) of VaR breaches. Similar to Kupiec (1995)'s test, a VaR breach is defined as:

$$I_t = \begin{cases} 1 & if \ \ R_t < VaR_t^{\alpha} \\ 0 & if \ \ R_t > VaR_t^{\alpha} \end{cases} \tag{25}$$

The independence likelihood ratio ($LR_{ind}$) of Conditional Coverage test is based on whether $VaR_{\alpha}$ breaches are serially independent random variables against the alternative of first-order Markov dependence.

$$LR_{ind} = -2\ln\left[\frac{(1-\pi_2)^{(n_{00}+n_{10})}\ \pi_2^{(n_{01}+n_{11})}}{\pi_{01}^{n_{01}}\ \pi_{11}^{n_{11}}\ (1-\pi_{01})^{n_{00}}\ (1-\pi_{11})^{n_{10}}}\right] \tag{26}$$

where $LR_{ind} \sim \chi^2(1)$, $\pi_2 = (n_{01}+n_{11})/(n_{00}+n_{01}+n_{10}+n_{11})$, $\pi_{01} = n_{01}/(n_{00}+n_{01})$, $\pi_{11} = n_{11}/(n_{10}+n_{11})$ and $n_{ij}$ is the number of observations with value $i$ followed by $j$. Conditional Coverage test combines $LR_{UC}$ and $LR_{ind}$ by jointly testing correct unconditional coverage and independence of VaR breaches. As a result, the null of the unconditional coverage is tested against the alternative of the independence test. The likelihood ratio is defined by ($LR_{CC} \sim \chi^2(2)$):

$$LR_{CC} = -2\ln\left[\frac{(1-p)^{n_0}\,p^{n_1}}{\pi_{01}^{n_{01}}\,\pi_{11}^{n_{11}}\,(1-\pi_{01})^{n_{00}}\,(1-\pi_{11})^{n_{10}}}\right] \tag{27}$$

For the backtests of daily portfolio ES forecasts, *Exceedance Residual Backtesting* approach of McNeil and Frey (2000) is applied. The test is based on new series ($r_{t+1}$) called exceedance residuals that are obtained from:

$$r_{t+1} = \frac{x_{t+1} - ES_t^{\alpha}}{\sigma_{t+1}}, \quad \{r_{t+1} : t \in T,\ x_{t+1} < VaR_t^{\alpha}\} \tag{28}$$

where $x_{t+1}$ is the return at time t+1 conditional on $x_{t+1}$ is smaller than $VaR_t^{\alpha}$ value at time t (the condition is $x_{t+1}$ is bigger than $VaR_t^{\alpha}$ if the upper quantile of the loss distribution is used). The exceedance residuals are tested for the null hypothesis of i.i.d. and zero mean with one-sided t-test. For a given significance level, a rejected null hypothesis indicates systematically underestimated expected shortfall forecasts.

Additional to the ES backtest of McNeil and Frey (2000), a combination of two measures as defined by Embrechts et al. (2005) are used to assess ES forecasting performance of the models. With the first measure $V_1^{ES}$, average of the differences of return series from the estimated ES values is obtained with a condition of being in excess of VaR ($R_t < \mathrm{VaR}_t^{\alpha}$). Taking into account the probability of misspecified VaR estimates, the second measure $V_2^{ES}$ considers only the average difference of the returns from ES conditional on being in excess of empirical $\alpha$-quantile of the ES differenced series.

$$V_1^{ES} = \frac{\sum (R_t - \mathrm{ES}_t^{\alpha})\, \mathbb{1}_{(R_t < \mathrm{VaR}_t^{\alpha})}}{\sum \mathbb{1}_{(R_t < \mathrm{VaR}_t^{\alpha})}} \tag{29}$$

$$V_2^{ES} = \frac{\sum (R_t - \mathrm{ES}_t^{\alpha})\, \mathbb{1}_{(D_t < D^{\alpha})}}{\sum \mathbb{1}_{(D_t < D^{\alpha})}} \tag{30}$$

where $\mathbb{1}$ is the indicator function, $D_t = R_t - ES_t^{\alpha}$ and $D^{\alpha}$ is the empirical $\alpha$-quantile of $D_t$. The average of absolute values of $V_1^{ES}$ and $V_2^{ES}$ gives the third measure $V^{ES} = (|V_1^{ES}| + |V_2^{ES}|)/2$. It is the final measure used to backtest model ES forecasts and a lower value of $V^{ES}$ indicates a better ES forecasting performance.

## 4. Empirical results

### 4.1. Marginal modelling

As mentioned previously, Univariate return series are modelled by using GARCH(1,1) and ARMA(1,1)-GARCH(1,1) processes with Normal and Student's t residual distribution assumptions. For this purpose, *rugarch* (Ghalanos, 2020) package of R software (R Core Team, 2019) is employed. Best fitting model for the first window of each return is determined according to Bayesian Information Criterion (BIC) and parameter significance values of the models. In case of persistent heteroscedasticity, GARCH orders are increased by one (see vakbn). Following the model selection, absolute returns bigger than five conditional standard deviations ($|r_t| > 5\sigma_t$) are treated as outliers and filtered as mentioned in Carnero et al. (2012) by replacing outliers with $5\sigma_t sign(r_t)$. If necessary, using the outlier filtered series, univariate conditional volatility models are refitted. While outlier filter is applied to all estimation windows, parameters of the selected volatility models for the June 2010–January 2012 sample period are given in Table 3.

**Table 3.** Univariate symmetric GARCH parameter estimates of stock returns/ June 2010 – January 2012.

| Stock Garch Fit | $\omega$ | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\nu$ | LB(10) | LM(7) |
|---|---|---|---|---|---|---|---|
| AKBNK GARCH(1,1)-norm | 0.614 | 0.130*** | - | 0.749*** | - | 0.837 | 0.821 |
| ARCLK GARCH(1,1)-norm | 0.491** | 0.124*** | - | 0.753*** | - | 0.069 | 0.363 |
| GARAN GARCH(1,1)-norm | 1.022 | 0.081* | - | 0.72*** | - | 0.356 | 0.376 |
| ISCTR GARCH(1,1)-std | 0.683* | 0.097** | - | 0.759*** | 6.8*** | 0.990 | 0.793 |
| KCHOL GARCH(1,1)-norm | 0.448 | 0.146*** | - | 0.758*** | - | 0.298 | 0.930 |
| KRDMD GARCH(1,1)-std | 0.917 | 0.102* | - | 0.679*** | 7.38*** | 0.528 | 0.950 |
| SAHOL GARCH(1,1)-std | 0.827* | 0.163** | - | 0.662*** | 6.40*** | 0.538 | 0.981 |
| SISE GARCH(1,1)-std | 0.86 | 0.191** | - | 0.661*** | 8.37** | 0.644 | 0.863 |
| TUPRS GARCH(1,1)-norm | 0.627** | 0.161*** | - | 0.712*** | - | 0.819 | 0.985 |
| TTKOM GARCH(1,1)-std | 1.07* | 0.148** | - | 0.577*** | 5.37*** | 0.567 | 0.435 |
| VAKBN GARCH(2,1)-norm | 0.892** | 0.00 | 0.163 | 0.652*** | - | 0.612 | 0.707 |
| YKBNK GARCH(1,1)-std | 0.177 | 0.087* | - | 0.891*** | 6.19*** | 0.290 | 0.441 |

Notes: p values of the estimated Ljung-Box (LB) (Ljung and Box, 1978) and ARCH LM (LM) (Engle, 1984) statistics are reported. ***, ** and * indicate significant parameter estimates for 1%, 5% and 10%, respectively. Model specific residual distribution type is shown with the -norm or -std suffixes for Normal and Student's t distributions.

Additional to the copula based GARCH(1,1), one-day ahead portfolio risk forecasting performance of copula based ARMA(1,1)-GARCH(1,1) is also evaluated. While the applied conditional mean specification for all returns is ARMA(1,1), *p,q* orders and residual distributions of the applied conditional variance specifications are the same as determined on the first window of each stock. Similar to GARCH(1,1), an outlier filter is also applied to ARMA(1,1)-GARCH(1,1). Research period, starting from June 2010 till December 2018, consisted 1760 windows with a size of 400 daily observations. While the parameters of the univariate models (GARCH(1,1) and ARMA(1,1)-GARCH(1,1)) are re-estimated in each window, only their *p,q* orders and residual distribution assumptions are kept fixed.

### *4.2. Regular Vine copula structure*

Following the marginal modelling, return specific pseudo-observations are obtained from the GARCH filtered series. Applied Kolmogorov-Smirnov test confirmed that pseudo-observations follow the uniform distribution between zero and one. First, twelve dimensional Normal and Student-t copulas are fitted to the sample and for this purpose, R package of *copula* (Hofert et al., 2020) is used. 66 parameters for Normal copula (bivariate rho estimates between the series) and 67 parameters (additional one degrees of freedom parameter to the bivariate rho estimates) for Student-t copula are estimated by using maximum pseudo-likelihood estimation (mpl). Obtained AIC values of the copula models fitted to GARCH(1,1) filtered series are given on Figure 1.

According to the model log-likelihood and AIC values, R-vine copula obtained smaller AIC values and found better at describing dependency structure of the returns for the considered time period. On the next step, with the obtained pseudo-uniform series, mixed R-vine copula structure is determined according to Dißmann et al. (2013) algorithm that uses maximum spanning trees with respect to absolute values of empirical kendall's tau. At each tree level, the algorithm selects the nodes maximizing the sum of absolute kendall's tau or in other words, selects the pairs with the strongest dependencies and sequentially constructs the vine trees. Conditional and unconditional pair-copula types are selected by AIC criteria and the parameters are estimated by using maximum likelihood estimation (mle) method. For this purpose, R package of *VineCopula* (Nagler et al., 2019b) is employed. The first tree level of the constructed R-vine structure from the GARCH(1,1) filtered series for the period of June 2010

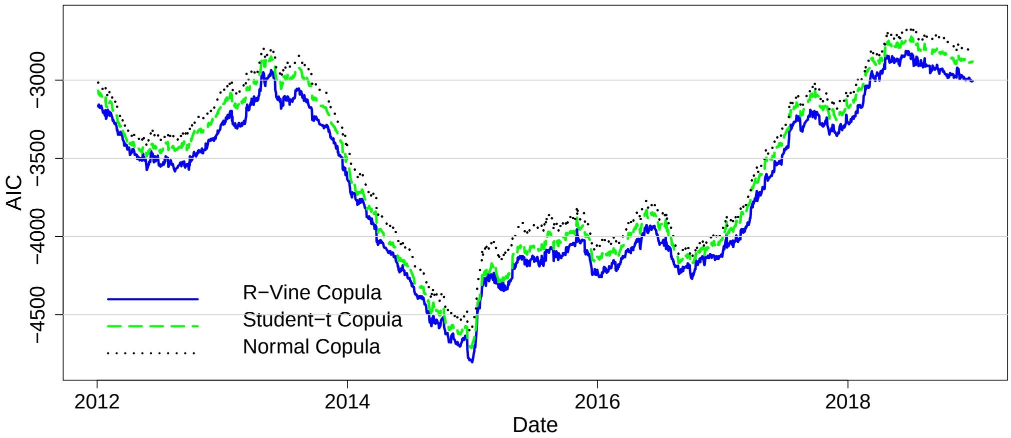


**Figure 1.** AIC values of copula models obtained in each re-estimation period.

and January 2012 is given on Figure 2. According to Figure 2, returns from the financial sector are connected at the centre of the R-vine tree with the strongest tau values. The tree also clearly shows the tendency of clustering towards highly related companies, such as sahol, akbnk and arclk, kchol, ykbnk.

### *4.3. Results*

In order to evaluate the performance of daily portfolio VaR and ES forecasts, several tests are applied for three different VaR and ES levels (5%, 2.5% and 1%). In Table 4, VaR backtest results of applied Unconditional Coverage (LRuc) test of Kupiec (1995) and Conditional Coverage (LRcc) test of Christoffersen (1998) are reported for both univariate marginals.

**Table 4.** Equally weighted portfolio VaR backtest results.

| | $VaR_{5\%}$ | | | $VaR_{2.5\%}$ | | | $VaR_{1\%}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Model | Actual Exc. | LRuc Stat. | LRcc Stat. | Actual Exc. | LRuc Stat. | LRcc Stat. | Actual Exc. | LRuc Stat. | LRcc Stat. |
| Var-Covar | 83 | 0.31 | 4.01 | 54 | 2.18 | 4.76 | 35 | 13.50* | 13.62* |
| DCC-GARCH | 77 | 1.51 | 1.63 | 44 | 0 | 0.63 | 28 | 5.26 | 5.8 |
| DCC-ARMA(1,1)-GARCH | 78 | 1.24 | 2.95 | 47 | 0.21 | 0.60 | 27 | 4.36 | 4.98 |
| NC GARCH | 86 | 0.05 | 0.23 | 48 | 0.36 | 0.7 | 26 | 3.53 | 4.24 |
| NC ARMA(1,1)-GARCH | 87 | 0.01 | 0.72 | 48 | 0.36 | 2.08 | 27 | 4.36 | 4.98 |
| StC GARCH | 87 | 0.01 | 0.72 | 47 | 0.21 | 0.6 | 24 | 2.11 | 3.04 |
| StC ARMA(1,1)-GARCH | 85 | 0.11 | 0.33 | 49 | 0.56 | 2.13 | 24 | 2.11 | 3.04 |
| R-vine GARCH | 89 | 0.01 | 1.41 | 49 | 0.56 | 2.13 | 24 | 2.11 | 3.04 |
| R-vine ARMA(1,1)-GARCH | 89 | 0.01 | 0.56 | 46 | 0.09 | 0.56 | 24 | 2.11 | 3.04 |

Notes: * indicates rejection of a model at the respective $\alpha$ level. Expected exceedances for the levels of 5%, 2.5% and 1% are 88, 44 and 17, respectively. Actual Exc. is the short form of actual exceedances. NC and StC are the abbreviations of Normal copula and Student-t copula, respectively.

According to the VaR backtest results, only parametric Var-Covar method failed to pass the tests at 1% level. All other models performed well. On the other hand, a VaR forecasting model is said to be performing better when the difference between the expected and actual exceedances is smaller. In this case, most of the copula models performed better than Var-Covar, DCC-GARCH and DCC-ARMA-GARCH at 5% level. On the other hand, R-vine and Student-t copula (ARMA)

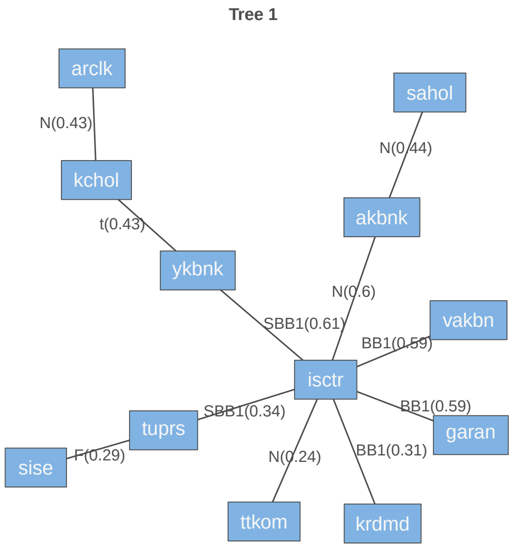


**Figure 2.** The first tree level (T1) of the first estimated R-vine structure. Stock tickers are given in each node. Each edge includes the information of fitted bivariate copula type with estimated kendall's tau. N represents Normal copula, t is Student-t copula, F is Frank copula and SBB1 is survival BB1 copula.

GARCH models clearly performed better at 1% level, while DCC-GARCH model performed better at 2.5% level.

ES backtest results of Embrechts et al. (2005) and McNeil and Frey (2000) are reported in Table 5. On the other hand, for the considered portfolio not all models performed well at ES backtests. Only R-vine copula GARCH model is not rejected at any of the three $\alpha$ levels and obtained the lowest $V^{ES}$ values at the test of Embrechts et al. (2005). As can be observed from the Table, including a conditional mean filter to the models did not have a positive effect on the model specific results either by improving the $p$ values obtained from the Exceedance Residuals test or by decreasing the $V^{ES}$ values.

**Table 5.** ES backtest results.

| Model | $ES_{5\%}$ ER | $ES_{5\%}$ $V^{ES}$ | $ES_{2.5\%}$ ER | $ES_{2.5\%}$ $V^{ES}$ | $ES_{1\%}$ ER | $ES_{1\%}$ $V^{ES}$ |
|---|---|---|---|---|---|---|
| Var-Covar | 0.00 | 0.52 | 0.00 | 0.82 | 0.00 | 1.19 |
| DCC-GARCH | 0.00 | 0.36 | 0.00 | 0.61 | **0.02** | 0.96 |
| DCC-ARMA(1,1)-GARCH | 0.00 | 0.37 | 0.00 | 0.62 | 0.01 | 1.03 |
| NC GARCH | 0.01 | 0.33 | 0.01 | 0.60 | **0.02** | 1.01 |
| NC ARMA(1,1)-GARCH | 0.01 | 0.33 | 0.01 | 0.63 | **0.01** | 1.07 |
| StC GARCH | 0.03 | 0.28 | 0.02 | 0.54 | **0.04** | 0.92 |
| StC ARMA(1,1)-GARCH | 0.02 | 0.30 | 0.02 | 0.53 | **0.03** | 0.94 |
| R-vine GARCH | **0.07** | 0.23* | **0.04** | 0.46* | **0.07** | 0.78* |
| R-vine ARMA(1,1)-GARCH | **0.07** | 0.23 | 0.02 | 0.51 | **0.05** | 0.82 |

Notes: Models that can not be rejected at the respective $\alpha$ level are shown in bold. p values of the Exceedance Residuals test (ER) (McNeil and Frey, 2000) is reported. * Shows the model with the smallest $V^{ES}$ value (Embrechts et al., 2005). NC and StC are the abbreviations of Normal copula and Student-t copula, respectively.

## 5. Conclusions

In this paper, multivariate Normal, Student-t and R-vine copulas are employed to model the co-dependence of twelve stocks listed in BIST30 index of Istanbul Stock Exchange for the purpose to obtain daily portfolio VaR and ES forecasts of an equally weighted portfolio. While there are many types of copulas for bivariate dependency modelling, in higher dimensions the available options are scarce. As a result, flexible R-vine copulas constructed only from bivariate copula functions gained a well deserved attention in finance. Even though they are not as flexible as Regular Vines, elliptical copulas are also applied frequently in multivariate dependency modelling. Furthermore, Dynamic Conditional Correlation Multivariate GARCH (DCC-GARCH) and the traditional Var-Covar methods are employed as an alternative to copula-based modelling.

The accuracy of portfolio risk forecasts of the multivariate models is assessed by various suggested backtesting procedures for three different $\alpha$ levels (5%, 2.5% and 1%). According to the results of Unconditional and Conditional Coverage backtests of Kupiec (1995) and Christoffersen (1998), portfolio VaR forecasts of the models are found to be accurate with only one model rejection at 1% level. Nevertheless, neither the VaR forecasts nor the applied backtests are sufficient to evaluate the magnitude of a portfolio's losses and the resulting optimal capital requirements, since as a risk measure VaR doesn't take into account losses below a specified level and VaR backtests consider only the number of VaR violations without their magnitude.

On the other hand, ES is not only a loss value obtained from a specific point of a return distribution. It also considers the magnitude of losses below a specified VaR level. From the applied ES backtests, Exceedance Residuals Backtesting of McNeil and Frey (2000) is a traditional ES backtest, that either accepts or rejects a model as an accurate risk estimation method. In this paper, the test rejected all models except R-vine copula-GARCH at 5% and 2.5% levels. While four of the models, other than Var-Covar, passed the test at 1% level, R-vine copula-GARCH obtained a higher p value indicating a better forecasting accuracy for the losses below 1% level. ES backtest of Embrechts et al. (2005) is a comparative backtesting approach. It ranks the models depending on their forecasting accuracy by evaluating a combination of average distances of ES forecasts from the realized losses. This approach also confirmed the better ES forecasting ability of univariate GARCH models when combined with R-vine copula functions by yielding lowest $V^{ES}$ values in any of the three alpha levels.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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