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Homogenized limits of Stokes flow and advective transport in thin perforated domains

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Abstract: We deal with the rigorous homogenization and dimension reduction of flow and transport problems posed in thin ε-periodic perforated layers wit a thickness of order ε^α with $\alpha \in (0,1)$. Therefore the thickness of the layer is large compared with its porosity. The aim is the derivation of effective models for $\varepsilon \to 0$, when the thickness of the layer tends to zero. For the flow problem, we consider incompressible Stokes equations with a pressure boundary condition on the top/bottom of the layer. The transport problem is given by reaction–diffusion–advection problem with advective flow governed by the fluid velocity from the Stokes model. Furthermore, we treat different scalings for the diffusion coefficient modelling low and fast diffusion in the horizontal direction. In the limit, a Darcy-type law is obtained for the Stokes flow with the Darcy velocity depending only on the derivative of the Darcy pressure in the vertical direction. The effective equation for the transport problem is again one of the diffusion advection-type including homogenized coefficients, and with advective flow given by the Darcy velocity and only taking place in the vertical direction. In the case of slow diffusion in the vertical direction, effective diffusion only takes place in the vertical direction, where, in the case of high diffusion in the horizontal direction, we obtain effective diffusion in all space directions. To pass to the limit, we use the method of two-scale convergence adapted to our microscopic geometry, which is based on uniform a priori estimates. Critical parts in the derivation of the macro-models are the control of the fluid pressure, for which we construct a Bogovskii operator for thin perforated domains with arbitrary boundary conditions on the top/bottom, and the strong two-scale convergence for the microscopic solution of the transport equation, which is necessary to pass to the limit in the advective term. This strong convergence is established by using a Kolmogorov–Simon compactness argument.

Keywords: homogenization; dimension reduction; Stokes equation; two-scale convergence; reaction–diffusion–advection equation

1. Introduction

The study of fluid flow and the transport of chemical substances or heat through thin, heterogeneous layers is crucial for numerous applications, ranging from medicine to geosciences and materials science. The different scalings in the microscopic geometry, such as the thickness and the porosity of the layer, lead to high computational challenges. To overcome this problem, effective models for $\varepsilon \to 0$ are derived, carrying information about the processes on the microscopic scale in homogenized coefficients. The present work deals with the rigorous homogenization and dimension reduction of transport and flow problems posed in thin ε-periodic perforated layers with a thickness of order ε^α with $\alpha \in (0, 1)$. Here, $0 < \varepsilon \ll 1$ is a small parameter which describes the ratio between the macroscopic size (the diameter) of the thin layer and its heterogeneity. Since $\alpha < 1$, we are dealing with layers whose thickness is much greater than their internal heterogeneity, and therefore, we have a periodic structure in all space directions. However, for $\varepsilon \to 0$, the thickness of the layer tends to zero, and therefore, we are dealing with a simultaneous homogenization and dimension reduction problem. The fluid flow is described by the (quasi-) stationary incompressible Stokes equations. The transport is given by a reaction–diffusion advection equation, with advection given by the velocity field of the Stokes problem, and different scalings of the diffusion coefficient with respect to ε and α. Using two-scale compactness methods, we derive, for $\varepsilon \to 0$, limit problems on the macroscopic scale. For the transport equation, again, a reaction–diffusion–advection is obtained; for the Stokes problem, we obtain a Darcy-type equation.

To pass to the limit $\varepsilon \to 0$, we make use of the two-scale convergence adapted to thin layers with thickness of order ε^α. This method captures both, the homogenization in the horizontal direction, and the dimension reduction in the vertical direction. This type of two-scale convergence was introduced in [1]; and is an extension of the two-scale convergence from the seminal works [2, 3] in domains; see also [4] for a first definition of two-scale convergence in thin homogeneous domains and [5] for thin heterogeneous layers. On the basis of uniform a priori estimates for the microscopic solutions with respect to ε and α, we obtain two-scale compactness results for these solutions. More precisely, for the fluid velocity (and pressure) and the solution of the transport equation, we get different scalings of the gradient with respect to ε and α, leading to a different structure of the limit functions. Hence, in a first step, we show general two-scale compactness results for different types of scalings of the gradient, which generalizes the results from [2] to the thin layer. Although the thickness of the layer goes to zero for $\varepsilon \to 0$, and therefore the thin layer reduces to a lower dimensional manifold, the macroscopic variable of the two-scale limit depends on n variables. In other words, the limit function is defined on a thick layer of order 1. This is a crucial difference compared with the case $\alpha = 1$, when the thin layer only consists of one micro-cell in the vertical direction (with no periodicity in the vertical direction).

The fluid flow is described by the incompressible Stokes equations. On the top/bottom of the thin layer, we impose a pressure boundary condition, and on the lateral part of the layer and the perforations inside the layer, a no-slip condition is assumed. In a first step, we derive uniform a priori estimates for the fluid velocity and the fluid pressure. Here, the crucial part is the bound for the pressure. For this, we construct a Bogovskii operator (for vector fields having arbitrary boundary values on the top/bottom of the thin layer) with suitable scalings of its operator norm with respect to ε and α adapted to the microscopic geometry. We solve the divergence equation in a thin homogeneous layer with thickness of order ε^α. Now, by applying the restriction operator from [6] for ε-periodic domains, we obtain the

Bogovskii operator for the perforated thin layer. On the basis of these a priori estimates and the general two-scale convergence results, we get the compactness of the microscopic Stokes solutions. As usual, the two-scale limit of the velocity is depending on the macroscopic and the microscopic variables, while the limit pressure, the so-called Darcy pressure, only depends on the macroscopic variable. However, in contrast to the classical case in perforated domains (see [6, 7]), we only obtain H^1-regularity in the x_n component of the Darcy pressure. Hence, the resulting Darcy equation does not depend on the whole gradient of the pressure; but only on the derivative with respect to the x_n component. This is also a significant difference to the case where $\alpha = 1$; see, for example, [8, 9]. In that case, the macroscopic variable for the Darcy pressure (and also the fluid limit) is given on a lower-dimensional manifold, and the full gradient (with respect to the horizontal direction) also contributes to the Darcy velocity.

There is extensive literature on the homogenization of Stokes flow in perforated domains. Here, we have to mention the seminal work of Tartar in [6, 7], where a restriction operator for connected perforations is constructed. We also refer to the pioneering works [10–12] in which Stokes flows through perforated domains with inclusions of different sizes were considered. In particular, reference [11] dealt with flow across a perforated layer including tiny holes of critical size, leading to a Brinkman law in the limit across the effective interface. In this paper, we consider perforations of order ε, usually leading to a Darcy law. The critical scaling for the thin layer is an open question. The homogenization and simultaneous dimension reduction of the Stokes equations on a thin, periodically perforated layer has received less attention than perforated bulk domains, except for heterogeneities of the thin layer with a specific structure. For example, with a rough surface given as a graph (see [13, 14]) or the perforations with a cylindrical shape (see [8], and also [15] for a formal treatment). In [9], the case $\alpha = 1$ (only one layer of micro cells in the vertical direction) with Navier slip boundary conditions on the perforations and the top/bottom of the thin layer is considered. Both the specific choice of α and the slip condition (particularly on the top/bottom) lead to qualitatively distinct effective equations compared with our problem. More precisely, the effective model only takes place on a $(n-1)$ dimensional manifold and contains an additional force term for the Darcy velocity, due to the Navier slip boundary condition on the perforations. Furthermore, full H^1-control of the macroscopic pressure is obtained. Anguiano et al. [8] dealt with the homogenization and dimension reduction of the Navier–Stokes equations with no-slip boundary conditions in a periodically perforated thin domain, where the periodicity scale differs from the thickness scale. Compared with the present work, for homogenization and dimension reduction the unfolding method is used, which gives an equivalent characterization of the two-scale convergence. Furthermore, their analysis is limited to the case of cylindrical solids, without oscillations in the vertical direction. We emphasize, that this has a significant influence on the limit model. Further, in our case of a pressure boundary condition on the top/bottom of the thin layer, the a priori estimate for the pressure is of order $\varepsilon^{\frac{3\alpha}{2}}$ instead of order $\varepsilon^{\frac{\alpha}{2}}$ in the no-slip case. As a special case of our results for arbitrary perforations, we also consider the case of cylindrical inclusions. This leads to a Darcy flow depending only on the vertical direction of the Darcy pressure (and the vertical forces), where the horizontal flow depends only on the horizontal forces multiplied by the permeability tensor. In our paper, we use the two-scale convergence defined in [1], where this method was used for the homogenization and dimension reduction of a linearized fluid structure interaction problem coupling instationary Stokes flow with linear elasticity for different scalings. As in our case, the thickness of the layer tends to zero for $\varepsilon \to 0$, but the periodic oscillations within the layer are much smaller than the thickness. In the limit, a Biot law is obtained, where the generalized Darcy velocity also only depends on the n-th derivative of the Darcy pressure. The crucial difference is in the proof of the a priori estimates for the microscopic pressure. As usual in the derivation of the Biot law, the continuity condition between the fluid flow and the time-derivative of the displacement at the fluid structure interface allows the control of the pressure. In our case, we have to construct a restriction operator adapted to the microscopic geometry. We also refer to [16] for the derivation of a Biot plate equation in the case where $\alpha = 1$.

The last part of our paper deals with the homogenization and dimension reduction of a transport problem, modeling, for instance, the evolution of a chemical species' concentration (as well as heat transfer), given by a reaction–diffusion–advection equation with advection governed by the Stokes velocity. Additionally, we consider different scalings for the diffusion coefficient depending on both ε and α. We cover the cases of fast and slow diffusion in the horizontal direction. On the top and bottom of the layer, we consider Dirichlet boundary conditions, and on the lateral part of the layer and on the perforations, we consider homogeneous Neumann boundary conditions. As for the fluid flow, the first step involves deriving ε-uniform a priori estimates for the concentration. Naturally, these depend on the scaling of the diffusion coefficient. In order to deal with the advection term, strong two-scale convergence of the microscopic concentration is required, for which we need control of the time-derivative. For this, additional L^∞-estimates are needed. In the case of slow diffusion, standard energy bounds for the time-derivative and Sobolev norms (depending on the scaling for the diffusion) are insufficient to guarantee the strong two-scale convergence of the concentration. Therefore, further control of the spatial variable is needed. This is achieved by estimating the differences in the shifts of the microscopic concentration, which finally allows an application of Kolmogorov–Simon type compactness results. The different diffusion coefficients lead to two distinct limit models; as $\varepsilon \to 0$. In the case of fast diffusion, the homogenized model exhibits effective diffusion in both the horizontal and vertical direction, while advection only takes place in the vertical x_n direction. It is worth emphasizing that; even though the layer reduces to a lower dimensional manifold, we still get an effect in the vertical direction. Conversely, in the case of slow diffusion, the weaker estimates only ensure diffusive and convective flow in the vertical direction.

The homogenization of reaction–diffusion–advection equations for slow and fast diffusion is nowadays well understood. We refer to the seminal works [17, 18]. The latter deals particularly with the case of slow diffusion with a specific nonlinear reaction term for the scalar case. More general nonlinearities and systems are considered in [19, 20]. As well as in [21], where a general two-scale compactness result of thhe Kolmogorov–Simon type is shown for problems with low diffusion. Rigorous results for the derivation of effective models via simultaneous dimension reduction and homogenization via two-scale compactness for reaction–diffusion problems including nonlinearities for the case where $\alpha = 1$ can be found in [5, 22–24] for different scalings of the diffusion coefficient (for dimension reduction problems including nonlinearities, see, for example, references [25, 26] for different scalings of the diffusion coefficient). In these contributions the thin layer is coupled to bulk regions, but the mathematical methods used for the homogenization process are similar to the methods used in this paper. Linear parabolic problems for perforated thin layers, again for $\alpha = 1$, are treated in [27] via asymptotic expansions and error estimates for different boundary conditions. We also refer to [28] for a poroelastic layer, coupling the linearized fluid–structure interaction, leading to a Biot system in the limit. A reaction–diffusion–advection equation modeling heat flow with the advective term given by the solution of a Stokes equation, was recently treated in [29] for a thin layer with a rough surface, given as a graph. For $\alpha \in (0,1)$ rigorous results seem to be missing. Our paper is a first essential step, and we treat two critical scalings. The principal ideas to establish the strong two-scale convergence of the concentration in our transport problem are similar to those used in the aforementioned papers, particularly for regarding two-scale compactness results (for thin domains) of the Kolmogorov–Simon type, which we generalized to our geometrical setting.

The paper is structured as follows. In Section 2, we introduce the microscopic formulations of both the Stokes and transport problems, formulate the macroscopic models, outline the key steps in their derivation, and present the main results of our analysis. We also provide a detailed description of the underlying microscopic geometry. Section 3 gives an introduction to the two-scale convergence adapted to thin, heterogeneous layers with a thickness of order ε^α. We further establish compactness results for H^1–functions, depending on different scalings of the gradient. The macroscopic models for the fluid and transport problem are derived in Sections 4 and 5, respectively. For both, we proceed in the following way: First, we establish ε-uniform a priori estimates, then, we show the two-scale compactness results, and, finally, we derive the macroscopic models.

1.1. Notations

Let $n \in \mathbb{N}$. For $\Omega \subset \mathbb{R}^n$, a bounded Lipschitz domain, we use $L^p(\Omega)$ and $W^{1,p}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with $p \in [1,\infty]$. In particular, for $p = 2$, we write $H^1(\Omega)^d := W^{1,2}(\Omega)^d$. With S being a subset of $\partial\Omega$, we let $H^1(\Omega, S)$ denote the $H^1(\Omega)$ functions vanishing on S in the sense of traces. For norms defined on vector–valued function spaces X^d with $d \in \mathbb{N}$, we omit the upper index and write $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$. For a Banach space X and $p \in [1,\infty]$, we denote the usual Bochner spaces by $L^p(\Omega, X)$ and, in particular, $L^p((0,T), X)$ when time is involved. For the dual space of X, we use the notation X'. The duality pairing between X' and X is denoted by $\langle\cdot,\cdot\rangle_X$.

We consider the following periodic function spaces. Let $Y := (0,1)^n$. Then $C^\infty_{\mathrm{per}}(Y)$ is the space of smooth functions on $\mathbb{R}^n$, which are Y-periodic, and $H^1_{\mathrm{per}}(Y)$ is the closure of $C^\infty_{\mathrm{per}}(Y)$ with respect to the norm on $H^1(Y)$. Further, for a subset $Y^* \subset Y$ with $\partial Y \subset \partial Y^*$, we use $H^1_{\mathrm{per}}(Y^*)$ to denote the space of functions from $H^1_{\mathrm{per}}(Y)$ restricted to Y^*. For $y \in Y$, we use the notation $\bar{y} := (y_1, \dots, y_{n-1})$.

For a function $f \in H^1(\Sigma \times (a,b))$ with $a < b$ and $\Sigma \subset \mathbb{R}^{n-1}$, we write $\nabla_{\bar{x}} f(x) := (\partial_1 f(x), \dots, \partial_{n-1} f(x))$ (with $\bar{x} := (x_1, \dots, x_{n-1})$ for $x \in \Omega$) and also identify this vector in a natural way with a vector in $\mathbb{R}^n$ by $\nabla_{\bar{x}} f(x) := (\nabla_{\bar{x}} f(x), 0)$. If Σ is a rectangular domain with an integer side length, we use $H^1_\#(\Sigma \times (a,b))$ to denote the space of Σ-periodic functions in $\bar{x}$-direction, and similar; we use $C^\infty_\#(\overline{\Omega})$ to denote the space of smooth and Σ-periodic functions. Finally, we define the Frobenius product $B : C := \mathrm{tr}(B^\top C) = \sum_{i,j=1}^n B_{ij} C_{ij}$ for $B, C \in \mathbb{R}^{n\times n}$.

2. The microscopic models and main results

In this section, we briefly introduce the microscopic problems for the fluid flow and the transport problem, explain the essential steps used for the derivation of the macroscopic models, and formulate the main results of this paper. The aim of this paper is twofold: First, we study a Stokes problem subject to no-slip boundary conditions on the oscillating boundary and pressure boundary conditions on the upper and lower surfaces of the thin layer. We then perform rigorous homogenization and dimension reduction for this setting. Here, we only deal with the stationary problem. In the next step, we treat a reaction–diffusion–advection problem, where the advective velocity is given as the solution of the Stokes problem considered before (here, we assume that the Stokes problem is quasi–stationary, which does not influence the previous results). For this, we assume different scalings for the diffusion coefficient, leading to a different macroscopic behavior. To pass to the limit $\varepsilon \to 0$, we use the method of two-scale convergence adapted to thin layers of order ε^{α}. The different scalings lead to different bounds for (the gradient of) the concentration, and we prove general two-scale compactness results for Sobolev functions to deal with these different cases.

2.1. The microscopic domain

Let $n \in \mathbb{N}$ with $n \geq 2$ (for the treatment of the transport problem in Section 5, we will restrict this assumption to $n \leq 4$) and $\Sigma := (a, b) \subset \mathbb{R}^{n-1}$ with $a, b \in \mathbb{Z}^{n-1}$ and $a_i < b_i$ for $i = 1, ..., n-1$. Additionally, we assume that $\varepsilon > 0$ with $\varepsilon^{-1} \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\varepsilon^{\alpha-1} \in \mathbb{N}$. This is necessary to construct the perforated layer via suitable reference cells, such that no micro-cells are intersected by the outer boundary.

We emphasize that all our results and proofs can be easily transferred to the case of a curved domain Σ, as long as we consider a kind of safety zone around $\partial\Sigma \times (-\varepsilon^{\alpha}, \varepsilon^{\alpha})$ with a width of order ε. More precisely, micro–cells intersecting the lateral boundary do not include perforations. This choice is, of course, quite common in the homogenization theory.

Example 2.1. *We want to show that it is possible to construct such ε and α. Choose a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n = n^{-q}$ and $\alpha = 1 - p/q$, where $p, q \in \mathbb{N}$ with $q > p$ and* $\gcd(q, p) = 1$. *It obviously holds that $\varepsilon_n^{-1} \in \mathbb{N}$ and $\alpha \in (0, 1)$. Furthermore, it holds $\varepsilon_n^{\alpha-1} = n^{q(1-\alpha)} = n^p \in \mathbb{N}$.*

We define the whole layer by

$$\Omega_{\varepsilon,\alpha} := \Sigma \times (\varepsilon^{-\alpha}, \varepsilon^{\alpha}),$$

together with its top/bottom $S_{\varepsilon}^{\pm} := \Sigma \times \{\pm\varepsilon^{\alpha}\}$. Within the layer, we have a fluid part $\Omega_{\varepsilon,\alpha}^{f}$ and a solid part $\Omega_{\varepsilon,\alpha}^{s}$, which are both non–empty and have a periodical microscopic structure. More precisely, we define the reference cell

$$Y := (0, 1)^n.$$

The cell consists again of a fluid part Y_f and a solid part Y_s with a common interface $\Gamma := \mathrm{int}(\overline{Y_f} \cap \overline{Y_s})$ (here, we use the relative topology on the intersection). Hence, we have

$$Y = Y_f \cup Y_s \cup \Gamma.$$

We assume that Y_f and Y_s are non–empty, open and connected with a Lipschitz boundary and fulfill $Y_f \cap Y_s = \emptyset$. Now, we introduce the set

$$K_{\varepsilon} := \{k \in \mathbb{Z}^n \ : \ \varepsilon(Y + k) \subset \Omega_{\varepsilon,\alpha}\}.$$

In particular, we have

$$\Omega_{\varepsilon,\alpha} = \mathrm{int}\left(\bigcup_{k\in K_{\varepsilon}} \varepsilon(\overline{Y} + k)\right).$$

We define the fluid part of the layer via

$$\Omega_{\varepsilon,\alpha}^{f} := \mathrm{int}\left(\bigcup_{k\in K_{\varepsilon}} \varepsilon(\overline{Y}_f + k)\right).$$

The fluid–structure interface between the fluid and the solid part is denoted by

$$\Gamma_{\varepsilon,\alpha} := \operatorname{int}\left(\bigcup_{k\in K_\varepsilon} \varepsilon(\Gamma + k)\right).$$

We assume that $\Omega^f_{\varepsilon,\alpha}$ is a connected Lipschitz domain. Furthermore, we denote the upper and lower part of the boundary of $\Omega^f_{\varepsilon,\alpha}$ as

$$S^{\pm}_{\varepsilon,f} := \partial\Omega^f_{\varepsilon,\alpha} \cap (\partial\Sigma \times \{\pm\varepsilon^\alpha\}),$$

and the lateral part of $\Omega^f_{\varepsilon,\alpha}$ is defined by

$$\partial_D\Omega^f_{\varepsilon,\alpha} := \partial\Omega^f_{\varepsilon,\alpha} \cap (\partial\Sigma \times (-\varepsilon^\alpha, \varepsilon^\alpha)).$$

A diagram illustrating the reference cell alongside the microscopic layer can be found in Figure 1. Finally, we introduce the macroscopic domain (the thick layer)

$$\Omega := \Sigma \times (-1, 1),$$

which can also be obtained by rescaling the domain $\Omega_{\varepsilon,\alpha}$. We denote the top/bottom of Ω as

$$S^{\pm}_1 := \Sigma \times \{\pm 1\},$$

and the lateral boundary as

$$\partial_D\Omega := \partial\Omega \setminus (S^+_1 \cup S^-_1).$$

Here, the notation D is related to the no-slip (Dirichlet) boundary condition for the fluid problem. However, we consider Neumann boundary or periodic boundary conditions on this part of the boundary in the transport problem.

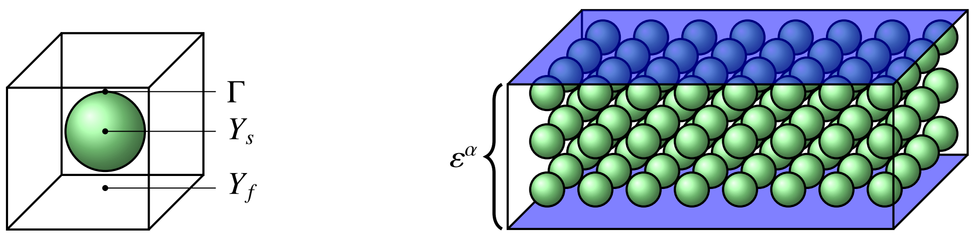

Figure 1. Left: The reference cell Y. Right: The layer $\Omega^f_{\varepsilon,\alpha}$ with $S^{\pm}_{\varepsilon,f}$ (blue part) and $\partial_D\Omega^f_{\varepsilon,\alpha}$ (the rest of the boundary).

2.2. *The flow problem*

We begin by formulating the microscopic problem for the fluid flow. To this end, we consider the stationary, linear Stokes system describing an incompressible, Newtonian, and isothermal fluid. The equations are written in dimensionless form; for simplicity, all physical constants are set equal to 1.

More precisely, we are looking for a fluid velocity $u_{\varepsilon,\alpha}\colon \Omega^f_{\varepsilon,\alpha} \to \mathbb{R}^n$ and a fluid pressure $p_{\varepsilon,\alpha}\colon \Omega^f_{\varepsilon,\alpha} \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(e(u_{\varepsilon,\alpha})) + \nabla p_{\varepsilon,\alpha} &= f_{\varepsilon,\alpha} && \text{in } \Omega^f_{\varepsilon,\alpha}, \\ \nabla\cdot u_{\varepsilon,\alpha} &= 0 && \text{in } \Omega^f_{\varepsilon,\alpha}, \\ u_{\varepsilon,\alpha} &= 0 && \text{on } \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha}, \\ (-e(u_{\varepsilon,\alpha}) + p_{\varepsilon,\alpha}\mathrm{Id})\nu &= p^b_{\varepsilon,\alpha}\nu && \text{on } S^{\pm}_{\varepsilon,f}. \end{aligned} \tag{2.1}$$

Here, $e(u_{\varepsilon,\alpha}) \coloneqq \frac{1}{2}(\nabla u_{\varepsilon,\alpha} + \nabla u^{\mathrm{T}}_{\varepsilon,\alpha})$ denotes the symmetric gradient of $u_{\varepsilon,\alpha}$, $f_{\varepsilon,\alpha}$ is a volume force, and $p^b_{\varepsilon,\alpha}$ is a pressure boundary condition; see Section 4 for the precise assumptions on the data. In a first step, we show the ε-uniform a priori estimates (depending additionally on the parameter α) for the fluid velocity $u_{\varepsilon,\alpha}$ and the fluid pressure $p_{\varepsilon,\alpha}$, where, as usual, the critical point is the derivation of the estimate for the pressure. For this, we first show a Bogovskii result for the whole layer $\Omega_{\varepsilon,\alpha}$ (without perforation), and then apply the restriction operator to obtain a Bogovskii result in the perforated layer $\Omega^f_{\varepsilon,\alpha}$, which allows us to control the pressure. More precisely, we get

$$\varepsilon^{-2}\left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^{-1}\left\|\nabla u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^{-\alpha}\|p_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\varepsilon^{\alpha/2}.$$

On the basis of this estimate and general two-scale compactness results, we obtain limit functions $u_0 \in L^2(\Omega, H^1_{\mathrm{per}}(Y))^n$ with $u_0 = 0$ in $Y \setminus Y_f$ and $\nabla_y\cdot u_0 = 0$ in $\Omega\times Y$; and $p_0 \in L^2(\Omega)$ (with $\Omega = \Sigma\times(-1,1)$ being the thick layer), such that up to a subsequence (we refer to Section 3 for the definition of the two-scale convergence), we have

$$\varepsilon^{-2}u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} u_0, \qquad \varepsilon^{-1}\nabla u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \nabla_y u_0, \qquad \varepsilon^{-\alpha}p_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} p_0.$$

Here and in the following, if not stated otherwise, we extend the functions by zero to the whole layer. We emphasize that the limit fluid velocity u_0 depends on both; the macroscopic variable $x \in \Omega$ and the microscopic variable $y \in Y$. In the next step, we derive a two-scale homogenized Stokes problem (see Eq (4.2)), which includes all the necessary information of the limit problem. From this, we find that u_0 can be expressed as

$$u_0(x,y) = \sum_{i=1}^{n-1} f^i_0 w_i + (f^n_0 - \partial_{x_n}p_0)w_n, \tag{2.2}$$

where (w_i, q_i) for $i = 1,\dots,n$ are the solutions of the cell problem (4.3). Compared with homogenization results for Stokes flow in porous media, here, only the n-th derivative of the (Darcy) pressure p_0 contributes to the macroscopic fluid velocity. We define the average of u_0 as the Darcy velocity

$$\bar{u}(x) \coloneqq \int_{Y_f} u_0(x,y)\,\mathrm{d}y.$$

It follows from the divergence-free condition of $u_{\varepsilon,\alpha}$ that the n-th component of the Darcy velocity is constant in the x_n-direction; that is, $\partial_{x_n}\bar{u}^n = 0$. Hence, with the permeability tensor K defined in Eq (4.4), we obtain

$$\begin{aligned} \bar{u} &= K(f_0 - e_n\partial_{x_n}p_0) && \text{in } \Omega, \\ \partial_{x_n}\bar{u} &= 0 && \text{in } \Omega. \end{aligned}$$

As usual for the homogenization of Stokes problems, we will see that the pressure boundary condition on $S^{\pm}_{\varepsilon,\alpha}$ leads to the Dirichlet boundary condition $p_0 = p_0^b$ for the limit pressure. In total, we get the Darcy equation

$$\begin{aligned} \partial_{x_n}\left[K(f_0 - e_n\partial_{x_n}p_0)\right]_n &= 0 \quad \text{in } \Omega, \\ p_0 &= p_0^b \quad \text{on } S_1^{\pm}. \end{aligned} \tag{2.3}$$

In summary, we obtain the following result:

Theorem 2.2. *Let* $(u_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})$ *be the weak solution of the microscopic problem* (2.1). *In this case, under Assumptions (S1) and (S2),* $u_0 \in L^2(\Omega, H^1_{\mathrm{per}}(Y))^n$ *with* $u_0 = 0$ *in* $Y \setminus Y_f$ *and* $\nabla_y \cdot u_0 = 0$ *in* $\Omega \times Y$*, and* $p_0 \in L^2(\Omega)$ *exists such that* $\varepsilon^{-2}u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} u_0$ *and* $\varepsilon^{-\alpha}p_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} p_0$. *In addition, the Darcy pressure* p_0 *is the unique weak solution of the Darcy equation in Eq* (2.3) *and* u_0 *is given by Eq* (2.2).

We emphasize that in the limit $\varepsilon \to 0$, the macroscopic quantities are given in the thick layer $\Omega = \Sigma \times (-1, 1)$, although the thin layer reduces to a lower dimensional manifold Σ. Here, we have an essential different behavior compared with the case when the layer is of thickness ε, where the limit functions only depend on the macroscopic variable $\bar{x} \in \Sigma$.

2.3. The transport problem

In the next step, we consider the transport problem for a concentration given by a reaction–diffusion–advection equation with the advection $u_{\varepsilon,\alpha}$; given as the solution of Eq (2.1) (now depending smoothly on time). More precisely, we are looking for a $c_{\varepsilon,\alpha} : (0,T) \times \Omega^f_{\varepsilon,\alpha} \to \mathbb{R}$ which is the solution of

$$\begin{aligned} \frac{1}{\varepsilon^\alpha}\partial_t c_{\varepsilon,\alpha} - \nabla\cdot\Big(D^\alpha_\varepsilon \nabla c_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\Big) &= \frac{1}{\varepsilon^\alpha}g_{\varepsilon,\alpha} \quad \text{in } (0,T)\times\Omega^f_{\varepsilon,\alpha}, \\ -\Big(D^\alpha_\varepsilon \nabla c_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\Big)\cdot\nu &= 0 \quad \text{on } (0,T)\times\Gamma_{\varepsilon,\alpha}, \\ c_{\varepsilon,\alpha} &= c^b_\varepsilon \quad \text{on } (0,T)\times(S^+_{\varepsilon,f}\cup S^-_{\varepsilon,f}), \\ c_{\varepsilon,\alpha}(0) &= 0 \quad \text{in } \Omega^f_{\varepsilon,\alpha}, \end{aligned} \tag{2.4a}$$

with a source term $g_{\varepsilon,\alpha}$ and a boundary concentration c^b_ε. The system is closed with suitable boundary conditions on $\partial_D\Omega^f_{\varepsilon,\alpha}$, which depend on the choice of the diffusion coefficient. More precisely, for the diffusion coefficient D^α_ε, we consider two different scalings with respect to ε and α as follows:

(D1) $D^\alpha_\varepsilon = \varepsilon^\alpha D I \in \mathbb{R}^{n\times n}$,

(D2) $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha}, \dots, \varepsilon^{-\alpha}, \varepsilon^\alpha) \in \mathbb{R}^{n\times n}$,

with a fixed constant $D > 0$ and the unit matrix I in $\mathbb{R}^{n\times n}$ (since in the first case the diffusion matrix D^α_ε acts as a scalar, we will often just write $D^\alpha_\varepsilon = \varepsilon^\alpha D$). On the lateral boundary, we consider the following boundary condition:

$$\begin{aligned} -\Big(D^\alpha_\varepsilon \nabla c_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\Big)\cdot\nu = 0 \quad \text{on } (0,T)\times\partial_D\Omega^f_{\varepsilon,\alpha}, \ \text{if } D^\alpha_\varepsilon = \varepsilon^\alpha D, \\ c_{\varepsilon,\alpha} \text{ is } \Sigma\text{-periodic, if } D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha}, \dots, \varepsilon^{-\alpha}, \varepsilon^\alpha). \end{aligned} \tag{2.4b}$$

Hence, in Case (D1), we consider homogeneous Neumann boundary conditions, and in Case (D2), periodic boundary conditions. Although, we are particularly interested in the macroscopic behavior inside the domain, the effects at the lateral boundary are also important for applications. We emphasize that the different choices are elemental for the derivation of the limit problem. While for $D_\varepsilon^\alpha = \varepsilon^\alpha D$, it would be no problem to consider periodic boundary conditions, our proof fails for Neumann boundary conditions in Case (D2); see also Remark 5.9.

From a physical point of view, the first case (D1) treats slow diffusion in the $\bar{x}$-direction, where the second case (D2) deals with fast diffusion in the horizontal direction. We will see that in the first case, the diffusion in the macroscopic limit is only in the vertical direction; in the case of fast diffusion, we get diffusion in all space directions.

We proceed in the same way as for the Stokes equation and first establish uniform a priori estimates with respect to ε. More precisely, we get

$$\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} + \|\sqrt{D^\alpha_\varepsilon}\nabla c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C, \tag{2.5}$$

which immediately implies the existence of a limit function $c_0 \in L^2((0,T)\times\Omega)$, specifically independent of the microscopic variable y, such that, up to a subsequence, we have

$$c_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} c_0.$$

Further, we obtain a bound for the time-derivative on the dual space of the Sobolev space carrying the norm induced by the left-hand side of the previous inequality. For this, we need, in particular, an L^∞-bound for the concentration $c_{\varepsilon,\alpha}$; to control the advective term. Using an Kolmogorov–Simon type compactness argument; based on additional estimates for the differences of the shifts of the microscopic solutions, we can then establish also the strong two-scale convergence of the sequence $c_{\varepsilon,\alpha}$. These are necessary to pass to the limit $\varepsilon \to 0$ in the advective term (since we only get the weak two-scale convergence of the fluid velocity $u_{\varepsilon,\alpha}$).

From inequality (2.5), we see that the difference between the two cases lies in the scaling for the gradient $\nabla_{\bar{x}} c_{\varepsilon,\alpha}$ with respect to the first $n-1$ components, leading to different regularity results (weak differentiability) of the limit function with respect to the spatial variable.

The case $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha},\dots,\varepsilon^{-\alpha},\varepsilon^\alpha)$: This leads to

$$\|\nabla_{\bar{x}} c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} + \varepsilon^\alpha\|\partial_{x_n} c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

We obtain $c_0 \in H^1(\Omega)$ and also the existence of the corrector functions $\bar{c}_1 \in L^2((0,T)\times\Omega\times Y_f)$ with $\nabla_{\bar{y}}\bar{c}_1 \in L^2((0,T)\times\Omega\times Y_f)^{n-1}$ and being $(0,1)^{n-1}$-periodic with respect to $\bar{y}$, and $c_1 \in L^2((0,T)\times\,\Omega, H^1_{\mathrm{per}}(0,1))$ (only depending on the y_n variable), such that (up to a subsequence) we have

$$(\nabla_{\bar{x}} c_{\varepsilon,\alpha}, \varepsilon^\alpha\partial_{x_n} c_{\varepsilon,\alpha}) \overset{2,\alpha}{\rightharpoonup} \nabla c_0 + (\nabla_{\bar{y}}\bar{c}_1, \partial_{y_n} c_1).$$

With these compactness results, we are able to pass to the limit in the weak variational equation associated with Eq (2.4). Here, we modify the standard homogenization approach based on the two-scale convergence to our thin structure. By choosing suitable test functions, we first derive cell problems for $\bar{c}_1$ and c_1; (see Eqs (5.10) and (5.13)), which allow us to express $\bar{c}_1$ and c_1 in terms of ∇c_0 and suitable cell solutions independent of the macroscopic quantities. More precisely, we have

$$\bar{c}_1(t,x,y) = \sum_{i=1}^{n-1} \partial_{x_i} c_0(t,x)\bar{\chi}_i(y), \qquad c_1(t,x,y_n) = \partial_{x_n} c_0(t,x)\chi_n(y_n),$$

where $\bar{\chi}_i$ for $i = 1,\dots,n-1$, and χ_n are the solutions of the cell problems in Eqs (5.12) and (5.15). In the next step, we choose macroscopic test functions, also capturing the dimension reduction to find through the expression of $\bar{c}_1$ and c_1 that c_0 is a unique solution of the macroscopic problem

$$\begin{aligned} \partial_t c_0 - \nabla\cdot(D^*\nabla c_0 - c_0\bar{u}e_n) &= \bar{g}_0 && \text{in } (0,T)\times\Omega, \\ c_0 &= c_0^b && \text{on } (0,T)\times S_1^{\pm}, \\ c_0(0) &= 0, \\ c_0 \text{ is } \Sigma\text{-periodic.} \end{aligned} \tag{2.6}$$

Here, D^* is an effective diffusion coefficient; (see Eq (5.18)), and $\bar{u}$ is the Darcy velocity obtained in Theorem 2.2. Here, $\bar{g}_0$ is the average of the (two-scale) limit of $g_{\varepsilon,\alpha}$. First of all, we notice that, macroscopically, there is also an effect in the x_n-direction, although the layer reduces for $\varepsilon \to 0$ to a lower–dimensional manifold. The effective diffusion takes place in the horizontal and the vertical direction, where the advective flux only takes place in the vertical direction. Finally, let us summarize our results in the following main theorem.

Theorem 2.3. *Let $c_{\varepsilon,\alpha}$ be the microscopic solution of Eq* (2.4) *and $D_\varepsilon^\alpha = D\mathrm{diag}(\varepsilon^{-\alpha},\dots,\varepsilon^{-\alpha},\varepsilon^{\alpha})$. Under Assumptions (T1), (T2), and (T3), $c_0 \in L^2((0,T),H^1(\Omega))$ exists such that $c_{\varepsilon,\alpha} \overset{2,\alpha}{\to} c_0$ and c_0 is the unique weak solution of the macroscopic problem* (2.6)*. Furthermore, the advection term $u_{\varepsilon,\alpha}$ behaves as in Theroem 2.2*

The proof of the compactness result, together with some additional convergence of the gradient $\nabla c_{\varepsilon,\alpha}$, can be found in Section 5.2, and the derivation of the macroscopic model is presented in Section 5.3, where we also give the definition of a weak solution of the problem (2.6).

The case $D_\varepsilon^\alpha = \varepsilon^\alpha DI$: In the case of low diffusion in the horizontal direction, we obtain, for the gradient $\nabla_{\bar{x}} c_{\varepsilon,\alpha}$, a scaling of the form

$$\varepsilon^{\frac{\alpha}{2}}\|\nabla_{\bar{x}} c_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C.$$

In this case, we obtain no spatial regularity (differentiability) of c_0 with respect to $\bar{x}$. Though we again obtain the weak two-scale convergence of $c_{\varepsilon,\alpha}$ to a limit function $c_0 \in L^2((0,T)\times\Omega)$, we only obtain $\partial_{x_n} c_0 \in L^2((0,T)\times\Omega)$. Further, for the gradient, we obtain the convergence

$$\varepsilon^\alpha \nabla c_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \partial_{x_n} c_0 e_n + \nabla_y c_1.$$

Now, compared with the previous case, the scaled gradient $\nabla c_{\varepsilon,\alpha}$ does not convergence to the sum of the full gradient of c_0, but only the n-th component, and the rest is included in the gradient (with respect to y) of the corrector c_1. However, we can proceed in the same way as in the previous case, but this time, we get the expression

$$c_1(t,x,y) = \partial_{x_n} c_0(t,x)\chi_n(y),$$

again, using the cell solution χ_n of Eq (5.12). Finally, the macroscopic model reads as follows:

$$\begin{aligned} \partial_t c_0 - \partial_{x_n}(D^*_{nn}\partial_{x_n} c_0 - c_0\bar{u}^n) &= \bar{g}_0 \quad \text{in } (0,T)\times\Omega, \\ c_0 &= c_0^b \quad \text{on } (0,T)\times S_1^{\pm}, \\ c_0(0) &= 0. \end{aligned} \tag{2.7}$$

In this case, we only have diffusive and convective flow in the vertical direction. To pass to the limit in the advective term, again we need the strong (two-scale) convergence of the concentration. We summarize the main result in the following theorem.

Theorem 2.4. *Let $c_{\varepsilon,\alpha}$ be the microscopic solution of Eq* (2.4) *and $D^\alpha_\varepsilon = \varepsilon^\alpha D$. Under Assumptions (T1), (T2), and (T3), $c_0 \in L^2((0,T)\times\Omega)$ with $\partial_{x_n} c_0 \in L^2((0,T)\times\Omega)$ exists such that $c_{\varepsilon,\alpha} \overset{2,\alpha}{\to} c_0$ and c_0 is the unique weak solution of the macroscopic problem* (2.7)*. Furthermore, the advection term $u_{\varepsilon,\alpha}$ behaves as in Theorem 2.2*

For the proof of this result, we again refer to Sections 5.2 and 5.3.

3. The two-scale convergence for thin heterogeneous layers

In this section, we introduce the two-scale convergence adapted to thin layers with a thickness of order ε^α, with $\alpha \in (0,1)$. Compared with the classical two-scale convergence (see [2, 3]), we introduce an additional variable capturing the dimension reduction. Such a two-scale convergence was introduced in [1]. Here, we use a slightly different notation. More precisely, in [1], they work with the rescaled thin layer (in the fixed domain Ω), whereas we work in the physical domain $\Omega_{\varepsilon,\alpha}$ (and later in $\Omega^f_{\varepsilon,\alpha}$). Our aim is the derivation of several compactness results for functions with weak derivatives with different bounds with respect to the scaling parameter ε (and α).

Definition 1. *We say a sequence $v_{\varepsilon,\alpha} \in L^2(\Omega_{\varepsilon,\alpha})$ converges (weakly) in the two-scale sense to a limit function $v_0 \in L^2(\Omega\times Y)$, if, for all $\phi \in L^2(\Omega, C^0_{\text{per}}(Y))$, it holds that*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha}\int_{\Omega_{\varepsilon,\alpha}} v_{\varepsilon,\alpha}(x)\cdot\phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}, \frac{x_n}{\varepsilon}\right) \mathrm{d}x = \int_\Omega\int_Y v_0(x,y)\cdot\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

We write $v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0$.

We say that a two-scale convergent sequence $v_{\varepsilon,\alpha}$ converges strongly in the two-scale sense, if additionally it holds that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{\alpha}{2}}\|v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} = \|v_0\|_{L^2(\Omega\times Y)}.$$

We write $v_{\varepsilon,\alpha} \overset{2,\alpha}{\to} v_0$.

Remark 3.1.

(i) For $w_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} w_0$, it holds that

$$\|w_0\|_{L^2(\Omega\times Y)} \le \liminf_{\varepsilon\to 0} \varepsilon^{-\frac{\alpha}{2}}\|w_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})}.$$

(ii) As in the usual two-scale convergence (see [2, 3], it is straightforward to show, that a product between a strongly and weakly two-scale convergent sequences converges in the distributional sense. More precisely, for $w_{\varepsilon,\alpha} \in L^2(\Omega_{\varepsilon,\alpha})$ and $v_{\varepsilon,\alpha} \in L^2(\Omega_{\varepsilon,\alpha})$ with $w_{\varepsilon,\alpha} \xrightarrow{2,\alpha} w_0$ and $v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0$, it holds for every $\phi \in C^\infty(\overline{\Omega})$ that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha}} w_{\varepsilon,\alpha} v_{\varepsilon,\alpha} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x = \int_\Omega \int_Y w_0 v_0 \phi \, \mathrm{d}y \, \mathrm{d}x.$$

In our case, we need this result for the case where $w_0 \in L^2(\Omega)$ depends only on the macroscopic variable, which simplifies the proof (no density argument for the approximation of w_0 is necessary).

In the following, we provide several compactness results for sequences in $u_{\varepsilon,\alpha}$ in $H^1(\Omega_{\varepsilon,\alpha})$ for different scalings of the gradient. First of all, we quote the standard compactness result for suitable bounded sequences in $L^2(\Omega_{\varepsilon,\alpha})$, which can be obtained by similar arguments as in the proofs of [2].

Lemma 3.2. *For every sequence $v_{\varepsilon,\alpha} \in L^2(\Omega_{\varepsilon,\alpha})$ such that*

$$\|v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}},$$

$v_0 \in L^2(\Omega \times Y)$ exists such that (up to a subsequence) $v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0$.

Now, our first compactness result for the Sobolev functions treats the case when the gradient is of one ε-order lower then the function itself, leading to the case that the limit function depends on the macroscopic and the microscopic variable.

Proposition 3.3. *Let $v_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha})$ with*

$$\left\|v_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla v_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\alpha/2}.$$

Then $v_0 \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ exists such that

$$v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0, \quad \varepsilon\nabla v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \nabla_y v_0.$$

This result was shown in [1, Lemma B.4] for the two-scale convergence on the rescaled domain Ω and, in our notation, it follows directly by the transformation rule.

Proposition 3.4. *Let $v_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha})$ be a sequence such that*

$$\|v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} + \varepsilon^\alpha \|\nabla v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Then $v_0 \in L^2(\Omega)$ with $\partial_{x_n} v_0 \in L^2(\Omega)$ and $v_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y)/\mathbb{R})$ exists such that up to a subsequence

$$v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0, \qquad \varepsilon^\alpha \nabla v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \partial_{x_n} v_0 e_n + \nabla_y v_1.$$

Proof. By the compactness result in Lemma 3.2, $v_0 \in L^2(\Omega \times Y)$ and $\xi_0 \in L^2(\Omega \times Y)^n$ exists such that up to a subsequence

$$v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0, \qquad \varepsilon^\alpha \nabla v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \xi_0.$$

Since $\alpha \in (0,1)$, we obtain

$$\varepsilon \|\nabla v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}$$

and Proposition 3.3 immediately implies $\nabla_y v_0 = 0$, and therefore $v_0(x,y) = v_0(x)$ is independent of y. Next, we show $\partial_{x_n} v_0 \in L^2(\Omega)$. Choose $\phi \in C_0^\infty(\Omega)$ and use the two-scale compactness of $v_{\varepsilon,\alpha}$ and $\varepsilon^\alpha \nabla v_{\varepsilon,\alpha}$ to obtain

$$\begin{aligned}\int_\Omega \int_Y \xi_0^n \phi \,\mathrm{d}y\,\mathrm{d}x &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha}} \varepsilon^\alpha \partial_{x_n} v_{\varepsilon,\alpha} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha}} v_{\varepsilon,\alpha} \partial_{x_n} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x = -\int_\Omega \int_Y v_0 \partial_{x_n} \phi \,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

This implies $\partial_{x_n} v_0 = \int_Y \xi_0^n \,\mathrm{d}y$ (remember that v_0 is independent of y). It remains to identify the limit ξ_0. For this, we choose $\phi \in C_0^\infty(\Omega, C_{\text{per}}^\infty(Y))^n$ such that $\nabla_y \cdot \phi = 0$ and obtain, by similar arguments as given above,

$$\begin{aligned}\int_\Omega \int_Y \xi_0 \cdot \phi \,\mathrm{d}y\,\mathrm{d}x &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha}} \varepsilon^\alpha \nabla v_{\varepsilon,\alpha} \cdot \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha}} v_{\varepsilon,\alpha} \left[\varepsilon^\alpha \nabla_{\bar{x}} \cdot \phi + \partial_{x_n} \phi_n\right]\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x \\ &= -\int_\Omega \int_Y v_0 \partial_{x_n} \phi_n \,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \int_Y \partial_{x_n} v_0 e_n \cdot \phi \,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

By the Helmholtz decomposition, we obtain the existence of $v_1 \in L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$ such that $\xi_0 = \partial_{x_n} w_0 e_n + \nabla_y v_1$, which gives the desired result.

It is well known (and can be easily shown by adapting the proof of the trace inequality from [30]), that the functions $w \in L^2(\Omega)$ with a weak derivative $\partial_{x_n} w \in L^2(\Omega)$ have traces on $L^2(S_1^\pm)$. Hence, under the conditions of Proposition 3.4, we find that v_0 has traces on the top/bottom $S_1^\pm$ of Ω. The following result shows that the trace of $v_{\varepsilon,\alpha}$ on $S_\varepsilon^\pm$ is preserved under the two-scale convergence.

Proposition 3.5. *Under the assumptions of Proposition 3.4, it holds that*

$$\|v_{\varepsilon,\alpha}\|_{L^2(S_\varepsilon^\pm)} \leq C.$$

Further, up to a subsequence, it holds that $v_{\varepsilon,\alpha}|_{S_\varepsilon^\pm} \overset{2}{\rightharpoonup} v_0|_{S_1^\pm}$ *weakly in* $L^2(\Sigma)$ *(in the standard two-scale sense, see [2]).*

Proof. Define $\tilde{v}_{\varepsilon,\alpha} \in H^1(\Omega)$ by $\tilde{v}_{\varepsilon,\alpha}(x) := v_{\varepsilon,\alpha}(\bar{x}, \varepsilon^\alpha x_n)$. Then, a simple rescaling argument gives

$$\|v_{\varepsilon,\alpha}\|_{L^2(S_\varepsilon^\pm)} = \|\tilde{v}_{\varepsilon,\alpha}\|_{L^2(S_1^\pm)} \leq C\left(\|\tilde{v}_{\varepsilon,\alpha}\|_{L^2(\Omega)} + \|\partial_{x_n} \tilde{v}_{\varepsilon,\alpha}\|_{L^2(\Omega)}\right)$$

$$= C\left(\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} + \varepsilon^{\frac{\alpha}{2}}\|\partial_{x_n} v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})}\right) \leq C.$$

In particular, $\eta_0^\pm \in L^2(\Sigma \times (0,1)^{n-1})$ exists such that up to a subsequence, $v_{\varepsilon,\alpha}|_{S_\varepsilon^\pm} \overset{2}{\rightharpoonup} \eta_0^\pm$. Choosing $\phi \in C_0^\infty(\Omega \cup S_1^\pm, C^1_{\text{per}}((0,1)^{n-1}))$ (constantly extended in the y_n-direction), we obtain (with $\nu_n = \pm 1$ on $S_\varepsilon^\pm$)

$$\begin{aligned}
\int_\Sigma \int_{(0,1)^{n-1}} \eta_0^\pm \phi(\bar{x}, \pm 1, \bar{y}) d\bar{x} &= \lim_{\varepsilon\to 0} \int_\Sigma v_{\varepsilon,\alpha}|_{S_\varepsilon^\pm} \phi\left(\bar{x}, \pm 1, \frac{\bar{x}}{\varepsilon}\right) d\bar{x} \\
&= \lim_{\varepsilon\to\infty} \int_{S_\varepsilon^\pm} v_{\varepsilon,\alpha} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}\right) d\sigma \\
&= \lim_{\varepsilon\to\infty} \pm \int_{\Omega_{\varepsilon,\alpha}} \partial_{x_n} v_{\varepsilon,\alpha} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}\right) + \frac{1}{\varepsilon^\alpha} v_{\varepsilon,\alpha} \partial_{x_n} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}\right) \mathrm{d}x \\
&= \pm \int_\Omega \int_Y (\partial_{x_n} v_0 + \partial_{y_n} v_1)\phi + v_0 \partial_{x_n}\phi \,\mathrm{d}y\,\mathrm{d}x = \int_{S_1^\pm} \int_{(0,1)^{n-1}} v_0 \phi \,\mathrm{d}\sigma,
\end{aligned}$$

where, at the end, we used integration by parts, $\int_Y \partial_{y_n} v_1 \,\mathrm{d}y = 0$, and the fact that v_0 is independent of y. This gives the desired result.

Next, we give a two-scale compactness result when the components of the gradient are scaled differently. Here, we directly show the result for the perforated domain $\Omega_{\varepsilon,\alpha}^f$, for which we have to use a special Helmholtz decomposition. Let us compare the situation with the scaling in Proposition 3.4. A function $v_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}^f)$ fulfilling the estimate in this proposition (with $\Omega_{\varepsilon,\alpha}$ replaced by $\Omega_{\varepsilon,\alpha}^f$), can be extended with the extension operator E_ε from Lemma 5.7 to a function $E_\varepsilon v_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha})$ fulfilling the same a priori estimate. Hence, we immediately obtain, from Proposition 3.4, the following (here, we use the characteristic function to denote the zero extension of a function):

$$\chi_{\Omega_{\varepsilon,\alpha}^f} v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} v_0, \qquad \varepsilon^\alpha \chi_{\Omega_{\varepsilon,\alpha}^f} \nabla v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f}\left(\partial_{x_n} v_0 e_n + \nabla_y v_1\right). \tag{3.1}$$

In other words, the extension operator allows us to treat the perforated layer as a homogeneous layer (this is a common approach in the homogenization theory for porous media). However, for different scalings for the gradient $\nabla_{\bar{x}}$ with respect to the horizontal variable and the n-th derivative ∂_{x_n}, as given in the following proposition (related to the case of high diffusion in th horizontal direction), such an argument is not possible, since the extension operator from Lemma 5.7 only allows us to control the partial derivatives of the extended function by the full gradient of the function itself. This gives another estimate for the extended function; see also Section 5 for more details. We introduce the space

$$H^1_{\text{per},\nabla_{\bar{y}}}(Y_f) := \{p \in L^2(Y_f) \;:\; \nabla_{\bar{y}} p \in L^2(Y_f)^{n-1},\ p \text{ is } (0,1)^{n-1}\text{-periodic}\},$$

together with the norm

$$\|p\|^2_{H^1_{\text{per},\nabla_{\bar{y}}}(Y_f)} := \|p\|^2_{L^2(Y_f)} + \|\nabla_{\bar{y}} p\|^2_{L^2(Y_f)}.$$

Proposition 3.6. *Let $v_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha})$ be a sequence such that*

$$\|v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \|\nabla_{\bar{x}} v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^\alpha \|\partial_{x_n} v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Then $v_0 \in H^1(\Omega)$, $v_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y_f)/\mathbb{R})$ with $\nabla_{\bar{y}} v_1 = 0$, and $\bar{v}_1 \in L^2(\Omega, H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f))$ exists such that up to a subsequence

$$\chi_{\Omega^f_{\varepsilon,\alpha}} v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} v_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} (\nabla_{\bar{x}} v_{\varepsilon,\alpha}, \varepsilon^\alpha \partial_{x_n} v_{\varepsilon,\alpha}) \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} \left(\nabla v_0 + (\nabla_{\bar{y}} \bar{v}_1, \partial_{y_n} v_1) \right).$$

The result is also valid for $\Omega_{\varepsilon,\alpha}$ instead of $\Omega^f_{\varepsilon,\alpha}$.

Proof. From the assumed a priori estimates on $v_{\varepsilon,\alpha}$, we get the existence of $v_0 \in L^2(\Omega \times Y)$ and $\xi_0 \in L^2(\Omega \times Y)^n$ (both vanishing in Y_s), such that up to a subsequence

$$\chi_{\Omega^f_{\varepsilon,\alpha}} v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} v_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} (\nabla_{\bar{x}} v_{\varepsilon,\alpha}, \varepsilon^\alpha \partial_{x_n} v_{\varepsilon,\alpha}) \overset{2,\alpha}{\rightharpoonup} \xi_0.$$

Since we also have

$$\varepsilon^\alpha \|\nabla v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}},$$

we can apply Proposition 3.4 (see also Eq (3.1)), to obtain $v_0(x, y) = \chi_{Y_f}(y) v_0(x)$ with $v_0 \in L^2(\Omega)$ such that $\partial_{x_n} v_0 \in L^2(\Omega)$, and $\xi_0^n = \chi_{Y_f}(\partial_{x_n} v_0 + \partial_{y_n} v_1)$ for some $v_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y)/\mathbb{R})$. Since $\varepsilon^\alpha \nabla_{\bar{x}} v_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} 0$, we also have $\nabla_{\bar{y}} v_1 = 0$ in Y_f, and therefore v_1 is independent of $\bar{y}$. Now, for all $\phi \in C^\infty_0(\Omega, C^\infty_{\mathrm{per}}(Y))^{n-1}$ with $\nabla_{\bar{y}} \cdot \phi = 0$, we have (with $\bar{\xi}_0 := (\xi_0^1, \dots, \xi_0^{n-1})$)

$$\begin{aligned} \int_\Omega \int_{Y_f} \bar{\xi}_0 \cdot \phi \, dy \, dx &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} \nabla_{\bar{x}} v_{\varepsilon,\alpha} \cdot \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) dx \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} v_{\varepsilon,\alpha} \nabla_{\bar{x}} \cdot \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) dx = -\int_\Omega \int_{Y_f} v_0 \nabla_{\bar{x}} \cdot \phi \, dy \, dx. \end{aligned}$$

If we first choose a ϕ constant with respect to y and use that v_0 is independent of y, we get $v_0 \in H^1(\Omega)$. Then, with ϕ being arbitrary (still $\nabla_{\bar{y}} \cdot \phi = 0$), we get

$$\int_\Omega \int_{Y_f} (\bar{\xi}_0 - \nabla_{\bar{x}} v_0) \cdot \phi \, dy \, dx = 0,$$

which again implies, by the Helmholtz decomposition below, the existence of $\bar{v}_1 \in L^2(\Omega \times Y_f)$ and $\nabla_{\bar{y}} \bar{v}_1 \in L^2(\Omega \times Y_f)^{n-1}$ (unique up to an L^2-function only depending on y_n), such that $\bar{\xi}_0 = \nabla_{\bar{x}} v_0 + \nabla_{\bar{y}} \bar{v}_1$.

We define the space

$$L_{\sigma,\mathrm{per}} := \left\{ u \in L^2(Y_f)^{n-1} \ : \ \int_{Y_f} u \cdot \nabla_{\bar{y}} \phi \, dy = 0 \text{ for all } \phi \in C^\infty_{\mathrm{per}}(\overline{Y_f}) \right\}.$$

Since $L_{\sigma,\mathrm{per}}$ is closed, we get $L^2(Y_f)^{n-1} = L_{\sigma,\mathrm{per}} \perp L^\perp_{\sigma,\mathrm{per}}$. Obviously, we have $\nabla_{\bar{y}} H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f) \subset L^\perp_{\sigma,\mathrm{per}}$, since $C^\infty_{\mathrm{per}}(\overline{Y_f})$ is dense in $H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f)$. Next, we define the quotient space $\widetilde{H} := H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f)/\ker(\nabla_{\bar{y}})$. Now, for a given $v \in L_{\sigma,\mathrm{per}} \subset L^2(Y_f)^{n-1}$, we consider the problem

$$\widetilde{a}([p], [\phi]) := \int_{Y_f} \nabla_{\bar{y}} p \cdot \nabla_{\bar{y}} \phi \, dy = \int_{Y_f} v \cdot \nabla_{\bar{y}} \phi \, dy =: l([\phi])$$

for every $[p], [\phi] \in \widetilde{H}$, and $p \in [p]$, $\phi \in [\phi]$. This problem is well-defined, since the kernel of $\nabla_{\bar{y}}$ consists of L^2-functions only depending on y_n. By the Lax–Milgram lemma, we obtain the existence of a unique solution $[p] \in \tilde{H}$, and therefore the existence of $p \in H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f)$ (unique up to a function depending on only on y_n), such that

$$\int_{Y_f} (\nabla_{\bar{y}} p - v) \cdot \nabla_{\bar{y}} \phi \, \mathrm{d}y = 0,$$

for all $\phi \in H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f)$ (in particular, $\phi \in C^\infty_{\mathrm{per}}(\overline{Y_f})$). Hence, $\nabla_{\bar{y}} p - v \in L_{\sigma,\mathrm{per}}$ and since $\nabla_{\bar{y}} p, v \in L^\perp_{\sigma,\mathrm{per}}$, we obtain $\nabla_{\bar{y}} p - v \in L_{\sigma,\mathrm{per}} \cap L^{\mathrm{per}}_{\sigma,\mathrm{per}} = \{0\}$, which implies

$$L^2(Y_f)^{n-1} = L_{\sigma,\mathrm{per}} \perp \nabla_{\bar{y}} H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f).$$

This finishes the proof.

In the following, we also identify the space $\{\phi \in H^1_{\mathrm{per}}(Y_f) \ : \ \nabla_{\bar{y}} \phi = 0\}$ with the space $H^1_{\mathrm{per}}(0,1)$. Finally, we consider the asymptotic expansion of $v_{\varepsilon,\alpha}$ which is justified by the previous compactness results. We make the ansatz

$$v_{\varepsilon,\alpha}(x) = v_0\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \varepsilon^{1-\alpha} v_1\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \varepsilon \bar{v}_1\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \dots.$$

Hence, we obtain the following expansions for our three compactness results:

– Proposition 3.3: Oscillations already occur in the lowest order term and we get

$$v_{\varepsilon,\alpha}(x) = v_0\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right).$$

– Proposition 3.4: No oscillations in the zeroth order term. Oscillations occur in the term of order $\varepsilon^{1-\alpha}$

$$v_{\varepsilon,\alpha}(x) = v_0\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) + \varepsilon^{1-\alpha} v_1\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right).$$

– Proposition 3.6: In this case, the gradient in the horizontal direction is of the same order as the function itself, leading to a situation when the corrector of order $\varepsilon^{1-\alpha}$ is independent of the horizontal microscopic variable $\bar{y}$, and an additional corrector of order ε is necessary as follows

$$v_{\varepsilon,\alpha}(x) = v_0\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) + \varepsilon^{1-\alpha} v_1\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x_n}{\varepsilon}\right) + \varepsilon \bar{v}_1\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right).$$

Remark 3.7. *All the results can be generalized in an obvious way to the time-dependent case. More precisely, a sequence $v_{\varepsilon,\alpha} \in L^p((0,T), L^2(\Omega_{\varepsilon,\alpha}))$ with $p \in [1,\infty)$ converges weakly in the two-scale sense to a limit function $v_0 \in L^p((0,T), L^2(\Omega \times Y))$, if, for all $\phi \in L^p((0,T), L^2(\Omega, C^0_{\mathrm{per}}(Y)))$, it holds that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} \int_0^T \int_{\Omega_{\varepsilon,\alpha}} v_{\varepsilon,\alpha}(x) \cdot \phi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}, \frac{x_n}{\varepsilon}\right) \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \int_Y v_0(x,y) \cdot \phi(t,x,y) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t.$$

Our compactness results are valid for $p \in (1,\infty)$. The strong two-scale convergence can be generalized in a straightforward way. We use the same notation as in the time-independent case. It should be clear from the context for which regularity with respect to time can be obtained for the convergence.

4. The flow problem

In this section, we deal with the homogenization and dimension reduction of the microscopic Stokes problem (2.1). We show uniform a priori estimates for the fluid velocity and the fluid pressure with respect to the parameters ε and α. Using the general compactness results from Section 3, we get the two-scale convergence of $u_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$ to suitable limit functions, which allow us to pass to the limit in the microscopic problem by choosing test-functions adapted to the structure of the limit function.

We start with the weak formulation for the microscopic problem and state the assumptions on the data We say that $(u_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}) \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n \times L^2(\Omega^f_{\varepsilon,\alpha})$ is a weak solution of Eq (2.1), if $\nabla \cdot u_{\varepsilon,\alpha} = 0$ and for all $\phi_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n$, it holds that

$$\int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha}) : e(\phi_{\varepsilon,\alpha})\,\mathrm{d}x - \int_{\Omega^f_{\varepsilon,\alpha}} p_{\varepsilon,\alpha} \nabla \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x = \int_{\Omega^f_{\varepsilon,\alpha}} f_{\varepsilon,\alpha} \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x - \int_{S^\pm_{\varepsilon,f}} p^b_{\varepsilon,\alpha} \nu \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}\sigma$$

or, equivalently

$$\int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha}) : \nabla\phi_{\varepsilon,\alpha}\,\mathrm{d}x - \int_{\Omega^f_{\varepsilon,\alpha}} p_{\varepsilon,\alpha} \nabla \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x = \int_{\Omega^f_{\varepsilon,\alpha}} f_{\varepsilon,\alpha} \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x - \int_{S^\pm_{\varepsilon,f}} p^b_{\varepsilon,\alpha} \nu \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}\sigma.$$

Under the assumption that $p^b_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha})$ (see below for the assumptions on the data), we can use the divergence theorem in the last term on the right-hand side to obtain

$$\int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha}) : e(\phi_{\varepsilon,\alpha})\,\mathrm{d}x - \int_{\Omega^f_{\varepsilon,\alpha}} (p_{\varepsilon,\alpha} - p^b_{\varepsilon,\alpha}) \nabla \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x = \int_{\Omega^f_{\varepsilon,\alpha}} (f_{\varepsilon,\alpha} - \nabla p^b_{\varepsilon,\alpha}) \cdot \phi_{\varepsilon,\alpha}\,\mathrm{d}x. \tag{4.1}$$

Assumptions on the data:

(S1) For the volume force $f_{\varepsilon,\alpha} \in L^2(\Omega^f_{\varepsilon,\alpha})^n$, we assume

$$\left\| f_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\alpha/2}.$$

(S2) For the boundary pressure, we assume $p^b_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha})$ such that

$$\|p^b_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^\alpha \|\nabla p^b_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{3\alpha}{2}}.$$

Furthermore, $p^b_0 \in L^2(\Omega)$ with $\partial_{x_n} p^b_0 \in L^2(\Omega)$ exists such that $\varepsilon^{-\alpha} p^b_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} p^b_0$ and $p^b_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y)/\mathbb{R})$ such that $\nabla p^b_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \partial_{x_n} p^b_0 e_n + \nabla p^b_1$ (see Section 3 for the definition of the two-scale convergence).

A simple example of a non-zero boundary pressure $p^b_{\varepsilon,\alpha}$ fulfilling the previous assumptions is $p^b_{\varepsilon,\alpha} = \varepsilon^\alpha$. Although the pressure is pointwise small, it gives a contribution to the macroscopic behavior of the problem, since $p^b_0 = 1$.

Corollary 4.1. *The problem* (2.1) *admits a unique weak solution.*

For a fixed ε, this result is classical, and we skip the proof.

4.1. A priori estimates for the microscopic solutions $v_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$

We begin by deriving the estimates for the fluid velocity $u_{\varepsilon,\alpha}$. In order to do so, we introduce the following Poincaré and Korn inequalities on the layer $\Omega^f_{\varepsilon,\alpha}$. The proof is standard and can be obtained by decomposing $\Omega^f_{\varepsilon,\alpha}$ into reference cells and then applying the Poincaré and Korn inequality with zero boundary conditions on Γ.

Lemma 4.2. *Let $v_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha}, \Gamma_{\varepsilon,\alpha})$. Then a constant $C > 0$ not depending on ε exists such that*

$$\left\|v_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon \left\|\nabla v_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

Furthermore, it holds that

$$\left\|v_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla v_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon \left\|e(v_{\varepsilon,\alpha})\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

We are now ready to derive the estimate for the fluid velocity.

Proposition 4.3. *Let $u_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n$ be the weak solution of the Stokes problem* (2.1). *Then it holds that*

$$\left\|e(u_{\varepsilon,\alpha})\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq \varepsilon^{\alpha/2+1}$$

and, in particular,

$$\varepsilon^{-2} \left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^{-1} \left\|\nabla u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\alpha/2}.$$

Proof. We test the weak formulation (4.1) with the solution $u_{\varepsilon,\alpha}$ to obtain the following (the $(p_{\varepsilon,\alpha} - p^b_{\varepsilon,\alpha})$ term vanishes since $\nabla \cdot u_{\varepsilon,\alpha} = 0$):

$$\begin{aligned}\left\|e(u_{\varepsilon,\alpha})\right\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} &\leq \left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\|\nabla p^b_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\ &\leq C\varepsilon^{\alpha/2+1} \left\|\nabla u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.\end{aligned}$$

Hence, with the Korn inequality from Lemma 4.2, we achieve

$$\left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\alpha/2+2}.$$

The estimate of the fluid pressure $p_{\varepsilon,\alpha}$ is less elementary. The goal is to construct and estimate a Bogovskii operator on the thin perforated layer $\Omega^f_{\varepsilon,\alpha}$, in order to obtain test functions $\phi_{\varepsilon,\alpha}$ such that $\nabla \cdot \phi_{\varepsilon,\alpha} = p_{\varepsilon,\alpha}$ in $\Omega^f_{\varepsilon,\alpha}$. We begin by establishing the Bogovskii operator on the whole layer $\Omega_{\varepsilon,\alpha}$. We use the same techniques as in [31, Lemma 5 Step 4], now adapted to the layer of thickness ε^α.

Proposition 4.4. *For all $f_{\varepsilon,\alpha} \in L^2(\Omega_{\varepsilon,\alpha})$, $\psi_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n$ exists such that*

$$\nabla \cdot \psi_{\varepsilon,\alpha} = f_{\varepsilon,\alpha} \quad \text{in } \Omega_{\varepsilon,\alpha}$$

and

$$\left\|\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} + \varepsilon^\alpha \left\|\nabla\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^\alpha \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})}.$$

Proof. We set

$$\tilde{K}_{\varepsilon,\alpha} := \left\{k \in \mathbb{Z}^{n-1} \times \{0\} \,:\, \varepsilon^\alpha(Y+k) \subset \Omega_{\varepsilon,\alpha}\right\}.$$

Since $\varepsilon^\alpha/\varepsilon \in \mathbb{N}$, we have

$$\Omega_{\varepsilon,\alpha} = \operatorname{int}\left(\bigcup_{k\in\tilde{K}_{\varepsilon,\alpha}} \varepsilon^\alpha(Y+k)\right).$$

For $k \in \tilde{K}_{\varepsilon,\alpha}$ and $f \in L^2(\Omega_{\varepsilon,\alpha})$, we define

$$f^k_{\varepsilon,\alpha}\colon Y \to \mathbb{R}, \quad f^k_{\varepsilon,\alpha} = f_{\varepsilon,\alpha}(\varepsilon^\alpha(x+k)).$$

With the use of the Bogovskii operator (see [32, Theorem 5.4]), we obtain $\psi^k_{\varepsilon,\alpha} \in H^1(Y, \partial Y \setminus S^\pm)^n$ with $S^+ := (0,1)^{n-1} \times \{1\}$ and $S^- := (0,1)^{n-1} \times \{0\}$, such that

$$\nabla \cdot \psi^k_{\varepsilon,\alpha} = f^k_{\varepsilon,\alpha}, \quad \left\|\psi^k_{\varepsilon,\alpha}\right\|_{H^1(Y)} \leq C \left\|f^k_{\varepsilon,\alpha}\right\|_{L^2(Y)}.$$

Now, we define

$$\psi_{\varepsilon,\alpha}\colon \Omega_{\varepsilon,\alpha} \to \mathbb{R}^n, \quad \psi_{\varepsilon,\alpha}(x) := \varepsilon^\alpha \psi^k_{\varepsilon,\alpha}\left(\frac{x}{\varepsilon^\alpha} - k\right) \quad \text{for } x \in \varepsilon^\alpha(Y+k).$$

For $\psi_{\varepsilon,\alpha}$, we have

$$\nabla \cdot \psi_{\varepsilon,\alpha} = f_{\varepsilon,\alpha} \quad \text{in } \Omega_{\varepsilon,\alpha}$$

and

$$\begin{aligned}\left\|\nabla\psi_{\varepsilon,\alpha}\right\|^2_{L^2(\Omega_{\varepsilon,\alpha})} &= \sum_{k\in\tilde{K}_{\varepsilon,\alpha}} \int_{\varepsilon^\alpha(Y+k)} \left|\nabla\psi^k_{\varepsilon,\alpha}\left(\frac{x}{\varepsilon^\alpha} - k\right)\right|^2 \mathrm{d}x = \sum_{k\in\tilde{K}_{\varepsilon,\alpha}} \varepsilon^{n\alpha} \int_Y \left|\nabla\psi^k_{\varepsilon,\alpha}\right|^2 \mathrm{d}y \\ &\leq C \sum_{k\in\tilde{K}_{\varepsilon,\alpha}} \varepsilon^{n\alpha} \int_Y \left|f^k_{\varepsilon,\alpha}\right|^2 \mathrm{d}y = C \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})}.\end{aligned}$$

Hence, with the Poincaré inequality, we obtain

$$\|\psi_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^\alpha \|\nabla\psi_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})},$$

which can be obtained in the same way as the inequality in Lemma 4.2 by replacing ε with ε^α (we use the zero boundary conditions of $\psi_{\varepsilon,\alpha}$ on the lateral boundary of $\varepsilon^\alpha(Y+k)$)

$$\left\|\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} + \varepsilon^\alpha \left\|\nabla\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq \varepsilon^\alpha \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})}.$$

We now want to establish the Bogovskii operator on the perforated layer $\Omega^f_{\varepsilon,\alpha}$. This is done via the restiction operator introduced by Allaire in [6, Theorem 2.3]. There is an operator $R_\varepsilon\colon H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n \to H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n$ with

$$\begin{aligned} R_\varepsilon u_{\varepsilon,\alpha} &= u_{\varepsilon,\alpha} \quad \text{for all } u_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n \text{ with } u_{\varepsilon,\alpha} = 0 \text{ in } \Omega^s_{\varepsilon,\alpha}, \\ \nabla \cdot R_\varepsilon u_{\varepsilon,\alpha} &= \nabla \cdot u_{\varepsilon,\alpha} \quad \text{for all } u_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n \text{ with } \nabla \cdot u_{\varepsilon,\alpha} = 0 \text{ in } \Omega^s_{\varepsilon,\alpha},\end{aligned}$$

and

$$\left\|R_\varepsilon u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla R_\varepsilon u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\left(\left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|u_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}\right)$$

for all $u_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n$. Using this operator, we can restrict the Bogovskii operator, constructed in Proposition 4.4, to the perforated layer $\Omega^f_{\varepsilon,\alpha}$.

Corollary 4.5. *For $f_{\varepsilon,\alpha} \in L^2(\Omega^f_{\varepsilon,\alpha})$, $\phi_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n$ exists such that*

$$\nabla \cdot \phi_{\varepsilon,\alpha} = f_{\varepsilon,\alpha} \quad \text{in } \Omega^f_{\varepsilon,\alpha}$$

and

$$\left\|\phi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla \phi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\varepsilon^\alpha \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

Proof. For $f_{\varepsilon,\alpha} \in L^2(\Omega^f_{\varepsilon,\alpha})$ (extended to $\Omega_{\varepsilon,\alpha}$ by zero), we have $\psi_{\varepsilon,\alpha} \in H^1(\Omega_{\varepsilon,\alpha}, \partial_D\Omega_{\varepsilon,\alpha})^n$ with

$$\nabla \cdot \psi_{\varepsilon,\alpha} = f_{\varepsilon,\alpha} \quad \text{in } \Omega_{\varepsilon,\alpha}.$$

By setting $\phi_{\varepsilon,\alpha} := R_\varepsilon \psi_{\varepsilon,\alpha}$, we immediately obtain

$$\nabla \cdot \psi_{\varepsilon,\alpha} = f_{\varepsilon,\alpha} \quad \text{in } \Omega^f_{\varepsilon,\alpha}.$$

Further, we obtain

$$\begin{aligned}\left\|\nabla\phi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} &= \left\|\nabla R_\varepsilon \psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\ &\le C\left(\varepsilon^{-1}\left\|\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})} + \left\|\nabla\psi_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})}\right) \\ &\le C\varepsilon^{\alpha-1}\left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega_{\varepsilon,\alpha})}.\end{aligned}$$

In total, the Poincaré inequality from Lemma 4.2, we obtain

$$\left\|\phi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon \left\|\nabla \phi_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\varepsilon^\alpha \left\|f_{\varepsilon,\alpha}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

Remark 4.6. *We emphasize that the result from Corollary 4.5 is also valid for $\alpha = 1$. Further, we obtain the existence of a Bogovskii operator $\mathcal{B}_{\varepsilon,\alpha} : H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n \to L^2(\Omega^f_{\varepsilon,\alpha})$ with $\nabla \cdot \mathcal{B}_{\varepsilon,\alpha}(f_{\varepsilon,\alpha}) = f_{\varepsilon,\alpha}$ and*

$$\varepsilon\|\nabla\mathcal{B}_{\varepsilon,\alpha}(f_{\varepsilon,\alpha})\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\varepsilon^\alpha\|f_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

There is a crucial difference between considering a pressure boundary condition and a no-slip boundary condition on $S^\pm_{\varepsilon,f}$. In the latter (see, for example [8, 9]), the Bogovskii operator, here denoted $\mathcal{B}^0_{\varepsilon,\alpha}$, has to be defined on $H^1_0(\Omega^f_{\varepsilon,\alpha})^n$ and only fulfills

$$\varepsilon\|\nabla\mathcal{B}^0_{\varepsilon,\alpha}(f_{\varepsilon,\alpha})\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\|f_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

Hence, as can be seen in the following proposition, the pressure estimate can be improved to order $\varepsilon^{\frac{3\alpha}{2}}$ (instead of order $\varepsilon^{\frac{\alpha}{2}}$).

Now, we can prove the a priori estimate for the fluid pressure.

Proposition 4.7. *Let* $(u_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}) \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D\Omega^f_{\varepsilon,\alpha} \cup \Gamma_{\varepsilon,\alpha})^n \times L^2(\Omega^f_{\varepsilon,\alpha})$ *be the weak solution of the Stokes problem. Then the following holds:*

$$\left\| p_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{3\alpha/2}.$$

Proof. We test the weak formulation with $\phi_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha}, \partial_D \cup \Gamma_{\varepsilon,\alpha})^n$ such that $\nabla \cdot \phi_{\varepsilon,\alpha} = p_{\varepsilon,\alpha}$, obtained via Corollary 4.5, and get

$$\begin{aligned}
\left\| p_{\varepsilon,\alpha} \right\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq & \left\| e(u_{\varepsilon,\alpha}) \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\| \nabla\phi_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \left\| p^b_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\| p_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\
& + \left\| f_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\| \phi_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \left\| \nabla p^b_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \left\| \phi_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\
\leq & C\left(\varepsilon^{\alpha/2+1}\varepsilon^{\alpha-1} + \varepsilon^{3\alpha/2} + \varepsilon^{\alpha/2}\varepsilon^{\alpha} + \varepsilon^{\alpha/2}\varepsilon^{\alpha} \right) \left\| p_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\
\leq & C\varepsilon^{3\alpha/2} \left\| p_{\varepsilon,\alpha} \right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.
\end{aligned}$$

As mentioned in Remark 4.6 above, in the case of no-slip boundary conditions on $S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f}$, the pressure would be of order $\varepsilon^{\frac{\alpha}{2}}$. However, we will see later that our order is somehow optimal and allows us to pass to the limit $\varepsilon \to 0$, while the bound of order $\varepsilon^{\frac{\alpha}{2}}$ causes trouble; see also [1, Section 3.6.2].

4.2. Two-scale compactness for the microscopic solutions $v_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$

We use the uniform a priori estimates obtained in the previous section to show the compactness results for the weak microscopic solution $(u_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})$ of Eq (2.1). Further, we establish suitable properties obtained from the divergence-free condition of $u_{\varepsilon,\alpha}$ and the zero boundary conditions on $\Gamma_{\varepsilon,\alpha}$.

Proposition 4.8. *The weak solution* $(u_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})$ *of the Stokesm problem satisfies*

$$\varepsilon^{-2}u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} u_0, \quad \varepsilon^{-1}\nabla u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \nabla_y u_0, \quad \varepsilon^{-\alpha} p_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} p_0$$

with $u_0 \in L^2(\Omega, H^1_{\text{per}}(Y))^n$ *and* $p_0 \in L^2(\Omega)$. *Additionally, we have* $u_0 = 0$ *in* $\Omega \times (Y \setminus Y_f)$. *Further, we have* $\nabla_y \cdot u_0 = 0$ *and for the Darcy velocity*

$$\bar{u}(x) := \int_{Y_f} u_0 \, \mathrm{d}y,$$

we have $\partial_{x_n}\bar{u}^n = 0$, *and therefore* $\bar{u}^n$ *is constant in the* x_n*-direction.*

Proof. Due to the a priori estimates of the weak solution from Propositions 4.3 and 4.7, as well as the two-scale compactness result from Proposition 3.3, we obtain the existence of $u_0 \in L^2(\Omega, H^1_{\text{per}}(Y))^n$ and $p_0 \in L^2(\Omega \times Y)$ such that

$$\varepsilon^{-2}u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} u_0, \quad \varepsilon^{-1}\nabla u_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \nabla_y u_0, \quad \varepsilon^{-\alpha} p_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} p_0.$$

Here, we extend $u_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$ by zero to the whole reference cell Y. Now, let $\phi \in C_0^\infty(\Omega \times (Y \setminus Y_f))^n$ extended by zero to Y. We then have

$$0 = \frac{1}{\varepsilon^\alpha} \int_{\Omega_{\varepsilon,\alpha} \setminus \Omega^f_{\varepsilon,\alpha}} \varepsilon^{-2} u_{\varepsilon,\alpha}(x) \cdot \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{\bar{x}}{\varepsilon}, \frac{x_n}{\varepsilon}\right) \mathrm{d}x \overset{\varepsilon\to 0}{\to} \int_\Omega \int_{Y \setminus Y_f} u_0(x,y) \cdot \phi(x,y)\, \mathrm{d}y\, \mathrm{d}x.$$

Hence $u_0 = 0$ in $\Omega \times (Y \setminus Y_f)$. We now want to show $\nabla_y \cdot u_0 = 0$. For this, we choose $\phi \in C_0^\infty(\Omega, C^\infty_{\mathrm{per}}(Y_f))$. We then compute

$$\begin{aligned}
0 =& \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-1} \nabla \cdot u_{\varepsilon,\alpha}(x) \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x \\
=& \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-1} u_{\varepsilon,\alpha}(x) \cdot \left[\nabla_{\bar{x}} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + e_n \varepsilon^{-\alpha} \partial_{x_n} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)\right] \mathrm{d}x \\
&+ \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-2} u_{\varepsilon,\alpha}(x) \cdot \nabla_y \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x \\
&\overset{\varepsilon\to 0}{\to} \int_\Omega \int_{Y_f} u_0(x,y) \cdot \nabla_y \phi(x,y)\, \mathrm{d}y\, \mathrm{d}x.
\end{aligned}$$

In particular, it holds that $\nabla_y \cdot u_0 = 0$. Now, let $\phi \in C_0^\infty(\Omega)$. We then compute

$$\begin{aligned}
0 =& -\int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-2} \nabla \cdot u_{\varepsilon,\alpha}(x) \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x \\
=& \sum_{i=1}^{n-1} \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-2} u^i_{\varepsilon,\alpha}(x) \partial_i \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x + \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^{-2} u^n_{\varepsilon,\alpha}(x) \partial_{x_n} \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x \\
&\overset{\varepsilon\to 0}{\to} \int_\Omega \int_{Y_f} u^n_0(x,y)\, \mathrm{d}y\, \partial_{x_n} \phi(x)\, \mathrm{d}x.
\end{aligned}$$

This proves $\partial_{x_n} \bar{u}^n = 0$, and therefore, $\bar{u}^n$ is constant in x_n-direction. Lastly, we want to show that p_0 is, in fact, independent of the microscopic variable, i.e., $p_0 \in L^2(\Omega)$. For that, we choose

$$\phi_{\varepsilon,\alpha}(x) = \mathrm{diag}(\varepsilon^\beta, ..., \varepsilon^\beta, \varepsilon^\gamma) \phi\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)$$

with $\phi \in C_0^\infty(\Omega, C^\infty_{\mathrm{per}}(Y_f))^n$ and we compute

$$\begin{aligned}
\int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha}) : e(\phi_{\varepsilon,\alpha})\, \mathrm{d}x =& \int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha}) \colon \nabla \phi_{\varepsilon,\alpha}\, \mathrm{d}x \\
=& \sum_{i,j=1}^{n-1} \int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha})_{ji} \left(\varepsilon^\beta \partial_{x_i} \phi^j\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \varepsilon^{\beta-1} \partial_{y_i} \phi^j\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\
&+ \sum_{i=1}^{n-1} \int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha})_{ni} \left(\varepsilon^\gamma \partial_{x_i} \phi^n\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \varepsilon^{\gamma-1} \partial y_i \phi^n\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\
&+ \sum_{j=1}^{n-1} \int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha})_{jn} \left(\varepsilon^{\beta-\alpha} \partial_{x_n} \phi^j\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) + \varepsilon^{\beta-1} \partial_{y_n} \phi^j\left(\bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)\right) \mathrm{d}x
\end{aligned}$$

$$+\int_{\Omega^f_{\varepsilon,\alpha}} e(u_{\varepsilon,\alpha})_{nn}\left(\varepsilon^{\gamma-\alpha}\partial_{x_n}\phi^n\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)+\varepsilon^{\gamma-1}\partial_{y_n}\phi^n\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)\right)\mathrm{d}x,$$

$$\begin{aligned}\int_{\Omega^f_{\varepsilon,\alpha}}&(p_{\varepsilon,\alpha}-p^b_{\varepsilon,\alpha})\nabla\cdot\phi_{\varepsilon,\alpha}\,\mathrm{d}x\\&=\sum_{i=1}^{n-1}\int_{\Omega^f_{\varepsilon,\alpha}}\left(p_{\varepsilon,\alpha}-p^b_{\varepsilon,\alpha})\right)\left(\varepsilon^\beta\partial_{x_i}\phi^i\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)+\varepsilon^{\beta-1}\partial_{y_i}\phi^i\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)\right)\mathrm{d}x\\&\quad+\int_{\Omega^f_{\varepsilon,\alpha}}\left(p_{\varepsilon,\alpha}-p^b_{\varepsilon,\alpha}\right)\left(\varepsilon^{\gamma-\alpha}\partial_{x_n}\phi^n\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)+\varepsilon^{\gamma-1}\partial_{y_n}\phi^n\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)\right)\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\int_{\Omega^f_{\varepsilon,\alpha}}(f_{\varepsilon,\alpha}-\nabla p^b_{\varepsilon,\alpha})\cdot\phi_{\varepsilon,\alpha}\,\mathrm{d}x&=\sum_{i=1}^{n-1}\int_{\Omega^f_{\varepsilon,\alpha}}(f^i_{\varepsilon,\alpha}(x)-\partial_i p^b_{\varepsilon,\alpha}(x))\varepsilon^\beta\phi^i\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)\mathrm{d}x\\&\quad+\int_{\Omega^f_{\varepsilon,\alpha}}(f^n_{\varepsilon,\alpha}(x)-\partial_{x_n}p^b_{\varepsilon,\alpha}(x))\varepsilon^\gamma\phi^n\left(\bar{x},\frac{x_n}{\varepsilon^\alpha},\frac{x}{\varepsilon}\right)\mathrm{d}x.\end{aligned}$$

By choosing $\beta=\gamma=1-2\alpha$ and taking the limit $\varepsilon\to 0$, we obtain

$$0=\int_\Omega\int_Y(p_0(x,y)-p^b_0(x))\nabla_y\cdot\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x\quad\text{for all }\phi\in C^\infty_0(\Omega\times Y)^n.$$

This immediately implies that p_0 is independent of the microscopic variable, i.e., $p_0\in L^2_0(\Omega)$, since $p^b_0\in L^2(\Omega)$.

4.3. Derivation of the macroscopic model

For the derivation of the macroscopic model, we proceed as in the proof of Proposition 4.8, with the exception of choosing $\beta=\gamma=-\alpha$ and $\phi\in C^\infty_0(\overline{\Omega}\setminus\partial_D\Omega, C^\infty_{\mathrm{per}}(Y_f))^n$ such that $\nabla_y\cdot\phi=0$. By doing so, we obtain the following in the limit $\varepsilon\to 0$:

$$\begin{aligned}\int_\Omega\int_{Y_f}e_y(u_0(x,y))\colon\nabla_y\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x-\int_\Omega(p_0(x)-p^b_0(x))\partial_{x_n}\int_{Y_f}\phi^n(x,y)\,\mathrm{d}y\,\mathrm{d}x\\=\int_\Omega\int_{Y_f}(f_0(x)-e_n\partial_{x_n}p^b_0(x)-\nabla_y p^b_1(x,y))\cdot\phi(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}\tag{4.2}$$

for all $\phi\in C^\infty_0(\overline{\Omega}\setminus\partial_D\Omega, C^\infty_{\mathrm{per}}(Y_f))^n$ with $\nabla_y\cdot\phi=0$. It is easy to see that the term including $\nabla_y p^b_1$ vanishes via integration by parts. We now show, that $\partial_{x_n}p_0$ exists and is an $L^2(\Omega)$ function. In order to do this, we find $(w_i,q_i)\in H^1_{\mathrm{per}}(Y_f)^n\times L^2(Y_f)/\mathbb{R}$, solving, in the weak sense, the equation

$$\begin{aligned}-\nabla\cdot(e(w_i))-\nabla q_i&=e_i&&\text{in }Y_f,\\\nabla\cdot w_i&=0&&\text{in }Y_f,\\w_i&=0&&\text{on }\Gamma,\\&w_i,q_i\text{ are }Y\text{-periodic},\end{aligned}\tag{4.3}$$

for $i = 1, ..., n$. The existence and uniqueness of a solution is standard. We define the permeability tensor

$$K_{ij} = \int_{Y_f} e(w_i) : e(w_j)\,\mathrm{d}y = \int_{Y_f} e(w_i) : \nabla w_j\,\mathrm{d}y = \int_{Y_f} e_i \cdot w_j\,\mathrm{d}y. \tag{4.4}$$

Further, we define the test function $\phi \in C_0^\infty(\overline{\Omega} \setminus \partial_D\Omega, H^1_{\mathrm{per}}(Y_f))^n$ via

$$\phi(x, y) := K_{nn}^{-1} \eta(x) w_n(y),$$

with $\eta \in C_0^\infty(\overline{\Omega} \setminus \partial_D\Omega)$. We see that $\nabla_y \cdot \phi = 0$ and

$$\int_{Y_f} \phi^n\,\mathrm{d}y = \eta.$$

Via a density argument, we can test Eq (4.2) with ϕ and obtain

$$\left| \int_\Omega (p_0(x) - p_0^b(x))\partial_{x_n}\eta(x)\,\mathrm{d}x \right| \leq C(u_0, f_0, p_0^b, w) \, \|\eta\|_{L^2(\Omega)},$$

so, in particular, $\partial_{x_n} p_0 \in L^2(\Omega)$ and $p_0 = p_0^b$ on $S_1^\pm$. Hence, we can rewrite Eq (4.2) as

$$\begin{aligned} &\int_\Omega \int_{Y_f} e_y(u_0(x, y)) : \nabla_y \phi(x, y)\,\mathrm{d}y\,\mathrm{d}y \\ &\quad + \int_\Omega \int_{Y_f} \partial_{x_n} p_0(x)\phi(x, y)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y_f} f_0(x) \cdot \phi(x, y)\,\mathrm{d}y\,\mathrm{d}x = 0, \end{aligned} \tag{4.5}$$

for all $\phi \in C_0^\infty(\Omega, C^\infty_{\mathrm{per}}(Y_f))^n$ with $\nabla_y \cdot \phi = 0$. Here, the terms containing p_0^b cancel each other other out via integration by parts and due to the zero boundary condition of ϕ. Through the application of the Bogovskii operator, $p_1 \in L^2(\Omega, L^2_0(Y_f))$ exists such that

$$\begin{aligned} &\int_\Omega \int_{Y_f} e_y(u_0(x, y)) : \nabla_y \phi(x, y)\,\mathrm{d}y\,\mathrm{d}y \\ &\quad + \int_\Omega \int_{Y_f} \partial_{x_n} p_0(x)\phi(x, y) - p_1 \nabla_y \cdot \phi(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \int_{Y_f} f_0(x) \cdot \phi(x, y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

for all $\phi \in C_0^\infty(\Omega, C^\infty_{\mathrm{per}}(Y_f))^n$. In other words, (u_0, p_0, p_1) solves, in the weak sense, the equation

$$-\nabla_y \cdot \Big(e_y(u_0)\Big) - \nabla_y p_1 + e_n \partial_{x_n} p_0 = f_0 \quad \text{in } \Omega \times Y_f.$$

We rewrite this equation in the form

$$-\nabla_y \cdot \Big(e_y(u_0)\Big) - \nabla_y p_1 = \sum_{i=1}^{n-1} e_i f_0^i + e_n (f_0^n - \partial_{x_n} p_0) \quad \text{in } \Omega \times Y_f.$$

Together with $\nabla_y \cdot u_0 = 0$ in $\Omega \times Y_f$, the boundary condition $u_0 = 0$ on $\Omega \times \Gamma$ and the Y-periodicity of u_0. Since the equation is linear and the solution (u_0, p_1) is unique (for a given right-hand side), we obtain

$$u_0(x, y) = \sum_{i=1}^{n-1} f_0^i w_i + (f_0^n - \partial_{x_n} p_0) w_n,$$

$$p_1(x,y) = \sum_{i=1}^{n-1} f_0^i q_i + (f_0^n - \partial_{x_n} p_0) q_n,$$

where $(w_i, q_i) \in H^1_{\text{per}}(Y_f)^n \times L^2(Y_f)/\mathbb{R}$ is again the unique solution of Eq (4.3). We now define the Darcy velocity

$$\bar{u}(x) := \int_{Y_f} u_0(x,y)\, \mathrm{d}y.$$

For $j = 1, ..., n$, we obtain the following with the permeability tensor K, defined in Eq (4.4):

$$\bar{u}_j = \sum_{i=1}^{n-1} f_0^i \int_{Y_f} w_i \cdot e_j \,\mathrm{d}y + (f_0^n - \partial_{x_n} p_0) \int_{Y_f} w_n \cdot e_j \,\mathrm{d}y = \sum_{i=1}^{n-1} K_{ij} f_0^i \,\mathrm{d}y + K_{n,j}(f_0^n - \partial_{x_n} p_0)$$

and therefore

$$\bar{u} = K(f_0 - e_n \partial_{x_n} p_0) \quad \text{in } \Omega. \tag{4.6}$$

Hence, the tuple $(\bar{u}, p_0)$ satisfies

$$\begin{aligned} \bar{u} &= K(f_0 - e_n \partial_{x_n} p_0) && \text{in } \Omega, \\ \partial_{x_n} \bar{u}^n &= 0 && \text{in } \Omega. \end{aligned}$$

In other words, the Darcy pressure p_0 solves the equation

$$\begin{aligned} \partial_{x_n} \left[K(f_0 - e_n \partial_{x_n} p_0)\right]_n &= 0 && \text{in } \Omega, \\ p_0 &= p_0^b && \text{on } S_1^{\pm}. \end{aligned}$$

It is obvious that this problem admits a unique weak solution, as well as the problem (4.2). In particular, this implies that all convergence results are valid for the whole sequence, which completes the proof of Theorem 2.2.

4.4. The case of cylindrical inclusions Y_s

We comment on the case where the solid inclusions Y_s are given as cylinders; more precisely, we have $Y_s = Y_s' \times (0,1)$ with $Y_s' \subset (0,1)^{n-1}$ being strictly included. In the past, problems in this microscopic geometry have received considerable attention in the literature; see, for example, references [8] and also [15] for formal results. However, in both papers, a no-slip boundary condition on the top/bottom $S^{\pm}_{\varepsilon,f}$ on the thin layer was considered, which has a significant influence on the macroscopic model. In this case, it is easy to check that $\nabla_{\bar{x}} \cdot \bar{u} = 0$ and also that, the microscopic pressure $p_{\varepsilon,\alpha}$ is of order $\varepsilon^{\frac{\alpha}{2}}$. In particular, this implies (as can be seen from the calculations in the proof of Proposition 4.8, where we can now choose the test functions with $\partial_{y_n}\phi = \partial_{\bar{x}}\phi = 0$ and $\nabla_{\bar{y}} \cdot \phi = 0$, and therefore γ and β are independent of each other) that the limit pressure only depends on $\bar{x}$ and fulfills $\nabla_{\bar{x}} p_0 \in L^2(\Sigma)^{n-1}$. In the case of a pressure boundary condition on $S^{\pm}_{\varepsilon,f}$, these results seem to be impossible. However, we can simplify the representation for the Darcy velocity in Eq (4.6) by considering the structure of K in more detail.

It is easy to check (solve a similar equation on Y_f') that the cell solutions (w_i, q_i) for $i = 1, \ldots, n-1$ are constant with respect to y_n and, we have $w_i^n = 0$. In particular, we find for $i = 1, \ldots, n-1$ that

$$K_{in} = K_{ni} = \int_{Y_f} e_y(w_i) : \nabla_y w_n \,\mathrm{d}y = \int_{Y_f} w_i^n \,\mathrm{d}y = 0,$$

and K has the block structure

$$K = \begin{pmatrix} \bar{K} & 0 \\ 0 & K_{nn} \end{pmatrix},$$

where $\bar{K}$ is the submatrix of K consisting of the first $(n-1)$ columns and rows. This leads to ($\bar{f}_0 = (f_0^1, \dots, f_0^{n-1})$)

$$\bar{u} = (\bar{K}\bar{f}_0, 0)^T + (0, K_{nn}(f_0^n - \partial_{x_n} p_0))^T.$$

In particular, the horizontal part of the Darcy velocity is just given by $\bar{K}\bar{f}_0$, and only the vertical velocity depends on the Darcy pressure.

5. The transport problem

We now deal with simultaneous homogenization and dimension reduction for a reaction–diffusion–advection equation in Eq (2.4), where the advective velocity is obtained via the Stokes problem (2.1), which we assume now to be quasi-stationary. More precisely, we assume that $f_{\varepsilon,\alpha} \in L^\infty((0,T), L^2(\Omega^f_{\varepsilon,\alpha}))^n$ and $p^b_{\varepsilon,\alpha} \in L^\infty((0,T), H^1(\Omega^f_{\varepsilon,\alpha}))$ fulfill the same estimates as in Assumptions (S1) and (S2), with additional L^∞-regularity with respect to time. This leads to the same a priori estimates for $u_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$ as in Section 4.1, with additional L^∞-regularity in time. Furthermore, the compactness results from Section 4.2 remain valid, where the limit function are L^∞ with respect to time. We also assume that $n \leq 4$, guaranteeing the existence of a weak microscopic solution. For the diffusion coefficient D^α_ε, we consider the following cases with $D > 0$ being fixed (already given in Section 2):

(D1) $D^\alpha_\varepsilon = \varepsilon^\alpha D I \in \mathbb{R}^{n\times n}$,

(D2) $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha}, \dots, \varepsilon^{-\alpha}, \varepsilon^\alpha) \in \mathbb{R}^{n\times n}$.

Let us give the definition of a weak solution in Case (D1): We say that $c_{\varepsilon,\alpha}$ is a weak solution of the problem (2.4) (for the diffusion coefficient given in Case (D1)) if $c_{\varepsilon,\alpha} \in L^2((0,T), H^1(\Omega^f_{\varepsilon,\alpha}))$ with $\partial_t c_{\varepsilon,\alpha} \in L^2((0,T), H^1(\Omega^f_{\varepsilon,\alpha}, S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f})')$ such that $c_{\varepsilon,\alpha} = c^b_\varepsilon$ on $S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f}$, and for all $\psi_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha})$ with $\psi_{\varepsilon,\alpha} = 0$ on $S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f}$, it holds almost everywhere in $(0,T)$ that

$$\frac{1}{\varepsilon^\alpha}\langle \partial_t c_{\varepsilon,\alpha}, \psi_{\varepsilon,\alpha}\rangle_{H^1(\Omega^f_{\varepsilon,\alpha})} + \int_{\Omega^f_{\varepsilon,\alpha}} D^\alpha_\varepsilon \nabla c_{\varepsilon,\alpha} \cdot \nabla\psi_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c_{\varepsilon,\alpha} \nabla\psi_{\varepsilon,\alpha}\, \mathrm{d}x = \frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha}\psi_{\varepsilon,\alpha}\, \mathrm{d}x, \tag{5.1}$$

together with the initial condition $c_{\varepsilon,\alpha}(0) = 0$. Introducing the quantity

$$w_{\varepsilon,\alpha} := c_{\varepsilon,\alpha} - c^b_\varepsilon,$$

we obtain $w_{\varepsilon,\alpha} = 0$ on $S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f}$, and this function fulfills

$$\begin{aligned} \frac{1}{\varepsilon^\alpha}\langle \partial_t w_{\varepsilon,\alpha}, \psi_{\varepsilon,\alpha}\rangle_{H^1(\Omega^f_{\varepsilon,\alpha})} &+ \int_{\Omega^f_{\varepsilon,\alpha}} D^\alpha_\varepsilon \nabla w_{\varepsilon,\alpha} \cdot \nabla\psi_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} w_{\varepsilon,\alpha} \nabla\psi_{\varepsilon,\alpha}\, \mathrm{d}x \\ &= \frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}} \left(g_{\varepsilon,\alpha} - \partial_t c^b_\varepsilon\right)\psi_{\varepsilon,\alpha}\, \mathrm{d}x + \int_{\Omega^f_{\varepsilon,\alpha}} \left(D^\alpha_\varepsilon \nabla c^b_\varepsilon - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c^b_\varepsilon\right) \cdot \nabla\psi_{\varepsilon,\alpha}\, \mathrm{d}x. \end{aligned} \tag{5.2}$$

In the case of high diffusion in the horizontal direction (D2), we have to consider, in the definition above, $c_{\varepsilon,\alpha} \in L^2((0,T), H^1_{\#}(\Omega^f_{\varepsilon,\alpha}))$ and $\partial_t c_{\varepsilon,\alpha} \in L^2((0,T), H^1_{\#}(\Omega^f_{\varepsilon,\alpha}, S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f})')$ and the test functions $\psi_{\varepsilon,\alpha} \in H^1_{\#}(\Omega^f_{\varepsilon,\alpha})$ (recall that # indicates the Σ-periodicity).

Assumptions on the data

(T1) The source term $g_{\varepsilon,\alpha} \in L^\infty((0,T) \times \Omega^f_{\varepsilon,\alpha})$ fulfills

$$\|g_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C.$$

Further, $g_0 \in L^2((0,T) \times \Omega \times Y_f)$ exists such that $\chi_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} g_0$.

(T2) The boundary value c^b_ε fulfills

$$c^b_\varepsilon \in L^2((0,T), H^1(\Omega^f_{\varepsilon,\alpha})) \cap H^1((0,T), L^2(\Omega^f_{\varepsilon,\alpha})),$$

with $c^b_\varepsilon(0) = 0$, such that

$$\|c^b_\varepsilon\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha})} + \varepsilon^{-\frac{\alpha}{2}}\|\partial_t c^b_\varepsilon\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C.$$

In Case (D2), we additionally assume $c^b_{\varepsilon,\alpha} \in L^2((0,T), H^1_{\#}(\Omega^f_{\varepsilon,\alpha}))$. For the bound of the gradient we consider two different cases.

- For $D^\alpha_\varepsilon = \varepsilon^\alpha D$, we assume

$$\varepsilon^\alpha \|\nabla c^b_\varepsilon\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Morover, (see also Proposition 3.4) $c^b_0 \in L^2((0,T) \times \Omega)$ with $\partial_{x_n} c^b_0 \in L^2((0,T) \times \Omega)$ and $c^b_1 \in L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$ exsist such that

$$\chi_{\Omega^f_{\varepsilon,\alpha}} c^b_\varepsilon \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} c^b_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^\alpha \nabla c^b_\varepsilon \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f}\left(\partial_{x_n} c^b_0 e_n + \nabla_y c^b_1\right).$$

- For $D^\alpha_\varepsilon = D\text{diag}(\varepsilon^{-\alpha}, \ldots, \varepsilon^{-\alpha}, \varepsilon^\alpha) \in \mathbb{R}^{n\times n}$, we assume

$$\|\nabla_{\bar{x}} c^b_\varepsilon\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} + \varepsilon^\alpha \|\partial_{x_n} c^b_\varepsilon\|_{L^\infty((0,T)\times\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Moreover (see also Proposition 3.6) $c^b_0 \in L^2((0,T), H^1(\Omega))$ and $c^b_1 \in L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$ exist such that

$$\chi_{\Omega^f_{\varepsilon,\alpha}} c^b_\varepsilon \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} c^b_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} (\nabla_{\bar{x}} c^b_\varepsilon, \varepsilon^\alpha \partial_{x_n} c^b_\varepsilon) \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f}\left(\nabla c^b_0 + \nabla_y c^b_1\right).$$

(T3) It holds that

$$\begin{aligned} &\|\delta f_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} + \|\nabla \delta p^b_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} + \|\delta g_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\quad + \|\partial_t \delta c^b_\varepsilon\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} + \|\delta c^b_\varepsilon\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} + \varepsilon^\alpha \|\nabla \delta g_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} \leq \kappa(|l\varepsilon|)\varepsilon^{\frac{\alpha}{2}}, \end{aligned}$$

where $\delta\phi$ for a function ϕ denotes the difference in the shifts $\delta\phi(x) = \phi(x + l\varepsilon) - \phi(x)$ for given $l \in \mathbb{Z}^{n-1} \times \{0\}$; see also Eq (5.4) for more details.

Remark 5.1. *(i) Our focus is the treatment of the advective term and the different scalings for the diffusion coefficient. Therefore, we have chosen a homogeneous initial condition and a linear reaction term. However, it is straightforward to extend our results to more general data.*

(ii) Of course, due to our assumptions, we can expect more regularity for the time-derivative. However, we show homogenization and dimension reduction (in particular the strong two-scale compactness results) for the time-derivative being a functional in the dual space of $H^1(\Omega^f_{\varepsilon,\alpha}, S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f})$ (respectively, with Σ-periodic boundary conditions), to provide methods for more general data.

Corollary 5.2. *A unique weak solution of the microscopic problem* (2.4) *exists.*

This result is standard for fixed a ε and can be obtained by the Galerkin method; see, for example, [33].

5.1. A priori estimates for the microscopic solution $c_{\varepsilon,\alpha}$

We derive uniform a priori estimates with respect to ε and α. Of course, we will obtain different estimates for the gradient depending on the choice of D^α_ε for Cases (D1) and (D2). However, the ideas are the same for both cases and we can follow a standard procedure for energy estimates to obtain L^2-bounds for $c_{\varepsilon,\alpha}$ and its gradient. A more critical part is to obtain a uniform bound for the time-derivative. First of all, we need L^∞-bounds for the concentration $c_{\varepsilon,\alpha}$ to control the advective term uniformly in ε. Next, to establish later strong (two-scale) convergence for $c_{\varepsilon,\alpha}$ via a Kolmogorov-Simon type compactness argument, we need bounds for the time-derivative in the dual spaces of Sobolev functions with weighted (with respect to ε and α) norms adapted to the a priori bounds for $c_{\varepsilon,\alpha}$ and $\nabla c_{\varepsilon,\alpha}$ in L^2. Finally, for the strong convergence of $c_{\varepsilon,\alpha}$, an additional control with respect to the spatial variable is necessary; therefore, we give an additional estimate for the differences in shifts of the microscopic solutions $c_{\varepsilon,\alpha}$ and $u_{\varepsilon,\alpha}$.

The case of $D^\alpha_\varepsilon = \varepsilon^\alpha D$

We test Eq (5.2) with $\psi_{\varepsilon,\alpha} = w_{\varepsilon,\alpha}$ and obtain

$$\frac{1}{2\varepsilon^\alpha}\frac{d}{dt}\|w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^\alpha D\|\nabla w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} - \frac{1}{2}\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla w^2_{\varepsilon,\alpha}\,\mathrm{d}x$$
$$= \frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}}\left(g_{\varepsilon,\alpha} - \partial_t c^b_\varepsilon\right) w_{\varepsilon,\alpha}\,\mathrm{d}x + \int_{\Omega^f_{\varepsilon,\alpha}}\left(\varepsilon^\alpha D\nabla c^b_\varepsilon - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c^b_\varepsilon\right)\cdot\nabla w_{\varepsilon,\alpha}\,\mathrm{d}x.$$

For the convective term, we can use integration by parts together with the zero boundary conditions of $u_{\varepsilon,\alpha}$ and $w_{\varepsilon,\alpha}$, and also the divergence-free condition of $u_{\varepsilon,\alpha}$, to obtain

$$\frac{1}{2}\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla w^2_{\varepsilon,\alpha}\,\mathrm{d}x = 0.$$

It remains to estimate the terms on the right-hand side. For the first term, we obtain the following with the assumptions on $g_{\varepsilon,\alpha}$ and $\partial_t c^b_\varepsilon$:

$$\frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}}\left(g_{\varepsilon,\alpha} - \partial_t c^b_\varepsilon\right) w_{\varepsilon,\alpha}\,\mathrm{d}x \le \frac{C}{\varepsilon^{\frac{\alpha}{2}}}\|w_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \le C\left(1 + \frac{1}{\varepsilon^\alpha}\|w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})}\right).$$

For the second term on the right-hand side in the above equation, we consider the diffusive and advective term separately. For the first one, we have

$$\int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^\alpha D\nabla c^b_\varepsilon \cdot \nabla w_{\varepsilon,\alpha}\,\mathrm{d}x \leq C\varepsilon^{\frac{\alpha}{2}}\|\nabla w_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C(\theta) + \theta\varepsilon^\alpha\|\nabla w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} \tag{5.3}$$

for an arbitrary $\theta > 0$. For the advective term, we use the essential boundedness of c^b_ε and the a priori bound for $u_{\varepsilon,\alpha}$ from Proposition 4.3 to obtain

$$\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c^b_\varepsilon \cdot \nabla w_{\varepsilon,\alpha}\,\mathrm{d}x \leq C\left\|\frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}\|\nabla w_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C(\theta) + \theta\varepsilon^\alpha\|\nabla w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

Altogether, choosing θ to be small enough, we can use an absorption argument, then we integrate with respect to time and use the Gronwall inequality to obtain

$$\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|w_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha}))} + \varepsilon^{\frac{\alpha}{2}}\|\nabla w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C.$$

Due to the assumptions on c^b_ε, we obtain the same estimate also for $c_{\varepsilon,\alpha}$. We summarize our results in the following Proposition.

Proposition 5.3. *For $D^\alpha_\varepsilon = \varepsilon^\alpha D$, it holds that*

$$\|w_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha}))} + \varepsilon^\alpha\|\nabla w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

The same estimate is valid for $c_{\varepsilon,\alpha}$ instead of $w_{\varepsilon,\alpha}$.

The case where $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha},\dots,\varepsilon^{-\alpha},\varepsilon^\alpha) \in \mathbb{R}^{n\times n}$

We proceed in the same way as in the previous case, testing Eq (2.4) with $w_{\varepsilon,\alpha}$ to obtain

$$\begin{aligned}\frac{1}{2\varepsilon^\alpha}\frac{d}{dt}\|w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} + \frac{1}{\varepsilon^\alpha}D\|\nabla_{\bar{x}} w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^\alpha D\|\partial_{x_n} w_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} - \frac{1}{2}\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla w^2_{\varepsilon,\alpha}\,\mathrm{d}x \\ =\frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}}\left(g_{\varepsilon,\alpha} - \partial_t c^b_\varepsilon\right) w_{\varepsilon,\alpha}\,\mathrm{d}x - \int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c^b_\varepsilon \cdot \nabla w_{\varepsilon,\alpha}\,\mathrm{d}x \\ + \int_{\Omega^f_{\varepsilon,\alpha}} \frac{1}{\varepsilon^\alpha} D\nabla_{\bar{x}} c^b_\varepsilon \cdot \nabla_{\bar{x}} w_{\varepsilon,\alpha} + \varepsilon^\alpha D\partial_{x_n} c^b_\varepsilon \partial_{x_n} w_{\varepsilon,\alpha}\,\mathrm{d}x.\end{aligned}$$

Now, the main difference from the proof of Proposition 5.3 lies in the estimate of the last term on the right-hand side. Here, we can argue in the same way as for Eq (5.3) separately for $\nabla_{\bar{x}}$ and ∂_{x_n}. In summary, we get the following proposition

Proposition 5.4. *For $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha},\dots,\varepsilon^{-\alpha},\varepsilon^\alpha) \in \mathbb{R}^{n\times n}$, it holds that*

$$\|w_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega_{\varepsilon,\alpha}))} + \|\nabla_{\bar{x}} w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} + \varepsilon^\alpha\|\partial_{x_n} w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

The same estimate is valid for $c_{\varepsilon,\alpha}$ instead of $w_{\varepsilon,\alpha}$.

Estimates for the time-derivative $\partial_t w_{\varepsilon,\alpha}$: To control the time-derivative, we need control for the convective term. As usual when dealing with flow in porous medium, the embedding H^1 into L^4 (at least for $n \le 4$) is not applicable, since the gradient of $u_{\varepsilon,\alpha}$ scales badly with respect to ε. To overcome this problem, it is a standard approach to show L^∞-bounds for the concentration $c_{\varepsilon,\alpha}$. The proof follows the same lines as in the case of full (perforated domains), so we only give a brief sketch.

Lemma 5.5. *For both Cases (D1) and (D2) for the diffusion coefficient D_ε^α, it holds that*

$$\|c_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha})} \le C.$$

Proof. We emphasize that now we work directly with the weak formulation of $c_{\varepsilon,\alpha}$ instead of $w_{\varepsilon,\alpha}$. We define $W := c_{\varepsilon,\alpha}$ and put $W_k := W - k$ for $k \in \mathbb{N}$ and $W_k^+ := (W-k)^+$ with $(\cdot)^+ := \max\{0,\cdot\}$, and use W_k^+ as a test function in Eq (5.1). This is an admissible test function for $k > \|c_\varepsilon^b\|_{L^\infty((0,T)\times\Omega_{\varepsilon,\alpha})}$, since then $W_k^+ = 0$ on $S^\pm_{\varepsilon,f}$. We obtain the following after integration in time from 0 to $t \in [0,T]$:

$$\frac{1}{2\varepsilon^\alpha}\|W_k^+(t)\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} + \int_0^t\int_{\Omega^f_{\varepsilon,\alpha}} D_\varepsilon^\alpha \nabla W_k^+ \cdot \nabla W_k^+ \,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_0^t\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} W\cdot\nabla W_k^+ \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon^\alpha}\int_0^t\int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha} W_k^+ \,\mathrm{d}x\,\mathrm{d}s.$$

For the convective term, we use integration by parts to get

$$\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} W\cdot\nabla W_k^+\,\mathrm{d}x = \frac{1}{2}\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla (W_k^+)^2\,\mathrm{d}x + k\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla W_k^+\,\mathrm{d}x = 0,$$

where in the last equality, we used the zero boundary condition of $u_{\varepsilon,\alpha}$ and W_k^+. The force term can be estimated in the following way by using the L^∞ bound for $g_{\varepsilon,\alpha}$:

$$\frac{1}{\varepsilon^\alpha}\int_0^t\int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha}W_k^+\,\mathrm{d}x\,\mathrm{d}s \le \frac{C}{\varepsilon^\alpha}\left(\int_0^t\int_{\{W_k>k\}}\mathrm{d}x\,\mathrm{d}s + \|W_k^+\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha})}\right).$$

Now, the Gronwall inequality and [34, II Theorem 6.1 and Remark 6.2] imply the desired result.

We are now able to estimate the time-derivative $\partial_t c_{\varepsilon,\alpha}$. Here, it is necessary to estimate the norm in the dual space of the functions spaces suitably scaled with respect to ε and α, and therefore depending on the choice of D_ε^α in particular. For this, we introduce the space $\mathcal{H}_{D_\varepsilon^\alpha}$ consisting of functions in $H^1(\Omega^f_{\varepsilon,\alpha}, S^+_{\varepsilon,f}\cup S^-_{\varepsilon,f})$ in Case (D1) and $H^1_\#(\Omega^f_{\varepsilon,\alpha}, S^+_{\varepsilon,f}\cup S^-_{\varepsilon,f})$ in Case (D2), together with the norm

$$\|\psi_{\varepsilon,\alpha}\|^2_{\mathcal{H}_{D_\varepsilon^\alpha}} := \frac{1}{\varepsilon^\alpha}\|\psi_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})} + \|\sqrt{D_\varepsilon^\alpha}\nabla\psi_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

We emphasize that from our a priori estimates on $c_{\varepsilon,\alpha}$ above, we have

$$\|c_{\varepsilon,\alpha}\|_{L^2((0,T),\mathcal{H}_{D_\varepsilon^\alpha})} \le C.$$

Proposition 5.6. *It holds that*

$$\frac{1}{\varepsilon^\alpha}\|\partial_t c_{\varepsilon,\alpha}\|_{L^2((0,T),\mathcal{H}'_{D_\varepsilon^\alpha})} \le C.$$

Proof. We test Eq (5.1) with $\psi_{\varepsilon,\alpha} \in \mathcal{H}_{D_\varepsilon^\alpha}$ such that $\|\psi_{\varepsilon,\alpha}\|_{\mathcal{H}_{D_\varepsilon^\alpha}} \leq 1$. In particular, we have

$$\|\nabla\psi_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{-\frac{\alpha}{2}}.$$

We get the following almost everywhere in $(0,T)$:

$$\begin{aligned}
&\frac{1}{\varepsilon^\alpha}\left|\langle\partial_t c_{\varepsilon,\alpha}, \psi_{\varepsilon,\alpha}\rangle_{H^1(\Omega^f_{\varepsilon,\alpha})}\right| \\
=&\left|-\int_{\Omega^f_{\varepsilon,\alpha}} D_\varepsilon^\alpha \nabla c_{\varepsilon,\alpha}\cdot\nabla\psi_{\varepsilon,\alpha} - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\nabla\psi_{\varepsilon,\alpha}\,\mathrm{d}x + \frac{1}{\varepsilon^\alpha}\int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha}\psi_{\varepsilon,\alpha}\,\mathrm{d}x\right| \\
\leq& C\|\sqrt{D_\varepsilon^\alpha}\nabla c_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}\|\sqrt{D_\varepsilon^\alpha}\nabla\psi_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\
&+\left\|\frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})}\|c_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha})}\|\nabla\psi_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + \varepsilon^{-\alpha}\|g_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}\|\psi_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} \\
\leq& C\|\sqrt{D_\varepsilon^\alpha}\nabla c_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + C\varepsilon^{-\frac{\alpha}{2}}\left\|\frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2(\Omega^f_{\varepsilon,\alpha})} + C,
\end{aligned}$$

where we used the L^∞ bound for $g_{\varepsilon,\alpha}$ and $c_{\varepsilon,\alpha}$ as obtained in Lemma 5.5. Taking the supremum over $\psi_{\varepsilon,\alpha}$, squaring, integrating with respect to time, and using the a priori estimates for $c_{\varepsilon,\alpha}$ and $u_{\varepsilon,\alpha}$, we get the desired result.

Estimates for the shifts: To obtain strong convergence (in the two-scale sense), more control on the spatial variable is necessary. For this, we introduce the following notation for the differences in the shifted functions. Let $\psi_{\varepsilon,\alpha} : \mathbb{R}^{n-1}\times(-\varepsilon^\alpha,\varepsilon^\alpha)\to\mathbb{R}$ and $l\in\mathbb{Z}^{n-1}\times\{0\}$. We define

$$\delta\psi_{\varepsilon,\alpha} := \psi_{\varepsilon,\alpha}(\cdot + l\varepsilon) - \psi_{\varepsilon,\alpha}, \tag{5.4}$$

where, in this notation, we neglect the dependence on l and ε, which should be clear from the context. In the following, we extend the function $u_{\varepsilon,\alpha}$ by zero to $\mathbb{R}^{n-1}\times(-\varepsilon^\alpha,\varepsilon^\alpha)$ and the function $c_{\varepsilon,\alpha}$ first with the extension operator from Lemma 5.7 below to $\Omega_{\varepsilon,\alpha}$. Then we proceed in an arbitrary smooth way to $\mathbb{R}^{n-1}\times(-\varepsilon^\alpha,\varepsilon^\alpha)$, such that the a priori estimates, particularly the L^∞–estimate, remain valid (this can be done by mirroring). We use the same notation for both extensions as before.

Lemma 5.7 (Extension operator)**.** *There is an extension operator $E_\varepsilon : H^1(\Omega^f_{\varepsilon,\alpha})\to H^1(\Omega_{\varepsilon,\alpha})$ such that for all $v_{\varepsilon,\alpha}\in H^1(\Omega^f_{\varepsilon,\alpha})$, it holds that*

$$\|E_\varepsilon v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\|v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}, \qquad \|\nabla E_\varepsilon v_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})} \leq C\|\nabla v_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha})}.$$

If we also have $v_{\varepsilon,\alpha}\in L^\infty(\Omega^f_{\varepsilon,\alpha})$, it holds that

$$\|E_\varepsilon v_{\varepsilon,\alpha}\|_{L^\infty(\Omega_{\varepsilon,\alpha})} \leq C\|v_{\varepsilon,\alpha}\|_{L^\infty(\Omega^f_{\varepsilon,\alpha})}.$$

This result can be shown as in [35]. The fact that we deal with a thin layer with a thickness of order ε^α has no influence. For the inequality for the L^∞-bound, we refer to [36, Lemma A.3], where the L^∞-bound was shown for the case of perforations being strictly included. However, since the construction was also based on a standard local extension operator obtained, for example, via a mirroring argument, it is straightforward to extend the results from [36] via the global construction from [35] to the case of connected perforations.

Next, we construct a domain $\Omega^f_{\varepsilon,\alpha,h}$ for small values of h, which is obtained by cutting of micro-cells from $\Omega^f_{\varepsilon,\alpha}$ near to the lateral boundary (a distance smaller than h). More precisely, we introduce the following notation. For $0 < h \ll 1$ let $\Sigma_h := \{x \in \Sigma \, : \, \mathrm{dist}\{x, \partial\Sigma\} > h\}$ and we set

$$K_{\varepsilon,h} := \left\{k \in \mathbb{Z}^{n-1} \, : \, \varepsilon(k + (0,1)^{n-1}) \subset \Sigma_h\right\},$$

and define

$$\Sigma_{\varepsilon,h} := \mathrm{int}\left\{\bigcup_{k\in K_{\varepsilon,h}} \varepsilon([0,1]^{n-1} + k)\right\}.$$

In other words, $\Sigma_{\varepsilon,h}$ consists of all points in x with a distance greater than h and included in a microscopic cell $\varepsilon(k + [0,1]^{n-1})$ strictly contained in Σ_h. We now define

$$\Omega_{\varepsilon,\alpha,h} := \Sigma_{\varepsilon,h} \times (-\varepsilon^\alpha, \varepsilon^\alpha), \qquad \Omega^f_{\varepsilon,\alpha,h} := \Omega_{\varepsilon,\alpha,h} \cap \Omega^f_{\varepsilon,\alpha}.$$

Proposition 5.8. *We obtain, for every $0 < h \ll 1$, a constant $C_h > 0$ depending on h (but independent of ε), such that for every $l \in \mathbb{Z}^{n-1} \times \{0\}$ and $|l\varepsilon| < h$, it holds that*

$$\varepsilon^{-\frac{\alpha}{2}} \|\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha}))} \le C\sqrt{h} + C_h \varepsilon^{\frac{\alpha}{2}} + \kappa(|l\varepsilon|),$$

with $\kappa(s) \to 0$ for $s \to 0$. In Case (D2) with periodic boundary conditions (after extending $c_{\varepsilon,\alpha}$ periodically in Σ-direction), the inequality is valid for $h = 0$ and arbitrary l and ε, and the constant on the right-hand side is independent of h.

Proof. We first consider Case (D1) with $D^\alpha_\varepsilon = \varepsilon^\alpha D$. We use similar ideas as in the proof of [24, Lemma 4.3], where we also have to estimate the convective term. We define the space

$$\mathcal{H}_{\varepsilon,h} := \left\{\phi \in H^1(\Omega^f_{\varepsilon,\alpha,h}) \, : \, \phi = 0 \text{ on } \partial\Omega^f_{\varepsilon,\alpha,h} \setminus \Gamma_{\varepsilon,\alpha}\right\}.$$

Let $l \in \mathbb{Z}^{n-1} \times \{0\}$, such that $|l\varepsilon| < h$. It is easy to check, that for all $\psi_{\varepsilon,\alpha} \in \mathcal{H}_{\varepsilon,h}$ it holds almost everywhere in $(0,T)$ that

$$\begin{aligned} &\frac{1}{\varepsilon^\alpha}\langle \partial_t \delta c_{\varepsilon,\alpha}, \psi_{\varepsilon,\alpha}\rangle_{\mathcal{H}_{\varepsilon,h}} + \int_{\Omega^f_{\varepsilon,\alpha,h}} \varepsilon^\alpha D \nabla \delta c_{\varepsilon,\alpha} \cdot \nabla \psi_{\varepsilon,\alpha} \, \mathrm{d}x \\ &\quad - \int_{\Omega^f_{\varepsilon,\alpha,h}} \frac{\delta(u_{\varepsilon,\alpha} c_{\varepsilon,\alpha})}{\varepsilon^2} \cdot \nabla \psi_{\varepsilon,\alpha} \, \mathrm{d}x = \frac{1}{\varepsilon^\alpha} \int_{\Omega^f_{\varepsilon,\alpha,h}} \delta g_{\varepsilon,\alpha} \psi_{\varepsilon,\alpha} \, \mathrm{d}x. \end{aligned} \tag{5.5}$$

First, we assume that $c^b_\varepsilon = 0$. We choose a cut-off function $\eta \in C^\infty_0(\Sigma_h)$ with $0 \le \eta \le 1$ and $\eta = 1$ in Σ_{2h}. We emphasize that η depends on h and, in particular, the gradient is of order $\frac{1}{h}$. In the following, we use C_h to denote constants which depend on h (and might grow to ∞ for $h \to 0$). We now choose $\psi_{\varepsilon,\alpha} = \eta^2 \delta c_{\varepsilon,\alpha} \in \mathcal{H}_{\varepsilon,\alpha}$ and we have

$$\nabla\psi_{\varepsilon,\alpha} = \eta\nabla(\eta\delta c_{\varepsilon,\alpha}) + \eta\delta c_{\varepsilon,\alpha}\nabla\eta = 2\eta\delta c_{\varepsilon,\alpha}\nabla\eta + \eta^2\nabla\delta c_{\varepsilon,\alpha}.$$

We get the following for all $t \in [0,T]$:

$$\begin{aligned} &\frac{1}{2\varepsilon^\alpha}\|\eta\delta c_{\varepsilon,\alpha}(t)\|^2_{L^2(\Omega^f_{\varepsilon,\alpha,h})} + \varepsilon^\alpha D\|\eta\nabla\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + 2\varepsilon^\alpha D\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\nabla\delta c_{\varepsilon,\alpha}\cdot\nabla\eta\eta\delta c_{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad -\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\frac{\delta(u_{\varepsilon,\alpha}c_{\varepsilon,\alpha})}{\varepsilon^2}\cdot\nabla(\eta^2\delta c_{\varepsilon,\alpha})\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{\varepsilon^\alpha}\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\delta g_{\varepsilon,\alpha}\eta^2\delta c_{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

For the third term on the left-hand side, we get

$$\begin{aligned} \left|\varepsilon^\alpha\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\nabla\delta c_{\varepsilon,\alpha}\cdot\nabla\eta\eta\delta c_{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}s\right| &\le \frac{1}{\varepsilon^\alpha}\|\eta\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + C_h\varepsilon^{3\alpha}\|\nabla\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\le \frac{1}{\varepsilon^\alpha}\|\eta\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + C_h\varepsilon^{2\alpha}, \end{aligned}$$

where, in the last inequality, we used the a priori estimate for $\nabla c_{\varepsilon,\alpha}$ from Proposition 5.3. For the convective term, we use

$$\int_{\Omega^f_{\varepsilon,\alpha,h}}\frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\nabla(\eta\delta c_{\varepsilon,\alpha})^2\,\mathrm{d}x = 0,$$

to obtain the following with $\delta(u_{\varepsilon,\alpha}c_{\varepsilon,\alpha}) = c_{\varepsilon,\alpha}\delta u_{\varepsilon,\alpha} + u_{\varepsilon,\alpha}(\cdot + l\varepsilon)\delta c_{\varepsilon,\alpha}$ and the a priori estimates from Propositions 4.3, 5.3 and 5.4:

$$\begin{aligned} &\left|\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\frac{\delta(u_{\varepsilon,\alpha}c_{\varepsilon,\alpha})}{\varepsilon^2}\cdot\nabla\psi_{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\le\left|\int_0^t\int_{\Omega^f_{\varepsilon,\alpha,h}}\frac{\delta u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\cdot\left[2\eta\delta c_{\varepsilon,\alpha}\nabla\eta + \eta^2\nabla\delta c_{\varepsilon,\alpha}\right] + \frac{u_{\varepsilon,\alpha}(\cdot + l\varepsilon)}{\varepsilon^2}\delta c_{\varepsilon,\alpha}\left[\eta\nabla(\eta\delta c_{\varepsilon,\alpha}) + \eta\delta c_{\varepsilon,\alpha}\nabla\eta\right]\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\le C_h\left\|\frac{\delta u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})}\|c_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha,h})}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\quad + C\left\|\frac{\eta\delta u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})}\|c_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha,h})}\|\eta\nabla\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\quad + C_h\left\|\frac{u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})}\|\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T)\times\Omega^f_{\varepsilon,\alpha,h})}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\le C_h\varepsilon^{\frac{\alpha}{2}}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + C\left\|\frac{\eta\delta u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})}\|\eta\nabla\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\quad + C_h\varepsilon^{\frac{\alpha}{2}}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\le C_h\varepsilon^\alpha + \frac{1}{\varepsilon^\alpha}\|\eta\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + \frac{C}{\varepsilon^\alpha}\left\|\frac{\eta\delta u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + \frac{D\varepsilon^\alpha}{2}\|\eta\nabla\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})}. \end{aligned}$$

The term on the right-hand side of Eq (5.6) can be estimated in a similar way. In total, we get the following with an absorption argument and the Gronwall inequality:

$$\begin{aligned} &\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha,h})} + \varepsilon^{\frac{\alpha}{2}}\|\eta\nabla\delta c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} \\ &\quad\le C_h\varepsilon^{\frac{\alpha}{2}} + \frac{C}{\varepsilon^{\frac{\alpha}{2}}}\left\|\frac{\eta\delta u_{\varepsilon,\alpha}}{\varepsilon^2}\right\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} + \frac{C}{\varepsilon^{\frac{\alpha}{2}}}\|\delta g_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})}. \end{aligned} \tag{5.7}$$

It remains to estimate the term including $\delta u_{\varepsilon,\alpha}$. For this, we consider the equation for $\delta u_{\varepsilon,\alpha}$. More precisely, for all $\phi_{\varepsilon,\alpha} \in H^1(\Omega^f_{\varepsilon,\alpha,h})^n$, such that $\phi_{\varepsilon,\alpha} = 0$ on $\partial\Omega^f_{\varepsilon,\alpha,h} \setminus (S^+_{\varepsilon,f} \cup S^-_{\varepsilon,f})$, it holds that

$$\int_{\Omega^f_{\varepsilon,\alpha,h}} \nabla\delta u_{\varepsilon,\alpha} : \nabla\phi_{\varepsilon,\alpha}\,\mathrm{d}x - \int_{\Omega^f_{\varepsilon,\alpha,h}} (\delta p_{\varepsilon,\alpha} - \delta p^b_{\varepsilon,\alpha})\nabla\cdot\phi_{\varepsilon,\alpha}\,\mathrm{d}x = \int_{\Omega^f_{\varepsilon,\alpha,h}} (\delta f_{\varepsilon,\alpha} - \nabla\delta p^b_{\varepsilon,\alpha})\cdot\phi_{\varepsilon,\alpha}\,\mathrm{d}x.$$

Now, we choose $\phi_{\varepsilon,\alpha} = \eta^2\delta u_{\varepsilon,\alpha}$ to obtain the following with $\nabla\phi_{\varepsilon,\alpha} = \eta^2\nabla\delta u_{\varepsilon,\alpha} + 2\eta\nabla\eta\otimes\delta u_{\varepsilon,\alpha}$ and $\nabla\cdot\phi_{\varepsilon,\alpha} = 2\eta\nabla\eta\cdot\delta u_{\varepsilon,\alpha}$:

$$\begin{aligned}
\|\eta\nabla\delta u_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha,h})} =& \int_{\Omega^f_{\varepsilon,\alpha,h}} (\delta p_{\varepsilon,\alpha} - \delta p^b_{\varepsilon,\alpha})2\eta\nabla\eta\cdot\delta u_{\varepsilon,\alpha}\,\mathrm{d}x + \int_{\Omega^f_{\varepsilon,\alpha,h}} (\delta f_{\varepsilon,\alpha} - \nabla\delta p^b_{\varepsilon,\alpha})\cdot\delta u_{\varepsilon,\alpha}\eta^2\,\mathrm{d}x \\
&- 2\int_{\Omega^f_{\varepsilon,\alpha,h}} \eta\nabla\delta u_{\varepsilon,\alpha} : (\nabla\eta\otimes\delta u_{\varepsilon,\alpha})\,\mathrm{d}x \\
\le& C_h\|\delta p_{\varepsilon,\alpha} - \delta p^b_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})}\|\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} \\
&+ C\|\delta f_{\varepsilon,\alpha} - \nabla\delta p^b_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})}\|\eta\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} \\
&+ C_h\|\eta\nabla\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})}\|\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} \\
\le& C_h\varepsilon^{2+2\alpha} + C\varepsilon^{2+\alpha}\kappa(|l\varepsilon|) + C_h\varepsilon^{2+\frac{\alpha}{2}}\|\eta\nabla\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})},
\end{aligned}$$

where, at the end, we used the a priori estimates for $u_{\varepsilon,\alpha}$ and $p_{\varepsilon,\alpha}$ from Propositions 4.3 and 4.7, and Assumption (T3). For the last term, we can use the Young inequality to obtain the following with an absorption argument:

$$\|\eta\nabla\delta u_{\varepsilon,\alpha}\|^2_{L^2(\Omega^f_{\varepsilon,\alpha,h})} \le C_h\varepsilon^{2+2\alpha} + C\varepsilon^{2+\alpha}\kappa(|l\varepsilon|).$$

Using the Poincaré inequality from Lemma 4.2, we get

$$\begin{aligned}
\|\eta\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} &\le C\varepsilon\left(\|\delta u_{\varepsilon,\alpha}\nabla\eta\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} + \|\eta\nabla\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})}\right) \\
&\le C_h\varepsilon^{3+\frac{\alpha}{2}} + C_h\varepsilon^{2+\alpha} + C\varepsilon^{2+\frac{\alpha}{2}}\kappa(|l\varepsilon|).
\end{aligned}$$

In summary, we obtain

$$\varepsilon^{-2}\|\eta\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} + \varepsilon^{-1}\|\eta\nabla\delta u_{\varepsilon,\alpha}\|_{L^2(\Omega^f_{\varepsilon,\alpha,h})} \le C_h\varepsilon^{\alpha} + C\varepsilon^{\frac{\alpha}{2}}\kappa(|l\varepsilon|). \tag{5.8}$$

Using this estimate in the inequality (5.7), we get the following with Assumption (T3):

$$\frac{1}{\varepsilon^{\frac{\alpha}{2}}}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha,h}))} + \varepsilon^{\frac{\alpha}{2}}\|\eta\nabla\delta c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,h})} \le C_h\varepsilon^{\frac{\alpha}{2}} + \kappa(|l\varepsilon|).$$

Using the L^∞-estimate for $c_{\varepsilon,\alpha}$ from Lemma 5.5 (and the fact that here, we consider here a smooth extension of $c_{\varepsilon,\alpha}$ to the whole layer $\mathbb{R}^{n-1}\times(-\varepsilon^\alpha,\varepsilon^\alpha)$), we obtain

$$\begin{aligned}
\varepsilon^{-\frac{\alpha}{2}}\|\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha}))} &\le \varepsilon^{-\frac{\alpha}{2}}\|\eta\delta c_{\varepsilon,\alpha}\|_{L^\infty((0,T),L^2(\Omega^f_{\varepsilon,\alpha}\setminus\Omega^f_{\varepsilon,\alpha,2h}))} + C\sqrt{h} \\
&\le C\sqrt{h} + C_h\varepsilon^{\frac{\alpha}{2}} + \kappa(|l\varepsilon|).
\end{aligned}$$

This is the desired result for $c^b_\varepsilon = 0$. For the general case, we consider $w_{\varepsilon,\alpha}$ in the previous calculations instead of $c_{\varepsilon,\alpha}$. In Eq (5.6), we then have to consider the function $g_{\varepsilon,\alpha} - \partial_t c^b_\varepsilon$ instead of $g_{\varepsilon,\alpha}$, and we obtain the following on the right-hand side the additional term:

$$\int_{\Omega^f_{\varepsilon,\alpha}} \left(D^\alpha_\varepsilon \nabla c^b_\varepsilon - \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c^b_\varepsilon \right) \cdot \nabla(\eta^2 c_{\varepsilon,\alpha})\, \mathrm{d}x.$$

This term can be estimated in the same way as the respective terms in the calculation above with c^b_ε instead of $c_{\varepsilon,\alpha}$ and using Assumption (T3). This finishes the proof in Case (D1).

Now, we consider Case (D2). Here, we can choose $\psi_{\varepsilon,\alpha} = \delta c_{\varepsilon,\alpha}$ directly in Eq (5.5) and, we can work with the full domain $\Omega^f_{\varepsilon,\alpha}$ instead of $\Omega^f_{\varepsilon,\alpha,h}$ (formally, we can choose $h = 0$), since we can extend all the functions periodically in the horizontal direction. In particular, in Eq (5.6), we can put $\eta = 0$ and all terms including $\nabla\eta$ vanish. The remaining terms can be estimated as in Case (D2).

Remark 5.9.

(i) We also showed that

$$\varepsilon^{\frac{\alpha}{2}} \|\nabla \delta c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha,2h})} \leq C_h \varepsilon^{\frac{\alpha}{2}} + \kappa(|l\varepsilon|)$$

and an estimate for $\delta u_{\varepsilon,\alpha}$ and $\nabla \delta u_{\varepsilon,\alpha}$. However, for the proof of the strong convergence of $c_{\varepsilon,\alpha}$ this estimate is not necessary and, therefore, we only formulated the result for $\delta c_{\varepsilon,\alpha}$.

(ii) The proof of Proposition 5.8 simplifies for the case of periodic boundary conditions. However, this assumption seems to be necessary in Case (D2). Otherwise (assuming also a Neumann boundary condition), we get in Eq (5.6) *the critical term*

$$2\varepsilon^{-\alpha} D \int_0^t \int_{\Omega^f_{\varepsilon,\alpha,h}} \nabla_{\bar{x}} \delta c_{\varepsilon,\alpha} \cdot \nabla\eta\eta\delta c_{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}s.$$

Using the same estimate as in the proof above, we only get

$$\left| 2\varepsilon^{-\alpha} D \int_0^t \int_{\Omega^f_{\varepsilon,\alpha,h}} \nabla_{\bar{x}} \delta c_{\varepsilon,\alpha} \cdot \nabla\eta\eta\delta c_{\varepsilon,\alpha}\, \mathrm{d}x\, \mathrm{d}s \right| \leq \frac{1}{\varepsilon^\alpha} \|\eta\delta c_{\varepsilon,\alpha}\|^2_{L^2((0,t)\times\Omega^f_{\varepsilon,\alpha,h})} + C_h.$$

As we will see later in the proof of Proposition 5.11, this is not enough to guarantee the strong two-scale convergence of $c_{\varepsilon,\alpha}$.

5.2. Two-scale compactness for the microscopic solutions $c_{\varepsilon,\alpha}$

We formulate the (two-scale) compactness results for the microscopic solution $c_{\varepsilon,\alpha}$ for the different choices of D^α_ε. The weak convergence results are direct consequences of the general two-scale compactness results obtained in Section 3 and the a priori estimates in Section 5.1. However, to deal with the nonlinear advective term, we also need strong two-scale compactness results. For this, we use the additional bound for the differences in the shifts with respect to the spatial variable.

In the following, we extend the functions $c_{\varepsilon,\alpha}$ with the extension operator E_ε from Lemma 5.7 to the whole thin layer $\Omega_{\varepsilon,\alpha}$ and use the same notation $c_{\varepsilon,\alpha}$ for the extension. We emphasize that the a priori estimates in Case (D2) for the extended function do not remain, since we only have

$$\|\nabla_{\bar{x}} E_\varepsilon c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} \leq C\|\nabla c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{-\frac{\alpha}{2}}.$$

Hence, due to the (arbitrary) shape of the perforations Y_s, we can only control the horizontal gradient $\nabla_{\bar{x}} E_\varepsilon c_{\varepsilon,\alpha}$ by the full gradient $\nabla c_{\varepsilon,\alpha}$, including, in particular, the n-th component of $c_{\varepsilon,\alpha}$, scaling badly with respect to ε. We start with the formulation of the weak compactness results for the microscopic solution.

Proposition 5.10.

(i) For $D^\alpha_\varepsilon = \varepsilon^\alpha D$, we have the following.
The maps $c_0 \in L^2((0,T)\times\Omega)$ with $\partial_{x_n} c_0 \in L^2((0,T)\times\Omega)$ and $c_1 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y)/\mathbb{R})$ exist such that, up to a subsequence,

$$\chi_{\Omega^f_{\varepsilon,\alpha}} c_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} c_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^\alpha \nabla c_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f}\left(\partial_{x_n} c_0 e_n + \nabla_y c_1\right).$$

Further, we have $c_0 = c^b_0$ on $S^\pm_1$ in the (generalized) trace sense.

(ii) For $D^\alpha_\varepsilon = D\mathrm{diag}(\varepsilon^{-\alpha},\dots,\varepsilon^{-\alpha},\varepsilon^\alpha) \in \mathbb{R}^{n\times n}$ we have the following.
The maps $c_0 \in L^2((0,T),H^1(\Omega))$ and $c_1 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(0,1)/\mathbb{R})$ (the microscopic variable is the y_n-component), and $\bar{c}_1 \in L^2((0,T)\times\Omega, H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f))$ exist such that, up to a subsequence,

$$\chi_{\Omega^f_{\varepsilon,\alpha}} c_{\varepsilon,\alpha} \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f} c_0, \qquad \chi_{\Omega^f_{\varepsilon,\alpha}} (\nabla_{\bar{x}} c_{\varepsilon,\alpha}, \varepsilon^\alpha \partial_{x_n} c_{\varepsilon,\alpha}) \overset{2,\alpha}{\rightharpoonup} \chi_{Y_f}\left(\nabla c_0 + (\nabla_{\bar{y}}\bar{c}_1, \partial_{y_n} c_1)\right).$$

Further, we have $c_0 = c^b_0$ on $S^\pm_1$ in the trace sense.

Proof. The convergence results are a direct consequence of the a priori estimates from Propositions 5.3 and 5.4, and the compactness results from Propositions 3.4 and 3.6. It remains to establish the trace condition $c_0 = c^b_0$ on $S^\pm_1$. We use again the notation $w_{\varepsilon,\alpha} := E_\varepsilon(c_{\varepsilon,\alpha} - c^b_\varepsilon)$ (here, we explicitly use the extension operator E_ε to better distinguish between the extended functions and the function itself). We emphasize that, in general, we do not have $w_{\varepsilon,\alpha} = 0$ on the whole boundary $S^\pm_\varepsilon$. However, denoting $\Omega^{b,\pm}_{\varepsilon,\alpha}$ as the subset of $\Omega_{\varepsilon,\alpha}$ consisting of micro–cells touching the outer boundary $S^\pm_\varepsilon$, we obtain the following using the standard trace inequality for ε-periodic domains as well as the Poincaré inequlity (we use $w_{\varepsilon,\alpha} = 0$ on $S^\pm_{\varepsilon,f}$):

$$\begin{aligned}\|w_{\varepsilon,\alpha}\|_{L^2((0,T)\times S^\pm_\varepsilon)} &\leq C\left(\frac{1}{\sqrt{\varepsilon}}\|w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^{b,\pm}_{\varepsilon,\alpha})} + \sqrt{\varepsilon}\|\nabla w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^{b,\pm}_{\varepsilon,\alpha})}\right)\\ &\leq C\sqrt{\varepsilon}\|\nabla w_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} \leq C\varepsilon^{\frac{1}{2}-\frac{\alpha}{2}}.\end{aligned}$$

Since $\alpha < 1$, we get the strong two-scale convergence of $w_{\varepsilon,\alpha}$ to 0 on $S^\pm_\varepsilon$. Obviously, we have $E_\varepsilon c^b_\varepsilon \overset{2,\alpha}{\rightharpoonup} c^b_0$, and therefore, Proposition 3.5 implies the desired result.

It remains to establish the strong two-scale convergence of $c_{\varepsilon,\alpha}$. First, we consider the case $D_\varepsilon^\alpha = D\mathrm{diag}(\varepsilon^{-\alpha}, \dots, \varepsilon^{-\alpha}, \varepsilon^\alpha)$, which can be seen as the more simple case, since no additional assumptions on the data (see Assumption (T3)) are necessary. Furthermore, the argument is less technical, because we can apply directly the Simon compactness result from [37].

Proposition 5.11. *It holds up to a subsequence*

$$c_{\varepsilon,\alpha} \overset{2,\alpha}{\to} c_0.$$

Proof. We define, for almost every $(t, x) \in (0, T) \times \Omega$, the rescaled function $\tilde{c}_{\varepsilon,\alpha}(t, x) := \tilde{c}_{\varepsilon,\alpha}(t, \bar{x}, \varepsilon^\alpha x_n)$. With the properties of the extension operator from Lemma 5.7 (again, to illustrate the use of this lemma, we explicitly write $E_\varepsilon c_{\varepsilon,\alpha}$ for the extended function), we find for $0 < h \ll 1$ that (see [16, Proposition 5] for similar arguments)

$$\begin{aligned}
\|\tilde{c}_{\varepsilon,\alpha}(\cdot_t + h, \cdot_x) - \tilde{c}_{\varepsilon,\alpha}\|^2_{L^2((0,T-h),L^2(\Omega))} &= \frac{1}{\varepsilon^\alpha}\|E_\varepsilon c_{\varepsilon,\alpha}(\cdot_t + h, \cdot_x) - E_\varepsilon c_{\varepsilon,\alpha}\|^2_{L^2((0,T-h),L^2(\Omega_{\varepsilon,\alpha}))} \\
&\leq \frac{C}{\varepsilon^\alpha}\|c_{\varepsilon,\alpha}(\cdot_t + h, \cdot_x) - c_{\varepsilon,\alpha}\|^2_{L^2((0,T-h),L^2(\Omega^f_{\varepsilon,\alpha}))} \\
&\leq \frac{C\sqrt{h}}{\varepsilon^\alpha}\|\partial_t c_{\varepsilon,\alpha}\|_{L^2((0,T),\mathcal{H}'_{D_\varepsilon^\alpha})}\|c_{\varepsilon,\alpha}\|_{L^2((0,T),\mathcal{H}_{D_\varepsilon^\alpha})} \\
&\leq C\sqrt{h},
\end{aligned}$$

where, at the end, we use the a priori estimates from Propositions 5.3, 5.4, and 5.6. Next, we control differences in the shifts in the spatial variable. We extend $c_{\varepsilon,\alpha}$ to a function in $(0, T) \times \mathbb{R}^n$ preserving, in particular, the L^∞-estimates. Hence, due to the essential bound from Lemma 5.5, it is enough to show the estimate above for $\Omega_h := \{x \in \Omega : \mathrm{dist}(\partial\Omega, x) > h\}$ for $0 < h \ll 1$. Let $\xi \in \mathbb{R}^n$ such that $|\xi| < h$. In the following, we use similar arguments as in the proof of [16, Proposition 6] and therefore we skip some details. Let $\bar{\xi}_\varepsilon := \varepsilon\left[\frac{\bar{\xi}}{\varepsilon}\right]$. We now have

$$\begin{aligned}
\|\tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + \xi) - \tilde{c}_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_h)} \leq &\|\tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + (\bar{\xi}, 0)) - \tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + \xi)\|_{L^2((0,T)\times\Omega_h)} \\
&+ \|\tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + (\bar{\xi}, 0)) - \tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + (\bar{\xi}_\varepsilon, 0))\|_{L^2((0,T)\times\Omega_h)} \\
&+ \|\tilde{c}_{\varepsilon,\alpha} - \tilde{c}_{\varepsilon,\alpha}(\cdot_t, \cdot_x + (\bar{\xi}_\varepsilon, 0))\|_{L^2((0,T)\times\Omega_h)} =: I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon.
\end{aligned}$$

For the first term, we use the mean value theorem, to obtain

$$I^1_\varepsilon \leq C|\xi_n|\|\partial_{x_n}\tilde{c}_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega)} \leq C|\xi_n|.$$

For the second term, we proceed in a similar way, to get

$$I^2_\varepsilon \leq C|\bar{\xi} - \bar{\xi}_\varepsilon|\|\nabla_{\bar{x}}\tilde{c}_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega)} \leq C\varepsilon^{1-\frac{\alpha}{2}}\|\nabla c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C\varepsilon^{1-\alpha}.$$

For the last term I^3_ε, we get the following with Proposition 5.8:

$$I^3_\varepsilon \leq \frac{C}{\varepsilon^{\frac{\alpha}{2}}}\|c_{\varepsilon,\alpha}(\cdot_t, \cdot_x + (\bar{\xi}_\varepsilon, 0)) - c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \leq C\sqrt{h} + C_h\varepsilon^{\frac{\alpha}{2}} + \kappa(|\bar{\xi}_\varepsilon|).$$

We can now apply the Fréchet–Kolmogorov–Riesz compactness theorem to obtain the strong convergence of $\tilde{c}_{\varepsilon,\alpha}$ to some limit function $\tilde{c}_0$ in $L^2((0,T)\times\Omega)$. It is easy to check that $\tilde{c}_0 = c_0$. Hence, we get (since c_0 is independent of y)

$$\|c_0\|_{L^2(\Omega\times Y)} = \|c_0\|_{L^2(\Omega)} = \lim_{\varepsilon\to 0}\|\tilde{c}_{\varepsilon,\alpha}\|_{L^2(\Omega)} = \varepsilon^{-\frac{\alpha}{2}}\|c_{\varepsilon,\alpha}\|_{L^2(\Omega_{\varepsilon,\alpha})},$$

and, therefore, the strong two-scale convergence of $c_{\varepsilon,\alpha}$.

Strong two-scale compactness results in thin layers were also obtained, for example, in [23, 24] (see also [38] for similar ideas in the case of perforated domains). Compared with our situation, a crucial difference lies in the different scaling of the diffusion in different (horizontal and vertical) directions. In the aforementioned contributions, in the case of fast diffusion, an additional bound for the differences in shifts was not necessary. Let us explain, why we cannot avoid this bound in our situation, as long as we consider arbitrary domains. The rescaled (extended) function $\tilde{c}_{\varepsilon,\alpha}$ fulfills

$$\begin{aligned}\|\tilde{c}_{\varepsilon,\alpha}\|^2_{L^2((0,T),H^1(\Omega))} =& \frac{1}{\varepsilon^\alpha}\|E_\varepsilon c_{\varepsilon,\alpha}\|^2_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} + \frac{1}{\varepsilon^\alpha}\|\nabla_{\bar{x}}E_\varepsilon c_{\varepsilon,\alpha}\|^2_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})}\\ &+ \varepsilon^\alpha\|\partial_{x_n}E_\varepsilon c_{\varepsilon,\alpha}\|^2_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})}.\end{aligned}$$

On the right-hand side, we have the norms in the full layer $\Omega_{\varepsilon,\alpha}$ and have to consider the extended function $E_\varepsilon c_{\varepsilon,\alpha}$. Now, as already mentioned at the beginning of this section, we can only control the horizontal gradient $\nabla_{\bar{x}}E_\varepsilon c_{\varepsilon,\alpha}$ by the full gradient $\nabla c_{\varepsilon,\alpha}$, and only get

$$\|\tilde{c}_{\varepsilon,\alpha}\|_{L^2((0,T)\times H^1(\Omega))} \le C\varepsilon^{-\alpha}.$$

This is a consequence of the fact, that the estimate for the gradient for the extension operator is depending on the full norm. We can only avoid this problem in the case of a specific geometry for example, by considering cylindrical inclusions as in Section 4.4. In this case, we have

$$\|\nabla_{\bar{x}}E_\varepsilon c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega_{\varepsilon,\alpha})} \le C\|\nabla_{\bar{x}}c_{\varepsilon,\alpha}\|_{L^2((0,T)\times\Omega^f_{\varepsilon,\alpha})} \le C\varepsilon^{\frac{\alpha}{2}},$$

which implies that $\tilde{c}_{\varepsilon,\alpha}$ is bounded in $L^2((0,T),H^1(\Omega))$. Now, with the estimate for the differences in the shifts with respect to time in the proof of Proposition 5.11, we can directly apply [37, Theorem 1], without an additional estimate for the differences in the shifts in the spatial variable. However, in conclusion, we see that for an arbitrary shape of the perforations and different orders of diffusion in different directions, the usual a priori bounds in H^1 and for the time-derivative, as obtained in Propositions 5.4 and 5.6, are not enough to guarantee strong two-scale convergence. We emphasize, that this problem also occurs in the case of perforated domains which are not thin.

5.3. Derivation of the macroscopic model

On the basis of the compactness results obtained in the previous section, we are now able to proceed with the derivation of the macroscopic model with its effective coefficients. Here, we proceed in the usual way for homogenization problems, where we first derive the cell problems for the corrector functions c_1 and then derive the macroscopic equation. This has to be adapted to our situation including the dimension reduction, which is included in our definition of the two-scale convergence. We first deal with Case (D2) of high diffusion in the horizontal direction. The other case follows by similar arguments.

5.3.1. The case $D_\varepsilon^\alpha = D\mathrm{diag}(\varepsilon^{-\alpha}, \dots, \varepsilon^{-\alpha}, \varepsilon^\alpha)$

First of all, in Eq (5.1), we choose test functions of the form $\psi_{\varepsilon,\alpha}(t,x) := \psi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right)$ with $\psi \in C_0^\infty([0,T), C_\#^\infty(\overline{\Omega}, C_{\mathrm{per}}^\infty(Y)))$ and compact support in $(-1,1)$ with respect to the x_n variable (third component), and get the following after integration with respect to time and integration by parts in time:

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} c_{\varepsilon,\alpha}\partial_t\psi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x\,\mathrm{d}t \\
&-\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2} c_{\varepsilon,\alpha}\cdot\left[\nabla_{\bar{x}}\psi + \varepsilon^{-\alpha}\partial_{x_n}\psi e_n + \varepsilon^{-1}\nabla_y\psi\right]\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} \left[\varepsilon^{-\alpha}D\nabla_{\bar{x}}c_{\varepsilon,\alpha} + \varepsilon^\alpha D\partial_{x_n}c_{\varepsilon,\alpha}e_n\right]\cdot\left[\nabla_{\bar{x}}\psi + \varepsilon^{-\alpha}\partial_{x_n}\psi e_n + \varepsilon^{-1}\nabla_y\psi\right]\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x\,\mathrm{d}t \\
&= \varepsilon^{-\alpha}\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha}\psi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon}\right) \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.9}$$

First, we derive the cell problem for the limit function $\bar{c}_1$. For this, we multiply the equation above by ε and use the estimates from Proposition 5.10, to obtain (all terms except the diffusive term including $\nabla_{\bar{y}}\psi$ vanish for $\varepsilon \to 0$, see Proposition 5.4)

$$\int_0^T \int_\Omega \int_{Y_f} D(\nabla_{\bar{x}}c_0 + \nabla_{\bar{y}}\bar{c}_1)\cdot\nabla_{\bar{y}}\psi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By density, this is valid for all $\psi \in L^2((0,T)\times\Omega, H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f))$. In other words, $\bar{c}_1$ is the unique (up to a constant depending on y_n) weak solution of the cell problem

$$\begin{aligned}
-\nabla_{\bar{y}}\cdot(D(\nabla_{\bar{x}}c_0 + \nabla_{\bar{y}}\bar{c}_1)) &= 0 \quad \text{in } (0,T)\times\Omega\times Y_f, \\
-D(\nabla_{\bar{x}}c_0 + \nabla_{\bar{y}}\bar{c}_1))\cdot\nu &= 0 \quad \text{on } (0,T)\times\Omega\times\Gamma, \\
\bar{c}_1 \text{ is } Y\text{-periodic.}&
\end{aligned} \tag{5.10}$$

From this, we obtain the decomposition

$$\bar{c}_1(t,x,y) = \sum_{i=1}^{n-1}\partial_{x_i}c_0(t,x)\bar{\chi}_i(y) \tag{5.11}$$

for almost every $(t,x,y) \in (0,T)\times\Omega\times Y_f$, where $\bar{\chi}_i \in H^1_{\mathrm{per},\nabla_{\bar{y}}}(Y_f)$ is the unique (up to L^2 functions only depending on y_n) weak solution of the cell problem (for $i = 1,\dots,n-1$)

$$\begin{aligned}
-\nabla_y\cdot(D(e_i + \nabla_{\bar{y}}\bar{\chi}_i)) &= 0 \quad \text{in } Y_f, \\
-D(e_i + \nabla_{\bar{y}}\bar{\chi}_i)\cdot\nu &= 0 \quad \text{on } \Gamma, \\
\bar{\chi}_i \text{ is } Y\text{-periodic.}&
\end{aligned} \tag{5.12}$$

Next, we derive a cell problem for c_1. For this, in Eq (5.9), we choose test functions that are independent of $\bar{y}$, i.e., $\phi(t,x,y) = \phi(t,x,y_n)$ and multiply the equation by $\varepsilon^{1-\alpha}$. Now, the term including $\nabla_{\bar{y}}\phi$ vanishes, and we get

$$\int_\Omega \int_{Y_f} (\partial_{x_n}c_0 + \partial_{y_n}c_1)\partial_{y_n}\phi(y_n)\,\mathrm{d}y = 0,$$

and by density this equation is valid for all $\phi \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(0,1))$. We define

$$A(y_n) := \mathcal{H}^{n-1}\left(\left\{\bar{y} \in Y \,:\, (\bar{y}, y_n) \in Y_f\right\}\right).$$

Since Y_f and $\Omega^f_{\varepsilon,\alpha}$ are Lipschitz and connected, we have $A \in L^\infty(0,1)$ and $A \geq a_0 > 0$. Identifying c_1 with a function in $L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(0,1))$, we find that c_1 is a weak solution of the problem

$$\begin{aligned} -\partial_{y_n}(A(\partial_{x_n} c_0 + \partial_{y_n} c_1)) = 0 \quad &\text{in } (0,T)\times\Omega\times(0,1), \\ c_1 \text{ is 1-periodic.}& \end{aligned} \tag{5.13}$$

Since this problem has a unique weak solution, we get

$$c_1(t,x,y_n) = \partial_{x_n} c_0(t,x)\chi_n(y), \tag{5.14}$$

where $\chi_n \in H^1_{\mathrm{per}}(0,1)/\mathbb{R}$ is the unique weak solution of the cell problem

$$\begin{aligned} -\partial_{y_n}(A(1 + \partial_{y_n}\chi_n)) = 0 \quad &\text{in } (0,1), \\ \chi_n \text{ is 1-periodic.}& \end{aligned} \tag{5.15}$$

We are now able to derive the macroscopic model. In Eq (5.9), we choose test functions of the form $\psi(t,x) := \psi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}\right)$ (independent of the microscopic variable y) and obtain

$$\begin{aligned} &-\varepsilon^{-\alpha}\int_0^T\int_{\Omega^f_{\varepsilon,\alpha}} c_{\varepsilon,\alpha}\partial_t\psi\left(t,\bar{x},\frac{x_n}{\varepsilon^\alpha}\right)\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega^f_{\varepsilon,\alpha}} \frac{u_{\varepsilon,\alpha}}{\varepsilon^2}c_{\varepsilon,\alpha}\cdot[\nabla_{\bar{x}}\psi + \varepsilon^{-\alpha}\partial_{x_n}\psi e_n]\left(t,\bar{x},\frac{x_n}{\varepsilon^\alpha}\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad+\int_0^T\int_{\Omega^f_{\varepsilon,\alpha}} [\varepsilon^{-\alpha}D\nabla_{\bar{x}}c_{\varepsilon,\alpha} + \varepsilon^{\alpha}D\partial_{x_n}c_{\varepsilon,\alpha}e_n]\cdot[\nabla_{\bar{x}}\psi + \varepsilon^{-\alpha}\partial_{x_n}\psi e_n]\left(t,\bar{x},\frac{x_n}{\varepsilon^\alpha}\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad= \varepsilon^{-\alpha}\int_0^T\int_{\Omega^f_{\varepsilon,\alpha}} g_{\varepsilon,\alpha}\psi\left(t,\bar{x},\frac{x_n}{\varepsilon^\alpha}\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.16}$$

Using the compactness results from Propositions 4.8, 5.10, and 5.11 (in particular, we need the strong two-scale convergence of $c_{\varepsilon,\alpha}$ to pass to the limit in the convective term; see also Remark 3.1), we get

$$\begin{aligned} &-\int_0^T\int_\Omega\int_{Y_f} c_0\partial_t\psi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega\int_{Y_f} u_0 c_0\cdot\partial_{x_n}\psi e_n\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad+\int_0^T\int_\Omega\int_{Y_f} D(\nabla c_0 + (\nabla_{\bar{y}}\bar{c}_1, \partial_{y_n}c_1))\cdot\nabla\psi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega\int_{Y_f} g_0\psi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.17}$$

With the decompositions of $\bar{c}_1$ and c_1 from Eqs (5.11) and (5.14), we get

$$\int_{Y_f} D(\nabla c_0 + (\nabla_{\bar{y}}\bar{c}_1, \partial_{y_n}c_1))\cdot\nabla\psi\,\mathrm{d}y = D^*\nabla c_0\cdot\nabla\psi$$

almost everywhere in $(0,T)\times\Omega$ with the homogenized diffusion coefficient $D^* \in \mathbb{R}^{n\times n}$ given by

$$D_{ij} := \begin{cases} \int_{Y_f} D(e_i + \nabla_{\bar{y}}\bar{\chi}_i)\cdot(e_j + \nabla_{\bar{y}}\bar{\chi}_j)\,\mathrm{d}y & \text{for } i,j = 1,\ldots,n-1, \\ 0 & \text{for } i = n \text{ or } j = n, \\ \int_{Y_f} D(1 + \partial_{y_n}\chi_n)(1 + \partial_{y_n}\chi_n)\,\mathrm{d}y & \text{for } i = j = n. \end{cases} \tag{5.18}$$

Remark 5.12. *This formula is also valid in the case* $D = \begin{pmatrix} \tilde{D} & 0 \\ 0 & D_{nn} \end{pmatrix}$ *with* $\tilde{D} \in \mathbb{R}^{(n-1)\times(n-1)}$ *being positive and* $D_{nn} > 0$*.*

Altogether, we end up with

$$-|Y_f| \int_0^T \int_\Omega c_0 \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega c_0 \bar{u} e_n \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega D^* \nabla c_0 \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \bar{g}_0 \psi \, \mathrm{d}x \, \mathrm{d}t$$

with $\bar{g}_0 := \int_{Y_f} g_0 \, \mathrm{d}y$. By density, this is valid for all $\psi \in L^2((0,T), H^1_\#(\Omega))$ with $\psi = 0$ on $S_1^\pm$ and $\partial_t \psi \in L^2((0,T)\times\Omega)$. In particular, this implies that $\partial_t c_0 \in L^2((0,T), H^1_\#(\Omega, S_1^+ \cup S_1^-)')$ and we have almost everywhere in $(0,T)$

$$\langle \partial_t c_0, \psi \rangle_{H^1(\Omega, S_1^+ \cup S_1^-)} - \int_\Omega \bar{u} c_0 e_n \cdot \nabla\psi \, \mathrm{d}x + \int_\Omega D^* \nabla c_0 \cdot \nabla\psi \, \mathrm{d}x = \int_\Omega \bar{g}_0 \psi \, \mathrm{d}x$$

for all $\psi \in H^1_\#(\Omega, S_1^+ \cup S_1^-)$ and $c_0(0) = 0$. In other words, c_0 is a weak solution of the macroscopic problem (2.6). Obviously, a weak solution of this problem is unique and, in particular, we get the convergence of the whole sequence. This finishes the proof of Theorem 2.3.

5.3.2. The case $D_\varepsilon^\alpha = \varepsilon^\alpha D$

In this case, we proceed in a similar way as before. The only difference occurs in the diffusive term, where this term in Eq (5.9) has to be replaced by

$$\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^\alpha D \nabla c_{\varepsilon,\alpha} \cdot \left[\nabla_{\bar{x}} \psi + \varepsilon^{-\alpha} \partial_{x_n} \psi e_n + \frac{1}{\varepsilon} \nabla_y \psi \right] \left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}, \frac{x}{\varepsilon} \right) \mathrm{d}x \, \mathrm{d}t.$$

Multiplication by $\varepsilon^{1-\alpha}$ and $\varepsilon \to 0$ gives the following with the same arguments as given above:

$$\int_0^T \int_\Omega \int_{Y_f} D \left(\partial_{x_n} c_0 e_n + \nabla_y c_1 \right) \cdot \nabla_y \psi \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t = 0,$$

and, by density, this is valid for all $\phi \in L^2((0,T)\times\Omega, H^1_{\mathrm{per}}(Y_f))$. In other words, c_1 is the unique (up to constant) weak solution of the cell problem

$$\begin{aligned} -\nabla_y \cdot (D(\partial_{x_n} c_0 e_n + \nabla_y c_1)) &= 0 && \text{in } (0,T)\times\Omega\times Y_f, \\ -D(\partial_{x_n} c_0 e_n + \nabla_y c_1) \cdot \nu &= 0 && \text{on } (0,T)\times\Omega\times\Gamma, \\ c_1 \text{ is } Y\text{-periodic.} \end{aligned}$$

Hence, we obtain the following for almost every $(t,x,y) \in (0,T)\times\Omega\times Y_f$:

$$c_1(t,x,y) = \partial_{x_n} c_0(t,x) \chi_n(y),$$

where χ_n is the cell solution of Eq (5.12) for $i = n$. In Eq (5.1) again choosing test functions of the form $\psi(t,x) := \psi\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}\right)$ with $\psi \in C_0^\infty([0,T)\times(\Omega\cup\partial_D\Omega))$, we obtain Eq (5.16) with the diffusive term replaced by

$$\begin{aligned}
&\int_0^T \int_{\Omega^f_{\varepsilon,\alpha}} \varepsilon^\alpha D\nabla c_{\varepsilon,\alpha} \cdot [\nabla_{\bar{x}}\psi + \varepsilon^{-\alpha}\partial_{x_n}\psi e_n]\left(t, \bar{x}, \frac{x_n}{\varepsilon^\alpha}\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad \overset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega \int_{Y_f} D(\partial_{x_n} c_0 e_n + \nabla_y c_1)\cdot e_n \partial_{x_n}\psi \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega D^*_{nn}\partial_{x_n} c_0 \partial_{x_n}\psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Arguing, as in the previous case, we obtain $\partial_t c_0 \in L^2((0,T)\times\Sigma, H^1_0(-1,1))$, and almost everywhere in $(0,T)$, we have

$$\langle \partial_t c_0, \psi\rangle_{L^2(\Sigma,H^1_0(-1,1))} - \int_\Omega \bar{u} c_0 \partial_{x_n}\psi \,\mathrm{d}x + \int_\Omega D^*_{nn}\partial_{x_n} c_0 \cdot \partial_{x_n}\psi \,\mathrm{d}x = \int_\Omega \bar{g}_0 \psi \,\mathrm{d}x$$

for all $\psi \in L^2(\Sigma, H^1_0(-1,1))$. In other words, c_0 is a weak solution of the problem (2.7). It is easy to check that a weak solution of this problem is unique. In particular, all the convergence results are valid for the whole sequence. This finishes the proof of Theorem 2.4.

6. Conclusions

In summary, we rigorously derived effective models on the macroscopic scale, which describe Stokes flow and transport in thin perforated layers. We considered different scalings for the diffusion in the transport equation, leading to different bounds with respect to ε for the H^1-norms of the microscopic solutions. On the basis of these uniform a priori estimates, we established several compactness results for the microscopic solutions in the sense of two-scale convergence adapted to thin heterogeneous layers. Compared with previous homogenization results for the Stokes equation in perforated domains and thin structures of order ε (no oscillations in the vertical direction), we obtain, in the limit, a Darcy equation which only includes the n-th derivative of the Darcy pressure. This is also a consequence of the pressure boundary condition on the top/bottom of the layer. This boundary condition is, of course, important for applications. However, we emphasize that imposing the (also natural) no-slip condition on this part of the boundary causes serious difficulties. In this case, we are unable to derive a limit problem and even doubt whether such a derivation is possible at all. This is a consequence of the arbitrary shape of the perforations within the thin layer and does not occur for cylindrical inclusions (with no change in the geometry in the vertical direction). Another interesting choice would be a Navier slip boundary condition on the top/bottom, for which we expect similar estimates for the microscopic pressure as for the no-slip boundary condition. Here, we are not sure about the homogenization process, and a more detailed analysis is necessary. We emphasize that in this case, we can also consider a Neumann boundary condition for the transport equation, which causes difficulties for the pressure boundary condition in our situation, since we are not able to control the advective term in the reaction–diffusion–advection equation. Another important question, related to the works in [10, 11], is the treatment of Stokes flow through a thin perforated layer with tiny perforations having a size of order $a(\varepsilon)$ with $a(\varepsilon) \ll \varepsilon$. For the critical scaling $a_\varepsilon = \varepsilon^3$ for bulk problems and $a_\varepsilon = \varepsilon^2$ for obstacles on a hyperplane see [11], this leads to a Brinkman law (smaller objects are invisible for the fluid). Here, it would be interesting how this scaling is influenced by the parameter α and a rigorous proof is missing.

For the transport problem, we again obtain, in the limit $\varepsilon \to 0$, a reaction–diffusion–advection equation with homogenized diffusion coefficients and the advection given by the n-th component of the Darcy velocity. Hence, the horizontal component of the Darcy velocity has no influence on the macroscopic evolution of the concentration (or heat, in the case of heat flow). The effective diffusion depends on the scaling in the microscopic equation for the diffusive term. For fast diffusion in the horizontal direction, we obtain effective diffusion in all space dimensions, where, in the case of slow diffusion, the macroscopic diffusion only occurs in the vertical direction. In the latter case, we end up in a partial differential equation with spatial derivatives only in the vertical direction, and the horizontal components occur as a parameter. Here, it would be interesting to consider other scalings for the transport equation to obtain macroscopic models depending on the full Darcy velocity or an additional micro–macro coupling with the cell problems (very slow diffusion). We are also interested in the case of the coupling of the thin perforated layer with bulk regions, to obtain effective interface conditions between the two bulk domains, which is part of our ongoing work.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

Author contributions

All authors contributed equally to this study.

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