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Research article

Multi-parameter families of higher-order Bernoulli polynomials and extended Stirling numbers linked to the generalized Bessel-Struve kernel

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Abstract: In this paper, we introduce a new class of generalized Bernoulli polynomials and numbers by making use of the generalized Bessel–Struve kernel function as a generating mechanism. This combination enriches the structural and analytical characteristics of the resulting polynomial family and provides a broader extension of classical Bernoulli-type sequences. By choosing specific values of the free parameters, the classical Bernoulli polynomials appear as special cases, which confirms that the proposed framework has a unifying nature. We carry out a detailed investigation of the basic structural properties, which includes generating function representations, recurrence and differential relations, and implicit summation formulas. In addition, we extend the Stirling numbers of the second kind in connection with the new polynomial families, which adds a combinatorial dimension to the study. Explicit expressions for the higher-order generalized sequence $X_{n,\mu}^{b,c,(\alpha)}(x)$ are provided for $n = 0, 1, 2, 3$. Numerical experiments include the verification of the addition formula to machine-precision accuracy across positive, negative, and complex inputs, systematic tracking of the real zeros as the kernel parameters vary, and a graphical analysis of the root locus and complex root distribution for degrees up to $n = 20$. We hope that the results obtained here will open new directions of research in mathematical analyses and related areas of applied mathematics and theoretical physics.

Keywords: Bernoulli polynomials; generalized Bernoulli polynomials; Bessel–Struve kernel function; Stirling numbers

1. Introduction

Special polynomial sequences and their associated numbers have been widely studied over the past few decades, and contributions to this area are distributed across many branches of mathematics (see [1–5]). Further developments and related studies are available in [6–11]. Such polynomials and numbers are useful in several areas including combinatorics, the approximation theory, and mathematical physics (see [12–16]). More advances and applications have also been reported in [17–22]. In particular, Bernoulli polynomials and Bernoulli numbers occupy a central place in this theory, mainly because of their importance in generating function methods, numerical analyses, and the study of analytic functions [23–29].

On the other hand, the Bessel–Struve kernel function, which combines key features of classical Bessel and Struve functions [30–33], has received growing attention in recent years. This is largely due to its interesting analytic properties and its role in solving certain differential equations and in the integral transform theory.

In many physical and engineering applications, Bessel-type functions naturally arise in heat transfer, wave propagation, and electromagnetic boundary-value problems. The parameters introduce additional flexibility and affect the structural properties of the proposed polynomial family, as illustrated by the numerical results.

Motivated by the importance of Bernoulli polynomials (see, for example, [34–39]) on one side and the analytic richness of the Bessel–Struve kernel on the other, in this paper, we define two new families of generalized Bernoulli polynomials and numbers associated with the generalized Bessel–Struve kernel function, and we study their fundamental properties in detail.

Classical Bernoulli polynomials $\mathbb{B}_n(x)$ and Bernoulli numbers $\mathbb{B}_n$ are characterized via their exponential generating functions (see [40, 41]):

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} \mathbb{B}_n(x)\frac{t^n}{n!}, \qquad (|t| < 2\pi),$$

and

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathbb{B}_n\frac{t^n}{n!}, \qquad (|t| < 2\pi). \tag{1.1}$$

Higher-order analogues of $\mathbb{B}_n(x)$ and $\mathbb{B}_n$ are obtained through the following generating relations (see [27–29]):

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad (|t| < 2\pi,\ 1^{\alpha} = 1),$$

and

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(\alpha)}\frac{t^n}{n!}, \qquad (|t| < 2\pi,\ 1^{\alpha} = 1). \tag{1.2}$$

Upon specializing $\alpha = 1$, one recovers $\mathbb{B}_n^{(1)}(x) = \mathbb{B}_n(x)$ and $\mathbb{B}_n^{(1)}(0) = \mathbb{B}_n$.

Further extensions and refinements of Bernoulli numbers and polynomials may be found in [14, 25, 35, 42]. The central aim of the current investigation is to put forward two new generalizations of Bernoulli polynomials and numbers by incorporating the generalized Bessel–Struve kernel function into the generating mechanism.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ refer to the natural, real, and complex number systems, respectively, and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Bessel–Struve kernel function $S_{\mu}(\xi t)$, defined for $\xi_{\mu}, \mu > -\frac{1}{2}$, on $\mathbb{C}$, arises as the solution of the differential initial value problem

$$f''(t) + \frac{2\mu + 1}{t}(f'(t) - f'(0)) = \xi_{\mu} f(t),$$

which is subject to the conditions $f(0) = 1$ and $f'(0) = \frac{\xi\,\Gamma(\mu+1)}{\sqrt{\pi}\,\Gamma(\mu+\frac{3}{2})}$. It admits the following power series expansion (see [30, 43]):

$$S_\mu(\xi t) = \frac{\Gamma(\mu+1)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(\xi t)^n\,\Gamma(\frac{n+1}{2})}{n!\,\Gamma(\frac{n}{2}+\mu+1)}. \tag{1.3}$$

Remark: The term $\Gamma\left(\frac{n+1}{2}\right)$ is well defined for all $n \geq 0$. In particular, for $n = 2k$, $\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(k+\frac{1}{2}\right)$, whereas for $n = 2k+1$, $\Gamma\left(\frac{n+1}{2}\right) = \Gamma(k+1)$. Hence, the series representation remains valid for both even and odd indices.

Its connections with the exponential function are given by

$$S_{-\frac{1}{2}}(t) = e^t, \tag{1.4}$$

and

$$S_{\frac{1}{2}}(t) = \frac{e^t - 1}{t}. \tag{1.5}$$

The generalized Bessel function $J_\mu^{b,c}$ and the generalized Struve function $B_\mu^{b,c}$, whose introductions are attributed to [5] and [33], respectively, each satisfy the differential equation

$$x^2 f''(x) + bxf'(x) + [cx^2 - \mu^2 + (1-b)\mu]f(x) = 0,$$

while $B_\mu^{b,c}$ satisfies the inhomogeneous counterpart

$$x^2 f''(x) + bxf'(x) + [cx^2 - \mu^2 + (1-b)\mu]f(x) = \frac{4(\frac{x}{2})^{\mu+1}}{\sqrt{\pi}\Gamma(\mu+1)}.$$

Their respective series expansions take the following forms:

$$J_\mu^{b,c}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\,\Gamma(n+\ell)}\left(\frac{x}{2}\right)^{2n+\mu},$$
$$B_\mu^{b,c}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n+\frac{3}{2})\,\Gamma(n+\ell+\frac{1}{2})}\left(\frac{x}{2}\right)^{2n+\mu+1}, \quad \ell = \mu + \frac{b+1}{2}.$$

For $\mu > -\frac{1}{2}$, the generalized Bessel–Struve kernel function (see [32]) possesses the following power series representation:

$$S_\mu^{b,c}(t) = \sum_{n=0}^{\infty} \frac{\Gamma(\ell)(c)^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\ell+\frac{n}{2})n!} t^n. \tag{1.6}$$

The generalized Bessel–Struve kernel $S_\mu^{b,c}(t)$ introduces the additional parameters b and c, which provide extra degrees of freedom beyond the classical kernel. Here, b modifies the differential structure, while c acts as a scaling parameter. Consequently, different choices of b and c generate broader families of functions, thus providing greater flexibility in the study of boundary-value problems and related mathematical structures.

The following index-shifting identities for double series (see [40]) are employed throughout the subsequent analysis:

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\psi(q,p)=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\psi(q,p-q), \tag{1.7}$$

and

$$\sum_{p=0}^{\infty}\sum_{q=0}^{p}\delta(q,p)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta(q,p+q).$$

Here, $\psi(q,p)$ and $\delta(q,p)$ denote arbitrary functions (or sequences) of the nonnegative integer indices q and p, for which the involved double series are convergent.

The remainder of the paper proceeds as follows: Section 2 introduces the generalized polynomial and number families along with their connections to classical Bernoulli and Stirling structures via the generalized Bessel–Struve kernel; Section 3 establishes key analytic properties, including differential formulas, recurrence identities, and implicit summation relations; Section 4 presents computational experiments and graphical analyses that illuminate the effect of the governing parameters on the polynomial behavior and zero distribution; and concluding observations and prospective research directions are offered in Section 5.

2. Generalized polynomials and numbers

This section is devoted to constructing generalized counterparts of Bernoulli polynomials and numbers by means of the generalized Bessel–Struve kernel function. Moreover, a new variant of Stirling numbers of the second kind is introduced.

Definition 2.1. Let $n\in\mathbb{N}_0$ and $\Re(\mu)>-1$. The generalized polynomial sequence $\mathbb{X}_{n,\mu}^{b,c}(x)$ is introduced via the following generating relation:

$$\frac{e^{xt}}{S_{\mu}^{b,c}(t)}=\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c}(x)\frac{t^n}{n!}, \tag{2.1}$$

where $S_{\mu}^{b,c}(t)$ denotes the generalized Bessel–Struve kernel function given in Eq (1.6).

The function (2.1) is analytic in a neighborhood of the origin. The radius of convergence of the above mentioned function is R, which denotes the modulus of the nearest nonzero zero of the generalized Bessel–Struve kernel $S_{\mu}^{b,c}(z)$. Here, $R=\min\{|t|:S_{\mu}^{b,c}(t)=0,\ t\neq 0\}$, and consequently, the series converges for $|t|<R$.

Then, the corresponding generalized numbers $\mathbb{X}_{n,\mu}^{b,c}$ are recovered by setting $x=0$ in Eq (2.1), yielding the following:

$$\frac{1}{S_{\mu}^{b,c}(t)}=\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c}\frac{t^n}{n!}. \tag{2.2}$$

Particular parameter choices in $\mathbb{X}_{n,\mu}^{b,c}(x)$ lead to noteworthy special cases, as described below.

Case I (Reduction to Bernoulli polynomials) Taking $b = c = 1$ in Eq (2.1) gives the following:

$$\frac{e^{xt}}{S_\mu(t)} = \sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}(x)\frac{t^n}{n!}. \tag{2.3}$$

Inserting $\mu = \frac{1}{2}$ and invoking Eq (1.5) transforms Eq (2.3) into the following:

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} \mathbb{X}_{n,\frac{1}{2}}(x)\frac{t^n}{n!}.$$

A comparison with the standard definition of Bernoulli polynomials (see [31]) reveals the following:

$$\mathbb{X}_{n,\frac{1}{2}}(x) = \mathbb{B}_n(x).$$

In particular, $x = 0$ yields $\mathbb{X}_{n,\frac{1}{2}}(0) = \mathbb{B}_n$, which are the classical Bernoulli numbers defined by Eq (1.1).

Case II Choosing $\mu = -\frac{1}{2}$ in Eq (2.3) and applying Eq (1.4) produces the following:

$$e^{(x-1)t} = \sum_{n=0}^{\infty} \mathbb{X}_{n,-\frac{1}{2}}(x)\frac{t^n}{n!}.$$

Matching coefficients of $\frac{t^n}{n!}$ on both sides leads to the following:

$$\mathbb{X}_{n,-\frac{1}{2}}(x) = (x-1)^n. \tag{2.4}$$

From Eq (2.4), for $n \in \mathbb{N}_0$,

$$\mathbb{X}_{n,-\frac{1}{2}}(0) = (-1)^n,$$

so that $\mathbb{X}_{2n,-\frac{1}{2}}(0) = 1$ and $\mathbb{X}_{2n+1,-\frac{1}{2}}(0) = -1$ [31].

Definition 2.2. For $\alpha \in \mathbb{R}$ or $\mathbb{C}$, the higher-order generalized polynomial family $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ is defined by

$$\frac{e^{xt}}{[S_\mu^{b,c}(t)]^\alpha} = \sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}, \tag{2.5}$$

and the radius of convergence for this function remains the same as that of Eq (2.1).

When $b = c = 1$, Eq (2.5) specializes to the following:

$$\frac{e^{xt}}{[S_\mu(t)]^\alpha} = \sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{2.6}$$

The theory of Appell and Sheffer polynomial sequences plays an important role in the study of generating functions, operational identities, and differential properties of special polynomials. Since the generating function of the generalized Bernoulli polynomials associated with the generalized Bessel–Struve kernel function is of exponential type, it is natural to investigate whether this family admits an Appell–Sheffer characterization or not.

From the generating function (2.5),

$$\sum_{n=0}^{\infty} X_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!} = \left(S_{\mu}^{b,c}(t)\right)^{-\alpha} e^{xt}.$$

By comparing this with the standard Sheffer generating function $A(t)e^{xH(t)}$ (see [8]),

$$A(t) = \left(S_{\mu}^{b,c}(t)\right)^{-\alpha}, \qquad H(t) = t.$$

Hence, $X_{n,\mu}^{b,c,(\alpha)}(x)$, is a Sheffer sequence for the pair $\left(\left(S_{\mu}^{b,c}(t)\right)^{-\alpha}, t\right)$. Since $H(t) = t$, the sequence belongs to the Appell class (see [7]). Therefore, the polynomial family admits an Appell–Sheffer characterization. This connection provides a natural operational interpretation of the introduced polynomials.

For $\alpha = 0$ in Eq (2.5), it follows that

$$e^{xt} = \sum_{n=0}^{\infty} X_{n,\mu}^{b,c,(0)}(x)\frac{t^n}{n!}.$$

$$\sum_{n=0}^{\infty} X_{n,\mu}^{b,c,(0)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!},$$

and we get

$$X_{n,\mu}^{b,c,(0)}(x) = x^n.$$

Thus, the polynomial family reduces to the standard monogenic basis ($\{x^n\}_{n\geq 0}$).

Setting $\alpha = 1$ recovers Eq (2.3). Additionally, one verifies that $\mathbb{X}_{n,\frac{1}{2}}^{(\alpha)}(x) = \mathbb{B}_n^{(\alpha)}(x)$, with $\mathbb{B}_n^{(\alpha)}(x)$ as in Eq (1.2), and $\mathbb{X}_{n,-\frac{1}{2}}^{(\alpha)}(x) = (x-\alpha)^n$. Setting $x = 0$ gives a new family of extended numbers, denoted as $\mathbb{X}_{n,\mu}^{(\alpha)}$.

Explicit expressions for the first few higher-order generalized polynomials are as follows:

$$\begin{aligned}
\mathbb{X}_{0,\mu}^{b,c,(\alpha)}(x) =& 1,\\
\mathbb{X}_{1,\mu}^{b,c,(\alpha)}(x) =& x - \frac{\alpha\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+1/2)}, \quad \ell = \mu + \frac{b+1}{2},\\
\mathbb{X}_{2,\mu}^{b,c,(\alpha)}(x) =& x^2 - \frac{2x\alpha\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+1/2)} + \left\{\frac{-\alpha c}{2\ell} + \alpha(\alpha+1)\left(\frac{\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+1/2)}\right)^2\right\},\\
\mathbb{X}_{3,\mu}^{b,c,(\alpha)}(x) =& x^3 - x^2\frac{3\alpha\,\Gamma(\ell)\,c^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})} + 3x\left\{\frac{-\alpha c}{2\ell} + \alpha(\alpha+1)\left(\frac{\Gamma(\ell)\,c^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})}\right)^2\right\} - \frac{\alpha\,\Gamma(\ell)c^{\frac{3}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{3}{2})}\\
&+ \frac{3\alpha(\alpha+1)\Gamma(\ell)c^{\frac{3}{2}}}{2\sqrt{\pi}\,\ell\,\Gamma(\ell+\frac{1}{2})} - \alpha(\alpha+1)(\alpha+2)\left(\frac{\Gamma(\ell)\,c^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})}\right)^3,
\end{aligned}$$

and the associated number sequences are as follows:

$$\begin{aligned}
\mathbb{X}_{0,\mu}^{b,c,(\alpha)} &= 1, \\
\mathbb{X}_{1,\mu}^{b,c,(\alpha)} &= -\frac{\alpha\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+1/2)}, \quad \ell = \mu + \frac{b+1}{2}, \\
\mathbb{X}_{2,\mu}^{b,c,(\alpha)} &= -\frac{\alpha c}{2\ell} + \alpha(\alpha+1)\left(\frac{\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+1/2)}\right)^2, \\
\mathbb{X}_{3,\mu}^{b,c,(\alpha)} &= -\frac{\alpha\Gamma(\ell)\, c^{\frac{3}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{3}{2})} + \frac{3\alpha(\alpha+1)\Gamma(\ell)\, c^{\frac{3}{2}}}{2\sqrt{\pi}\,\ell\,\Gamma(\ell+\frac{1}{2})} - \alpha(\alpha+1)(\alpha+2)\left(\frac{\Gamma(\ell)\, c^{\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})}\right)^3.
\end{aligned}$$

For the reader's convenience, Table 1 summarizes the principal parameters that appear in the generating function (2.1) and their admissible ranges.

Table 1. Summary of principal parameters and ranges.

Parameter	Description	Admissible range
x	Polynomial argument	$x \in \mathbb{R}$ (or $\mathbb{C}$)
n	Polynomial degree	$n \in \mathbb{N}_0$
α	Order parameter (deformation)	$\alpha \in \mathbb{R}$ (or $\mathbb{C}$)
μ	Kernel index	$\Re(\mu) > -\frac{1}{2}$
b	Bessel-type parameter	$b \in \mathbb{R}$
c	Scaling parameter	$c \in \mathbb{R}$
ℓ	Derived parameter	$\ell = \mu + \frac{b+1}{2}$
K	Kernel constant	$K = \frac{\Gamma(\ell)c^{1/2}}{\sqrt{\pi}\Gamma(\ell+\frac{1}{2})}$

For simplicity throughout the manuscript, we consider two specific parameter regimes, Sets A and B, which correspond to $\{\alpha = 0.5, b = 1, c = 1, \mu = 1, \ell = 2\}$ and $\{\alpha = 1, b = 2, c = 2, \mu = 1, \ell = \frac{5}{2}\}$, respectively. The values of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}$ computed under these sets for $n = 0, 1, \ldots, 10$ are listed in Table 2. Both sequences start from $\mathbb{X}_{0,\mu}^{b,c,(\alpha)} = 1$ and decrease in magnitude as n increases, with sign alternations appearing for a larger n. The values under Set B are generally a larger in magnitude than those under Set A, thus reflecting the influence of the elevated kernel parameters b, c, and the order α.

Remark 2.3. The classical Bernoulli numbers $\mathbb{B}_n$ are known to satisfy the asymptotic relation [1]

$$|\mathbb{B}_{2n}| \sim 2\,\frac{(2n)!}{(2\pi)^{2n}}, \quad n \to \infty,$$

which exhibits factorial growth. By contrast, the numerical values in Table 2 suggest that the generalized numbers $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}$ remain bounded and oscillate in sign for the parameter configurations considered, thus indicating a significantly slower growth rate. This difference can be attributed to the regularizing effect of the generalized Bessel–Struve kernel $S_{\mu}^{b,c}(t)$ in the denominator of the generating function in Eq (2.2), whose entire functional nature suppresses the factorial divergence present in the classical case. A rigorous asymptotic analysis of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}$ as $n \to \infty$ remains an interesting open problem for future work.

Table 2. Values of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)} = \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(0)$ for $n = 0, 1, \ldots, 10$ under Sets A and B.

n	Set A ($\alpha = 0.5,\ b = 1,\ c = 1,\ \mu = 1, \ell = 2$)	Set B ($\alpha = 1,\ b = 2,\ c = 2,\ \mu = 1, \ell = \frac{5}{2}$)
0	1.00000000	1.00000000
1	−0.21220659	−0.53033009
2	0.01009491	0.16250000
3	0.01050963	0.02430680
4	0.00059502	−0.03441964
5	−0.00307890	−0.02489868
6	−0.00106550	0.02220331
7	0.00193182	0.04441169
8	0.00161516	−0.01749157
9	−0.00199017	−0.11778641
10	−0.00332145	−0.02571422

The graphs of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ for $n = 0, 1, \ldots, 5, 7, 10, 20$ are depicted in Figure 1 under the parameter configurations of Set A (left) and Set B (right). At $\alpha = 1$, the polynomials $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ collapse to the standard family $\mathbb{X}_{n,\mu}^{b,c}(x)$ defined in Eqs (2.1) and (2.2). An extension of the Stirling numbers of the second kind is introduced through the following:

$$\sum_{n=0}^{\infty} \mathbb{S}_{2,\mu}^{b,c}(n,m)\frac{t^n}{n!} = \frac{(tS_{\mu}^{b,c}(t))^m}{m!}, \tag{2.7}$$

where $\Re(\mu) > -1$. Restricting to $b = c = 1$ in Eq (2.7) yields

$$\sum_{n=0}^{\infty} \mathbb{S}_{2,\mu}(n,m)\frac{t^n}{n!} = \frac{(tS_{\mu}(t))^m}{m!}, \tag{2.8}$$

as found in [31]. Moreover, setting $\mu = 1/2$ in Eq (2.8) recovers the classical Stirling numbers of the second kind as follows (see [24]):

$$\sum_{n=0}^{\infty} \mathbb{S}_{2}(n,m)\frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!}. \tag{2.9}$$

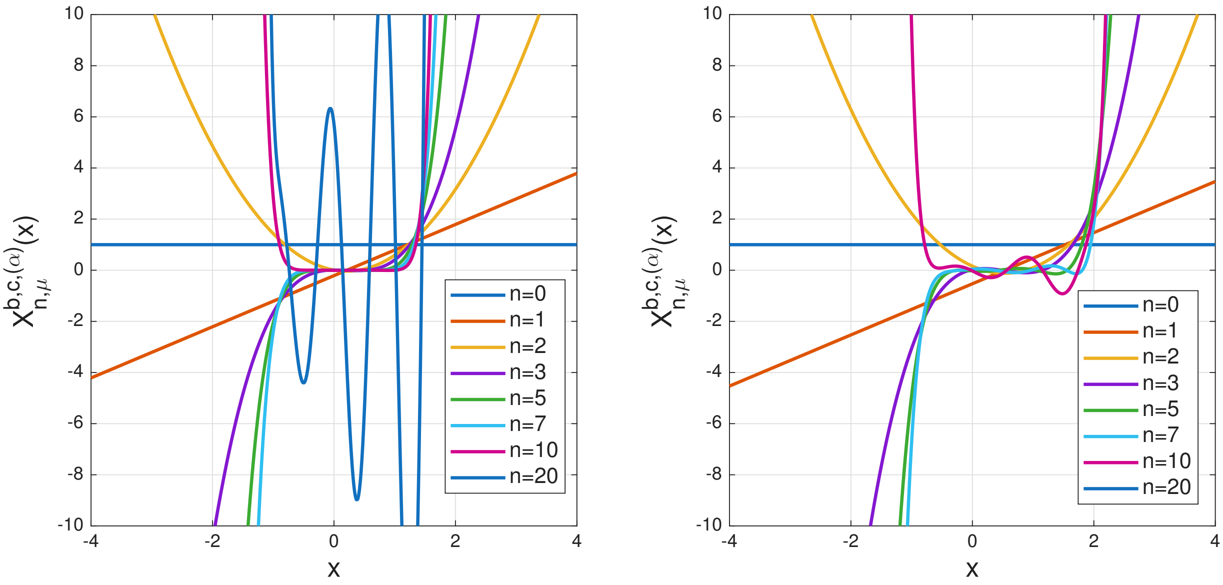


Figure 1. Graphs of the higher-order generalized polynomials $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$.

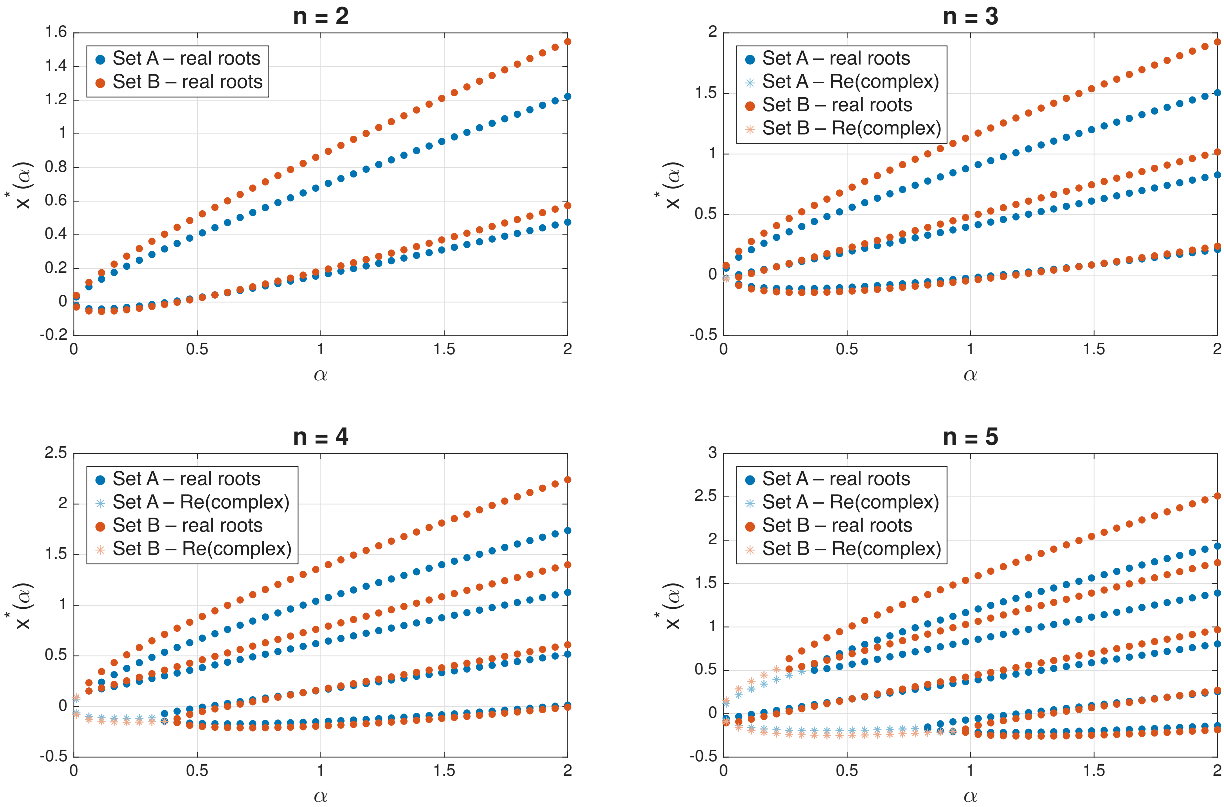


Figure 2. Root locus of the higher-order generalized polynomials $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ as a function of the deformation parameter $\alpha \in (0, 2]$, for degrees $n = 2, 3, 4, 5$.

Figure 2 displays the root locus of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ for $n = 2, 3, 4, 5$ as the deformation parameter α varies over $(0, 2]$, under both Set A and Set B. Three observations are worth noting. First, for all degrees n and both parameter sets, every real root monotonically migrates toward larger positive values of x as

α increases. This is consistent with the closed-form expression for $n = 1$, namely $x^*(\alpha) = \alpha K$, where $K = \Gamma(\ell)c^{1/2}/(\sqrt{\pi}\,\Gamma(\ell + \frac{1}{2}))$, which predicts a linear drift of the root with a slope entirely controlled by the kernel parameters b, c, and μ. Second, the polynomial $\mathbb{X}_{2,\mu}^{b,c,(\alpha)}(x)$ remains real-rooted throughout $\alpha \in (0, 2]$ for both parameter sets, whereas for $n \geq 3$, complex-conjugate root pairs emerge near $\alpha = 0$ and subsequently return to the real axis as α increases past a degree-dependent threshold. This bifurcation behavior, visible as starred markers in Figure 2, indicates that real-rootedness is recovered in the large-α regime for all degrees considered. Third, Set B consistently produces roots of a larger magnitude than Set A at the same α, thus confirming that the parameters b and c govern the rate of root migration without altering the qualitative bifurcation structure.

3. Characteristics of generalized polynomials

This section derives a collection of important properties for the higher-order generalized polynomials and numbers built upon the generalized Bessel–Struve kernel function. The results encompass differential formulas and various summation identities.

Theorem 3.1. *Let $r \in \mathbb{N}$ and $n \geq r$. The families $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ and $\mathbb{X}_{n,\mu}^{b,c}(x)$ satisfy the differentiation rules*

$$\frac{d^r}{dx^r}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x) = \frac{n!}{(n-r)!}\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x), \tag{3.1}$$

and

$$\frac{d^r}{dx^r}\mathbb{X}_{n,\mu}^{b,c}(x) = \frac{n!}{(n-r)!}\mathbb{X}_{n-r,\mu}^{b,c}(x). \tag{3.2}$$

Proof. Starting from the generating relation (2.5), we have the following:

$$\left[S_{\mu}^{b,c}(t)\right]^{-\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}. \tag{3.3}$$

By applying $\frac{d^r}{dx^r}$ to both sides of Eq (3.3) and using $\frac{d^r}{dx^r}e^{xt} = t^r e^{xt}$, one obtains the following:

$$t^r\left[S_{\mu}^{b,c}(t)\right]^{-\alpha} e^{xt} = \sum_{n=0}^{\infty} \frac{d^r}{dx^r}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}. \tag{3.4}$$

Substituting Eq (3.3) into the left-hand side of Eq (3.4) gives the following:

$$t^r\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{d^r}{dx^r}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}. \tag{3.5}$$

Re-indexing the left side of Eq (3.5) by replacing n with $n - r$ yields the following:

$$\sum_{n=r}^{\infty} \mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{(n-r)!} = \sum_{n=0}^{\infty} \frac{d^r}{dx^r}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}.$$

Equating coefficients of t^n on both sides establishes Eq (3.1). Identity (3.2) follows by setting $\alpha = 1$ in Eq (3.1).

Remark: By substituting $r = 1$ in the above Theorem, we get the following:

$$\frac{d}{dx}\left\{X_{n,\mu}^{b,c,(\alpha)}(x)\right\} = \frac{n!}{(n-1)!}X_{n-1,\mu}^{b,c,(\alpha)}(x).$$

Hence, the generalized Bernoulli polynomials $X_{n,\mu}^{b,c,(\alpha)}(x)$ admits the lowering operator

$$\widehat{L} = D_x = \frac{d}{dx},$$

which satisfies

$$\widehat{L}\left\{X_{n,\mu}^{b,c,(\alpha)}(x)\right\} = n\left\{X_{n-1,\mu}^{b,c,(\alpha)}(x)\right\}.$$

Theorem 3.2. *Let $r \in \mathbb{N}$ and $n \geq r$. The higher-order generalized polynomials $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ satisfy the order-shifting recurrence*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha+1)}(x) = \sum_{k=0}^{n}\binom{n}{k}\mathbb{X}_{n-k,\mu}^{b,c,(\alpha)}(x)\mathbb{X}_{k,\mu}^{b,c}, \tag{3.6}$$

where $\mathbb{X}_{k,\mu}^{b,c}$ are the numbers defined in Eq (2.2).

Proof. By decomposing the generating function at order $\alpha + 1$ as a product,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha+1)}(x)\frac{t^n}{n!} &= \frac{e^{xt}}{[S_{\mu}^{b,c}(t)]^{\alpha+1}}\\ &= \left(\frac{e^{xt}}{[S_{\mu}^{b,c}(t)]^{\alpha}}\right)\left(\frac{1}{S_{\mu}^{b,c}(t)}\right)\\ &= \left(\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\mathbb{X}_{k,\mu}^{b,c}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathbb{X}_{n-k,\mu}^{b,c,(\alpha)}(x)\mathbb{X}_{k,\mu}^{b,c}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficients of t^n on both sides yields Eq (3.6).

Theorem 3.3. *Let $n \in \mathbb{N}_0$ and $y \in \mathbb{R}$. The higher-order generalized polynomials obey the addition rules*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x+y) = \sum_{r=0}^{n}\binom{n}{r}y^r\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x), \tag{3.7}$$

and

$$\mathbb{X}_{n,\mu}^{b,c}(x+y) = \sum_{r=0}^{n}\binom{n}{r}y^r\mathbb{X}_{n-r,\mu}^{b,c}(x). \tag{3.8}$$

Proof. By replacing x with $x + y$ in the generating function (2.5),

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x+y)\frac{t^n}{n!} &= \left[S_{\mu}^{b,c}(t)\right]^{-\alpha}e^{(x+y)t}\\ &= \left[S_{\mu}^{b,c}(t)\right]^{-\alpha}e^{xt}e^{yt}\\ &= \left(\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\frac{(yt)^r}{r!}\right).\end{aligned}$$

By invoking the Cauchy product formula for power series,

$$\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x+y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x)\frac{t^{n-r}}{(n-r)!}\frac{(yt)^r}{r!}. \tag{3.9}$$

Rearranging the double sum in Eq (3.9) and invoking the binomial coefficient $\binom{n}{r}$, then equating t^n-coefficients, gives Eq (3.7). Identity (3.8) is obtained from Eq (3.7) by setting $\alpha = 1$.

Corollary 3.4. *Setting $y = 1$ in Eqs* (3.7) *and* (3.8) *produces*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x+1) = \sum_{r=0}^{n}\binom{n}{r}\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x),$$

and

$$\mathbb{X}_{n,\mu}^{b,c}(x+1) = \sum_{r=0}^{n}\binom{n}{r}\mathbb{X}_{n-r,\mu}^{b,c}(x).$$

Theorem 3.5. *Let $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$, $n \in \mathbb{N}_0$ and $y \in \mathbb{R}$. The higher-order generalized polynomials satisfy the following:*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha+\beta)}(x+y) = \sum_{r=0}^{n}\binom{n}{r}\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x)\mathbb{X}_{r,\mu}^{b,c,(\beta)}(y).$$

Proof. Factor the generating function at combined order $\alpha+\beta$ as

$$\left[S_{\mu}^{b,c}(t)\right]^{-(\alpha+\beta)} e^{(x+y)t} = \left[S_{\mu}^{b,c}(t)\right]^{-\alpha} e^{xt}\cdot\left[S_{\mu}^{b,c}(t)\right]^{-\beta} e^{yt},$$

and substitute the respective series expansions for each factor on the right,

$$\left(\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\mathbb{X}_{r,\mu}^{b,c,(\beta)}(y)\frac{t^r}{r!}\right) = \sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha+\beta)}(x+y)\frac{t^n}{n!}. \tag{3.10}$$

Applying the Cauchy product on the left side of Eq (3.10) yields the stated identity.

Theorem 3.6. *For $n \in \mathbb{N}_0$ and $p \in \mathbb{C}$, the following scaling identities hold:*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(px) = \sum_{r=0}^{n}\binom{n}{r}x^r(p-1)^r\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x), \tag{3.11}$$

and

$$\mathbb{X}_{n,\mu}^{b,c}(px) = \sum_{r=0}^{n}\binom{n}{r}x^r(p-1)^r\mathbb{X}_{n-r,\mu}^{b,c}(x). \tag{3.12}$$

Proof. Writing $e^{pxt} = e^{xt}e^{(p-1)xt}$ within the generating function,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(px)\frac{t^n}{n!} &= \left[S_{\mu}^{b,c}(t)\right]^{-\alpha} e^{pxt}\\ &= \left[S_{\mu}^{b,c}(t)\right]^{-\alpha} e^{xt}e^{(p-1)xt}\\ &= \left(\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\frac{\{(p-1)xt\}^r}{r!}\right).\end{aligned}$$

A further application of the Cauchy product formula gives the following:

$$\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(px)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}\{(p-1)x\}^r\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x)\right)\frac{t^n}{n!}. \tag{3.13}$$

Extracting the coefficient of t^n on both sides of Eq (3.13) establishes Eq (3.11). Then formula (3.12) is immediate from Eq (3.11) upon setting $\alpha = 1$.

Lemma: The series involved in the proofs of Theorems 3.2–3.6 are absolutely convergent in the common domain $|t| < \min(2\pi, |r_1|)$; therefore, the Cauchy product formula is applicable.

Theorem 3.7. *For $n \in \mathbb{N}_0$, the extended Stirling numbers satisfy the following convolution identity:*

$$\mathbb{S}_{2,\mu}^{b,c}(n,r+p)=\frac{r!p!}{(r+p)!}\sum_{m=0}^{n}\binom{n}{m}\mathbb{S}_{2,\mu}^{b,c}(n-m,r)\mathbb{S}_{2,\mu}^{b,c}(m,p). \tag{3.14}$$

Proof. Starting from the defining relation (2.7) and splitting the power $(tS_{\mu}^{b,c}(t))^{r+p}$ as a product,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{S}_{2,\mu}^{b,c}(n,r+p)\frac{t^n}{n!} &= \frac{[tS_{\mu}^{b,c}(t)]^{r+p}}{(r+p)!}\\ &= \frac{r!p!}{(r+p)!}\frac{[tS_{\mu}^{b,c}(t)]^{r}}{r!}\frac{[tS_{\mu}^{b,c}(t)]^{p}}{p!}\\ &= \frac{r!p!}{(r+p)!}\left(\sum_{n=0}^{\infty}\mathbb{S}_{2,\mu}^{b,c}(n,r)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\mathbb{S}_{2,\mu}^{b,c}(m,p)\frac{t^m}{m!}\right)\\ &= \frac{r!p!}{(r+p)!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathbb{S}_{2,\mu}^{b,c}(n-m,r)\mathbb{S}_{2,\mu}^{b,c}(m,p)\right)\frac{t^n}{n!}.\end{aligned}$$

The identification of t^n-coefficients on both sides completes the proof.

Theorem 3.8. *For $n \in \mathbb{N}_0$, the following mixed-order identities hold:*

$$\mathbb{X}_{n-\alpha,\mu}^{b,c,(\alpha)}=\sum_{r=0}^{n}\frac{\alpha!(n-\alpha)!}{r!(n-r)!}\mathbb{X}_{n-r,\mu}^{b,c,(2\alpha)}\mathbb{S}_{2,\mu}^{b,c}(r,\alpha), \tag{3.15}$$

and

$$\mathbb{X}_{n-1,\mu}^{b,c}=\sum_{r=0}^{n}\frac{(n-1)!}{r!(n-r)!}\mathbb{X}_{n-r,\mu}^{b,c,(2)}\mathbb{S}_{2,\mu}^{b,c}(r,1). \tag{3.16}$$

Proof. Starting from the reciprocal kernel representation,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}\frac{t^n}{n!} &= \frac{1}{[S_{\mu}^{b,c}(t)]^{\alpha}}\\ &= \frac{\alpha![tS_{\mu}^{b,c}(t)]^{\alpha}t^{-\alpha}}{\alpha![S_{\mu}^{b,c}(t)]^{2\alpha}}\\ &= \alpha!\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(2\alpha)}\frac{t^n}{n!}\sum_{r=0}^{\infty}\mathbb{S}_{2,\mu}^{b,c}(r,\alpha)\frac{t^{r-\alpha}}{r!}\\ &= \alpha!\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathbb{X}_{n-r,\mu}^{b,c,(2\alpha)}\mathbb{S}_{2,\mu}^{b,c}(r,\alpha)\frac{t^{n-\alpha}}{n!}.\end{aligned}$$

Comparing the powers of t^n on both sides yields Eq (3.15). Taking $\alpha = 1$ in Eq (3.15) gives Eq (3.16).

Theorem 3.9. *Let $n, r \in \mathbb{N}_0$. The following representations in terms of Fubini polynomials hold:*

$$\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{X}_{n-r,\mu}^{b,c,(\alpha)} \left[(1+y)F_r(x,y) - yF_r(x+1,y)\right], \tag{3.17}$$

and

$$\mathbb{X}_{n,\mu}^{b,c}(x) = \sum_{r=0}^{n} \binom{n}{r} \mathbb{X}_{n-r,\mu}^{b,c} \left[(1+y)F_r(x,y) - yF_r(x+1,y)\right], \tag{3.18}$$

where $F_r(x,y)$ denotes the two-variable Fubini polynomials, which satisfy

$$\sum_{r=0}^{\infty} F_r(x,y)\frac{t^r}{r!} = \frac{e^{xt}}{1-y(e^t-1)},$$

provided that $1 - y(e^t - 1) \neq 0$.

Proof. By introducing the factor $[1 - y(e^t - 1)]$ into the generating function,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!} &= \frac{e^{xt}}{[S_\mu^{b,c}(t)]^\alpha} \\
&= \frac{1}{[S_\mu^{b,c}(t)]^\alpha}\frac{e^{xt}}{[1-y(e^t-1)]}[1-y(e^t-1)] \\
&= \frac{(1+y)}{[S_\mu^{b,c}(t)]^\alpha}\frac{e^{xt}}{[1-y(e^t-1)]} - \frac{y}{[S_\mu^{b,c}(t)]^\alpha}\frac{e^{(x+1)t}}{[1-y(e^t-1)]}.
\end{aligned}$$

By inserting the series representations for the generalized polynomials and the Fubini polynomials,

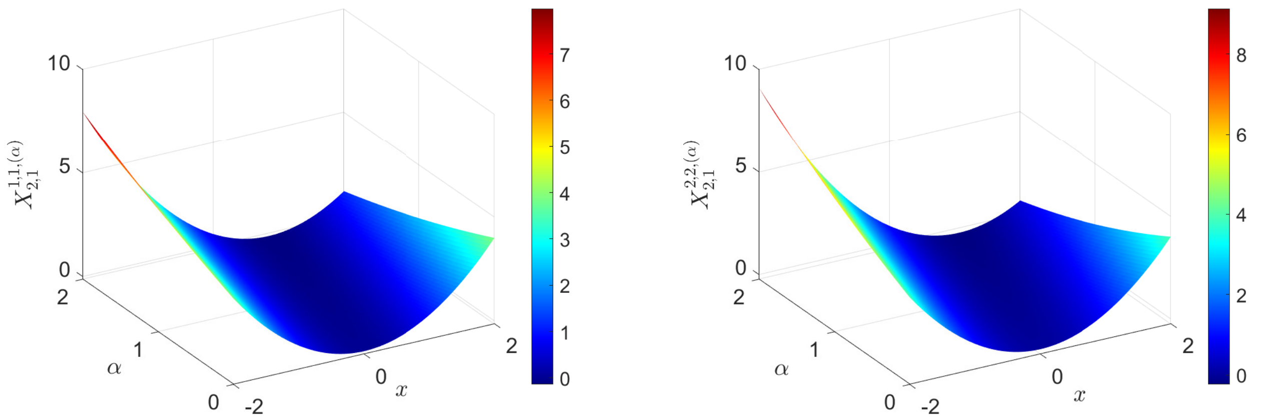
$$\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!} = (1+y)\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}\frac{t^n}{n!}\sum_{r=0}^{\infty} F_r(x,y)\frac{t^r}{r!} - y\sum_{n=0}^{\infty} \mathbb{X}_{n,\mu}^{b,c,(\alpha)}\frac{t^n}{n!}\sum_{r=0}^{\infty} F_r(x+1,y)\frac{t^r}{r!},$$

which, after collecting terms according to the Cauchy product, yields Eq (3.17). Setting $\alpha = 1$ gives Eq (3.18).

4. Numerical illustrations

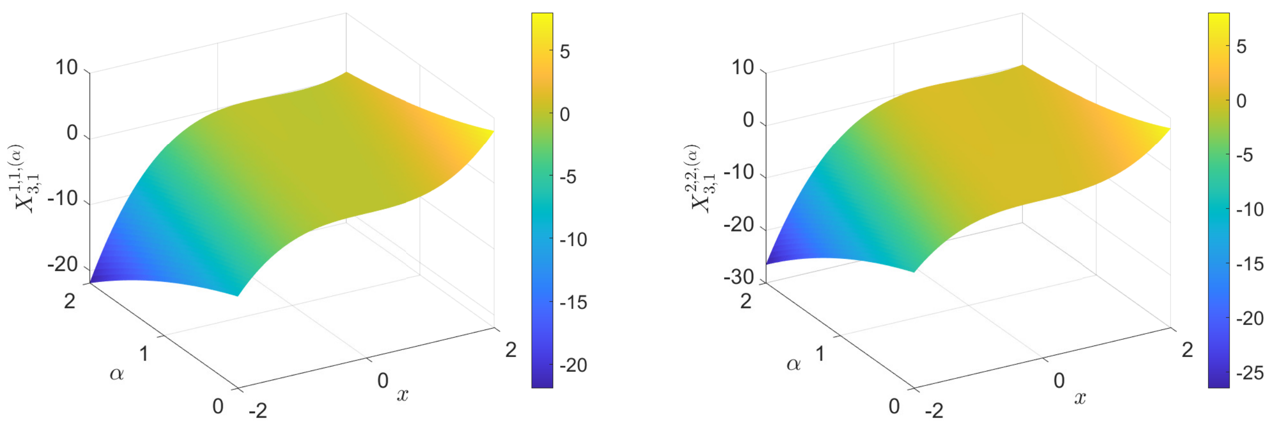
Numerical experiments and graphical displays for the higher-order generalized polynomials $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ are presented in this section. The objective is to clarify how the parameters α, b, c, and μ shape both the root distribution and the growth profile of the polynomials. All results reported here were produced using MATLAB 2024b.

To track the evolution of these polynomials as n grows, we adopted parameter Sets A and B introduced in Section 2, which corresponds to $\{\alpha = 0.5, b = 1, c = 1, \mu = 1, \ell = 2\}$ and $\{\alpha = 1, b = 2, c = 2, \mu = 1, \ell = \frac{5}{2}\}$, respectively. As shown in Figure 1, these configurations illustrate the curves traced by $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ near the origin for $n \in \{0, 1, \ldots, 5\}$.



(**a**) Surface of $\mathbb{X}_{2,1}^{1,1,(\alpha)}(x)$ (**b**) Surface of $\mathbb{X}_{2,1}^{2,2,(\alpha)}(x)$

Figure 3. Three-dimensional distribution of the generalized polynomials for $n = 2$ as a function of x and the order α for Set A (left) and Set B (right).



(**a**) Surface of $\mathbb{X}_{3,1}^{1,1,(\alpha)}(x)$ (**b**) Surface of $\mathbb{X}_{3,1}^{2,2,(\alpha)}(x)$

Figure 4. Three-dimensional distribution of the generalized polynomials for $n = 3$ as a function of x and the order α for Set A (left) and Set B (right).

To examine how the order α and the spatial variable x interact, the polynomials are plotted as surfaces over the rectangle $(x, \alpha) \in [-2, 2] \times [0, 2]$ in $\mathbb{R}^3$.

The three-dimensional plots in Figures 3 and 4 make clear that α acts as a deformation parameter: larger values of α produce increasingly pronounced surface curvatures. To analytically see why the vertex of the surfaces shifts along the x-axis as α increases, we appeal to the explicit forms established in Section 2. For $n = 2$, the explicit form

$$\mathbb{X}_{2,\mu}^{b,c,(\alpha)}(x) = x^2 - 2\alpha K\, x - \frac{\alpha c}{2\ell} + \alpha(\alpha+1)K^2, \quad K = \frac{\Gamma(\ell)\, c^{1/2}}{\sqrt{\pi}\,\Gamma(\ell + \frac{1}{2})},$$

shows that the vertex of the parabolic surface is located at $x = \alpha K$, which linearly migrates in α with slope $K > 0$ entirely determined by the kernel parameters b, c, and μ through $\ell = \mu + \frac{b+1}{2}$. For $n = 3$, the explicit form

$$\begin{aligned}\mathbb{X}_{3,\mu}^{b,c,(\alpha)}(x) = x^3 - 3\alpha K\, x^2 + 3\Big(\alpha(\alpha+1)K^2 - \frac{\alpha c}{2\ell}\Big)x \\ - \frac{\alpha\,\Gamma(\ell)\, c^{3/2}}{\sqrt{\pi}\,\Gamma(\ell + \frac{3}{2})} + \frac{3\alpha(\alpha+1)\Gamma(\ell)\, c^{3/2}}{2\sqrt{\pi}\,\ell\,\Gamma(\ell + \frac{1}{2})} - \alpha(\alpha+1)(\alpha+2)K^3\end{aligned}$$

confirms that the leading asymmetry of the cubic surface is again governed by the term $3\alpha K\, x^2$, so that the inflection structure of the surface proportionally shifts to αK as α varies. This confirms that the linear dependence of the surface geometry on α is a structural feature of the generating mechanism, which is persistent across degrees, as evidenced by Figures 3 and 4.

A numerical verification of the addition identity in Theorem 3.3 is carried out for $\alpha = 0.5$, $b = 1$, $c = 1$, and $\mu = 1$ (giving $\ell = 2$ and $K \approx 0.5642$). Both sides of Eq (3.7) are evaluated for $n = 1, 2, 3$ across six input configurations, including positive, negative, and complex values of x and y, and the results are recorded in Table 3.

Table 4 tracks the real zeros of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ for $\alpha = 1$ and $n = 2, 3, 4$ across twelve parameter configurations. First, increasing c at fixed b and μ shifts all zeros toward larger positive values, which is consistent with the role of c in $K = \Gamma(\ell)c^{1/2}/(\sqrt{\pi}\,\Gamma(\ell + \frac{1}{2}))$ that governs the zero locations. Second, increasing b or μ at fixed c enlarges ℓ, which reduces K and shifts the zeros toward the origin. Third, configurations that share the same values of ℓ and c produce identical zeros regardless of the individual values of b and μ, thus confirming that $\ell = \mu + \frac{b+1}{2}$ acts as the effective single control parameter of the zero distribution. For $n = 3$ and $n = 4$, one real zero is consistently negative across all configurations while the remaining zeros are positive, a pattern that persists as the kernel parameters vary.

Figure 5 displays the distribution of all roots of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ in the complex plane for $n = 5, 6, \ldots, 20$ under Sets A and B. Real roots lie on the real axis and complex roots appear as conjugate pairs symmetric about the real axis, reflecting the fact that the polynomial coefficients are real. Furthermore, as n increases, the complex roots migrate outward from the real axis, with their imaginary parts growing in magnitude while their real parts remain clustered in a bounded region near the origin. In particular, the real roots continue to interlace along the positive real axis in a regular pattern consistent with the behavior observed for smaller n in Table 4. These observations suggest that the proposed polynomial family exhibits an organized root geometry that becomes increasingly structured as the degree grows, which is a property that warrants further theoretical investigation.

Table 3. Numerical verification of the addition formula (3.7) for $\alpha = 0.5$, $b = 1$, $c = 1$, $\mu = 1$.

x	y	n	$\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x+y)$	$\sum_{r=0}^{n}\binom{n}{r}y^r\,\mathbb{X}_{n-r,\mu}^{b,c,(\alpha)}(x)$	Absolute error
		1	1.787793	1.787793	0.000×10^{0}
1.5	0.5	2	3.161269	3.161269	0.000×10^{0}
		3	5.524600	5.524600	0.000×10^{0}
		1	−1.212207	−1.212207	0.000×10^{0}
−1.5	0.5	2	1.434508	1.434508	2.220×10^{-16}
		3	−1.656395	−1.656395	6.661×10^{-16}
		1	0.787793	0.787793	1.110×10^{-16}
1.5	−0.5	2	0.585682	0.585682	2.220×10^{-16}
		3	0.404175	0.404175	5.551×10^{-16}
		1	−2.212207	−2.212207	0.000×10^{0}
−1.5	−0.5	2	4.858921	4.858921	0.000×10^{0}
		3	−10.596539	−10.596539	0.000×10^{0}
		1	1.487793	1.487793	2.220×10^{-16}
1.5	$0.2+0.5i$	2	1.928593	1.928593	4.965×10^{-16}
		3	2.019317	2.019317	0.000×10^{0}
		1	−1.512207	−1.512207	2.220×10^{-16}
−1.5	$0.2+0.5i$	2	2.001832	2.001832	2.220×10^{-16}
		3	−2.167593	−2.167593	9.930×10^{-16}

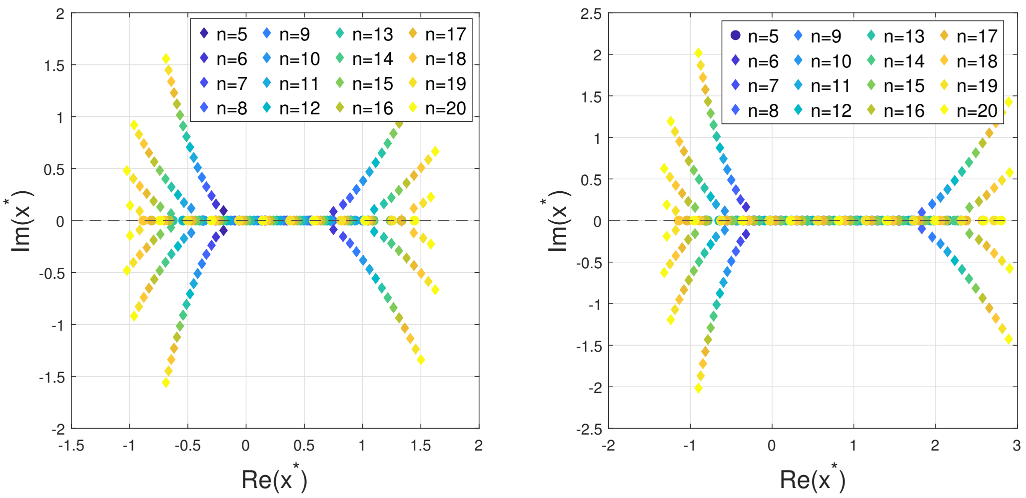


Figure 5. Distribution of all roots of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ in the complex plane for $n = 5, 6, \ldots, 20$ under Set A (left) and Set B (right).

Table 4. Real zeros of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ for $\alpha = 1$ and $n = 2, 3, 4$.

b	c	μ	ℓ	$n = 2$	$n = 3$	$n = 4$
	1			0.16008, 0.68875	−0.02267, 0.40361, 0.89230	−0.14938, 0.16136, 0.63085, 1.05483
1	2	1	2.0	0.22638, 0.97404	−0.03207, 0.57079, 1.26191	−0.21126, 0.22820, 0.89215, 1.49175
	3			0.27726, 1.19295	−0.03927, 0.69907, 1.54551	−0.25874, 0.27948, 1.09266, 1.82702
	1			0.13133, 0.61867	−0.03077, 0.34413, 0.81163	−0.13677, 0.11834, 0.54771, 0.97073
2	2	1	2.5	0.18573, 0.87493	−0.04351, 0.48668, 1.14782	−0.19342, 0.16735, 0.77457, 1.37281
	3			0.22747, 1.07157	−0.05329, 0.59606, 1.40579	−0.23689, 0.20496, 0.94865, 1.68135
	1			0.11285, 0.56621	−0.03377, 0.30338, 0.74899	−0.12512, 0.09177, 0.48888, 0.90259
3	2	1	3.0	0.15959, 0.80075	−0.04776, 0.42904, 1.05923	−0.17695, 0.12979, 0.69138, 1.27646
	3			0.19546, 0.98071	−0.05849, 0.52546, 1.29728	−0.21672, 0.15896, 0.84676, 1.56333
		0.5	2.0	0.22638, 0.97404	−0.03207, 0.57079, 1.26191	−0.21126, 0.22820, 0.89215, 1.49175
2	2	1.5	3.0	0.15959, 0.80075	−0.04776, 0.42904, 1.05923	−0.17695, 0.12979, 0.69138, 1.27646
		2.0	3.5	0.14127, 0.74261	−0.04904, 0.38676, 0.98811	−0.16288, 0.10464, 0.62890, 1.19711

In conclusion, the numerical and graphical results reveal several structural properties of the proposed polynomial family. The degree n uniquely determines the number of real roots, thus preserving the classical interlacing property characteristic of orthogonal polynomials under variations of the kernel parameters μ, b, and c. The fractional order α serves as a global control parameter: an increase in α induces a monotonic migration of the zeros along the real axis in a predictable manner governed by the constant K. This behavior is analytically established for $n = 2$ and numerically verified for $n \leq 20$ in Figures 2 and 5, respectively. Ultimately, the generalized Bessel–Struve kernel $S_{\mu}^{b,c}(t)$ provides a unifying generating function that yields the classical Bernoulli polynomials as a special case, where the kernel parameters b, c, and μ provide precise control over both the zero distribution and the asymptotic growth profiles.

5. Conclusions

In this paper, a new class of higher-order generalized polynomials and their associated number sequences was established by incorporating the Bessel–Struve kernel function $S_{\mu}^{b,c}(t)$ into the generating mechanism. Through the systematic application of series rearrangement and algebraic decomposition, the fundamental identities that govern these extended structures were derived. The coherence of the framework was confirmed by observing that imposing $b = c = 1$ in the derived identities (5.2)–(5.4) recovers the polynomial family of Ghayasuddin et al. [31], thus establishing the current results as a genuine generalization.

A notable contribution of this study is the provision of explicit representations for the generating functions of the proposed polynomials. By invoking the Wright hypergeometric function ${}_p\Psi_q(t)$ [14], the kernel $S_{\mu}^{b,c}(t)$ can be written as follows:

$$S_{\mu}^{b,c}(t)=\frac{c^{\frac{n}{2}}\Gamma(\mu+\frac{b+1}{2})}{\sqrt{\pi}}\;{}_1\Psi_1\left[\begin{array}{c}\left(\frac{1}{2},\frac{1}{2}\right)\\ (\mu+\frac{b+1}{2},\ \frac{1}{2})\end{array}\middle|\ t\right].$$

Decomposing the series into its even and odd parts gives the following:

$$S_{\mu}^{b,c}(t)=\Gamma\Big(\mu+\frac{b+1}{2}\Big)c^{n}\left\{{}_0\Psi_1\left[\begin{array}{c}\overline{}\\ (\mu+\frac{b+1}{2},\ 1)\end{array}\middle|\ \frac{t^2}{4}\right]+\frac{t}{2}\,c^{\frac{1}{2}}\;{}_1\Psi_2\left[\begin{array}{c}(1,1)\\ (\mu+\frac{b}{2}+1,1),\ (\frac{b}{2}+1,1)\end{array}\middle|\ \frac{t^2}{4}\right]\right\}. \tag{5.1}$$

By exploiting the connection between the Wright and generalized hypergeometric functions ${}_pF_q$ (see [40,41,44]). Expression (5.1) simplifies to the following:

$$S_{\mu}^{b,c}(t)=c^{n}\;{}_0F_1\left[\begin{array}{c}\overline{}\\ \mu+\frac{b+1}{2}\end{array}\middle|\ \frac{t^2}{4}\right]+\frac{t\,\Gamma(\mu+\frac{b+1}{2})\,c^{n+\frac{1}{2}}}{2\,\Gamma(\mu+\frac{b}{2}+1)\,\Gamma(\frac{b}{2}+1)}\;{}_1F_2\left[\begin{array}{c}1\\ \mu+\frac{b}{2}+1,\ \frac{b}{2}+1\end{array}\middle|\ \frac{t^2}{4}\right].$$

Substituting these equivalent representations into the master generating function (2.5) produces the following alternative forms for $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$:

$$\left\{\frac{c^{\frac{n}{2}}\Gamma(\mu+\frac{b+1}{2})}{\sqrt{\pi}}\;{}_1\Psi_1\left[\begin{array}{c}\left(\frac{1}{2},\frac{1}{2}\right)\\ (\mu+\frac{b+1}{2},\ \frac{1}{2})\end{array}\middle|\ t\right]\right\}^{-\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}, \tag{5.2}$$

$$\begin{aligned}\left\{c^{n}\,\Gamma\Big(\mu+\frac{b+1}{2}\Big)\left({}_0\Psi_1\left[\begin{array}{c}\overline{}\\ (\mu+\frac{b+1}{2},\ 1)\end{array}\middle|\ \frac{t^2}{4}\right]+\frac{t}{2}\,c^{\frac{1}{2}}{}_1\Psi_2\left[\begin{array}{c}(1,1)\\ (\mu+\frac{b}{2}+1,1),\ (\frac{b}{2}+1,1)\end{array}\middle|\frac{t^2}{4}\right]\right)\right\}^{-\alpha}e^{xt}\\ =\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!},\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\left\{c^{n}\;{}_0F_1\left[\begin{array}{c}\overline{}\\ \mu+\frac{b+1}{2}\end{array}\middle|\ \frac{t^2}{4}\right]+\frac{t\,\Gamma(\mu+\frac{b+1}{2})\,c^{n+\frac{1}{2}}}{2\,\Gamma(\mu+\frac{b}{2}+1)\Gamma(\frac{b}{2}+1)}\;{}_1F_2\left[\begin{array}{c}1\\ \mu+\frac{b}{2}+1,\ \frac{b}{2}+1\end{array}\middle|\ \frac{t^2}{4}\right]\right\}^{-\alpha}e^{xt}\\ =\sum_{n=0}^{\infty}\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)\frac{t^n}{n!}.\end{aligned} \tag{5.4}$$

In summary, the generalized polynomial families constructed here provide a unifying framework that subsumes several previously known polynomial classes and offers a versatile platform to address problems in the approximation theory and mathematical physics. The identities and representations established in this work constitute a solid foundation for further inquiry, particularly with respect to the asymptotic structure of the zero sets and potential combinatorial applications of the generalized Stirling numbers. From a computational perspective, the generalized Bessel–Struve kernel $S_{\mu}^{b,c}(t)$ admits the explicit power series representation (1.6), whose finite truncation allows for an evaluation to any prescribed precision at a modest cost. Moreover, the order-shifting recurrence established in Theorem 3.2 enables the higher-order polynomial family to be incrementally built from lower-order members, thereby avoiding redundant kernel evaluations and reducing the computational overhead in potential applications to numerical schemes such as collocation or Galerkin methods. A systematic study of the deployment of $\mathbb{X}_{n,\mu}^{b,c,(\alpha)}(x)$ in such schemes, including a rigorous analysis of computational cost and convergence rates, constitutes a well-motivated direction for future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Conceptualization, A.R. and M.A.; Methodology, A.R. and M.A.; Formal analysis, A.R.; Investigation, A.R. and M.A.; Validation, M.A. and D.K.; Visualization, D.K.; Writing—original draft, A.R.; Writing—review and editing, M.A. and D.K.; Supervision, M.A. All authors have read and agreed to the published version of the manuscript.

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