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Research article

A study of algebraic and operational characteristics of hybrid special polynomials in the context of Sheffer sequences

Mohra Zayed[1], **Shahid Ahmad Wani**[2,*], **Waseem Ahmad Khan**[3,*], **Ketan Kotecha**[2] and **Mdi Begum Jeelani**[4]

[1] Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia
[2] Symbiosis Institute of Technology, Pune Campus, Symbiosis International (Deemed University), Pune, India
[3] Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
[4] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11564, Saudi Arabia

* **Correspondence:** Email: shahid.wani@sitpune.edu.in, shahidwani177@gmail.com; wkhan1@pmu.edu.sa.

Abstract: This paper addresses the problem of constructing and analyzing a new four-variable family of hybrid special polynomials that unifies several classical polynomial systems within the Sheffer sequence framework. Specifically, we introduce the Legendre-Laguerre-Gould-Hopper-based Sheffer polynomials (LeLGHSP) by embedding the Legendre-Laguerre–Gould–Hopper hybrid structure defined via Bessel-Tricomi functions and higher-order exponential terms into the Sheffer generating function governed by the pair $(\mu(\tau), \sigma(\tau))$. Using operational calculus, inverse derivative operators, and Riordan array methods, we derive explicit series representations, determinantal forms via Cramer's rule, four operational identities expressing the new family in terms of known Sheffer and Gould–Hopper polynomials, and shift relations of binomial type. The quasi-monomial structure of the family is established by constructing multiplicative and derivative operators satisfying the Weyl algebra commutation relation, from which differential equations governing the polynomials are obtained directly. Three important special cases (generalized Legendre-Laguerre-Gould-Hopper-Hermite, Laguerre, and Pidduck-type Sheffer polynomials) are investigated in a unified manner. Surface plots illustrate the geometric behavior of the family. The framework generalizes numerous classical polynomial families and has potential applications in quantum mechanics, heat-type equations, approximation theory, and combinatorics.

Keywords: Sheffer sequences; Legendre-Laguerre-Gould-Hopper-Sheffer sequences; monomiality principle; surface plots; operational rules; determinant approach

1. Introduction

The central problem addressed in this paper is the construction of a new family of hybrid special polynomials that simultaneously generalizes several classical polynomial systems—including Hermite, Laguerre, Legendre, and Gould–Hopper polynomials—within the unifying algebraic framework of Sheffer sequences. Although each of these polynomial families has been studied extensively in isolation, their joint embedding into a single four-variable generating function governed by the Sheffer pair $(\mu(\tau), \sigma(\tau))$ has not been previously explored. The specific family introduced here, the Legendre–Laguerre–Gould–Hopper-based Sheffer polynomials (LeLGHSP), incorporates Bessel–Tricomi functions to encode the Legendre and Laguerre components operationally, and a higher-order exponential term to encode the Gould–Hopper structure.

Hybrid special polynomial families of this type are important for several reasons. First, they reveal structural connections between classical polynomial systems that are hidden when those systems are studied in isolation: A single generating function encodes all classical families as special cases, providing a unified analytical tool. Second, the operational calculus framework, based on inverse derivative operators and the monomiality principle, enables the derivation of differential equations, shift relations, and recurrence formulas by algebraic means without case-by-case computation. Third, the resulting polynomial families have direct applications in mathematical physics (quantum mechanics, heat-type equations), approximation theory (polynomial interpolation, quadrature), and combinatorics (lattice path enumeration, moment problems).

The special-functions theory is one of the pillars of mathematical physics, forming the basis for analysis of a wide variety of physical phenomena. Throughout this paper, we use the following standard notation: $\xi_1, \xi_2, \xi_3, \xi_4$ denote real or complex variables; ∂_{ξ_i} denotes the partial derivative with respect to ξ_i; $\mathcal{D}_{\xi_i}^{-1}$ denotes the inverse (antiderivative) operator; $\delta_{n,k}$ is the Kronecker delta; and $m \geq 1$ is a positive integer controlling the order of the Gould–Hopper extension.

The Sheffer sequences [1] constitute one of the most significant classes of special polynomial families in modern mathematics, and the algebraic foundations of their construction rest on the umbral calculus developed by Roman [2]. Closely related degenerate and λ-extensions of this umbral framework, including λ-Bell polynomials and degenerate Sheffer/λ-Sheffer sequences, have been developed by [3,4]. Recent developments in unified polynomial frameworks—including Apostol-type, U-type, and Bell-based families [5,6]—have further extended this landscape. More recent constructions have moved closer to the present setting by combining Gould–Hopper structures with Sheffer/Appell systems, operational rules, quasi-monomiality, and Riordan-array determinant methods [7, 8]; however, these studies do not simultaneously incorporate the two Bessel–Tricomi factors used here to encode the Legendre–Laguerre component. The monomiality principle, introduced by Steffensen [9] and reformulated by Dattoli [10], provides the operational backbone for deriving quasi-monomial operators and differential equations for these families; related recent Bell-based constructions also employ this principle systematically [11]. Gould–Hopper polynomials [12] extend the classical Hermite polynomials to arbitrary order m and arise naturally in combinatorics and quantum field theory. The Bessel–Tricomi function [13] enables the operational embedding of Legendre- and Laguerre-type structures; related q-deformed Legendre constructions based on q-Bessel–Tricomi kernels and operational calculus have recently been studied in [14]. Hybridisation of Appell and Sheffer systems with Hermite and truncated-exponential structures has been developed in several operational settings [15–18], while Gould–Hopper extensions and related amalgamations appear in [19]. Determinantal approaches to Sheffer sequences were developed in [20–23].

The present paper makes the following contributions. (i) It introduces the LeLGHSP as a new four-variable polynomial family; the underlying Legendre–Laguerre–Gould–Hopper kernel is recovered in the associated case $\mu(\tau) = 1$, while Hermite-, Laguerre-, and Pidduck-type Sheffer families arise as further specialisations. (ii) It derives explicit series representations, four operational identities, shift relations, and a determinantal definition via Riordan arrays. (iii) It establishes a quasi-monomial structure and obtains the associated differential equations from first principles. (iv) It provides a worked example, surface plots, and application notes connecting the framework to quantum mechanics, heat equations, and combinatorics.

A detailed comparison with existing literature is given at the end of this section below.

Note. All required background definitions—Sheffer sequences, Riordan arrays, the monomiality principle, Gould–Hopper polynomials, and the Bessel–Tricomi function—are collected in Section 2 to keep the introduction free of lengthy technical material and to make the paper accessible to readers not already familiar with these concepts.

A sequence $\mathcal{S}_n(\xi_1)_{n=0}^{\infty}$, with each $\mathcal{S}_n(\xi_1)$ being a polynomial having degree n, is called a Sheffer A-type zero sequence [1] (hereafter referred to as a Sheffer-type sequence) provided it satisfies an exponential generating function of the form

$$\mathcal{D}(\tau)\exp(\xi_1\mathbb{H}(\tau)) = \sum_{n=0}^{\infty}\mathcal{S}_n(\xi_1)\frac{\tau^n}{n!}, \tag{1.1}$$

where $\mathcal{D}(\tau)$ and $\mathbb{H}(\tau)$ are analytic functions near the origin with expansions

$$\mathcal{D}(\tau) = \sum_{n=0}^{\infty}\mathcal{D}_n\frac{\tau^n}{n!}, \quad \mathcal{D}_0 \neq 0, \tag{1.2}$$

and

$$\mathbb{H}(\tau) = \sum_{n=1}^{\infty}h_n\frac{\tau^n}{n!}, \quad h_1 \neq 0, \tag{1.3}$$

respectively. The condition $\mathcal{D}_0 \neq 0$ ensures invertibility of $\mathcal{D}(\tau)$, and $h_1 \neq 0$ ensures that $\mathbb{H}(\tau)$ is a delta series admitting a compositional inverse.

Alternatively, according to Roman [2] (see also the umbral calculus treatments of degenerate and λ-Sheffer sequences in [3,4]), the sequence $\mathcal{S}_n(\xi_1)$ is determined by formal power series

$$\sigma(\tau) = \sum_{n=0}^{\infty}\sigma_n\frac{\tau^n}{n!}, \quad \sigma_0 = 0, \sigma_1 \neq 0 \tag{1.4}$$

and

$$\mu(\tau) = \sum_{n=0}^{\infty}\mu_n\frac{\tau^n}{n!}, \quad \mu_0 \neq 0, \tag{1.5}$$

respectively. The exponential generating function of $\mathcal{S}_n(\xi_1)$ is then given by

$$\frac{1}{\mu(\sigma^{-1}(\tau))}\exp(\xi_1(\sigma^{-1}(\tau))) = \sum_{n=0}^{\infty}\mathcal{S}_n(\xi_1)\frac{\tau^n}{n!}, \tag{1.6}$$

where $\sigma^{-1}(\tau)$ is the compositional inverse of $\sigma(\tau)$, for all ξ_1 in $\mathbb{C}$. The correspondence between the two formulations in Eqs (1.1)–(1.3) and Eqs (1.4)–(1.6) is given by

$$\mathcal{D}(\tau) = \frac{1}{\mu(\sigma^{-1}(\tau))} \tag{1.7}$$

and

$$\mathbb{H}(\tau) = \sigma^{-1}(\tau). \tag{1.8}$$

The sequence $\mathcal{S}_n(\xi_1)$ in Eq (1.6) is the Sheffer sequence for the pair $(\mu(\tau), \sigma(\tau))$. Two important special cases deserve mention:

- The *associated Sheffer sequence* for $\sigma(\tau)$ (i.e., $\mu(\tau) = 1$):

$$\exp(\xi_1 \mathbb{H}(\tau)) = \sum_{n=0}^{\infty} \mathcal{P}_n(\xi_1) \frac{\tau^n}{n!}. \tag{1.9}$$

- The *Appell sequence* for $\mu(\tau)$ (i.e., $\sigma(\tau) = \tau$, so $\mathbb{H}(\tau) = \tau$):

$$\mathcal{D}(\tau) \exp(\xi_1 \tau) = \sum_{n=0}^{\infty} \mathcal{D}_n(\xi_1) \frac{\tau^n}{n!}. \tag{1.10}$$

According to Roman and Rota [24], the sequence $\mathcal{S}_n(\xi_1)$ is characterized by the Sheffer sequence associated with $\mathcal{P}_n(\xi_1)$ of binomial type [24] if it satisfies

$$\mathcal{S}_n(\xi_1 + \xi_2) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}_k(\xi_1) \mathcal{P}_{n-k}(\xi_2), \tag{1.11}$$

for all $n \geqslant 0$ and for every $\xi_2 \in \mathbb{R}$, where $\mathbb{R}$ is a field of characteristic zero.

Let $(\mathcal{S}_n(\xi_1))_{n\in\mathbb{N}}$ denote the Sheffer sequence corresponding to the pair $(\mu(\tau), \sigma(\tau))$, satisfying

$$\xi_1^n = \sum_{k=0}^{n} \mathcal{D}_{n,k}\, \mathcal{S}_k(\xi_1). \tag{1.12}$$

Then, $\mathcal{S}_n(\xi_1)$ admits the determinantal form [20, p 232]:

$$\begin{aligned} &\mathcal{S}_0(\xi_1) = \tfrac{1}{\mathcal{D}_{0,0}}, \\ &\mathcal{S}_n(\xi_1) = \tfrac{(-1)^n}{\mathcal{D}_{0,0}\ \mathcal{D}_{1,1}\ \dots\ \mathcal{D}_{n,n}} \begin{vmatrix} 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{n-1} & \xi_1^n \\ \mathcal{D}_{0,0} & \mathcal{D}_{1,0} & \mathcal{D}_{2,0} & \cdots & \mathcal{D}_{n-1,0} & \mathcal{D}_{n,0} \\ 0 & \mathcal{D}_{1,1} & \mathcal{D}_{2,1} & \cdots & \mathcal{D}_{n-1,1} & \mathcal{D}_{n,1} \\ 0 & 0 & \mathcal{D}_{2,2} & \cdots & \mathcal{D}_{n-1,2} & \mathcal{D}_{n,2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \mathcal{D}_{n-1,n-1} & \mathcal{D}_{n,n-1} \end{vmatrix}, \end{aligned} \tag{1.13}$$

where $\mathcal{D}_{n,k}$ denotes the (n,k)-th element of the Riordan array associated with the pair $(\mu(\tau), \sigma(\tau))$ [25].

A generalized Riordan array corresponding to a given sequence $(C_n)_{n\in\mathbb{N}_0}$ is defined as an ordered pair $(\mu(\tau), \sigma(\tau))$ constituting an infinite lower triangular matrix $(\mathcal{D}_{n,k})_{0\le k\le n<\infty}$ with entries given by

$$\mathcal{D}_{n,k} = \left[\frac{\tau^n}{C_n}\right]\mu(\tau)\frac{(\sigma(\tau))^k}{C_k},$$

where the functions $\mu(\tau)(\sigma(\tau))^k/C_k$ are the column generating functions of the Riordan array. They encode the full combinatorial and algebraic structure, enabling compact representation and computation of polynomial coefficients.

A productive avenue for addressing this objective is the combined application of operational methods and the monomiality principle, which together furnish a robust analytical framework for the treatment of a broad class of partial differential equations arising in physical contexts. The conceptual foundations of this approach trace back to Steffensen [9], who introduced the concept of poweroids and laid the groundwork for the notion of monomiality. This framework was later reformulated and substantially developed by Dattoli [10], who enhanced its analytical power and broadened its applicability.

A polynomial family $\{\mathcal{D}_n(\xi_1)\}_{n\ge0}$ is said to be *quasi-monomial* with respect to the pair $(\hat{\mathcal{M}}, \hat{\mathcal{P}})$ if $\hat{\mathcal{M}}$ (the multiplicative operator) and $\hat{\mathcal{P}}$ (the derivative operator) satisfy

$$\hat{\mathcal{M}}\{\mathcal{D}_n(\xi_1)\} = \mathcal{D}_{n+1}(\xi_1), \tag{1.14}$$

$$\hat{\mathcal{P}}\{\mathcal{D}_n(\xi_1)\} = n\,\mathcal{D}_{n-1}(\xi_1), \tag{1.15}$$

and the Weyl commutation relation

$$[\hat{\mathcal{P}}, \hat{\mathcal{M}}] = \hat{1}. \tag{1.16}$$

The quasi-monomial property thereupon leads to the basic operational identities

$$\hat{\mathcal{M}}\hat{\mathcal{P}}\{\mathcal{D}_n(\xi_1)\} = n\,\mathcal{D}_n(\xi_1), \tag{1.17}$$

$$\mathcal{D}_n(\xi_1) = \hat{\mathcal{M}}^n\{1\}, \tag{1.18}$$

$$\exp(\tau\hat{\mathcal{M}})\{1\} = \sum_{n=0}^{\infty}\mathcal{D}_n(\xi_1)\frac{\tau^n}{n!}. \tag{1.19}$$

In particular, Eq (1.18) provides a constructive method for generating all members of the family from the constant function 1 by repeated application of $\hat{\mathcal{M}}$, and Eq (1.17) shows that $\hat{\mathcal{M}}\hat{\mathcal{P}}$ acts as a degree operator.

The Kampé de Fériet (or Gould-Hopper) polynomials of order m are defined by [12, 19]

$$g_n^m(\xi_1, \xi_2) = \mathcal{G}_n^{[m]}(\xi_1, \xi_2) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{\xi_2^k\xi_1^{n-mk}}{k!(n-mk)!}. \tag{1.20}$$

The two-variable Gould-Hopper polynomials (2VGHP) $\mathcal{G}_n^{[m]}(\xi_1, \xi_2)$ are characterized by the generating function [13]

$$e^{\xi_1\tau+\xi_2\tau^m} = \sum_{n=0}^{\infty}\mathcal{G}_n^{[m]}(\xi_1, \xi_2)\frac{\tau^n}{n!} \tag{1.21}$$

and the operational identity

$$\mathcal{G}_n^{[m]}(\xi_1,\xi_2) = \exp\left(\xi_2 \frac{\partial^m}{\partial \xi_1^m}\right)\{\xi_1^n\}. \tag{1.22}$$

The quasi-monomial operators for $\mathcal{G}_n^{[m]}(\xi_1,\xi_2)$ are

$$\hat{M} := \xi_1 + m\xi_2 \frac{\partial^{m-1}}{\partial \xi_1^{m-1}}, \qquad \hat{P} := \frac{\partial}{\partial \xi_1}. \tag{1.23}$$

Here, $\hat{M}$ increases the degree by one (through the Gould-Hopper shift), and $\hat{P} = \partial/\partial\xi_1$ plays the role of the standard derivative operator.

Operational studies of hybrid special polynomials have combined classical Appell, Sheffer, Hermite, truncated-exponential, and Gould–Hopper structures in several ways [15–17, 19]. Motivated by this independent literature, we use the following Legendre–Laguerre-Gould-Hopper hybrid kernel as the basis of the present construction:

$$\exp(\xi_3\tau + \xi_4\tau^m)C_0(\xi_1\tau)C_0(-\xi_2\tau^2) = \sum_{n=0}^{\infty} {}_{S\mathcal{L}}\mathcal{G}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}, \tag{1.24}$$

where $C_0(\xi_1) = \mathcal{J}_0(2\sqrt{\xi_1})$ denotes the Bessel-Tricomi function of order zero [13], with series expansion

$$C_n(\xi_1) = \xi_1^{-\frac{n}{2}}\mathcal{J}_n(2\sqrt{\xi_1}) = \sum_{k=0}^{\infty}\frac{(-1)^k\xi_1^k}{k!(n+k)!}, \quad n = 0,1,2,\cdots. \tag{1.25}$$

The function C_0 satisfies the operational representation

$$C_0(\alpha\xi_1) = \exp\left(-\alpha\mathcal{D}_{\xi_1}^{-1}\right)\{1\}, \tag{1.26}$$

where the inverse derivative operator $\mathcal{D}_{\xi_1}^{-1}$ is defined by

$$\mathcal{D}_{\xi_1}^{-n}\{1\} = \frac{\xi_1^n}{n!}. \tag{1.27}$$

Equation (1.26) is central to all subsequent operational derivations: By replacing the variable ξ_1 in a polynomial by the inverse operator $\mathcal{D}_{\xi_3}^{-1}$, one effectively embeds the Legendre/Laguerre structure into the Gould-Hopper Sheffer framework.

The Legendre-Laguerre-Gould-Hopper polynomials ${}_{S\mathcal{L}}\mathcal{G}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ are quasi-monomial under

$$\hat{\mathcal{M}}_{{}_{S\mathcal{L}}\mathcal{G}_n^{[m]}} = \xi_3 + m\xi_4\frac{\partial^{m-1}}{\partial\xi_3^{m-1}} - \mathcal{D}_{\xi_1}^{-1} + 2\mathcal{D}_{\xi_2}^{-1}\frac{\partial}{\partial\xi_3}, \qquad \hat{\mathcal{P}}_{S\mathcal{L}} = \mathcal{D}_{\xi_3}. \tag{1.28}$$

These polynomials satisfy the operational representations

$${}_{S\mathcal{L}}\mathcal{G}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(\xi_4\partial_{\xi_3}^m - \mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{\xi_3^n\} \tag{1.29}$$

and

$${}_{S\mathcal{L}}\mathcal{G}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(\mathcal{D}_{\xi_2}^{-1}\frac{\partial^2}{\partial\xi_3^2} - \mathcal{D}_{\xi_1}^{-1}\frac{\partial}{\partial\xi_3}\right)\left\{\mathcal{G}_n^{[m]}(\xi_3,\xi_4)\right\}. \tag{1.30}$$

The principal novelty of the present work lies in the systematic integration of the Legendre-Laguerre-Gould-Hopper hybrid structure into the Sheffer framework through operational and Riordan array techniques. Although each of these components—Sheffer sequences, Gould-Hopper polynomials, and Bessel-Tricomi-based constructions—has been studied independently, their unified treatment within a single compositional generating scheme has not been explored in this form. By embedding the hybrid family into a Sheffer-type structure governed by the pair $(\mu(\tau), \sigma(\tau))$, we obtain a coherent algebraic setting in which operational identities, determinantal representations, and quasi-monomial properties emerge naturally and consistently.

This synthesis is not merely a formal extension; it reveals structural interconnections between classical and hybrid polynomial systems that remain hidden when treated separately. The approach provides a transparent mechanism for deriving differential equations, shift relations, and operator representations directly from the generating structure. Such a unified perspective enhances analytical flexibility and offers a constructive pathway for designing new hybrid families tailored to specific mathematical or physical models.

The polynomial families introduced here have potential applications in several concrete areas. In *quantum mechanics* and *mathematical physics*, Hermite and Laguerre polynomials form the eigenfunctions of the quantum harmonic oscillator and hydrogen atom, respectively; their Sheffer-type generalizations can model anharmonic corrections and multimode quantum systems. In the context of *approximation theory*, Sheffer sequences provide flexible bases for polynomial interpolation and numerical quadrature, and the hybrid structure introduced here extends this flexibility to multivariate settings. In *combinatorics and probability*, Sheffer sequences encode the umbral calculus framework of binomial-type sequences and arise naturally in the computation of moments of probability distributions and in lattice path enumeration. Finally, in *differential equations*, the quasi-monomial operators derived in Section 3 yield associated differential equations whose solutions are the polynomial families studied here, opening pathways for boundary value problems in applied mathematics.

It is instructive to place the present work explicitly in the context of related studies.

- In the associated case $\mu(\tau) = 1$, the Sheffer weight disappears and the present generating function reduces to the underlying Legendre-Laguerre-Gould-Hopper hybrid kernel. For arbitrary μ, the factor $1/\mu(\sigma^{-1}(\tau))$ adds the full Sheffer weighting.
- Classical Sheffer and Hermite-based Appell constructions [2, 15] are recovered as lower-dimensional specialisations, whereas the present family also retains the Bessel–Tricomi factors $C_0(\xi_1 \mathbb{H}(\tau))$ and $C_0(-\xi_2(\mathbb{H}(\tau))^2)$.
- Determinantal approaches to Sheffer sequences [20, 21] are extended here to the four-variable hybrid setting using Riordan arrays, which supplies a direct computational representation for the LeLGHSP.
- Recent Gould–Hopper Sheffer/Appell and Sheffer–λ frameworks [7, 8] share operational, determinantal, and monomiality ingredients with the present work. The distinguishing feature here is the simultaneous Legendre-Laguerre Bessel-Tricomi kernel together with the full four-variable Sheffer composition.
- Recent Apostol-type and U-type polynomial frameworks [5, 6, 26] operate within related Appell/Sheffer settings but do not combine both Gould-Hopper and Bessel-Tricomi components. The LeLGHSP therefore provide a distinct multicomponent hybridisation.

- The quasi-monomial operators derived here for the LeLGHSP (Theorem 3.1) reduce to the classical Gould–Hopper operators [12] when the additional hybrid and Sheffer components are suppressed, confirming the hierarchical nature of the construction.

The layout of the paper is as follows. Section 2 (Preliminaries) collects all necessary background definitions. In Section 3, we introduce LeLGHSP and derive their generating function, series representation, operational identities, shift relations, and determinantal form. Section 4 is devoted to the quasi-monomial structure and differential equations. Section 5 (Special Cases and Applications) investigates the Hermite-, Laguerre-, and Pidduck-type subclasses in a unified manner. Section 6 provides a discussion of the results, and Section 7 concludes the paper.

2. Legendre-Laguerre-Gould-Hopper-based Sheffer polynomials

To introduce the LeLGHSP, denoted ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$, we embed the LeLGHP defined by Eq (1.24) into the Sheffer framework of Eq (1.6) by substituting the multiplicative operator $\hat{M}_{S\mathcal{L}\mathcal{G}_n^{[m]}}$ for the variable ξ_1 in Eq (1.6). This substitution, combined with the Crofton identity in Eq (2.4), decouples the resulting exponential operator and yields the following definition.

Definition 2.1. The LeLGHSP ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ are defined by means of the generating function

$$\frac{1}{\mu(\sigma^{-1}(\tau))}C_0(\xi_1\sigma^{-1}(\tau))C_0(-\xi_2[\sigma^{-1}(\tau)]^2)\exp\left(\xi_3\sigma^{-1}(\tau)+\xi_4(\sigma^{-1}(\tau))^m\right)=\sum_{n=0}^{\infty}{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}, \tag{2.1}$$

or, equivalently (in terms of $\mathcal{D}(\tau)$ and $\mathbb{H}(\tau)$ via Eqs (1.7) and (1.8)),

$$\mathcal{D}(\tau)C_0(\xi_1\mathbb{H}(\tau))C_0(-\xi_2(\mathbb{H}(\tau))^2)\exp\left(\xi_3\mathbb{H}(\tau)+\xi_4(\mathbb{H}(\tau))^m\right)=\sum_{n=0}^{\infty}{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}. \tag{2.2}$$

Proof. We begin by substituting $\xi_1\to\hat{M}_{S\mathcal{L}\mathcal{G}_n^{[m]}}$ (given in Eq (1.28)) in the Sheffer generating function in Eq (1.6). This gives

$$\begin{aligned}&\frac{1}{\mu(\sigma^{-1}(\tau))}\exp(\hat{M}_{S\mathcal{L}\mathcal{G}}(\sigma^{-1}(\tau)))\{1\}\\&\qquad=\frac{1}{\mu(\sigma^{-1}(\tau))}\exp\left(\left(\xi_3-\mathcal{D}_{\xi_1}^{-1}+2\mathcal{D}_{\xi_2}^{-1}\frac{\partial}{\partial\xi_3}+m\xi_4\frac{\partial^{m-1}}{\partial\xi_3^{m-1}}\right)\sigma^{-1}(\tau)\right)=\sum_{n=0}^{\infty}\mathcal{S}_n(\hat{M})\frac{\tau^n}{n!}.\end{aligned} \tag{2.3}$$

To decouple the Gould–Hopper exponential, we apply the Crofton identity [27]

$$f\left(z+m\mu\frac{d^m}{dz^{m-1}}\right)\{1\}=\exp\left(\mu\frac{d^m}{dz^m}\right)\{f(z)\} \tag{2.4}$$

to the left-hand side of Eq (2.3). This separates the ξ_4-dependent term:

$$\frac{1}{\mu(\sigma^{-1}(\tau))}\exp\left(\xi_4\frac{\partial^m}{\partial\xi_3^m}\right)\exp\left(\left(\xi_3-\mathcal{D}_{\xi_1}^{-1}+2\mathcal{D}_{\xi_2}^{-1}\frac{\partial}{\partial\xi_3}\right)\sigma^{-1}(\tau)\right)=\sum_{n=0}^{\infty}\mathcal{S}_n(\hat{M})\frac{\tau^n}{n!}. \tag{2.5}$$

Expanding the remaining exponential using the operational identity in Eq (1.26) for the Bessel–Tricomi function—which converts $\exp(-\mathcal{D}_{\xi_i}^{-1}\cdot)$ into $C_0(\xi_i\cdot)$—and using the binomial-type expansion

$$(1-\tau)^{-\lambda} = \sum_{l=0}^{\infty}(\lambda)_l \frac{\tau^l}{l!}, \tag{2.6}$$

we obtain Eq (2.1). Equation (2.2) then follows from Eq (1.1) via the substitutions Eqs (1.7) and (1.8).

Theorem 2.1. *The LeLGHSP ${}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ admit the following explicit series representation:*

$${}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^r\,\xi_1^r\,\xi_2^s\,\xi_3^p\,\xi_4^q}{(r!)^2(s!)^2\,p!\,q!}\left[\frac{\tau^n}{n!}\right]\Big(\mathcal{D}(\tau)(\mathbb{H}(\tau))^{r+2s+p+mq}\Big). \tag{2.7}$$

Proof. We expand each factor in the generating function in Eq (2.2) as a power series. Using Eq (1.25) with $n=0$,

$$C_0(\xi_1\mathbb{H}(\tau)) = \sum_{r=0}^{\infty}\frac{(-1)^r\xi_1^r}{(r!)^2}(\mathbb{H}(\tau))^r, \qquad C_0\big(-\xi_2(\mathbb{H}(\tau))^2\big) = \sum_{s=0}^{\infty}\frac{\xi_2^s}{(s!)^2}(\mathbb{H}(\tau))^{2s},$$

and by the exponential series,

$$\exp\left(\xi_3\mathbb{H}(\tau)\right) = \sum_{p=0}^{\infty}\frac{\xi_3^p}{p!}(\mathbb{H}(\tau))^p, \qquad \exp\left(\xi_4(\mathbb{H}(\tau))^m\right) = \sum_{q=0}^{\infty}\frac{\xi_4^q}{q!}(\mathbb{H}(\tau))^{mq}.$$

Multiplying these four series with $\mathcal{D}(\tau)$ and collecting powers of $\mathbb{H}(\tau)$ yields the right-hand side of Eq (2.7). Comparing the coefficient of $\tau^n/n!$ with the right-hand side of Eq (2.2) completes the proof.

The first few members of the LeLGHSP family are:

$${}_{\mathcal{SLG}}\mathcal{S}_0^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \mathcal{D}_0,$$

$${}_{\mathcal{SLG}}\mathcal{S}_1^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \mathcal{D}_1 + \mathcal{D}_0h_1\Big(-\xi_1+\xi_3+\delta_{m,1}\xi_4\Big),$$

$$\begin{aligned}{}_{\mathcal{SLG}}\mathcal{S}_2^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) &= \mathcal{D}_2 + (\mathcal{D}_1h_1+\mathcal{D}_0h_2)\Big(-\xi_1+\xi_3+\delta_{m,1}\xi_4\Big)\\ &\quad + \mathcal{D}_0h_1^2\left(\frac{\xi_1^2}{4}+\xi_2-\xi_1\xi_3+\frac{\xi_3^2}{2}+\delta_{m,2}\xi_4+\delta_{m,1}\left(-\xi_1\xi_4+\xi_3\xi_4+\frac{\xi_4^2}{2}\right)\right),\end{aligned}$$

$$\begin{aligned}{}_{\mathcal{SLG}}\mathcal{S}_3^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) &= \mathcal{D}_3 + (\mathcal{D}_2h_1+3\mathcal{D}_1h_2+\mathcal{D}_0h_3)\Big(-\xi_1+\xi_3+\delta_{m,1}\xi_4\Big)\\ &\quad + \Big(3\mathcal{D}_1h_1^2+6\mathcal{D}_0h_1h_2\Big)\left(\frac{\xi_1^2}{4}+\xi_2-\xi_1\xi_3+\frac{\xi_3^2}{2}+\delta_{m,2}\xi_4+\delta_{m,1}\left(-\xi_1\xi_4+\xi_3\xi_4+\frac{\xi_4^2}{2}\right)\right)\\ &\quad + \mathcal{D}_0h_1^3\Bigg(-\frac{\xi_1^3}{36}-\xi_1\xi_2+\frac{\xi_1^2\xi_3}{4}-\xi_2\xi_3-\frac{\xi_1\xi_3^2}{2}+\frac{\xi_3^3}{6}+\delta_{m,3}\xi_4\\ &\qquad + \delta_{m,2}\left(-\xi_1\xi_4+\xi_3\xi_4\right)+\delta_{m,1}\left(\frac{\xi_1^2\xi_4}{4}-\xi_2\xi_4-\xi_1\xi_3\xi_4-\frac{\xi_1\xi_4^2}{2}+\frac{\xi_3^2\xi_4}{2}+\frac{\xi_3\xi_4^2}{2}+\frac{\xi_4^3}{6}\right)\Bigg).\end{aligned}$$

Example 2.1. *Explicit computation for $m = 1$ and $\mathcal{D}(\tau) = 1$, $\mathbb{H}(\tau) = \tau$ (associated Sheffer case).* In this case, $\mathcal{D}_0 = 1$, $h_1 = 1$, $\mathcal{D}_1 = 0$, $\mathcal{D}_2 = 0$, and $\delta_{m,1} = 1$, $\delta_{m,2} = 0$. The first few members reduce to:

$$\begin{aligned}
{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_0^{[1]} &= 1,\\
{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_1^{[1]}(\xi_1,\xi_2,\xi_3,\xi_4) &= -\xi_1 + \xi_3 + \xi_4,\\
{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_2^{[1]}(\xi_1,\xi_2,\xi_3,\xi_4) &= \frac{\xi_1^2}{4} + \xi_2 - \xi_1\xi_3 + \frac{\xi_3^2}{2} + (-\xi_1\xi_4 + \xi_3\xi_4 + \frac{\xi_4^2}{2}).
\end{aligned}$$

Setting additionally $\xi_2 = 0$, $\xi_4 = 0$, one recovers ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_2^{[1]}\big|_{\xi_2=\xi_4=0} = \xi_1^2/4 - \xi_1\xi_3 + \xi_3^2/2$, which is a quadratic polynomial in ξ_3 with coefficients depending on ξ_1, consistent with the Legendre-type structure.

The corresponding low-degree profiles are shown in Figure 1.

Remark 2.1. *Figure 1 illustrates the surface behavior of the first few members of the LeLGHSP family, plotted over the (ξ_1,ξ_3)-plane for fixed $\xi_2 = 1$, $\xi_4 = 1$ and the simplest Sheffer choice $\mathcal{D}(\tau) = 1$, $\mathbb{H}(\tau) = \tau$.*

The linear member ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_1^{[1]}$ yields a planar surface reflecting the linear dependence on ξ_1 and ξ_3. The quadratic member ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_2^{[1]}$ displays noticeable curvature from the mixed $\xi_1\xi_3$ terms. The cubic ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_3^{[1]}$ reveals multiple inflection regions and nonlinear coupling. A comparison between ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_3^{[1]}$ and ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_3^{[2]}$ demonstrates how the parameter m governs the degree of mixing between the Gould–Hopper component (encoded in ξ_4) and the base polynomial structure.

Remark 2.2. *For $\mu(\tau) = 1$ (equivalently $\mathcal{D}(\tau) = 1$), the LeLGHSP reduce to the Legendre-Laguerre Gould-Hopper-*associated *Sheffer polynomials (LeLGHASP) denoted by ${}_{S\mathcal{L}\mathcal{G}}\mathcal{P}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$.*

Corollary 2.1. *The Legendre-Laguerre Gould-Hopper-associated Sheffer polynomials (LeLGHASP) satisfy*

$$C_0(\xi_1\sigma^{-1}(\tau))C_0(-\xi_2[\sigma^{-1}(\tau)]^2)\exp\left(\xi_3\sigma^{-1}(\tau) + \xi_4(\sigma^{-1}(\tau))^m\right) = \sum_{n=0}^{\infty} {}_{S\mathcal{L}\mathcal{G}}\mathcal{P}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}. \tag{2.8}$$

Theorem 2.2. *The following operational identities for the LeLGHSP hold. These express ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}$ as the result of applying the combined Laguerre-Legendre operator to simpler Sheffer or Gould–Hopper polynomials, thereby clarifying the hierarchical structure of the new family:*

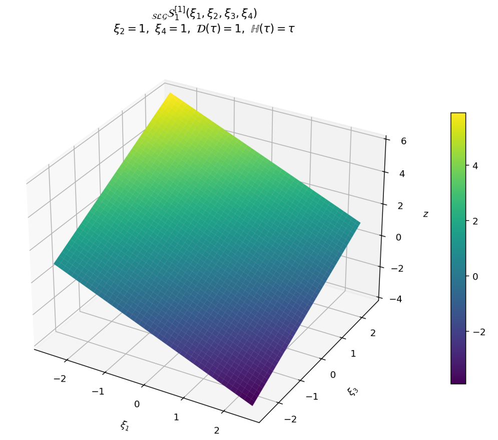
$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(\xi_4\partial_{\xi_3}^m - \mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{\mathcal{S}_n(\xi_3)\}, \tag{2.9}$$

$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(-\mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{{}_{H^{[m]}}\mathcal{S}_n(\xi_3,\xi_4)\}, \tag{2.10}$$

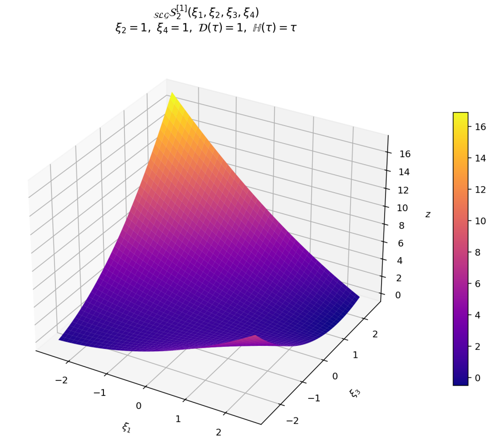
$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(\mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\left\{{}_{\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_3,\xi_4)\right\}, \tag{2.11}$$

$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = \exp\left(-\mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3}\right)\left\{{}_{S\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_2,\xi_3,\xi_4)\right\}. \tag{2.12}$$

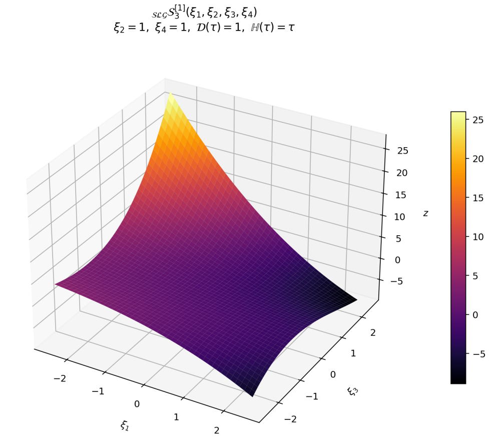
Proof. Equation (2.9) follows by applying the Crofton identity in Eq (2.4) to Eq (2.2) and comparing coefficients of $\tau^n/n!$. Equations (2.10)–(2.12) follow by analogous arguments, successively factoring out the Legendre ($\mathcal{D}_{\xi_1}^{-1}$) and Laguerre ($\mathcal{D}_{\xi_2}^{-1}$) components.



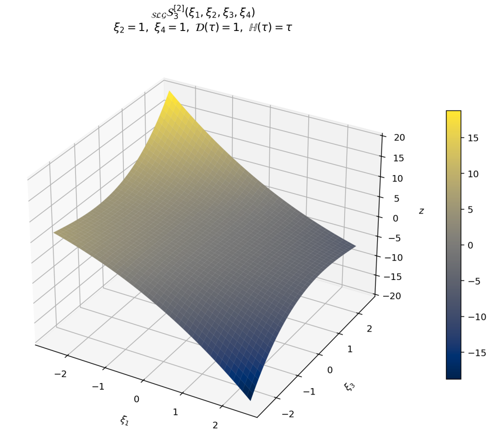
(a) ${}_{\mathcal{SLG}}\mathcal{S}_1^{[1]}(\xi_1,\xi_2,\xi_3,\xi_4)$ for $\xi_2 = 1$, $\xi_4 = 1$, $\mathcal{D}(\tau) = 1$, and $\mathbb{H}(\tau) = \tau$.



(b) ${}_{\mathcal{SLG}}\mathcal{S}_2^{[1]}(\xi_1,\xi_2,\xi_3,\xi_4)$ for $\xi_2 = 1$, $\xi_4 = 1$, $\mathcal{D}(\tau) = 1$, and $\mathbb{H}(\tau) = \tau$.



(c) ${}_{\mathcal{SLG}}\mathcal{S}_3^{[1]}(\xi_1,\xi_2,\xi_3,\xi_4)$ for $\xi_2 = 1$, $\xi_4 = 1$, $\mathcal{D}(\tau) = 1$, and $\mathbb{H}(\tau) = \tau$.



(d) ${}_{\mathcal{SLG}}\mathcal{S}_3^{[2]}(\xi_1,\xi_2,\xi_3,\xi_4)$ for $\xi_2 = 1$, $\xi_4 = 1$, $\mathcal{D}(\tau) = 1$, and $\mathbb{H}(\tau) = \tau$.

Figure 1. Surface plots of the first few members of the LeLGHSP family ${}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ for selected values of m and n, with $\xi_2 = 1$, $\xi_4 = 1$, $\mathcal{D}(\tau) = 1$, and $\mathbb{H}(\tau) = \tau$, plotted over the (ξ_1,ξ_3)-plane.

Theorem 2.3. *For the LeLGHSP, the following shift identity holds:*

$$
{}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3+u,\xi_4) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(u)\ {}_{\mathcal{SLG}}\mathcal{S}_k^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4), \qquad n \geq 0, \quad u \in \mathbb{R}. \tag{2.13}
$$

Proof. We substitute $\xi_3 + u$ in place of ξ_3 in the generating function in Eq (2.1) and decompose the resulting exponential as a product. Specifically, writing $\exp((\xi_3 + u)\sigma^{-1}(\tau)) = \exp(\xi_3\sigma^{-1}(\tau)) \cdot \exp(u\sigma^{-1}(\tau))$, we obtain

$$
\left[\frac{1}{\mu(\sigma^{-1}(\tau))}C_0\left(\xi_1\sigma^{-1}(\tau)\right)C_0\left(-\xi_2[\sigma^{-1}(\tau)]^2\right)\exp\left(\xi_3\sigma^{-1}(\tau) + \xi_4(\sigma^{-1}(\tau))^m\right)\right]\exp\left(u\sigma^{-1}(\tau)\right)
$$

$$= \sum_{n=0}^{\infty} {}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3 + u, \xi_4)\frac{\tau^n}{n!}. \quad (2.14)$$

Using Eqs (1.8) and (2.1) with the Cauchy product yields

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{n-k}(u)\,{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_k^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4)\right)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty} {}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3 + u, \xi_4)\frac{\tau^n}{n!}. \quad (2.15)$$

Equating coefficients of $\tau^n/n!$ gives Eq (2.13).

Determinants provide an independent and computationally effective tool in various areas of theoretical physics (reflectionless transmission in dielectrics, the Korteweg-de Vries (KdV) equation, relativistic gravitational fields). The determinant approach to Sheffer sequences [20–23, 28] motivates the following result.

Theorem 2.4. *The LeLGHSP ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4)$ satisfy the determinantal definition*

$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_0^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{a_{0,0}}\,{}_{S\mathcal{L}}\mathcal{G}_0^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4), \quad (2.16)$$

$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{(-1)^n}{a_{0,0}\,a_{1,1}\,\ldots\,a_{n,n}} \times \begin{vmatrix} {}_{S\mathcal{L}}\mathcal{G}_0^{[m]} & {}_{S\mathcal{L}}\mathcal{G}_1^{[m]} & \cdots & {}_{S\mathcal{L}}\mathcal{G}_{n-1}^{[m]} & {}_{S\mathcal{L}}\mathcal{G}_n^{[m]} \\ a_{0,0} & a_{1,0} & \cdots & a_{n-1,0} & a_{n,0} \\ 0 & a_{1,1} & \cdots & a_{n-1,1} & a_{n,1} \\ 0 & 0 & \cdots & a_{n-1,2} & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \quad (2.17)$$

where we have abbreviated ${}_{S\mathcal{L}}\mathcal{G}_k^{[m]} = {}_{S\mathcal{L}}\mathcal{G}_k^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4)$, and $a_{n,k}$ is the (n, k) entry of the Riordan array $(g(\tau), f(\tau))$.

Proof. Apply $\exp\left(\xi_4\partial_{\xi_3}^m - \mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)$ to both sides of Eq (1.12) and use Eqs (1.29) and (2.9) to obtain

$${}_{S\mathcal{L}}\mathcal{G}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4) = \sum_{k=0}^{n} a_{n,k}\,{}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4). \quad (2.18)$$

This yields an infinite lower-triangular system from which ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}$ can be extracted. Applying Cramer's rule to the first $n + 1$ equations and transposing the resulting determinant gives Eq (2.17).

Remark 2.3. *For $\mathbb{H}(\tau) = \tau$ (equivalently $\sigma^{-1}(\tau) = \tau$), the Sheffer polynomials $\mathcal{S}_n(\xi_3)$ reduce to the Appell polynomials $A_n(\xi_3)$, and the generating function in Eq (2.1) specializes accordingly. All results of this section carry over directly.*

3. Quasi-monomial properties

In this section, we establish the quasi-monomial structure of the LeLGHSP. The key idea is to differentiate the generating function in Eq (2.1) with respect to τ (to identify the multiplicative operator $\hat{M}$) and to substitute $\tau \to \sigma(\partial_{\xi_3})$ in the defining identity in Eq (3.4) (to identify the derivative operator $\hat{P}$). The resulting operators are expressed in terms of σ, μ and their derivatives, as well as the inverse operators $\partial_{\xi_i}^{-1}$.

Theorem 3.1. *The LeLGHSP ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4)$ are quasi-monomial under the multiplicative and derivative operators:*

$$\hat{M} = \left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\frac{\partial}{\partial \xi_3} + m\xi_4 \frac{\partial^{m-1}}{\partial \xi_3^{m-1}} - \frac{\mu^{'}(\partial_{\xi_3})}{\mu(\partial_{\xi_3})}\right)\frac{1}{\sigma^{'}(\partial_{\xi_3})}, \tag{3.1}$$

or equivalently,

$$\hat{M} = \left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\frac{\partial}{\partial \xi_3} + m\xi_4 \frac{\partial^{m-1}}{\partial \xi_3^{m-1}}\right)\mathbb{H}^{'}(\mathbb{H}^{-1}(\partial_{\xi_3})) + \frac{\mathcal{D}^{'}(\mathbb{H}^{-1}(\partial_{\xi_3}))}{\mathcal{D}(\mathbb{H}^{-1}(\partial_{\xi_3}))}, \tag{3.2}$$

$$\hat{P} = \sigma(\partial_{\xi_3}), \quad \textit{or equivalently,} \quad \hat{P} = \mathbb{H}^{-1}(\partial_{\xi_3}). \tag{3.3}$$

Proof. From the generating function, differentiating $\exp(\xi_3\sigma^{-1}(\tau) + \xi_4(\sigma^{-1}(\tau))^m)$ with respect to ξ_3 and applying $\sigma(\partial_{\xi_3})$ gives

$$\sigma(\partial_{\xi_3})\left(\exp\left(\xi_3\sigma^{-1}(\tau) + \xi_4(\sigma^{-1}(\tau))^m\right)\right) = \tau \exp\left(\xi_3\sigma^{-1}(\tau) + \xi_4(\sigma^{-1}(\tau))^m\right). \tag{3.4}$$

Differentiating Eq (2.1) with respect to τ and using the identity in Eq (3.4) leads (after replacing $\sigma^{-1}(\tau)$ by ∂_{ξ_3} in the coefficient functions) to

$$\left(\left(\hat{M} - \frac{g^{'}(\partial_{\xi_3})}{g(\partial_{\xi_3})}\right)\frac{1}{f^{'}(\partial_{\xi_3})}\right){}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4) = {}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_{n+1}^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4), \tag{3.5}$$

which, by the monomiality principle in Eq (1.14), gives Eq (3.1). Using the same identity in Eq (3.4) directly in Eq (2.1) and comparing with the monomiality principle in Eq (1.15) establishes Eq (3.3). The equivalent forms in Eq (3.2) follow from relations in Eqs (1.7) and (1.8).

Remark 3.1. *The explicit representation of the LeLGHSP follows directly from the monomiality identity in Eq* (1.18)*, which states $\mathcal{D}_n(\xi_1) = \hat{\mathcal{M}}^n\{1\}$. Substituting the multiplicative operator $\hat{M}$ from Eq* (3.1) *into this identity, and using the derivative operator $\hat{P}$ from Eq* (3.3)*, yields the closed-form expression in Corollary 4.1 below. This confirms that the LeLGHSP are completely determined by the action of $\hat{M}$ on the constant function* 1.

Corollary 3.1. *The LeLGHSP ${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4)$ have the explicit representation:*

$${}_{S\mathcal{L}\mathcal{G}}\mathcal{S}_n^{[m]}(\xi_1, \xi_2, \xi_3, \xi_4) = \hat{M}^n\{1\} = \left(\left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\frac{\partial}{\partial \xi_3} + m\xi_4 \frac{\partial^{m-1}}{\partial \xi_3^{m-1}} - \frac{g^{'}(\partial_{\xi_3})}{g(\partial_{\xi_3})}\right)\frac{1}{f^{'}(\partial_{\xi_3})}\right)^n\{1\}. \tag{3.6}$$

Theorem 3.2. *The LeLGHSP satisfy the differential equation:*

$$\left(\left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\frac{\partial}{\partial\xi_3} + m\xi_4\frac{\partial^{m-1}}{\partial\xi_3^{m-1}} - \frac{g'(\partial_{\xi_3})}{g(\partial_{\xi_3})}\right)\frac{\sigma(\partial_{\xi_3})}{\sigma'(\partial_{\xi_3})} - n\right){}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = 0. \tag{3.7}$$

Proof. Using Eqs (3.1) and (3.3) in Eq (1.17) ($\hat{M}\hat{P}\{{}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}\} = n\,{}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}$) gives Assertion (3.7).

Theorem 3.3. *The LeLGHSP satisfy the following additional shift formula:*

$${}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}(\xi_1,\xi_2,\xi_3+w,\xi_4) = \sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{SLG}}\mathcal{P}_{n-k}^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\,\mathcal{S}_k(w). \tag{3.8}$$

Proof. Write the generating function of the LeLGHSP in Eq (2.1) with $\xi_3 \to \xi_3 + w$, factoring the exponential as a product of the LeLGHASP generating function in Eq (2.8) and the Sheffer generating function in Eq (1.6). Applying the Cauchy product and equating coefficients of $\tau^n/n!$ gives Eq (3.8).

4. Special cases and applications

The Sheffer family encompasses a wide range of classical and modern polynomial sequences. Representative examples are summarised in Table 1, together with their defining pairs and generating functions. These systems include Hermite, Laguerre, Bernoulli, Poisson–Charlier, and Hahn polynomials and arise in areas such as mathematical physics, numerical analysis, and approximation theory.

Table 1. Classical Sheffer polynomial sequences.

S.No.	$\mathcal{D}(\tau)$; $\mathbb{H}(\tau)$	$\mu(\tau)$; $\sigma(\tau)$	Generating function	Polynomial family
I.	$e^{-\tau^m}$; $\nu\tau$	$e^{(\tau/\nu)^m}$; τ/ν	$\exp(\nu\vartheta_1\tau - \tau^m) = \sum_{n\geq 0} H_{n,m,\nu}(\vartheta_1)\tau^n/n!$	Generalized Hermite polynomials $H_{n,m,\nu}(\vartheta_1)$ [29]
II.	$(1-\tau)^{-\alpha-1}$; $\tau/(\tau-1)$	$(1-\tau)^{-\alpha-1}$; $\tau/(\tau-1)$	$(1-\tau)^{-\alpha-1}\exp(\vartheta_1\tau/(\tau-1)) = \sum_{n\geq 0} L_n^{(\alpha)}(\vartheta_1)\tau^n$	Generalized Laguerre polynomials $n!L_n^{(\alpha)}(\vartheta_1)$ [13, 30]
III.	$\tau/(1-\tau)$; $\ln((1+\tau)/(1-\tau))$	$2/(e^\tau - 1)$; $(e^\tau-1)/(e^\tau+1)$	$\frac{\tau}{1-\tau}\left(\frac{1+\tau}{1-\tau}\right)^{\vartheta_1} = \sum_{n\geq 0} P_n(\vartheta_1)\tau^n/n!$	Pidduck polynomials $P_n(\vartheta_1)$ [31, 32]
IV.	$e^{\beta\tau}$; $1-e^\tau$	$(1-\tau)^{-\beta}$; $\ln(1-\tau)$	$\exp(\beta\tau + \vartheta_1(1-e^\tau)) = \sum_{n\geq 0} a_n^{(\beta)}(\vartheta_1)\tau^n/n!$	Actuarial polynomials $a_n^{(\beta)}(\vartheta_1)$ [31]
V.	$e^{-\tau}$; $\ln(1+\tau/a)$	$\exp(a(e^\tau-1))$; $a(e^\tau-1)$	$e^{-\tau}(1+\tau/a)^{\vartheta_1} = \sum_{n\geq 0} c_n(\vartheta_1;a)\tau^n/n!$	Poisson-Charlier polynomials $c_n(\vartheta_1;a)$ [32–34]
VI.	$(1+(1+\tau)^\lambda)^{-\mu}$; $\ln(1+\tau)$	$(1+e^{\lambda\tau})^\mu$; $e^\tau-1$	$(1+(1+\tau)^\lambda)^{-\mu}(1+\tau)^{\vartheta_1} = \sum_{n\geq 0} s_n(\vartheta_1;\lambda,\mu)\tau^n/n!$	Peters polynomials $s_n(\vartheta_1;\lambda,\mu)$ [31]
VII.	$\tau/\ln(1+\tau)$; $\ln(1+\tau)$	$\tau/(e^\tau-1)$; $e^\tau-1$	$\frac{\tau}{\ln(1+\tau)}(1+\tau)^{\vartheta_1} = \sum_{n\geq 0} b_n(\vartheta_1)\tau^n/n!$	Bernoulli polynomials of the second kind $b_n(\vartheta_1)$ [33]
VIII.	$2/(2+\tau)$; $\ln(1+\tau)$	$(1+e^\tau)/2$; $e^\tau-1$	$\frac{2}{2+\tau}(1+\tau)^{\vartheta_1} = \sum_{n\geq 0} r_n(\vartheta_1)\tau^n/n!$	Related polynomials $r_n(\vartheta_1)$ [33]
IX.	$1/\sqrt{1+\tau^2}$; $\arctan(\tau)$	$\sec\tau$; $\tan\tau$	$\frac{1}{\sqrt{1+\tau^2}}\exp(\vartheta_1\arctan\tau) = \sum_{n\geq 0} R_n(\vartheta_1)\tau^n/n!$	Hahn polynomials $R_n(\vartheta_1)$ [35]
X.	$(1-4\tau)^{-1/2}\left(\frac{2}{1+\sqrt{1-4\tau}}\right)^{a-1}$; $-4\tau/(1+\sqrt{1-4\tau})^2$	$(1+\tau)/(1-\tau)^a$; $\frac{1}{4}-\frac{1}{4}\left(\frac{1+\tau}{1-\tau}\right)^2$	$(1-4\tau)^{-1/2}\left(\frac{2}{1+\sqrt{1-4\tau}}\right)^{a-1}\exp\left(\frac{-4\vartheta_1\tau}{(1+\sqrt{1-4\tau})^2}\right) = \sum_{n\geq 0} R_n(a,\vartheta_1)\tau^n$	Shively pseudo-Laguerre polynomials $R_n(a,\vartheta_1)$ [30, 36]

Applications of special cases. Each of the hybrid polynomial families introduced below inherits the Sheffer and Gould–Hopper structure, and accordingly carries potential applications: (i) In *solving heat-type and Schrödinger-type equations*, where the Hermite special case (Case I) and its operational form provide solutions via the shift $\xi_3 \to \xi_3 + it$ in the time variable; (ii) In *quantum optics and* *coherent states*, where the Laguerre special case (Case II) connects to radial wave functions; (iii) In *combinatorial enumeration*, where the Pidduck special case (Case III) links to permutation statistics. These interpretations are consistent with the classical roles of Hermite and Laguerre systems in mathematical physics [13, 35] and with the standard umbral/Sheffer viewpoint [2, 24].

Considering special case (I) from Table 1, we obtain the *generalized Legendre-Laguerre-Gould-Hopper-Hermite polynomials* (g$\mathbb{L}$e$\mathbb{L}\mathbb{G}\mathbb{H}\mathbb{H}\mathbb{P}$), denoted ${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}(\xi_1,\xi_2,\xi_3,\xi_4)$, defined via

$$e^{-\tau^m}C_0(\xi_1\nu\tau)C_0(-\xi_2(\nu\tau)^2)\exp\left(\xi_3\nu\tau+\xi_4(\nu\tau)^m\right)=\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}. \tag{4.1}$$

The first few explicit members are

$${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{0,m,\nu}=1,\quad {}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{1,m,\nu}=\nu\xi_3,\quad {}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{2,m,\nu}=\nu^2\left(\xi_3^2-\xi_1^2\right).$$

The corresponding degree-two, degree-three, and degree-four surfaces are displayed in Figure 2.

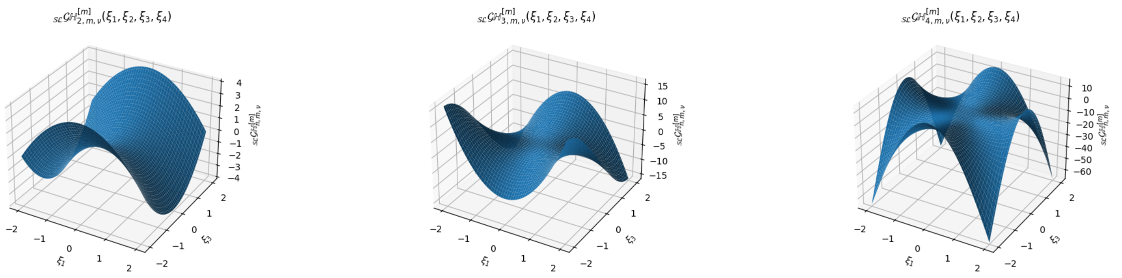


Figure 2. Three-dimensional surface plots of ${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}(\xi_1,\xi_2,\xi_3,\xi_4)$ for $n = 2,3,4$, illustrating the geometric behavior of the generalized Legendre–Laguerre–Gould–Hopper–Hermite polynomials.

Corollary 4.1. *The g$\mathbb{L}$e$\mathbb{L}\mathbb{G}\mathbb{H}\mathbb{H}\mathbb{P}$ ${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}(\xi_1,\xi_2,\xi_3,\xi_4)$ satisfy the following:*

(i) ***Operational identities:***

$${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}=\exp\left(\xi_4\partial^m_{\xi_3}-\mathcal{D}^{-1}_{\xi_1}\partial_{\xi_3}+\mathcal{D}^{-1}_{\xi_2}\partial^2_{\xi_3}\right)\{\mathbb{H}_{n,m,\nu}(\xi_3)\}, \tag{4.2}$$

$$=\exp\left(-\mathcal{D}^{-1}_{\xi_1}\partial_{\xi_3}+\mathcal{D}^{-1}_{\xi_2}\partial^2_{\xi_3}\right)\{{}_{H^{[m]}}\mathbb{H}_{n,m,\nu}(\xi_3,\xi_4)\}. \tag{4.3}$$

(ii) ***Shift identity:***

$${}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{n,m,\nu}(\xi_1,\xi_2,\xi_3+u,\xi_4)=\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(u)\,{}_{\mathcal{S}\mathcal{L}\mathcal{G}}\mathbb{H}^{[m]}_{k,m,\nu}(\xi_1,\xi_2,\xi_3,\xi_4). \tag{4.4}$$

(iii) ***Quasi-monomial operators:***

$$\hat{M}=\frac{1}{\nu}\left(\xi_3-\partial^{-1}_{\xi_1}+2\partial^{-1}_{\xi_2}\partial_{\xi_3}+m\xi_4\partial^{m-1}_{\xi_3}\right),\qquad \hat{P}=\nu\partial_{\xi_3}. \tag{4.5}$$

Note that for $m = 2$, this recovers the standard Hermite multiplicative operator, confirming consistency with the classical theory.

(iv) ***Differential equation:***

$$\Big(\xi_3\partial_{\xi_3} - \partial_{\xi_1}^{-1}\partial_{\xi_3} + 2\partial_{\xi_2}^{-1}\partial_{\xi_3}^2 + m\xi_4\partial_{\xi_3}^m - n\Big)_{\mathcal{SLG}}\mathbb{H}_{n,m,\nu}^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4) = 0. \tag{4.6}$$

This can be regarded as a generalized heat-type equation in the variable ξ_3, with m-order diffusion governed by ξ_4.

Considering special case (II) from Table 1, we obtain the *generalized Legendre–Laguerre–Gould–Hopper–Laguerre polynomials* (g$\mathbb{L}$e$\mathbb{LGHLP}$), denoted ${}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]}(\xi_1,\xi_2,\xi_3,\xi_4)$:

$$(1-\tau)^{-\alpha-1}C_0\left(\xi_1\frac{\tau}{\tau-1}\right)C_0\left(-\xi_2\Big(\frac{\tau}{\tau-1}\Big)^2\right)\exp\left(\xi_3\frac{\tau}{\tau-1} + \xi_4\Big(\frac{\tau}{\tau-1}\Big)^m\right) = \sum_{n=0}^{\infty} {}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}. \tag{4.7}$$

Corollary 4.2. *The g$\mathbb{L}$e$\mathbb{LGHLP}$ ${}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ possess the following properties.*

(i) ***Operational identities:***

$$\begin{aligned} {}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]} &= \exp\left(\xi_4\partial_{\xi_3}^m - \mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{\mathbb{L}_n^{[\alpha]}(\xi_3)\}, && (4.8)\\ &= \exp\left(-\mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{{}_{H^{[m]}}\mathbb{L}_n^{[\alpha]}(\xi_3,\xi_4)\}. && (4.9)\end{aligned}$$

(ii) ***Shift identity:***

$${}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]}(\xi_1,\xi_2,\xi_3+u,\xi_4) = \sum_{k=0}^{n}\binom{n}{k}p_{n-k}(u)\,{}_{\mathcal{SLG}}\mathbb{L}_k^{[\alpha,m]}(\xi_1,\xi_2,\xi_3,\xi_4). \tag{4.10}$$

(iii) ***Quasi-monomial operators:***

$$\hat{M} = \left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\partial_{\xi_3} + m\xi_4\partial_{\xi_3}^{m-1} - \frac{\alpha+1}{1-\partial_{\xi_3}}\right)\frac{1}{(1-\partial_{\xi_3})^2}, \qquad \hat{P} = \partial_{\xi_3}. \tag{4.11}$$

(iv) ***Differential equation:***

$$\left[\left(\xi_3 - \partial_{\xi_1}^{-1} + 2\partial_{\xi_2}^{-1}\partial_{\xi_3} + m\xi_4\partial_{\xi_3}^{m-1} - \frac{\alpha+1}{1-\partial_{\xi_3}}\right)(-\partial_{\xi_3} + \partial_{\xi_3}^2) - n\right]{}_{\mathcal{SLG}}\mathbb{L}_n^{[\alpha,m]}(\xi_1,\xi_2,\xi_3,\xi_4) = 0. \tag{4.12}$$

Considering the special case (III) from Table 1, we obtain the *Legendre-Laguerre-Gould-Hopper-Pidduck polynomials* ($\mathbb{L}$e$\mathbb{LGHPDP}$), denoted ${}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$:

$$\frac{\tau}{1-\tau}C_0\Big(\xi_1\ln\frac{1+\tau}{1-\tau}\Big)C_0\Big(-\xi_2[\ln\frac{1+\tau}{1-\tau}]^2\Big)\exp\left(\xi_3\ln\frac{1+\tau}{1-\tau} + \xi_4[\ln\frac{1+\tau}{1-\tau}]^m\right) = \sum_{n=0}^{\infty}{}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)\frac{\tau^n}{n!}. \tag{4.13}$$

Corollary 4.3. *The $\mathbb{L}$e$\mathbb{LGHPDP}$ ${}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)$ satisfy:*

(i) ***Operational identities:***

$$\begin{aligned} {}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]} &= \exp\left(\xi_4\partial_{\xi_3}^m - \mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{\mathbb{PD}_n(\xi_3)\}, && (4.14)\\ &= \exp\left(-\mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3} + \mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2\right)\{{}_{H^{[m]}}\mathbb{PD}_n(\xi_3,\xi_4)\}. && (4.15)\end{aligned}$$

(ii) ***Shift identity:***

$$ {}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]}(\xi_1,\xi_2,\xi_3+u,\xi_4)=\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(u)\,{}_{\mathcal{SLG}}\mathbb{PD}_k^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4). \tag{4.16} $$

(iii) ***Quasi-monomial operators:***

$$ \hat{M}=\left(\xi_3-\partial_{\xi_1}^{-1}+2\partial_{\xi_2}^{-1}\partial_{\xi_3}+m\xi_4\partial_{\xi_3}^{m-1}-\frac{1-\partial_{\xi_3}}{1+\partial_{\xi_3}}\right)\frac{1}{(1-\partial_{\xi_3}^2)},\qquad \hat{P}=\ln\frac{1+\partial_{\xi_3}}{1-\partial_{\xi_3}}. \tag{4.17} $$

(iv) ***Differential equation:***

$$ \left[\left(\xi_3-\partial_{\xi_1}^{-1}+2\partial_{\xi_2}^{-1}\partial_{\xi_3}+m\xi_4\partial_{\xi_3}^{m-1}-\frac{1-\partial_{\xi_3}}{1+\partial_{\xi_3}}\right)\frac{\ln\frac{1+\partial_{\xi_3}}{1-\partial_{\xi_3}}}{(1-\partial_{\xi_3}^2)}-n\right]{}_{\mathcal{SLG}}\mathbb{PD}_n^{[m]}(\xi_1,\xi_2,\xi_3,\xi_4)=0. \tag{4.18} $$

5. Discussion

The results established in this paper carry significance at several levels, which we discuss here.

Structural significance. The embedding of the LeLGHP family into the Sheffer framework via the pair $(\mu(\tau),\sigma(\tau))$ is not a mere formal generalization. It provides a unified algebraic setting in which quasi-monomial operators, differential equations, and determinantal representations emerge automatically from the generating structure, without requiring separate derivation for each subcase. In particular, all ten polynomial families listed in Table 1—including the generalized Hermite, Laguerre, Bernoulli, Hahn, Peters, and Poisson–Charlier polynomials—are embedded in the LeLGHSP hierarchy through appropriate choices of $\mu(\tau)$ and $\sigma(\tau)$.

The four operational representations (Theorem 2.3) reveal that the LeLGHSP are obtained from simpler polynomial families by applying combinations of the operators $\exp(\xi_4\partial_{\xi_3}^m)$ (Gould–Hopper shift), $\exp(-\mathcal{D}_{\xi_1}^{-1}\partial_{\xi_3})$ (Legendre/Bessel–Tricomi embedding), and $\exp(\mathcal{D}_{\xi_2}^{-1}\partial_{\xi_3}^2)$ (Laguerre embedding) to known Sheffer polynomials. This hierarchical decomposition is both computationally useful, as it reduces the study of LeLGHSP to that of simpler families, and it is conceptually illuminating, as it makes the role of each parameter (ξ_1,ξ_2,ξ_4,m) transparent.

The multiplicative and derivative operators $\hat{M}$ and $\hat{P}$ (Theorem 3.1) act as raising and lowering operators on the polynomial space: $\hat{M}$ maps ${}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}$ to ${}_{\mathcal{SLG}}\mathcal{S}_{n+1}^{[m]}$, and $\hat{P}$ maps it to $n\,{}_{\mathcal{SLG}}\mathcal{S}_{n-1}^{[m]}$. This ladder structure is the polynomial analog of creation and annihilation operators in quantum mechanics, and the Weyl commutation relation $[\hat{P},\hat{M}]=\hat{1}$ confirms that the LeLGHSP span an irreducible representation of the Weyl–Heisenberg algebra. The associated differential equation (Theorem 3.2) then appears as an eigenvalue equation $\hat{M}\hat{P}\{{}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}\}=n\,{}_{\mathcal{SLG}}\mathcal{S}_n^{[m]}$, with eigenvalue n.

Special cases and physical interpretation. The Hermite differential equation derived in Corollary 4.1(iv) can be regarded as a generalized heat equation in the variable ξ_3, with mth order diffusion governed by ξ_4. For $m=2$, it reduces to the standard Hermite equation. Similarly, the Laguerre case connects to radial Schrödinger equations in quantum mechanics, and the Pidduck case has combinatorial interpretations in terms of permutation statistics. These correspondences suggest that the LeLGHSP framework can serve as a computational tool in problems where multiple special function families interact.

Geometric interpretation of surface plots. The surface plots in Figures 1 and 2 confirm the analytical predictions. The linear degree-one member is planar; degree two exhibits the expected quadratic curvature from the mixed $\xi_1\xi_3$ interaction; degree three shows multiple inflection regions from the cubic Legendre–Laguerre coupling. Comparing $m = 1$ and $m = 2$ for degree three demonstrates how increasing m distorts the surface topology by introducing higher-order Gould–Hopper contributions through ξ_4.

Limitations of the current work. The present paper is primarily algebraic and operational in nature. Several aspects lie outside its current scope and are left for future work. No numerical experiments, convergence analysis of the generating series, or error estimates are included. The orthogonality of the LeLGHSP (if any) and their weight functions remain open questions; it is possible that the family is not orthogonal with respect to any standard measure, in which case biorthogonality or weak orthogonality would need to be investigated. Physical boundary conditions under which the derived differential equations define self-adjoint operators have not been studied. Finally, the generalization to q-deformed or fractional-order settings is proposed as future work rather than included here.

6. Conclusions

In this paper, a new class of Legendre–Laguerre–Gould–Hopper-based Sheffer polynomials (LeLGHSP) has been introduced and investigated. The main contributions are as follows.

- **Generating functions** (Theorem 2.1): A closed-form generating function governed by the Sheffer pair $(\mu(\tau), \sigma(\tau))$ was established in two equivalent forms, one in terms of $\sigma^{-1}(\tau)$ and one in terms of $\mathcal{D}(\tau)$ and $\mathbb{H}(\tau)$.
- **Series representation** (Theorem 2.2): An explicit quadruple-sum formula for the coefficients of the LeLGHSP was derived by expanding each factor of the generating function and collecting powers of $\mathbb{H}(\tau)$.
- **Operational identities** (Theorem 2.3): Four operational rules were established, expressing the LeLGHSP in terms of simpler Sheffer, Gould–Hopper, Laguerre-type Sheffer, and Legendre-type Sheffer polynomials via inverse-derivative exponential operators. These identities clarify the hierarchical structure of the family.
- **Shift relations** (Theorems 2.4 and 3.3): Two shift identities were proved, one expanding LeLGHSP with shifted argument in terms of associated Sheffer polynomials and LeLGHSP, and the other expanding them in terms of LeLGH associated Sheffer polynomials and Sheffer polynomials.
- **Determinantal definition** (Theorem 2.5): A representation via Riordan arrays and Cramer's rule was established, providing a computationally effective alternative to the generating function approach and linking the family to general linear interpolation theory.
- **Quasi-monomial operators and differential equations** (Theorems 3.1 and 3.2): Multiplicative and derivative operators satisfying the Weyl algebra commutation relation were constructed, and the associated differential equation was derived as a direct consequence of the monomiality principle.
- **Special cases** (Section 4): The Hermite-type ($g\mathbb{L}e\mathbb{LGHHP}$), Laguerre-type ($g\mathbb{L}e\mathbb{LGHLP}$), and Pidduck-type ($\mathbb{L}e\mathbb{LGHPDP}$) subclasses were investigated in a unified manner, recovering their generating functions, operational identities, shift formulas, quasi-monomial operators, and differential equations as corollaries of the general theory.

Motives and prospects for future research. The operational framework developed here naturally suggests several directions for further investigation.

1. **Fractional-order extensions.** Replacing the inverse derivative operator $\mathcal{D}_{\xi_i}^{-1}$ by Riemann–Liouville or Caputo fractional integral operators of order $\alpha \in (0, 1)$ would yield fractional-order LeLGHSP. This extension connects to the growing literature on fractional special functions and has potential applications in anomalous diffusion models.
2. ***q*-analog.** Constructing q-deformed analog via quantum calculus (q-derivatives and q-integrals) would link the LeLGHSP to q-Hermite and q-Laguerre polynomials, with applications in combinatorics and quantum groups.
3. **Orthogonality and zeros.** Although the present work is primarily algebraic and operational, the study of orthogonality relations, weight functions, and zero distributions of the LeLGHSP is of interest for numerical analysis and quadrature rules.
4. **Asymptotic analysis.** Large-n asymptotic behavior of the LeLGHSP coefficients and zeros, using steepest-descent or Plancherel–Rotach methods, is an open problem with implications for approximation theory.
5. **Boundary value problems and integral transforms.** The differential equations obtained in Theorems 3.2 and Corollaries 4.1–4.3 can be studied as boundary value problems on specific domains. Connections to Laplace, Fourier, and Mellin transform pairs should be explored.
6. **Probabilistic and umbral-combinatorial interpretations.** Sheffer sequences have natural interpretations as moments of probability distributions and as counting sequences for combinatorial structures. Identifying the probabilistic model whose moments are the LeLGHSP is an interesting open question.
7. **Multivariate extensions.** The four-variable LeLGHSP can potentially be extended to k-variable families by nesting additional Bessel–Tricomi or Gould–Hopper structures, further enriching the landscape of multivariate special polynomials.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflict of interest.

Author contributions

Conceptualization: SAW, WAK, MZ, MBJ, KK; Writing – original draft: SAW, WAK; Writing – review & editing: SAW, WAK, MZ, MBJ, KK; Software: SAW, WAK, MZ, KK, MBJ; Validation: MZ, KK; Project supervision: SAW, WAK. All authors have read and approved the final manuscript.

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