Networks and
Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(2): 693–724.
DOI: 10.3934/nhm.2026030
Received: 28 January 2026
Revised: 14 April 2026
Accepted: 15 April 2026
Published: 24 April 2026

Research article

Algebraic properties of central Bell-based type 2 Bernoulli and Euler polynomials of complex variable

Waseem Ahmad Khan[1], **Francesco Aldo Costabile**[2], **Khidir Shaib Mohamed**[3,*], **Ugur Duran**[4], **Abdulghani Muhyi**[5], **Azhar Iqbal**[6] **and Wei Sin Koh**[7]

[1] Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Saudi Arabia

[2] Department of Mathematics and Computer Science, University of Calabria, Rende 87036, CS, Italy

[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

[4] Department of Basic Sciences of Engineering, Faculty of Engineering and Natural Sciences, Iskenderun Technical University, TR-31200 Hatay, Turkiye

[5] Department of Mechatronics Engineering, Faculty of Engineering and Smart Computing, Modern Specialized University, Sana'a, Yemen

[6] Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Saudi Arabia

[7] Faculty of Business and Communications, INTI International University, Nilai 71800, Negeri Sembilan, Malaysia

* **Correspondence:** Email: k.idris@qu.edu.sa.

Abstract: Recently, by combining type 2 Bernoulli and Euler polynomials with the central Bell polynomials, the central Bell-based type 2 Bernoulli and Euler polynomials of order α were considered, and many of their properties, formulas, and applications were investigated. The main aim of this work is to consider higher-order central Bell-based type 2 Bernoulli and Euler polynomials of complex variable, by which, both sine and cosine central Bell-based type 2 Bernoulli and Euler polynomials of order λ are introduced by treating the imaginary and real components separately. Then, diverse summation formulas, differential formulas, addition formulas, and correlation formulas with new and existing old polynomials and numbers are derived in a systematic way. Also, several intriguing connections of sine and cosine central Bell-based type 2 Bernoulli and Euler polynomials of order λ with the bivariate and one-variable central Bell polynomials, and the classical Stirling and central factorial numbers of the second kinds are investigated in detail. Moreover, the first few members of the new polynomials are provided by the lists, and the distributions of zeros of the new polynomials are illustrated by graphical representations, enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed.

Keywords: type 2 Euler polynomials; type 2 Bernoulli polynomials; central Bell polynomials; partial differential equations; differential equations; process innovation; sine and cosine polynomials

1. Introduction

Special numbers and polynomial sequences constitute one of the central research directions in contemporary mathematical analysis. Their structural properties, generating functions, and operator-based representations have found applications in combinatorics, number theory, matrix theory, and approximation theory. In particular, generating function techniques provides a powerful analytical framework for constructing and studying new polynomial families [1–4]. The construction of several variations of the special polynomials was achieved by combining the concepts of two or more special polynomials, numbers, or functions. For example, truncated-exponential-based Laguerre-Frobenius Euler and Hermite-type polynomials, Hermite-based Bernoulli, Genocchi, and Euler polynomials, the Gould-Hopper-based Apostol type Genocchi, Euler, and Bernoulli polynomials, Laguerre-based Euler, and Bernoulli polynomials, the degenerate Apostol-type Hermite polynomials, the Bell-based Euler, and Bernoulli polynomials, and Lagrange-based Apostol-Genocchi, Apostol-Bernoulli, and Apostol-Euler polynomials, were constructed by mixing the definitions of Bell, Genocchi, Frobenius-Euler, Hermite, Gould-Hopper, Euler, Lagrange, truncated-exponential, Bernoulli, and Laguerre polynomials and many of their properties and applications were investigated deeply; references [5–8] and see the references cited therein. In very recent times, the central Bell-based type 2 Bernoulli ${}_{CB}\mathrm{b}_n^{(\lambda)}(x,y)$ polynomials of order $\lambda \in \mathbb{C}$ [9] and central Bell-based type 2 Euler ${}_{CB}\mathbb{E}_n^{(\lambda)}(x,y)$ polynomials of order $\lambda \in \mathbb{C}$ [10] were considered by:

$$\sum_{n=0}^{\infty} {}_{CB}\mathrm{b}_n^{(\lambda)}(x,y)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt+y\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (|t| < 2\pi), \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_n^{(\lambda)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt+y\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)} \quad (|t| < \pi), \tag{1.2}$$

which are abbreviated with *CBBT2BPOλ* and *CBBT2EPOλ*, respectively. Then, for the polynomials given in Eqs (1.1) and (1.2), several relations, identities, and formulas, including partial differentiation rules, addition formulas, summation formulae, symmetric identities, recurrence relations, and correlations with the central Bell polynomials and the central factorial numbers of the second kind, were derived deeply [9–11]. Also, various curious formulas of *CBBT2EPOλ* stemming from umbral algebra to possess different manners of obtaining old and new formulas were acquired.

Inspired and motivated by the polynomials given in Eqs (1.1) and (1.2), in this work, we aim to define cosine and sine central Bell-based type 2 Bernoulli, and Euler polynomials of order λ by considering higher-order central Bell-based type 2 Euler, and Bernoulli polynomials of complex variable. Besides, we explore various analytical characteristics, such as summation formulae, addition formulae, differential formulae, and correlations with the new and existing old polynomials and numbers in a systematic manner. In addition, we derive exciting connections of our parametric central Bell-based type 2 Euler, and Bernoulli polynomials with the classical central factorial, and Stirling numbers of the second kind. Additionally, the first few members of the new polynomials are given by four lists, and the distributions of zeros of the new polynomials are shown by graphical representations.

The Stirling numbers of the second kind (abbreviated with *SNSK*) are defined as follows [12–14]:

$$\sum_{n=k}^{\infty} k!S_2(n,k)\frac{t^n}{n!} = (e^t - 1)^k, \tag{1.3}$$

which yields, for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

where $(x)_0 = 1$, and $(x)_n = (x-(n-1))(x-(n-2)) \cdots (x-1)x$ for $n \in \mathbb{N}$, [15,16].

For $k \in \mathbb{N}_0$, the central factorial polynomials of the second kind (abbreviated with *CFPSK*) and central factorial numbers of the second kind (abbreviated with *CFNSK*) are, respectively, introduced as follows [17–20]:

$$\sum_{n=0}^{\infty} T(n,k:x)\, k! \frac{t^n}{n!} = \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k e^{xt} \text{ and } \sum_{n=0}^{\infty} T(n,k)\, k! \frac{t^n}{n!} = \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k. \tag{1.4}$$

For $n \in \mathbb{N}_0$, $T(n,k)$, fulfill the following equality [21]:

$$x^n = \sum_{k=0}^{n} T(n,k)\, x^{[k]},$$

where the notation $x^{[k]}$, termed as the central factorial of x, equals to $\prod_{j=0}^{k-1}\left(x - \frac{k-1}{2} + j\right)$ with $x^{[0]} = 1$ [22–25].

The bivariate central Bell polynomials (abbreviated with *BCBP* and denoted by $\phi_n^{(C)}(x,y)$), one-variable central Bell polynomials (abbreviated with *CBP* and denoted by $\phi_n^{(C)}(x)$), and classical central Bell numbers (abbreviated with *CBN* and denoted by $\phi_n^{(C)}$), are defined, respectively, as follows (see [26–28]):

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \phi_n^{(C)}(x,y) = e^{xt} e^{y\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \phi_n^{(C)}(x) = e^{x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \phi_n^{(C)} = e^{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}.$$

We note that $\phi_n^{(C)}(0,y) =: \phi_n^{(C)}(y)$ and $\phi_n^{(C)}(0,1) =: \phi_n^{(C)}(1) =: \phi_n^{(C)}$. We observe from Eqs (1.4)–(1.6) that

$$\phi_n^{(C)}(x,y) = \sum_{k=0}^{n} y^k T(n,k:x),$$

and

$$\phi_n^{(C)}(y) = \sum_{k=0}^{n} y^k T(n,k).$$

Type 2 Bernoulli, and Euler polynomials of order $\lambda \in \mathbb{C}$ (abbreviated with *T2BPOλ* and *T2EPOλ* and denoted by $\mathfrak{b}_n^{(\lambda)}(x)$ and $\mathbb{E}_n^{(\lambda)}(x)$, respectively) are defined by (see [29–31]):

$$\sum_{n=0}^{\infty} \mathfrak{b}_n^{(\lambda)}(x) \frac{t^n}{n!} = e^{xt}\left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} \quad (|t| < 2\pi), \tag{1.7}$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}_n^{(\lambda)}(x) \frac{t^n}{n!} = e^{xt}\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} \quad (|t| < \pi). \tag{1.8}$$

The corresponding numbers, named type 2 Bernoulli and Euler numbers of order λ (abbreviated with *T2BNO*λ and *T2ENO*λ), are determined as $\mathfrak{b}_n^{(\lambda)}(0) =: \mathfrak{b}_n^{(\lambda)}$ and $\mathbb{E}_n^{(\lambda)}(0) =: \mathbb{E}_n^{(\lambda)}$, respectively. The usual forms of the polynomials and numbers above, named the classical type 2 Bernoulli and Euler polynomials and numbers (abbreviated with *T2BP* and *T2EP* with *T2BN* and *T2EN*), are determined as $\mathfrak{b}_n^{(1)}(x) =: \mathfrak{b}_n(x)$ and $\mathbb{E}_n^{(1)}(x) =: \mathbb{E}_n(x)$ with $\mathfrak{b}_n^{(1)} =: \mathfrak{b}_n$ and $\mathbb{E}_n^{(1)} =: \mathbb{E}_n$, respectively.

The cosine and sine polynomials are defined as follows (see [32–35]):

$$\sum_{n=0}^{\infty} \mathbb{C}_n(x,y)\frac{t^n}{n!} = \cos yte^{xt}, \tag{1.9}$$

and

$$\sum_{n=0}^{\infty} \mathbb{S}_n(x,y)\frac{t^n}{n!} = \sin yte^{xt}, \tag{1.10}$$

where

$$\mathbb{C}_n(x,y) = \sum_{k=0}^{[\frac{n}{2}]} y^{2k}(-1)^k x^{n-2k}\binom{n}{2k}, \tag{1.11}$$

and

$$\mathbb{S}_n(x,y) = \sum_{k=0}^{[\frac{n-1}{2}]} y^{2k+1}(-1)^k x^{n-2k-1}\binom{n}{2k+1}. \tag{1.12}$$

The contents of this paper are as follows.

- The next section introduces sine and cosine *CBBT2BPO*λ and acquires various formulae and relations of the new polynomials. It also derives exciting connections of our polynomials *CBBT2BPO*λ with *CFNSK* and *SNSK*.
- Section 3 introduces sine and cosine *CBBT2EPO*λ and examines miscellaneous formulae and relations of the new polynomials. Moreover, it investigates exciting connections of our polynomials *CBBT2EPO*λ with *CFNSK* and *SNSK*.
- Section 4 provides the first few members of sine and cosine *CBBT2BPO*λ, and gives the graphical representations, forming 2D and 3D structures, of the zeros of certain members of sine and cosine *CBBT2BPO*λ.
- Section 5 gives the first few members of sine and cosine *CBBT2EPO*λ, and presents the graphical representations, forming 2D and 3D structures, of the zeros of certain members of sine and cosine *CBBT2EPO*λ.
- The last section analyzes the consequences of this study.

2. Central Bell-based type 2 Bernoulli polynomials of complex variable

In this section, we define the *CBBT2BPO*λ of complex variable and achieve to obtain numerous identities, relations, and formulae. Our starting point is the definition that follows:

$$\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z)\frac{t^n}{n!} = e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{(x+iy)t}\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} \quad \left(i^2=-1;\ |t|<2\pi\right). \tag{2.1}$$

When $\lambda = 1$, we have ${}_{CB}\mathfrak{b}_n(x+iy,z) := {}_{CB}\mathfrak{b}_n^{(1)}(x+iy,z)$, termed as the central Bell-based type 2 Bernoulli polynomials of complex variable (abbreviated with *CBBT2BPCV*).

We observe from Eqs (2.1) and (2.4) that

$$\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z)\frac{t^n}{n!} = (\cos yt + i\sin yt)\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt+z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}, \tag{2.2}$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_n^{(\lambda)}(x-iy,z)\frac{t^n}{n!} = (\cos yt - i\sin yt)\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt+z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}, \tag{2.3}$$

where (see [12, 36–38]):

$$e^{iyt} = \cos yt + i\sin yt. \tag{2.4}$$

It can be deduced from the Eqs (2.2) and (2.3) that

$$2\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\cos yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty}\left({}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z) + {}_{CB}\mathfrak{b}_n^{(\lambda)}(x-iy,z)\right)\frac{t^n}{n!}, \tag{2.5}$$

and

$$2i\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\sin yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty}\left({}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z) - {}_{CB}\mathfrak{b}_n^{(\lambda)}(x-iy,z)\right)\frac{t^n}{n!}. \tag{2.6}$$

Now, we state one of the main definitions as follows.

Definition 1. *The cosine central Bell-based type 2 Bernoulli polynomials of order $\lambda \in \mathbb{C}$ and sine central Bell-based type 2 Bernoulli polynomials of order $\lambda \in \mathbb{C}$ (abbreviated with CCBBT2BPOλ and SCBBT2BPOλ and denoted by ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$, respectively) are defined by*

$$\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\cos yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!}, \tag{2.7}$$

and

$$\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\sin yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)\frac{t^n}{n!}. \tag{2.8}$$

It can be implied from Eqs (2.7) and (2.8) that

$$\begin{gathered}
{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,0,z) = {}_{CB}\mathfrak{b}_n^{(\lambda)}(x,z), \\
{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,0,0) = \mathfrak{b}_n^{(\lambda)}(x), \\
{}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,0,z) = 0 \text{ for } n \in \mathbb{N}_0.
\end{gathered}$$

From Eqs (2.5)–(2.8), it can be deduced that

$${}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z) + {}_{CB}\mathfrak{b}_n^{(\lambda)}(x-iy,z) = 2{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z), \tag{2.9}$$

and

$${}_{CB}\mathfrak{b}_n^{(\lambda)}(x+iy,z) - {}_{CB}\mathfrak{b}_n^{(\lambda)}(x-iy,z) = 2i{}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z). \tag{2.10}$$

We provide some special analysis of the new polynomials (2.7) and (2.8) as follows.

Remark 1. *The one-variable cosine and sine central Bell-based type 2 Bernoulli polynomials can be acquired by taking $x = z = 0$ in Eqs (2.7) and (2.8), respectively, as follows:*

$$\sum_{n=0}^{\infty} \mathfrak{b}_n^{(\lambda,c)}(y)\frac{t^n}{n!} = \cos yt \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda}, \tag{2.11}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{b}_n^{(\lambda,s)}(y)\frac{t^n}{n!} = \sin yt \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda}. \tag{2.12}$$

It is clear, for $n \geq 0$, that $\mathfrak{b}_n^{(\lambda,c)} := \mathfrak{b}_n^{(\lambda,c)}(0)$ and $\mathfrak{b}_n^{(\lambda,s)}(0) = 0$.

Remark 2. *The bivariate cosine and sine central Bell-based type 2 Bernoulli polynomials can be considered by choosing $z = 0$ in Eqs (2.7) and (2.8), respectively, as follows:*

$$\sum_{n=0}^{\infty} \mathfrak{b}_n^{(\lambda,c)}(x,y)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\cos yt, \tag{2.13}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{b}_n^{(\lambda,s)}(x,y)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\sin yt. \tag{2.14}$$

Remark 3. *The two-variable cosine and sine central Bell-based type 2 Bernoulli polynomials can be obtained by taking $x = 0$ in Eqs (2.7) and (2.8), respectively, as follows:*

$$\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(y,z)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\cos yt, \tag{2.15}$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(y,z)\frac{t^n}{n!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\sin yt. \tag{2.16}$$

Remark 4. *The trivariate cosine and sine central Bell polynomials can be considered by setting $\lambda = 0$ in Eqs (2.7) and (2.8), respectively, as follows:*

$$\sum_{n=0}^{\infty} \phi_{n,c}^{(C)}(x,y,z)\frac{t^n}{n!} = e^{xt}e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\cos yt, \tag{2.17}$$

and

$$\sum_{n=0}^{\infty} \phi_{n,s}^{(C)}(x,y,z)\frac{t^n}{n!} = e^{xt}e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\sin yt. \tag{2.18}$$

Remark 5. *The bivariate cosine and sine central Bell polynomials can be considered by choosing $\lambda = x = 0$ in Eqs (2.7) and (2.8), respectively, as follows:*

$$\sum_{n=0}^{\infty} \phi_{n,c}^{(C)}(y,z)\frac{t^n}{n!} = e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\cos yt,$$

and

$$\sum_{n=0}^{\infty} \phi_{n,s}^{(C)}(y,z)\frac{t^n}{n!} = e^{z(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\sin yt.$$

We commence our examination of these polynomials by examining their basic properties. We first give two summation formulas for *CCBBT2BPOλ* and *SCBBT2BPOλ*.

Theorem 1. The following equalities hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$ {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^{k} \binom{n}{2k} y^{2k} {}_{CB}\mathfrak{b}_{n-2k}^{(\lambda)}(x,z), \tag{2.19} $$

and

$$ {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^{k} y^{2k+1} {}_{CB}\mathfrak{b}_{n-2k-1}^{(\lambda)}(x,z) \binom{n}{2k+1}. \tag{2.20} $$

Proof. It can be implied from Eqs (1.1) and (2.7) that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) \frac{t^{n}}{n!} &= \cos yt e^{xt} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \right)^{\lambda} \\ &= \sum_{k=0}^{\infty} (-1)^{k} y^{2k} \frac{t^{k}}{2k!} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n}^{(\lambda)}(x,z) \frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n}{2}]} y^{2k} \, (-1)^{k} \; {}_{CB}\mathfrak{b}_{n-2k}^{(\lambda)}(x,z) \binom{n}{2k} \right) \frac{t^{n}}{n!}, \end{aligned} \tag{2.21} $$

and also, it can be deduced from Eqs (1.1) and (2.8) that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) \frac{t^{n}}{n!} &= \sin yt e^{xt} \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \right)^{\lambda} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n-1}{2}]} {}_{CB}\mathfrak{b}_{n-2k-1}^{(\lambda)}(x,z) (-1)^{k} y^{2k+1} \binom{n}{2k+1} \right) \frac{t^{n}}{n!}. \end{aligned} \tag{2.22} $$

Therefore, by Eqs (2.21) and (2.22), we get Eqs (2.19) and (2.20). So, we complete the proofs.

Here are two addition formulas for ${}_{CB}\mathfrak{b}_{n}^{(\lambda)}(x,z)$.

Theorem 2. The following equalities are valid for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$:

$$ {}_{CB}\mathfrak{b}_{n}^{(\lambda)}(x+iy,z) = \sum_{k=0}^{n} \; {}_{CB}\mathfrak{b}_{n-k}^{(\lambda)}(x,z) \binom{n}{k} (iy)^{k} = \sum_{k=0}^{n} \; {}_{CB}\mathfrak{b}_{n-k}^{(\lambda)}(z) \binom{n}{k} (x+iy)^{k}, $$

and

$$ {}_{CB}\mathfrak{b}_{n}^{(\lambda)}(x-iy,z) = \sum_{k=0}^{n} \; {}_{CB}\mathfrak{b}_{n-k}^{(\lambda)}(x,z) \binom{n}{k} (-iy)^{k} = \sum_{k=0}^{n} \; {}_{CB}\mathfrak{b}_{n-k}^{(\lambda)}(z) \binom{n}{k} (x-iy)^{k}. $$

Proof. By using Eqs (2.2) and (2.3), we can readily obtain the assertions. So, we omit the details.

We note that when $x = 0$ in (1.1), we obtain the one-variable central Bell-based type 2 Bernoulli polynomials [39–42] as follows:

$$ \sum_{n=0}^{\infty} \; {}_{CB}\mathfrak{b}_{n}^{(\lambda)}(y) \frac{t^{n}}{n!} = e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \right)^{\lambda}. \tag{2.23} $$

Now, we state two correlations for ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 3. The following equalities hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z) = \sum_{k=0}^{n} \mathbb{C}_{n-k}(x,y)\ {}_{CB}\mathfrak{b}^{(\lambda)}_{k}(z)\binom{n}{k}, \tag{2.24} $$

and

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x,y,z) = \sum_{k=0}^{n} \mathbb{S}_{n-k}(x,y)\ {}_{CB}\mathfrak{b}^{(\lambda)}_{k}(z)\binom{n}{k}. \tag{2.25} $$

Proof. It can be implied from Eqs (1.9), (2.7), and (2.23) that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} &= \cos yte^{xt}\left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\ &= \left(\sum_{n=0}^{\infty} \mathbb{C}_n(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}^{(\lambda)}_{n}\,(z)\,\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathbb{C}_{n-k}(x,y)\ {}_{CB}\mathfrak{b}^{(\lambda)}_{k}(z)\binom{n}{k}\right)\frac{t^n}{n!}, \end{aligned} $$

which proves Eq (2.24). The other Eq (2.25) can be shown in the same way.

We provide four summation formulas for *CCBBT2BPOλ* and *SCBBT2BPOλ*.

Theorem 4. The following summation equalities hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z) = \sum_{k=0}^{n} \phi^{(C)}_{n-k}(z)\mathfrak{b}^{(\lambda,c)}_{k}(x,y)\binom{n}{k}, \tag{2.26} $$

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x,y,z) = \sum_{k=0}^{n} \phi^{(C)}_{n-k}(z)\mathfrak{b}^{(\lambda,s)}_{k}(x,y)\binom{n}{k}, \tag{2.27} $$

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z) = \sum_{k=0}^{n} {}_{CB}\mathfrak{b}^{(\lambda)}_{k,c}(y,z)x^{n-k}\binom{n}{k}, \tag{2.28} $$

and

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x,y,z) = \sum_{k=0}^{n} {}_{CB}\mathfrak{b}^{(\lambda)}_{k,s}(y,z)x^{n-k}\binom{n}{k}. \tag{2.29} $$

Proof. It can be indicated from Eqs (1.6), (2.7), and (2.13) that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} &= e^{xt}\left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\lambda} \cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\ &= \left(\sum_{n=0}^{\infty} \phi^{(C)}_n(z)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \mathfrak{b}^{(\lambda)}_{n,c}(x,y)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \phi^{(C)}_{n-k}(z)\mathfrak{b}^{(\lambda)}_{k,c}(x,y)\binom{n}{k}\right)\frac{t^n}{n!}, \end{aligned} \tag{2.30} $$

which proves Eq (2.26). The other can be similarly shown using Eqs (1.5), (1.6), (2.7), (2.8), and (2.13)–(2.16).

Here, we provide two addition formulas for ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 5. The following summation equalities hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$ {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x_1+x_2,y,z_1+z_2) = \sum_{k=0}^{n} \phi_{n-k}^{(C)}(x_2,z_2)\binom{n}{k} {}_{CB}\mathfrak{b}_{k,c}^{(\lambda)}(x_1,y,z_1), \tag{2.31} $$

and

$$ {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x_1+x_2,y,z_1+z_2) = \sum_{k=0}^{n} \phi_{n-k}^{(C)}(x_2,z_2)\binom{n}{k} {}_{CB}\mathfrak{b}_{k,s}^{(\lambda)}(x_1,y,z_1). \tag{2.32} $$

Proof. It can be deduced from Eqs (1.5) and (2.7) that,

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x_1+x_2,y,z_1+z_2)\frac{t^n}{n!} &= \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda} e^{x_1 t}\cos yt e^{x_2 t} e^{(z_1+z_2)(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\ &= \left(\sum_{n=0}^{\infty} \phi_n^{(C)}(x_2,z_2)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x_1,y,z_1)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} {}_{CB}\mathfrak{b}_{k,c}^{(\lambda)}(x_1,y,z_1)\phi_{n-k}^{(C)}(x_2,z_2)\right)\frac{t^n}{n!}, \end{aligned} $$

which proves Eq (2.31). In a similar manner, the assertion Eq (2.32) can be yielded utilizing Eqs (1.5) and (2.8).

We give two particular cases of Eqs (2.31) and (2.32) given below.

Corollary 1. *The following summation equalities hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:*

$$ {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x_1+x_2,y,z) = \sum_{k=0}^{n} {}_{CB}\mathfrak{b}_{k,c}^{(\lambda)}(x_1,y,z)x_2^{n-k}\binom{n}{k}, $$

and

$$ {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x_1+x_2,y,z) = \sum_{k=0}^{n} {}_{CB}\mathfrak{b}_{k,s}^{(\lambda)}(x_1,y,z)x_2^{n-k}\binom{n}{k}. $$

Now, we analyze four derivative properties for *CCBBT2BPOλ* and *SCBBT2BPOλ*.

Theorem 6. The following derivative properties hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$ \begin{aligned} \frac{\partial}{\partial x} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) &= n {}_{CB}\mathfrak{b}_{n-1,c}^{(\lambda)}(x,y,z), \\ \frac{\partial}{\partial y} {}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) &= -n {}_{CB}\mathfrak{b}_{n-1,s}^{(\lambda)}(x,y,z), \\ \frac{\partial}{\partial x} {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) &= n {}_{CB}\mathfrak{b}_{n-1,s}^{(\lambda)}(x,y,z), \end{aligned} \tag{2.33} $$

and

$$ \frac{\partial}{\partial y} {}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = n {}_{CB}\mathfrak{b}_{n-1,c}^{(\lambda)}(x,y,z). $$

Proof. It can be shown from Eq (2.7) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= \cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}te^{xt}\\ &= \sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)\frac{t^{n+1}}{n!},\end{aligned}$$

proving Eq (2.33). The other assertions can be similarly derived.

Two more derivative properties for ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$ are examined below.

Theorem 7. The following derivative properties hold for $\lambda\in\mathbb{C}$ and $n\in\mathbb{N}$:

$$\frac{\partial}{\partial z}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) = {}_{CB}\mathfrak{b}_{n-1,c}^{(\lambda-1)}(x,y,z),$$

and

$$\frac{\partial}{\partial z}{}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = {}_{CB}\mathfrak{b}_{n-1,s}^{(\lambda-1)}(x,y,z).$$

Proof. It can be indicated from Eq (2.7) that

$$\begin{aligned}\frac{\partial}{\partial z}\sum_{n=1}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}e^{xt}\cos yt\frac{\partial}{\partial z}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}e^{xt}\cos yt(e^{\frac{t}{2}}-e^{-\frac{t}{2}})e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda-1}e^{xt}\cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}t,\end{aligned}$$

which provides the first property. The second can be done in a similar way.

Here, we give two correlations for *CCBBT2BPOλ* and *SCBBT2BPOλ*, including $\phi_{n,c}^{(C)}(x,y,z)$ and $\phi_{n,s}^{(C)}(x,y,z)$.

Theorem 8. The following correlations hold for $n\in\mathbb{N}_0$:

$${}_{CB}\mathfrak{b}_{n+1,c}(x+\frac{1}{2},y,z) = (n+1)\,\phi_{n,c}^{(C)}(x,y,z) + {}_{CB}\mathfrak{b}_{n+1,c}(x-\frac{1}{2},y,z), \tag{2.34}$$

and

$${}_{CB}\mathfrak{b}_{n+1,s}(x+\frac{1}{2},y,z) = (n+1)\,\phi_{n,s}^{(C)}(x,y,z) + {}_{CB}\mathfrak{b}_{n+1,s}(x-\frac{1}{2},y,z).$$

Proof. It can be indicated from Eqs (2.7) and (2.17) that

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_{n,c}^{(C)}(x,y,z)\frac{t^n}{n!} = e^{xt}\cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} &= \left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n.c}(x,y,z)\frac{t^{n-}}{n!} \\ &= \sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}(x+\frac{1}{2},y,z)\frac{t^{n-1}}{n!} - \sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}(x-\frac{1}{2},y,z)\frac{t^{n-1}}{n!}\\ &= \sum_{n=0}^{\infty}\left({}_{CB}\mathfrak{b}_{n+1,c}(x+\frac{1}{2},y,z) - {}_{CB}\mathfrak{b}_{n+1,c}(x-\frac{1}{2},y,z)\right)\frac{t^n}{(n+1)!},\end{aligned} \tag{2.35}$$

which proves the first assertion. The second can be done in a similar way, utilizing Eqs (2.8) and (2.18).

Remark 6. *Two special cases for Theorem 8 are given as follows:*

$$ {}_{CB}\mathfrak{b}_{n+1,c}(\frac{1}{2},y,z) = (n+1)\,\phi^{(C)}_{n,c}(y,z) + {}_{CB}\mathfrak{b}_{n+1,c}(-\frac{1}{2},y,z), $$

and

$$ {}_{CB}\mathfrak{b}_{n+1,s}(\frac{1}{2},y,z) = (n+1)\,\phi^{(C)}_{n,s}(y,z) + {}_{CB}\mathfrak{b}_{n+1,s}(-\frac{1}{2},y,z). $$

We now give two correlation formulas for ${}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)$ and ${}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x,y,z)$, including $T(k,\lambda)$, $\phi^{(C)}_{n,c}(x,y,z)$, and $\phi^{(C)}_{n,s}(x,y,z)$.

Theorem 9. The following summation equalities are valid for $n,\lambda \in \mathbb{N}_0$:

$$ \phi^{(C)}_{n,c}(x,y,z) = \frac{n!\lambda!}{(n+\lambda)!}\sum_{k=0}^{n+\lambda} T(k,\lambda)\binom{n+\lambda}{k}{}_{CB}\mathfrak{b}^{(\lambda)}_{n+\lambda-k,c}(x,y,z), $$

and

$$ \phi^{(C)}_{n,s}(x,y,z) = \frac{n!\lambda!}{(n+\lambda)!}\sum_{k=0}^{n+\lambda} T(k,\lambda)\binom{n+\lambda}{k}{}_{CB}\mathfrak{b}^{(\lambda)}_{n+\lambda-k,s}(x,y,z). $$

Proof. It can be indicated from Eqs (1.4), (2.7), and (2.17) that

$$ \begin{aligned} \sum_{n=0}^{\infty}\phi^{(C)}_{n,c}(x,y,z)\frac{t^n}{n!} &= e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt}\cos yt = t^{-\lambda}\lambda!\frac{(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{\lambda}}{\lambda!}\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} \\ &= \left(\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)\frac{t^{n-\lambda}}{n!}\right)\left(\sum_{n=0}^{\infty}\lambda!T(n,\lambda)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\lambda!\sum_{k=0}^{n}{}_{CB}\mathfrak{b}^{(\lambda)}_{n-k,c}(x,y,z)\binom{n}{k}T(k,\lambda)\right)\frac{t^{n-\lambda}}{n!}, \end{aligned} $$

which yields the first alleged outcome. Similarly, we can easily obtain the second utilizing Eqs (1.4), (2.8), and (2.18).

Here, we give two correlation formulas for *CCBBT2BPOλ* and *SCBBT2BPOλ*, including $S_2(l,k)$.

Theorem 10. The following correlations hold for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$:

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}{}_{CB}\mathfrak{b}^{(\lambda)}_{n-l,c}(y,z)(x)_k S_2(l,k), $$

and

$$ {}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}{}_{CB}\mathfrak{b}^{(\lambda)}_{n-l,s}(y,z)(x)_k S_2(l,k). $$

Proof. It can be observed from Eqs (1.3), (2.7), and (2.15) that

$$ \begin{aligned} \sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} &= e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\cos yt\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}(e^t-1+1)^x \qquad (2.36) \\ &= e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\cos yt\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}\sum_{k=0}^{\infty}\frac{(e^t-1)^k}{k!}(x)_k \end{aligned} $$

$$
\begin{aligned}
&= e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\cos yt\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}\sum_{k=0}^{\infty}(x)_k\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(x)_k S_2(n,k)\right)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(y,z)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}(x)_k\ {}_{CB}\mathfrak{b}_{n-l,c}^{(\lambda)}(y,z)S_2(l,k)\binom{n}{l}\frac{t^n}{n!},
\end{aligned}
$$

which yields the first asserted consequence. In a similar way, we can readily acquire the second utilizing Eqs (1.3), (2.8), and (2.16).

We provide two summation formulas for ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 11. The following correlations hold for $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$:

$$
{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(2x)_k T\left(n-l,k:\frac{k}{2}-x\right){}_{CB}\mathfrak{b}_{l,c}^{(\lambda)}(y,z),
$$

and

$$
{}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(2x)_k T\left(n-l,k:\frac{k}{2}-x\right){}_{CB}\mathfrak{b}_{l,s}^{(\lambda)}(y,z).
$$

Proof. It can be indicated from Eqs (1.4), (2.7), and (2.15) that

$$
\begin{aligned}
\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= (e^{\frac{t}{2}}-e^{-\frac{t}{2}}+e^{-\frac{t}{2}})^{2x}\left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\lambda}\cos yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\
&= \left(\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(y,z)\frac{t^n}{n!}\right)\sum_{k=0}^{\infty}(2x)_k\frac{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k}{k!}e^{-(2x-k)\frac{t}{2}}\\
&= \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(2x)_k T\left(n,k:\frac{k-2x}{2}\right)\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(y,z)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}(2x)_k T\left(n-l,k:\frac{k}{2}-x\right){}_{CB}\mathfrak{b}_{l,c}^{(\lambda)}(y,z)\frac{t^n}{n!},
\end{aligned}
$$

which gives the first assertion. In a similar manner, we can achieve the second using Eqs (1.4), (2.8), and (2.16).

3. Central Bell-based Type 2 Euler polynomials of complex variable

In this part, we define *CBBT2EPO*λ of complex variable and achieve to obtain several of their identities, relations, and formulae. Our starting point is the definition that follows:

$$
e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{(x+iy)t}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} = \sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)\frac{t^n}{n!}. \tag{3.1}
$$

Using Eqs (2.4) and (3.1), we have

$$
\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)\frac{t^n}{n!} = (\cos yt + i\sin yt)e^{xt}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}, \tag{3.2}
$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z)\frac{t^n}{n!} = (\cos yt - i\sin yt)e^{xt}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}. \tag{3.3}$$

From Eqs (3.2) and (3.3), we get

$$\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty}\left(\frac{{}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)+{}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z)}{2}\right)\frac{t^n}{n!}, \tag{3.4}$$

and

$$\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\sin yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty}\left(\frac{{}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)-{}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z)}{2i}\right)\frac{t^n}{n!}. \tag{3.5}$$

Definition 2. *The cosine central Bell-based type 2 Euler polynomials of order $\lambda \in \mathbb{C}$ and sine central Bell-based type 2 Euler polynomials of order $\lambda \in \mathbb{C}$ (abbreviated with CCBBT2EPOλ and SCBBT2EPOλ and denoted by ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$, respectively) are defined by*

$$\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!}, \tag{3.6}$$

and

$$\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\sin yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)\frac{t^n}{n!}. \tag{3.7}$$

From Eqs (3.4)–(3.7), it can be deduced that

$${}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)+{}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z) = 2{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z), \tag{3.8}$$

and

$${}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z)-{}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z) = 2i{}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z). \tag{3.9}$$

It can be observed from Eqs (3.6) and (3.7) that

$$\begin{gathered}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,0,z) = {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,z),\\ {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,0,0) = \mathbb{E}_n^{(\lambda)}(x),\\ {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,0,z) = 0 \text{ for } n \in \mathbb{N}_0.\end{gathered}$$

Remark 7. *For $x = 0$ in Eqs (3.6) and (3.7), we get two-variable cosine and sine central Bell-based type 2 Euler polynomials:*

$$\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(y,z)\frac{t^n}{n!} = e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}\cos yt, \tag{3.10}$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(y,z)\frac{t^n}{n!} = e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}\sin yt, \tag{3.11}$$

respectively.

Remark 8. *Letting* $z = 0$ *in Eqs* (3.6) *and* (3.7)*, we obtain bivariate cosine and sine central Bell-based type 2 Euler polynomials:*

$$\sum_{n=0}^{\infty} \mathbb{E}_n^{(\lambda,c)}(x,y)\frac{t^n}{n!} = e^{xt}\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} \cos yt, \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}_n^{(\lambda,s)}(x,y)\frac{t^n}{n!} = e^{xt}\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} \sin yt, \tag{3.13}$$

respectively.

Remark 9. *Setting* $x = z = 0$ *in Eqs* (3.6) *and* (3.7)*, we obtain one-variable cosine and sine central Bell-based type 2 Euler polynomials:*

$$\sum_{n=0}^{\infty} \mathbb{E}_n^{(\lambda,c)}(y)\frac{t^n}{n!} = \cos yt\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda}, \tag{3.14}$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}_n^{(\lambda,s)}(y)\frac{t^n}{n!} = \sin yt\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda}, \tag{3.15}$$

respectively.

We now analyze some properties of *CCBBT2EPO*λ and *SCBBT2EPO*λ. We first give the following theorem, covering two summation formulas for ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 12. The following formulae are valid for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$:

$${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}y^{2k}\,{}_{CB}\mathbb{E}_{n-2k}^{(\lambda)}(x,z),$$

and

$${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z) = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}y^{2k+1}\,{}_{CB}\mathbb{E}_{n-2k-1}^{(\lambda)}(x,z).$$

Proof. It can be shown from Eqs (1.2) and (3.6) that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= \cos yt\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt+z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \sum_{n=0}^{\infty}(-1)^k y^{2k}\frac{t^k}{2k!}\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,z)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(-1)^k y^{2k}\,{}_{CB}\mathbb{E}_{n-2k}^{(\lambda)}(x,z)\right)\frac{t^n}{n!},\end{aligned}$$

and

$$\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)\frac{t^n}{n!} = \sin yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}(-1)^k y^{2k+1}{}_{CB}\mathbb{E}_{n-2k-1}^{(\lambda)}(x,z)\right)\frac{t^n}{n!}.$$

Therefore, by the computations above, we get the first claimed outcome. The second can be indicated using Eqs (1.2) and (3.7).

Here, we give two addition formulas for ${}_{CB}\mathbb{E}_n^{(\lambda)}(x,z)$.

Theorem 13. The following formulae hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$${}_{CB}\mathbb{E}_n^{(\lambda)}(x+iy,z) = \sum_{k=0}^{n} {}_{CB}\mathbb{E}_{n-k}^{(\lambda)}(x,z)\binom{n}{k}(iy)^k = \sum_{k=0}^{n}{}_{CB}\mathbb{E}_{n-k}^{(\lambda)}(z)(x+iy)^k\binom{n}{k}, \tag{3.16}$$

and

$${}_{CB}\mathbb{E}_n^{(\lambda)}(x-iy,z) = \sum_{k=0}^{n} {}_{CB}\mathbb{E}_{n-k}^{(\lambda)}(x,z)(-iy)^k\binom{n}{k} = \sum_{k=0}^{n} {}_{CB}\mathbb{E}_{n-k}^{(\lambda)}(z)(x-iy)^k\binom{n}{k}. \tag{3.17}$$

Proof. By using Eqs (3.2) and (3.3), we can easily get Eqs (3.16) and (3.17). Hence, we omit the details.

Now, we state two correlations for *CCBBT2EPOλ* and *SCBBT2EPOλ*.

Theorem 14. The following formulas are valid for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z) = \sum_{k=0}^{n}{}_{CB}\mathbb{E}_k^{(\lambda)}(z)\mathbb{C}_{n-k}(x,y)\binom{n}{k},$$

and

$${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z) = \sum_{k=0}^{n}{}_{CB}\mathbb{E}_k^{(\lambda)}(z)\mathbb{S}_{n-k}(x,y)\binom{n}{k}.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt}\cos yt\\ &= \left(\sum_{k=0}^{\infty}\mathbb{C}_k(x,y)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_n^{(\lambda)}(z)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}{}_{CB}\mathbb{E}_{n-k}^{(\lambda)}(z)\mathbb{C}_k(x,y)\binom{n}{k}\right)\frac{t^n}{n!},\end{aligned}$$

which implies the first assertion. The other can be investigated in the same way.

Here, we provide four summation and correlation formulas for ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 15. The following correlations hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z) = \sum_{k=0}^{n}\phi_{n-k}^{(C)}(z)\mathbb{E}_k^{(\lambda,c)}(x,y)\binom{n}{k},$$

$${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z) = \sum_{k=0}^{n}\phi_{n-k}^{(C)}(z)\mathbb{E}_k^{(\lambda,s)}(x,y)\binom{n}{k},$$

$$_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)=\sum_{k=0}^{n}x^{n-k}\ {}_{CB}\mathbb{E}_{k,c}^{(\lambda)}(y,z)\binom{n}{k},$$

and

$$_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=\sum_{k=0}^{n}x^{n-k}\ {}_{CB}\mathbb{E}_{k,s}^{(\lambda)}(y,z)\binom{n}{k}.$$

Proof. Using Eqs (3.6) and (3.7), we obtain the assertions above. Hence, we omit the details of the proofs.

We now give two addition formulas for *CCBBT2EPOλ* and *SCBBT2EPOλ*.

Theorem 16. The following addition formulas hold for $\lambda\in\mathbb{C}$ and $n\in\mathbb{N}_0$:

$$_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x_1+x_2,y,z_1+z_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{CB}\mathbb{E}_{k,c}^{(\lambda)}(x_1,y,z_1)\phi_{n-k}^{(C)}(x_2,z_2), \tag{3.18}$$

and

$$_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x_1+x_2,y,z_1+z_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{CB}\mathbb{E}_{k,s}^{(\lambda)}(x_1,y,z_1)\phi_{n-k}^{(C)}(x_2,z_2). \tag{3.19}$$

Proof. By Eq (3.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x_1+x_2,y,z_1+z_2)\frac{t^n}{n!}&=\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}e^{z_1(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{x_1t}\cos yte^{z_2(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{x_2t}\\&=\left(\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x_1,y,z_1)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\phi_n^{(C)}(x_2,z_2)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_{CB}\mathbb{E}_{k,c}^{(\lambda)}(x_1,y,z_1)\phi_{n-k}^{(C)}(x_2,z_2)\right)\frac{t^n}{n!},\end{aligned}$$

which proves Eq (3.18). The result of Eq (3.19) can be derived similarly to that of Eq (3.18).

We provide four derivative properties for $_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)$ and $_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 17. The following derivative properties hold for $\lambda\in\mathbb{C}$ and $n\in\mathbb{N}$:

$$\frac{\partial}{\partial x}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)=n{}_{CB}\mathbb{E}_{n-1,c}^{(\lambda)}(x,y,z), \tag{3.20}$$

$$\frac{\partial}{\partial y}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)=-n{}_{CB}\mathbb{E}_{n-1,s}^{(\lambda)}(x,y,z), \tag{3.21}$$

and

$$\frac{\partial}{\partial x}{}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=n{}_{CB}\mathbb{E}_{n-1,s}^{(\lambda)}(x,y,z), \tag{3.22}$$

$$\frac{\partial}{\partial y}{}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=n{}_{CB}E_{n-1,c}^{(\lambda)}(x,y,z). \tag{3.23}$$

Proof. Equation (3.6) yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}\ {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!}&=\cos yte^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}te^{xt}\\&=\sum_{n=0}^{\infty}{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^{n+1}}{n!},\end{aligned}$$

proving Eq (3.20). The others in Eqs (3.21)–(3.23) can be obtained in the same manner.

We provide two more derivative properties for *CCBBT2EPOλ* and *SCBBT2EPOλ*.

Theorem 18. The following derivative properties hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$\frac{\partial}{\partial z} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z) = {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(\frac{1}{2}+x,y,z) - {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(-\frac{1}{2}+x,y,z), \tag{3.24}$$

and

$$\frac{\partial}{\partial z} {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z) = {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(\frac{1}{2}+x,y,z) - {}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(-\frac{1}{2}+x,y,z). \tag{3.25}$$

Proof. By Eq (3.6), we have

$$\begin{aligned}\frac{\partial}{\partial z}\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)\frac{t^n}{n!} &= \cos yt\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}\frac{\partial}{\partial z}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \cos yt(e^{\frac{t}{2}}-e^{-\frac{t}{2}})\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{xt}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\\ &= \sum_{n=0}^{\infty}\left[{}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x+\frac{1}{2},y,z) - {}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x-\frac{1}{2},y,z)\right]\frac{t^n}{n!},\end{aligned}$$

which completes the proof of Eq (3.24). That of Eq (3.25) is similar.

Now, we state two correlation formulas for ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x,y,z)$ and ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$.

Theorem 19. The following correlation formulas hold for $n \in \mathbb{N}_0$:

$$2\phi_{n,c}^{(C)}(x,y,z) = {}_{CB}\mathbb{E}_{n,c}(\frac{1}{2}+x,y,z) + {}_{CB}\mathbb{E}_{n,c}(-\frac{1}{2}+x,y,z), \tag{3.26}$$

and

$$2\phi_{n,s}^{(C)}(x,y,z) = {}_{CB}\mathbb{E}_{n,s}(\frac{1}{2}+x,y,z) + {}_{CB}\mathbb{E}_{n,s}(-\frac{1}{2}+x,y,z). \tag{3.27}$$

Proof. By Eq (3.6), it is deduced that

$$\begin{aligned}2\sum_{n=0}^{\infty}\phi_{n,c}^{(C)}(x,y,z)\frac{t^n}{n!} &= 2e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt}\cos yt = \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}(x,y,z)\frac{t^n}{n!}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right) \qquad (3.28)\\ &= \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}(\frac{1}{2}+x,y,z)\frac{t^n}{n!} + \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}_{n,c}(-\frac{1}{2}+x,y,z)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left({}_{CB}\mathbb{E}_{n,c}(\frac{1}{2}+x,y,z) + {}_{CB}\mathbb{E}_{n,c}(-\frac{1}{2}+x,y,z)\right)\frac{t^n}{n!}.\end{aligned}$$

In view of Eq (3.28), we get Eq (3.26). Similarly, we can easily obtain Eq (3.27).

Remark 10. *Two special cases for Theorem 19 are given as follows:*

$$2\phi_{n,c}^{(C)}(y,z) = {}_{CB}\mathbb{E}_{n,c}(\frac{1}{2},y,z) + {}_{CB}\mathbb{E}_{n,c}(-\frac{1}{2},y,z),$$

and

$$2\phi_{n,s}^{(C)}(y,z) = {}_{CB}\mathbb{E}_{n,s}(\frac{1}{2},y,z) + {}_{CB}\mathbb{E}_{n,s}(-\frac{1}{2},y,z).$$

Now, we state two summation and correlation formulas for *CCBBT2EPOλ* and *SCBBT2EPOλ*.

Theorem 20. The following correlation formulas hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$ {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l} {}_{CB}\mathbb{E}^{(\lambda)}_{n-l,c}(y,z) S_2(l,k)\binom{n}{l}(x)_k, $$

and

$$ {}_{CB}\mathbb{E}^{(\lambda)}_{n,s}(x,y,z) = \sum_{l=0}^{n}\sum_{k=0}^{l} {}_{CB}\mathbb{E}^{(\lambda)}_{n-l,s}(y,z) S_2(l,k)\binom{n}{l}(x)_k. $$

Proof. Using Eqs (1.3) and (3.6), we observe that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} &= (1+e^t-1)^x\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\cos yt \\ &= \sum_{k=0}^{\infty}(x)_k\frac{(e^t-1)^k}{k!}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda} e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\cos yt \\ &= \sum_{k=0}^{\infty}(x)_k\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}\cos yt \\ &= \left(\sum_{l=0}^{\infty}\sum_{k=0}^{l}S_2(l,k)(x)_k\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(y,z)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}S_2(l,k)(x)_k\binom{n}{l} {}_{CB}\mathbb{E}^{(\lambda)}_{n-l,c}(y,z)\right)\frac{t^n}{n!}, \end{aligned} $$

which provides the first claimed outcome. Similarly, we can smoothly obtain the second.

Here, we give two correlation and summation formulas for *CCBBT2EPOλ* and *SCBBT2EPOλ*.

Theorem 21. The following correlations hold for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$ {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x,y,z) = \sum_{k=0}^{n}\sum_{l=0}^{k} T\left(k,l:\frac{l-2x}{2}\right) {}_{CB}\mathbb{E}^{(\lambda)}_{n-k,c}(y,z)\binom{k}{l}(2x)_l, $$

and

$$ {}_{CB}\mathbb{E}^{(\lambda)}_{n,s}(x,y,z) = \sum_{k=0}^{n}\sum_{l=0}^{k} T\left(k,l:\frac{l-2x}{2}\right) {}_{CB}\mathbb{E}^{(\lambda)}_{n-k,s}(y,z)\binom{k}{l}(2x)_l. $$

Proof. It can be derived from Eqs (1.3) and (3.6) that

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x,y,z)\frac{t^n}{n!} &= (e^{\frac{t}{2}}-e^{-\frac{t}{2}}+e^{-\frac{t}{2}})^{2x}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}\cos yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\ &= \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\lambda}\cos yt e^{z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\sum_{l=0}^{\infty}(2x)_l\frac{(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^l}{l!}e^{t\frac{(l-2x)}{2}} \\ &= \left(\sum_{k=0}^{\infty}\sum_{l=0}^{k}(2x)_l T\left(k,l:\frac{l-2x}{2}\right)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(y,z)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}(2x)_l T\left(k,l:\frac{l-2x}{2}\right) {}_{CB}\mathbb{E}^{(\lambda)}_{n-k,c}(y,z)\right)\frac{t^n}{n!}, \end{aligned} $$

which provides the first argued outcome. The second alleged consequence can be derived in a similar proof method above.

4. Computational values and zeros representations of *CCBBT2BPOλ* and *SCBBT2BPOλ*

This part provides the first few members of *CCBBT2BPOλ* and *SCBBT2BPOλ* by two lists, and presents the distributions of zeros of *CCBBT2BPOλ* and *SCBBT2BPOλ* by graphs with 2D and 3D structures. The numerical analyzes in this part are used to validate theoretical results and reveal distinctive scattering patterns in the distributions of their zeros across the complex plane, offering insights into their underlying analytic structure and providing a foundation for future analytical work on their asymptotic distribution and potential applications. Also, these visualizations elegantly reveal the layered and organized structure of the zero distributions, providing compelling geometric insight into the algebraic richness and complexity of the underlying polynomials.

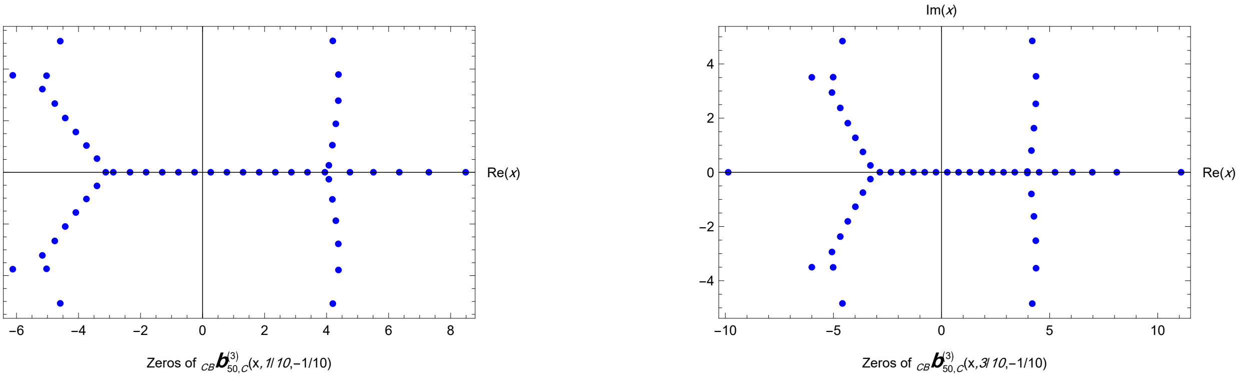
4.1. Computational values and zeros representations of CCBBT2BPOλ

Using Eq (2.7), we give a few members of *CCBBT2BPOλ* for $\lambda = 3$ as:

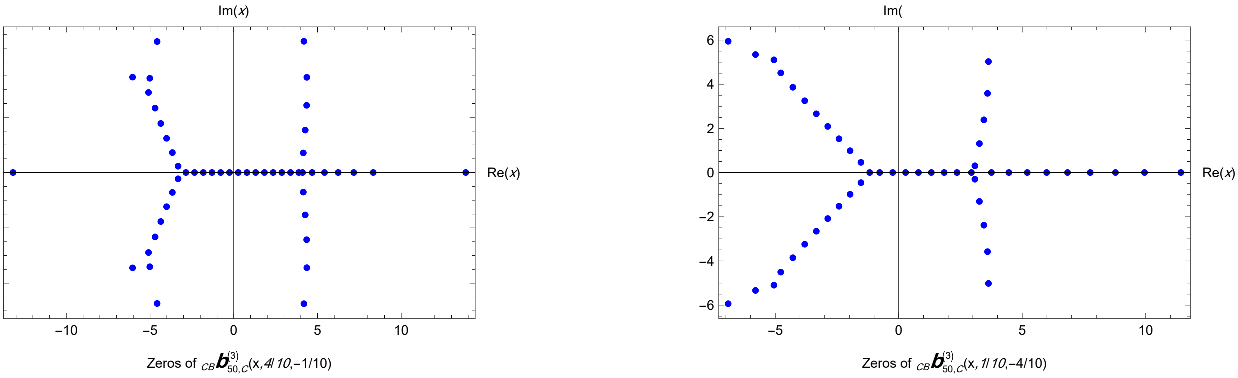
$$
\begin{aligned}
{}_{CB}\mathfrak{b}_{0,c}^{(3)}(x,y,z) &= 1,\\
{}_{CB}\mathfrak{b}_{1,c}^{(3)}(x,y,z) &= x+z,\\
{}_{CB}\mathfrak{b}_{2,c}^{(3)}(x,y,z) &= -\frac{1}{4}+x^2-y^2+2xz+z^2,\\
{}_{CB}\mathfrak{b}_{3,c}^{(3)}(x,y,z) &= -\frac{3x}{4}+x^3-3xy^2-\frac{z}{2}+3x^2z-3y^2z+3xz^2+z^3,\\
{}_{CB}\mathfrak{b}_{4,c}^{(3)}(x,y,z) &= \frac{17}{80}-\frac{3x^2}{2}+x^4+\frac{3y^2}{2}-6x^2y^2+y^4-2xz+4x^3z-12xy^2z-\frac{z^2}{2}+6x^2z^2-6y^2z^2+4xz^3+z^4,\\
{}_{CB}\mathfrak{b}_{5,c}^{(3)}(x,y,z) &= \frac{17x}{16}-\frac{5x^3}{2}+x^5+\frac{15xy^2}{2}-10x^3y^2+5xy^4+\frac{z}{2}-5x^2z+5x^4z+5y^2z-30x^2y^2z+5y^4z\\
&\quad -\frac{5xz^2}{2}+10x^3z^2-30xy^2z^2+10x^2z^3-10y^2z^3+5xz^4+z^5.
\end{aligned}
$$

Roots computational method: The computation involves the following steps:

1. We consider the above explicit forms of ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z)$ for $\lambda = 3$.
2. A root-finding algorithm (e.g., the Newton-Raphson method) is applied to the equation ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x,y,z) = 0$.
3. The algorithm is initialized with a grid of points in the real and complex regions of interest. All distinct zeros are marked when they fall within a specified range.



(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



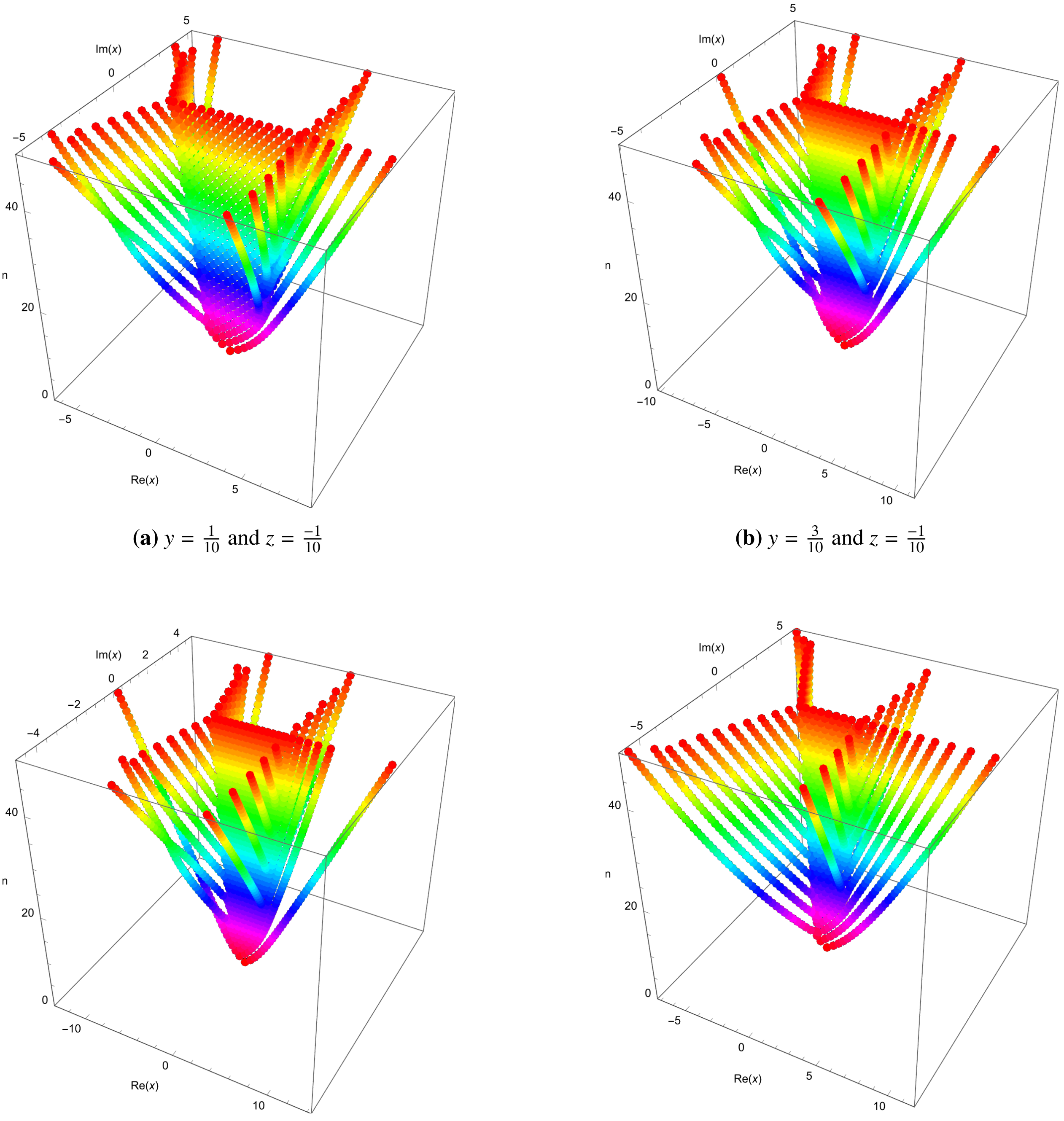
(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 1. Zeros with 2D structures of ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x, y, z)$.

We now perform to explore the fascinating distribution of zeros associated with *CCBBT2BPOλ*. Employing computational methods, we analyze the complex solutions of the equation ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x, y, z) = 0$, which exhibit a striking and structured scattering in the complex plane, offering both aesthetic appeal and deep analytical significance. Some interesting zeros of ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x, y, z) = 0$, for $\lambda = 3$ and $n = 50$ are shown in Figure 1, allowing for a detailed comparison of the function's behavior at different values of y and z. The figures show that the polynomials with the chosen special values have no reflection symmetry with respect to the imaginary axis. A range of values for n has been checked using computer simulations. However, it remains uncertain whether the following conjecture holds for all values of x (refer to Figures 1 and 2).

Conjecture 1. *For $y, z \in \mathbb{R}$, prove that the equation ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x, y, z) = 0$ has n distinct solutions.*

The stacking structures of approximation zeros of ${}_{CB}\mathfrak{b}_{n,c}^{(\lambda)}(x, y, z) = 0$, for $\lambda = 3$ and $1 \le n \le 50$, give 3D structures, which are presented in Figure 2. These allows for a detailed comparison of the function's behavior at different values of y and z.



(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$

(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 2. Stacking structure zeros of ${}_{CB}\mathrm{b}^{(\lambda)}_{n,c}(x, y, z)$ for $\lambda = 3$ and $1 \leq n \leq 50$.

4.2. *Computational values and zeros representations of SCBBT2BPOλ*

In view of Eq (2.8), we list the first six members of ${}_{CB}\mathrm{b}^{(\lambda)}_{n,s}(x, y, z)$ for $\lambda = 3$ as:

$$
\begin{aligned}
{}_{CB}\mathrm{b}^{(3)}_{0,s}(x, y, z) &= 0,\\
{}_{CB}\mathrm{b}^{(3)}_{1,s}(x, y, z) &= y,\\
{}_{CB}\mathrm{b}^{(3)}_{2,s}(x, y, z) &= 2xy + 2yz,\\
{}_{CB}\mathrm{b}^{(3)}_{3,s}(x, y, z) &= 3x^2y + 6xyz - y^3 + 3yz^2 - \frac{3y}{4},\\
{}_{CB}\mathrm{b}^{(3)}_{4,s}(x, y, z) &= 4x^3y + 12x^2yz - 4xy^3 + 12xyz^2 - 3xy - 4y^3z + 4yz^3 - 2yz,
\end{aligned}
$$

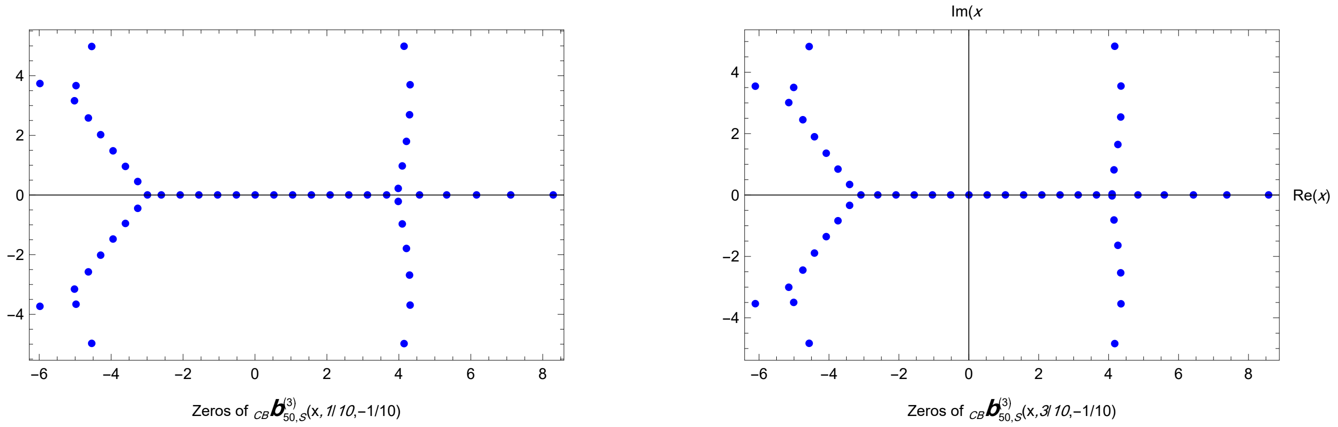
$$
{}_{CB}\mathfrak{b}_{5,s}^{(3)}(x,y,z) = 5x^4y + 20x^3yz - 10x^2y^3 + 30x^2yz^2 - \frac{15x^2y}{2} - 20xy^3z + 20xyz^3 - 10xyz + y^5 - 10y^3z^2 + \frac{5y^3}{2} + 5yz^4 - \frac{5yz^2}{2} + \frac{17y}{16}.
$$

Roots computational method: The computation involves the following steps:

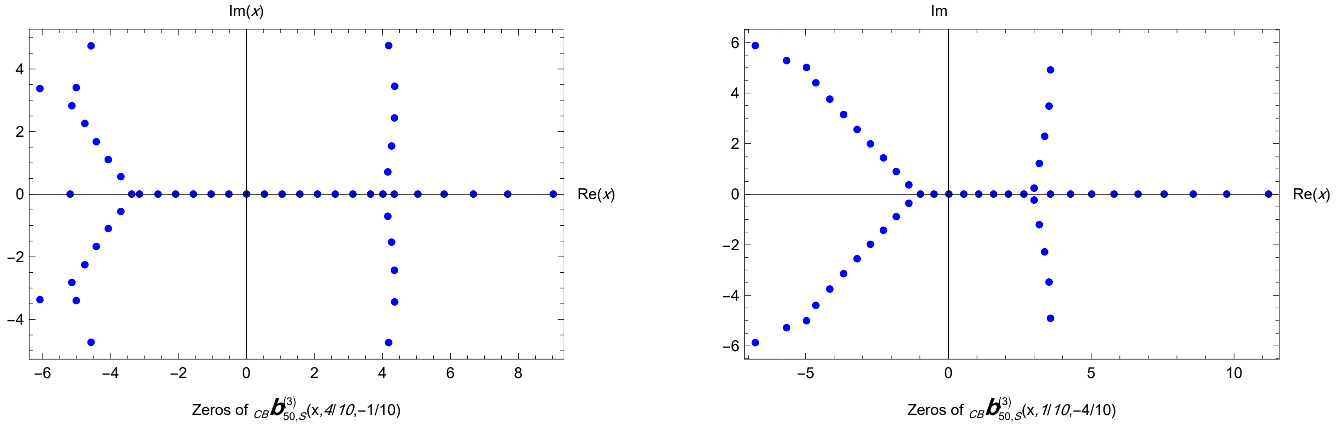
1. We consider the above explicit forms of ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z)$ for $\lambda = 3$.
2. A root-finding algorithm (e.g., the Newton-Raphson method) is applied to the equation ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = 0$.
3. The algorithm is initialized with a grid of points in the real and complex regions of interest. All distinct zeros are marked when they fall within a specified range.

Now, we explore the distributions of zeros and present graphical illustrations of *SCBBT2BPOλ* for specific parameter values and indices.

Some interesting zeros of ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = 0$, for $\lambda = 3$ and $n = 50$ are shown in Figure 3.



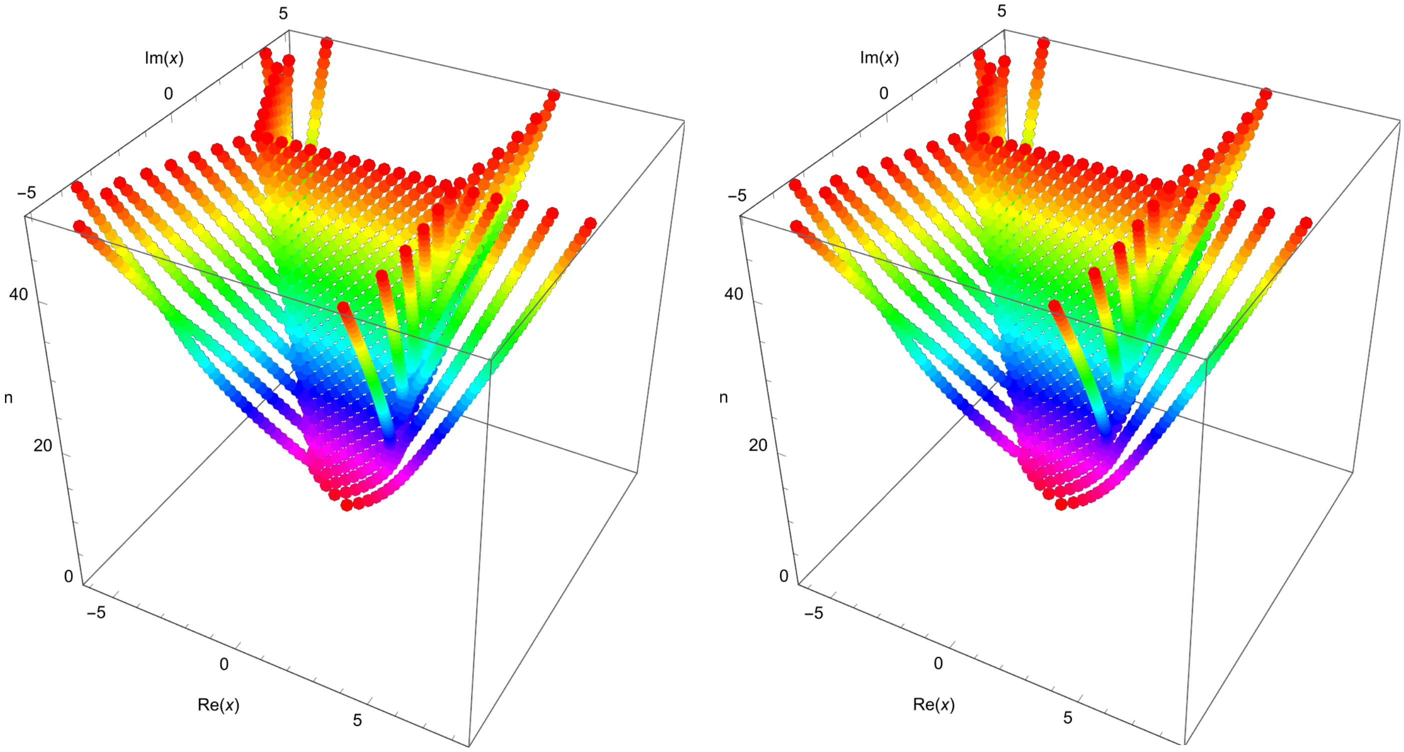
(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

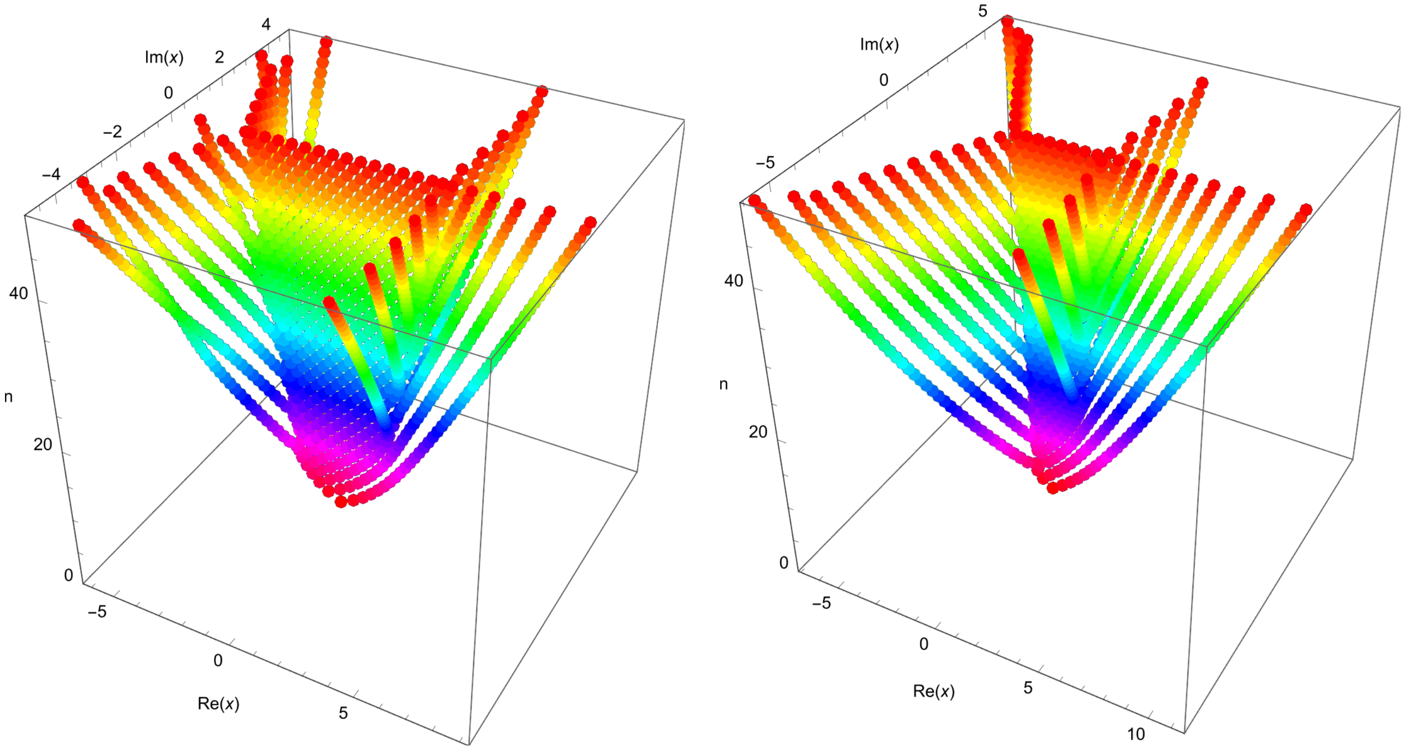
Figure 3. Zeros of ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = 0$.

The stacking structures of approximation zeros of ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x,y,z) = 0$, for $\lambda = 3$ and $1 \le n \le 50$, give 3D structures, which are presented in Figure 4.



(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$

(b) $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$

(d) $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 4. Stacking structure zeros of ${}_{CB}\mathfrak{b}_{n,s}^{(\lambda)}(x, y, z) = 0$ for $\lambda = 3$ and $1 \leq n \leq 50$.

The figures above show that the polynomials with the chosen special values have no reflection symmetry with respect to the imaginary axis. A range of values for n has been checked using computer simulations. However, it remains uncertain whether the following conjecture holds for all values of x (refer to Figures 3 and 4).

Conjecture 2. *For $y, z \in \mathbb{R}$, prove that the equation ${}_{CB}\mathfrak{b}^{(\lambda)}_{n,s}(x, y, z) = 0$ has n distinct solutions.*

5. Computational values and zeros representations of *CCBBT2EPOλ* and *SCBBT2EPOλ*

This part provides the first few members of *CCBBT2EPOλ* and *SCBBT2EPOλ* by a list, and presents the distributions of zeros of *CCBBT2EPOλ* and *SCBBT2EPOλ* by graphs with 2D and 3D structures. To validate theoretical results, numerical analysis is utilized in this part to reveal distinct scattering patterns in the distribution of zeros across the complex plane, providing insights into their underlying analytic structure and a foundation for future analytical work on their asymptotic distributions and potential applications. The structure of the zero distributions is beautifully revealed through these visualizations, providing compelling geometric insight into the algebraic richness and complexity of the underlying polynomials.

5.1. *Computational values and zeros representations of CCBBT2EPOλ*

In view of Eq (3.6), we give a few members of *CCBBT2EPOλ* for $\lambda = 3$ as:

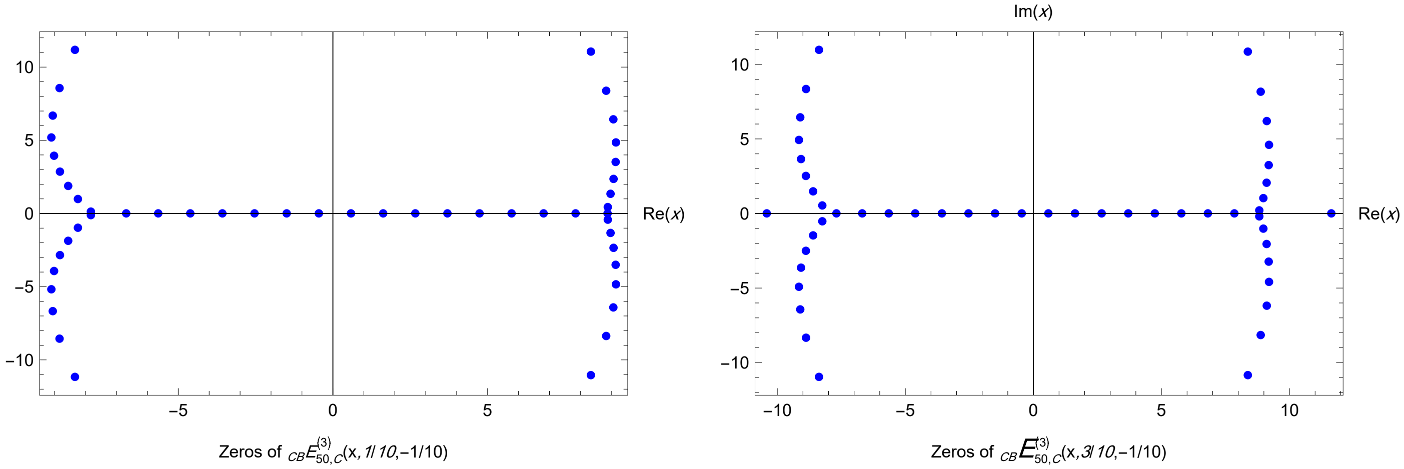
$$\begin{aligned}
{}_{CB}\mathbb{E}^{(3)}_{0,c}(x, y, z) &= 1,\\
{}_{CB}\mathbb{E}^{(3)}_{1,c}(x, y, z) &= x + z,\\
{}_{CB}\mathbb{E}^{(3)}_{2,c}(x, y, z) &= -\frac{3}{4} + x^2 - y^2 + 2xz + z^2,\\
{}_{CB}\mathbb{E}^{(3)}_{3,c}(x, y, z) &= -\frac{9x}{4} + x^3 - 3xy^2 - 2z + 3x^2z - 3y^2z + 3xz^2 + z^3,\\
{}_{CB}\mathbb{E}^{(3)}_{4,c}(x, y, z) &= \frac{33}{16} - \frac{9x^2}{2} + x^4 + \frac{9y^2}{2} - 6x^2y^2 + y^4 - 8xz + 4x^3z - 12xy^2z - \frac{7z^2}{2} + 6x^2z^2\\
&\quad - 6y^2z^2 + 4xz^3 + z^4,\\
{}_{CB}\mathbb{E}^{(3)}_{5,c}(x, y, z) &= \frac{165x}{16} - \frac{15x^3}{2} + x^5 + \frac{45xy^2}{2} - 10x^3y^2 + 5xy^4 + \frac{17z}{2} - 20x^2z + 5x^4z + 20y^2z\\
&\quad - 30x^2y^2z + 5y^4z - \frac{35xz^2}{2} + 10x^3z^2 - 30xy^2z^2 - 5z^3 + 10x^2z^3 - 10y^2z^3 + 5xz^4 + z^5.
\end{aligned}$$

Roots computational method: The computation involves the following steps:

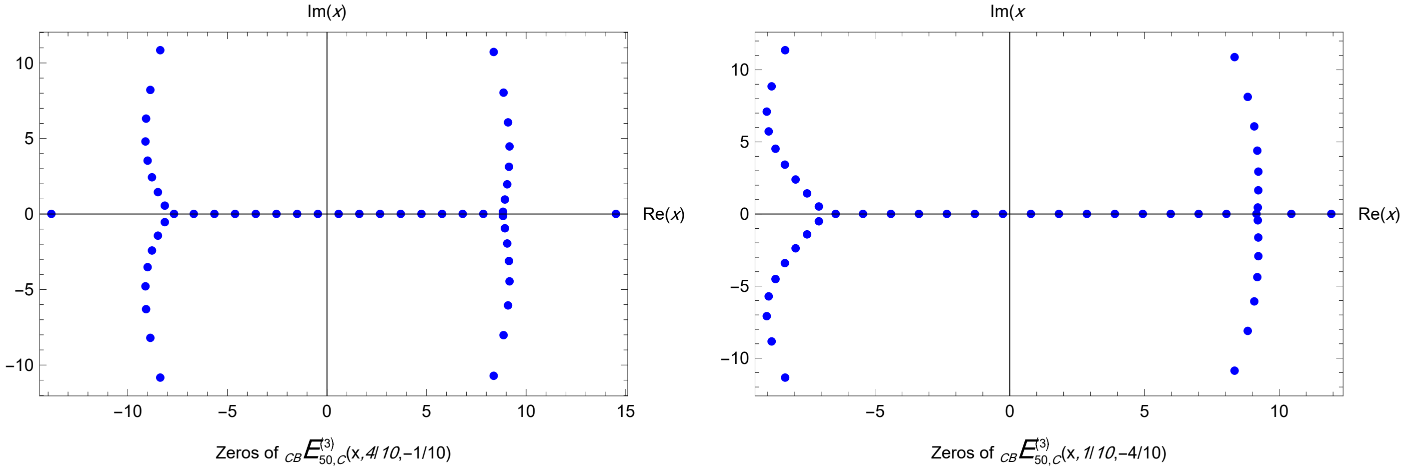
1. We consider the above explicit forms of *CCBBT2EPOλ* for $\lambda = 3$.
2. A root-finding algorithm (e.g., the Newton-Raphson method) is applied to the equation ${}_{CB}\mathbb{E}^{(3)}_{n,c}(x, y, z) = 0$.
3. The algorithm is initialized with a grid of points in the real and complex regions of interest. All distinct zeros are marked when they fall within a specified range.

We proceed to explore the fascinating distribution of zeros associated with *CCBBT2EPOλ*. Employing computational methods, we analyze the complex solutions of the equation ${}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x, y, z) = 0$, which exhibit a striking and structured scattering in the complex plane, offering both aesthetic appeal and deep analytical significance. Some interesting zeros of ${}_{CB}\mathbb{E}^{(\lambda)}_{n,c}(x, y, z) = 0$, for $\lambda = 3$ and $n = 50$ are shown in Figure 5, allowing for a detailed comparison of the function's behavior at different values of y and z. The figures show that the polynomials with the chosen special values have no reflection symmetry with respect to the imaginary axis. A range of values for n has been checked using computer simulations. However, it remains uncertain whether the following conjecture holds true for all values of x (refer to Figures 5 and 6).

Conjecture 3. *For $y, z \in \mathbb{R}$, prove that the equation ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x, y, z) = 0$ has n distinct solutions.*



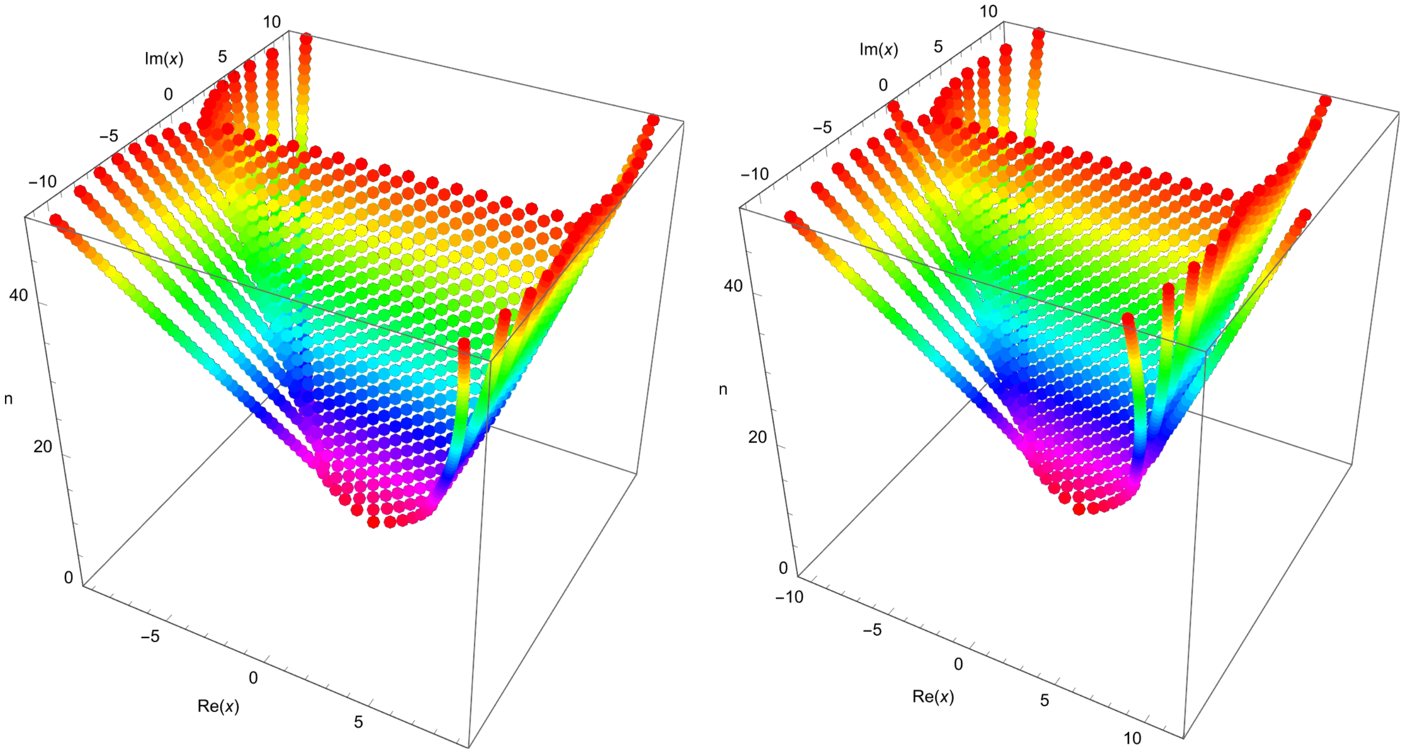
(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



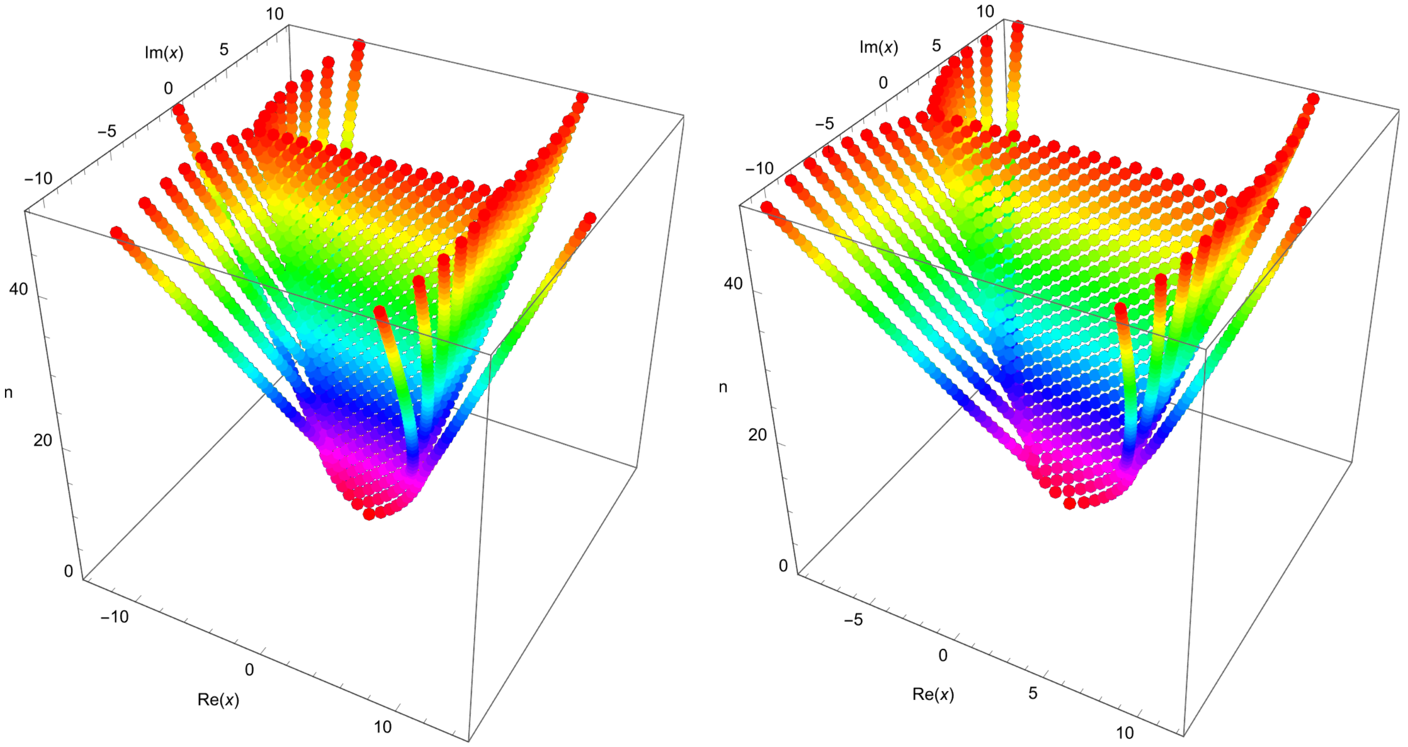
(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 5. Zeros with 2D structures of ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x, y, z)$.

The stacking structures of approximation zeros of ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x, y, z) = 0$, for $\lambda = 3$ and $1 \le n \le 50$, give 3D structures, which are presented in Figure 6. These allow for a detailed comparison of the function's behavior at different values of y and z.



(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



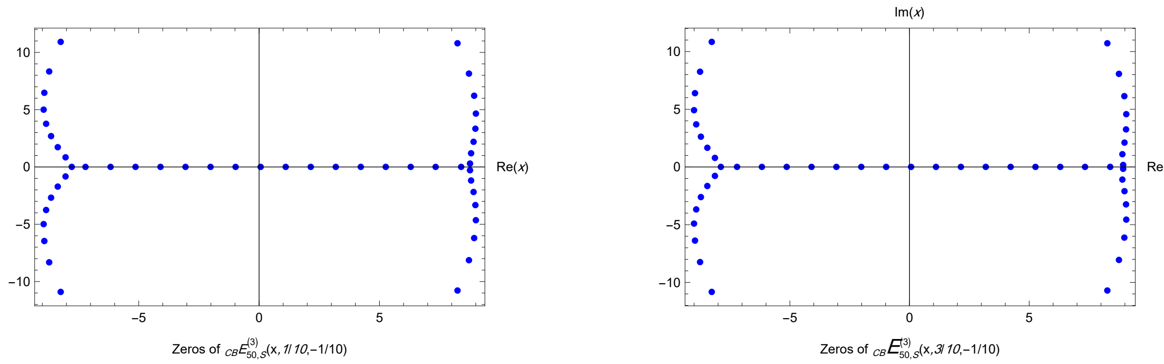
(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 6. Stacking structure zeros of ${}_{CB}\mathbb{E}_{n,c}^{(\lambda)}(x, y, z)$ for $\lambda = 3$ and $1 \leq n \leq 50$.

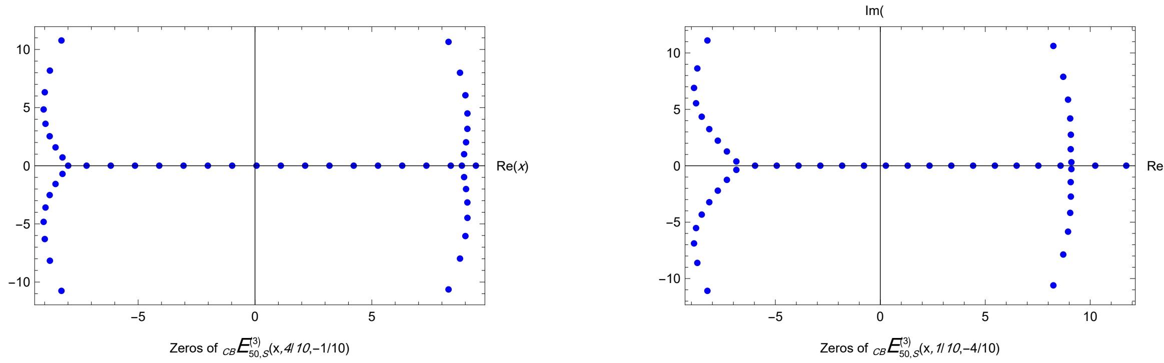
5.2. *Computational values and zeros representations of SCBBT2EPOλ*

In view of Eq (3.7), we list the first six members of ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$ for $\lambda = 3$ as:

$$
\begin{aligned}
{}_{CB}\mathbb{E}_{0,s}^{(3)}(x,y,z) &= 0,\\
{}_{CB}\mathbb{E}_{1,s}^{(3)}(x,y,z) &= y,\\
{}_{CB}\mathbb{E}_{2,s}^{(3)}(x,y,z) &= 2xy + 2yz,\\
{}_{CB}\mathbb{E}_{3,s}^{(3)}(x,y,z) &= 3x^2y + 6xyz - y^3 + 3yz^2 - \frac{9y}{4},\\
{}_{CB}\mathbb{E}_{4,s}^{(3)}(x,y,z) &= 4x^3y + 12x^2yz - 4xy^3 + 12xyz^2 - 9xy - 4y^3z + 4yz^3 - 8yz,\\
{}_{CB}\mathbb{E}_{5,s}^{(3)}(x,y,z) &= 5x^4y + 20x^3yz - 10x^2y^3 + 30x^2yz^2 - \frac{45x^2y}{2} - 20xy^3z + 20xyz^3 - 40xyz + y^5 - 10y^3z^2\\
&\quad + \frac{15y^3}{2} + 5yz^4 - \frac{35yz^2}{2} + \frac{165y}{16}.
\end{aligned}
$$



(a) $y = \frac{1}{10}$ and $z = \frac{-1}{10}$ **(b)** $y = \frac{3}{10}$ and $z = \frac{-1}{10}$



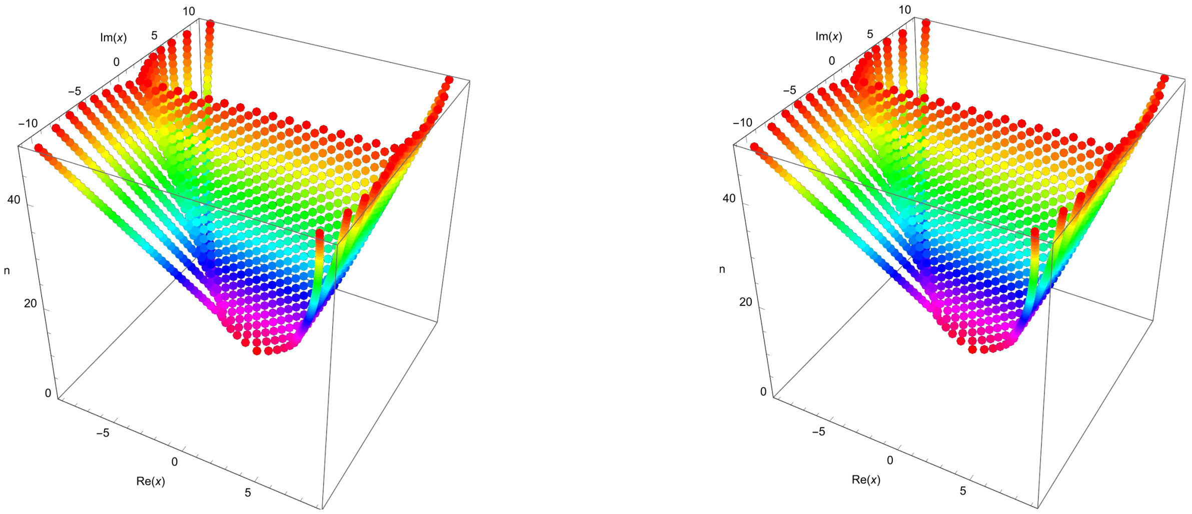
(c) $y = \frac{4}{10}$ and $z = \frac{-1}{10}$ **(d)** $y = \frac{1}{10}$ and $z = \frac{-4}{10}$

Figure 7. Zeros of ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z) = 0$.

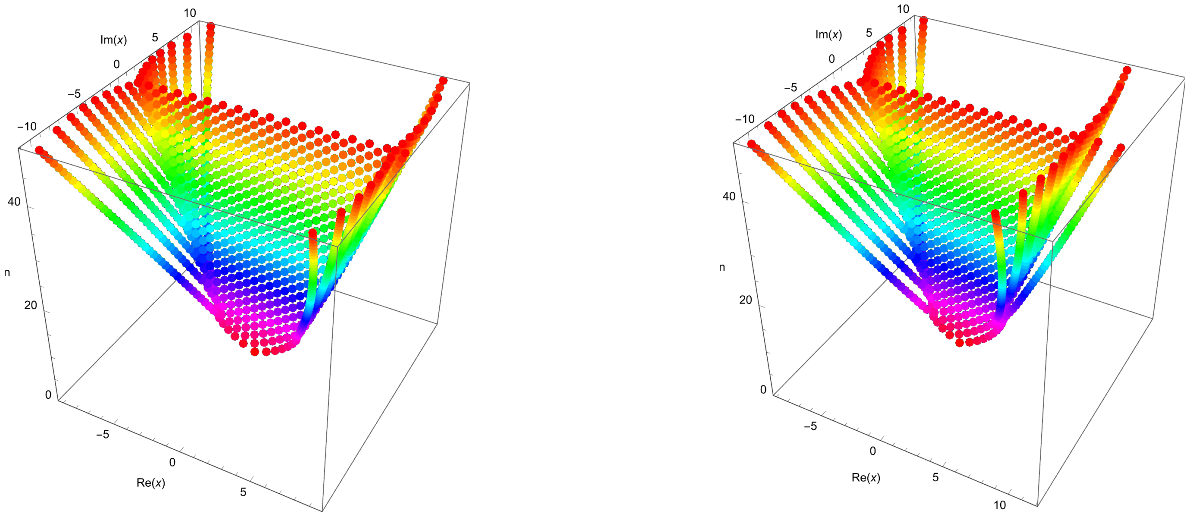
Roots computational method: The computation involves the following steps:

1. We consider the above explicit forms of *SCBBT2EPOλ* for $\lambda = 3$.
2. A root-finding algorithm (e.g., the Newton-Raphson method) is applied to the equation ${}_{CB}\mathbb{E}_{n,s}^{(3)}(x,y,z) = 0$.
3. The algorithm is initialized with a grid of points in the real and complex regions of interest. All distinct zeros are marked when they fall within a specified range.

Now, we explore the distributions of zeros and present graphical illustrations of central Bell-based Euler polynomials of complex variable ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)$ for specific parameter values and indices. Some interesting zeros of $\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=0$, for $\lambda=3$ and $n=50$ are shown in Figure 7. The stacking structures of approximation zeros of ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=0$, for $\lambda=3$ and $1\le n\le 50$, give 3D structures, which are presented in Figure 8.



(a) $y=\frac{1}{10}$ and $z=\frac{-1}{10}$ (b) $y=\frac{3}{10}$ and $z=\frac{-1}{10}$



(c) $y=\frac{4}{10}$ and $z=\frac{-1}{10}$ (d) $y=\frac{1}{10}$ and $z=\frac{-4}{10}$

Figure 8. Stacking structure zeros of ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=0$ for $\lambda=3$ and $1\le n\le 50$.

The figures show that the polynomials with the chosen special values have no reflection symmetry with respect to the imaginary axis. A range of values for n has been checked using computer simulations. However, it remains uncertain whether the following conjecture holds true for all values of x (refer to Figures 7 and 8).

Conjecture 4. *For $y,z\in\mathbb{R}$, prove that the equation ${}_{CB}\mathbb{E}_{n,s}^{(\lambda)}(x,y,z)=0$ has n distinct solutions.*

6. Conclusions

In this study, we have introduced parametric types of central Bell-based type 2 Euler and Bernoulli polynomials of order λ through defining two specific generating functions. Additionally, we have observed diverse analytical characteristics, such as addition formulae, summation formulae, differential formulae, and correlations with the new and existing old numbers and polynomials. Also, we have achieved exciting connections of sine and cosine central Bell-based type 2 Euler and Bernoulli polynomials with the central Bell polynomials, and the classical central factorial numbers and Stirling numbers of the second kind. Furthermore, the first few members of the new polynomials are given by the lists, and the distributions of zeros of the new polynomials are provided by graphical representations, enhancing the understanding of the numerical data and facilitating a more intuitive grasp of the concepts discussed. Future work will involve analyzing more detailed results and properties for the new polynomials in the context of the monomiality principle and umbral calculus. The outcomes of this work have potential applications, such as statistics, mathematical physics, probability, engineering, and mathematics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflict of interest

All authors declare that they have no competing interests.

Author contributions

Waseem Ahmad Khan: Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Francesco Aldo Costabile:** Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Khidir Shaib Mohamed:** Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Ugur Duran:** Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Abdulghani Muhyi:** Conceptualization; formal analysis; methodology; project administration; funding acquisition; supervision; validation; visualization; writing review and editing. **Azhar Iqbal:** Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Wei Sin Koh:**

Conceptualization; formal analysis; methodology; project administration; funding acquisition; supervision; validation; visualization; writing review and editing.

References

1. C. Cesarano, W. Ramírez, S. Khan, A new class of degenerate Apostol-type Hermite polynomials and applications, *Dolomites Res. Note. Approx.*, **15** (2022), 1–10. Available from: https://drna.padovauniversitypress.it/2022/1/1.
2. C. Cesarano, Integral representations and new generating functions of Chebyshev polynomials, *Hacet. J. Math. Stat.*, **44** (2015), 535–546. Available from: https://dergipark.org.tr/en/pub/hujms/issue/49486.
3. G. Dattoli, S. Lorenzutta, C. Cesarano, Generalized polynomials and new families of generating functions, *Ann. Univ. Ferrara*, **47** (2001), 57–61. https://doi.org/10.1007/BF02838175
4. M. A. Ozarslan, Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials, *Adv. Differ. Equ.*, **2013**, 116. https://doi.org/10.1186/1687-1847-2013-116
5. S. Khan, G.Yasmin, R. Khan, N. A. M. Hassan, Hermite-based Appell polynomials: Properties and applications, *J. Math. Anal. Appl.*, **351** (2009), 756–764. https://doi.org/10.1016/j.jmaa.2008.11.002
6. A. Kiran, M. Yaseen, A. Khan, T. Abdeljawad, M. A. Alqudah, R. Thinakaran, Solving time fractional diffusion-wave equation using hyperbolic polynomial B-splines: A uniform grid approach, *Ain Shams Eng. J.*, **17** (2026), 103868. https://doi.org/10.1016/j.asej.2025.103868
7. S. Khan, M. W. Al-Saad, R. Khan, Laguerre-based Appell polynomials: Properties and applications, *Math. Comput. Model.*, **52** (2010), 247–259. https://doi.org/10.1016/j.mcm.2010.02.022
8. Z. Ozat, B. Çekim, M. A. Ozarslan, F. A. Costabile, Truncated-exponential-based general-Appell polynomials, *Mathematics*, **13** (2025), 1266. https://doi.org/10.3390/math13081266
9. U. Duran, Central Bell-based type 2 Bernoulli polynomials of order β, *Fundam. J. Math. Appl.*, **8** (2025), 55–64. https://doi.org/10.33401/fujma.1630459
10. N. Alam, U. Duran, W. A. Khan, M. Sharma, On a novel class of special polynomials: Central Bell-based type 2 Euler polynomials associated with umbral calculus, *Contemp. Math.*, (2026), in press.
11. U. Duran, S. Araci, M. Acikgoz, Bell-based Bernoulli polynomials with applications, *Axioms*, **10** (2021), 29. https://doi.org/10.3390/axioms10010029
12. W. A. Khan, G. Muhiuddin, A. Muhyi, D. Al-Kadi, Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications, *Adv. Differ. Equ.*, **2021**, 420. https://doi.org/10.1186/s13662-021-03575-7
13. Y. Simsek, Some new families of special polynomials and numbers associated with finite operators, *Symmetry*, **12** (2020), 237. https://doi.org/10.3390/sym12020237
14. Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, *Appl. Anal. Discrete Math.*, **12** (2018), 1–35. https://doi.org/10.2298/AADM1801001S
15. W. Ramírez, C. Cesarano, S. DÍaz, New results for degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials, *WSEAS Trans. Math.*, **21** (2022), 604–608. https://doi.org/10.37394/23206.2022.21.69

16. H. M. Srivastava, M. A. Özarslan, C. Kaanoğlu, Some generalized Lagrange-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Russ. J. Math. Phys.*, **20** (2013), 110–120. https://doi.org/10.1134/S106192081301010X
17. M. S. Alatawi, W. A. Khan, U. Duran, Symmetric identities involving the extended degenerate central Fubini polynomials arising from the fermionic p-adic integral on $\mathbb{Z}_p$, *Axioms*, **3** (2024), 421. https://doi.org/10.3390/axioms13070421
18. D. S. Kim, D. V. Dolgy, D. Kim, T. Kim, Some identities on r-central factorial numbers and r-central Bell polynomials, *Adv. Differ. Equ.*, **2019**, 245. https://doi.org/10.1186/s13662-019-2195-0
19. D. S. Kim, J. Kwon, D. V. Dolgy, T. Kim, On central Fubini polynomials associated with central factorial numbers of the second kind, *Proc. Jangjeon Math. Soc.*, **21** (2018), 589–598. https://jangjeonopen.or.kr/public/upload/1542045161-pjms21-4-2.pdf
20. T. Kim, D. S. Kim, G. W. Jang, J. Kwon, Extended central factorial polynomials of the second kind, *Adv. Differ. Equ.*, **2019**, 1. https://doi.org/10.1186/s13662-019-1963-1
21. T. Kim, A note on central factorial numbers, *Proc. Jangjeon Math. Soc.*, **21** (2018), 575–588. Available from: https://jangjeonopen.or.kr/public/upload/1542044927-pjms21-4-1.pdf.
22. M. Acikgoz, U. Duran, Unified degenerate central Bell polynomials, *J. Math. Anal.*, **11** (2020), 18–33.
23. P. L. Butzer, K. Schmidt, E. Stark, E. Vogt, Central factorial numbers; their main properties and some applications, *Numer. Funct. Anal. Optim.*, **10** (1989), 419–488. https://doi.org/10.1080/01630568908816313
24. T. Komatsu, On s-Stirling transform and poly-Cauchy numbers of the second kind with level 2, *Aequationes Math.*, **97** (2022), 31–61. https://doi.org/10.1007/s00010-022-00931-0
25. T. Komatsu, J. L. Ramírez, D. Villamizar, A combinatorial approach to the generalized central factorial numbers, *Mediterr. J. Math.*, **18** (2021), 192. https://doi.org/10.1007/s00009-021-01830-5
26. D. S. Kim, H. Y. Kim, S. S. Pyo, T. Kim, Some identities of special numbers and polynomials arising from p-adic integrals on $\mathbb{Z}_p$, *Adv. Differ. Equ.*, **2019** (2019), 190. https://doi.org/10.1186/s13662-019-2129-x
27. T. Kim, D. S. Kim, A note on central Bell numbers and polynomials, *Russ. J. Math. Phys.*, **27** (2020), 76–81. https://doi.org/10.1134/S1061920820010070
28. T. Kim, D. S. Kim, A note on type 2 Changhee and Daehee polynomials, *Rev. R. Acad. Cienc. Exactas Fisicas. Nat. Ser. A Math. RACSAM*, **113** (2019), 2763–2771. https://doi.org/10.1007/s13398-019-00656-x
29. L. Chen, D. V. Dolgy, T. Kim, D. S. Kim, Probabilistic type 2 Bernoulli and Euler polynomials, *AIMS Math.*, **9** (2024), 14312–14324. https://doi.org/10.3934/math.2024696
30. G. W. Jang, T. Kim, A note on type 2 degenerate Euler and Bernoulli polynomials, *Adv. Stud. Contemp. Math.*, **29** (2019), 147–159. http://dx.doi.org/10.17777/ascm2019.29.1.147
31. D. S. Kim, H. Y. Kim, D. Kim, T. Kim, Identities of symmetry for type 2 Bernoulli and Euler polynomials, *Symmetry*, **11** (2019), 613. https://doi.org/10.3390/sym11050613
32. G. Muhiuddin, W. A. Khan, D. Al-Kadi, Construction on the degenerate poly-Frobenius-Euler polynomials of complex variable, *J. Funct. Spaces*, **2021** (2021), 3115424. https://doi.org/10.1155/2021/3115424
33. S. K. Sharma, W. A. Khan, C. S. Ryoo, A parametric kind of Fubini polynomials of a complex variable, *Mathematics*, **8** (2020), 643. https://doi.org/10.3390/math8040643

34. N. Alam, W. A. Khan, S. Obeidat, G. Muhiuddin, N. S. Diab, H. N. Zaidi, et al., A note on Bell-based Bernoulli and Euler polynomials of complex variable, *CMES Comput. Model. Eng. Sci.*, **135** (2023), 187–209. https://doi.org/10.32604/cmes.2022.021418
35. T. Kim, D. S. Kim, L. C. Jang, H. Y. Kim, On type 2 degenerate Bernoulli and Euler polynomials of complex variable, *Adv. Differ. Equ.*, **2019**, 490. https://doi.org/10.1186/s13662-019-2419-3
36. W. A. Khan, K. S. Nisar, M. Acikgoz, U. Duran, A. H. Abusufian, On unified Gould-Hopper based Apostol-type polynomials, *J. Math. Comput. Sci.*, **24** (2022), 287–298. https://dx.doi.org/10.22436/jmcs.024.04.01
37. M. Sharma, W. A. Khan, U. Duran, M. F. Ali, A. Sharma, A. Kumari, Bell-based Frobenius-type Eulerian polynomials and their applications, *Eur. J. Pure Appl. Math.*, **18** (2025), 1–19. https://doi.org/10.29020/nybg.ejpam.v18i3.6660
38. A. Aledamat, W. A. Khan, N. Ahmad, Bell-based partially degenerate Genocchi polynomials and their applications, *Bull. Math. Anal. Appl.*, **16** (2024), 61–76. https://doi.org/10.54671/BMAA-2024-4-4
39. N. Alam, S. A. Wani, W. A. Khan, K. Kotecha, H. N. Zaidi, F. Gassem, et al., On a class of generalized multivariate Hermite-Humbert polynomials via generalized Fibonacci polynomials, *Symmetry*, **16** (2024), 1415. https://doi.org/10.3390/sym16111415
40. A. Aledamat, W. A. Khan, C. S. Ryoo, Certain properties on Bell-based Apostol-Frobenius-Genocchi polynomials of complex variables, *J. Math. Comput. Sci.*, **33** (2024), 326–338. https://10.22436/jmcs.033.04.01
41. W. A. Khan, M. S. Alatawi, U. Duran, Applications and properties of bivariate Bell-based Frobenius-type Eulerian polynomials, *J. Funct. Spaces*, **2023**, 5205867. https://doi.org/10.1155/2023/5205867
42. M. Nadeem, W. A. Khan, K. A. H. Alzobydi, C. S. Ryoo, M. Shadab, R. Ali, Certain properties on Bell-based Apostol-type Frobenius-Genocchi polynomials and its applications, *Adv. Math. Models Appl.*, **1** (2023), 92–107. Available from: https://jomardpublishing.com/UploadFiles/Files/journals/AMMAV1N1/V8N1/Nadeem etal.pdf.

AIMS Press