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Research article

Degenerate Euler–Seidel method for degenerate Bernoulli, Euler, and Genocchi polynomials

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Abstract: This paper introduces a degenerate version of the Euler–Seidel method by incorporating a parameter λ into the classical recurrence relation. We define a degenerate Euler–Seidel matrix associated with an initial sequence and establish corresponding λ-generalized binomial identities and generating function relations. By applying this method to the degenerate Bernoulli, Euler, and Genocchi polynomials, we derive several new combinatorial identities. This work extends the classical Euler–Seidel method to the domain of degenerate special polynomials and numbers, thus providing a new framework to study their properties.

Keywords: degenerate Euler–Seidel matrix; degenerate Bernoulli polynomial; degenerate Euler polynomial; degenerate Genocchi polynomial

1. Introduction

The purpose of this paper is to extend and study a degenerate version of the Euler–Seidel method. This generalization introduces a parameter λ into the recurrence relation. The results are applied to study and derive new combinatorial identities for sequences such as the degenerate Bernoulli, Euler, and Genocchi polynomials.

In the degenerate version, the addition of the parameter λ creates a more flexible structure that collapses back into the classical version when $\lambda = 0$. The shift from the “Classical” to “Degenerate” method allows us to solve problems in asymptotic analyses where the standard Euler–Seidel method fails due to rigid step-sizes. By adjusting λ, we can model systems with “decaying” or “scaling” recursions, which is currently a hot topic in combinatorial physics. At its core, combinatorial physics is the intersection of theoretical physics and discrete mathematics. It is the art of using counting techniques, graph theory, and symmetry to solve complex physical problems, particularly those that involve many interacting parts. Instead of looking at a system as a smooth and continuous fluid, combinatorial physics often treats it as a collection of discrete "pieces" (such as particles on a grid or edges in a graph) and asks the following: "How many ways can these pieces be arranged to follow the laws of physics?"

For a given sequence $(a_n(x|\lambda))_{n\geq 0}$, we consider the degenerate Euler–Seidel matrix associated with this sequence, which is recursively determined by the following (see Eq (1.13)):

$$a_{0,n}(x \mid \lambda) = a_n(x \mid \lambda), \quad (n \geq 0),$$
$$a_{k,n}(x \mid \lambda) = (1-(k-n)\lambda)\, a_{k-1,n}(x \mid \lambda) + a_{k-1,n+1}(x \mid \lambda), \quad (n \geq 0,\ k \geq 1).$$

The degenerate Euler–Seidel matrix associated with $(a_n(x \mid \lambda))_{n\geq 0}$ is given by the following:

$$(a_{n,k}(x \mid \lambda))_{n,k\geq 0} = \begin{pmatrix} a_{0,0}(x \mid \lambda) & a_{0,1}(x \mid \lambda) & a_{0,2}(x \mid \lambda) & \cdots \\ a_{1,0}(x \mid \lambda) & a_{1,1}(x \mid \lambda) & a_{1,2}(x \mid \lambda) & \cdots \\ a_{2,0}(x \mid \lambda) & a_{2,1}(x \mid \lambda) & a_{2,2}(x \mid \lambda) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1.1}$$

The sequence $(a_{0,n}(x|\lambda))_{n\geq 0}$ is the *initial degenerate sequence*, and $(a_{n,0}(x|\lambda))_{n\geq 0}$ is the *final degenerate sequence*. The following λ-generalized binomial identities are established (see Theorems 3.1) using the generalized falling and rising factorials, $(1-\lambda)_{n-k,\lambda}$ and $\langle 1-\lambda\rangle_{n-k,\lambda}$, respectfully, (see Eq (1.3)) for the degenerate case:

$$a_{n,0}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)_{n-k,\lambda}\, a_{0,k}(x \mid \lambda),$$
$$a_{0,n}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \langle 1-\lambda\rangle_{n-k,\lambda}\, a_{k,0}(x \mid \lambda),$$

where the degenerate rising factorial sequence is given by

$$\langle x\rangle_{0,\lambda} = 1, \ \ \langle x\rangle_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1).$$

These correspond to the classical binomial identities in Eq (1.15), and they lead to the following degenerate version of Seidel's formula for the generating functions:

$$\overline{A}_\lambda(x,t) = e_\lambda^{1-\lambda}(t)\, A_\lambda(x,t),$$

where $A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x|\lambda)\frac{t^n}{n!}$, and $\overline{A}_\lambda(x,t) = \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!}$ (see Theorem 3.2). The results derived from the degenerate Euler–Seidel method are applied to the degenerate Bernoulli, Euler, and Genocchi polynomial sequences, yielding various combinatorial identities (see Theorems 3.3–3.5); for $n \geq 0$, we have the following identities:

$$\beta_{n,\lambda}(x+1-\lambda) = n(x-\lambda)_{n-1,\lambda} + \beta_{n,\lambda}(x-\lambda),$$
$$\mathcal{E}_{n,\lambda}(x+1-\lambda) = 2(x-\lambda)_{n,\lambda} - \mathcal{E}_{n,\lambda}(x-\lambda),$$
$$\mathcal{G}_{n,\lambda}(x+1-\lambda) = 2n(x-\lambda)_{n-1,\lambda} - \mathcal{G}_{n,\lambda}(x-\lambda).$$

The study of degenerate versions of special polynomials and numbers originated with Carlitz's pioneering work on degenerate Bernoulli and Euler numbers [1]. This field has since expanded to include transcendental functions, such as gamma functions [2]. Furthermore, the introduction of λ-umbral calculus [3] has established a more robust framework more robust than classical umbral calculus to analyze degenerate Sheffer polynomials. These investigations employ a diverse array of methodologies, including generating functions, combinatorial methods, a p-adic analysis, the operator theory, differential equations, and the probability theory (see references [2, 4–6] and the references therein).

Research into the Euler–Seidel method—a recursive algorithm used to transform sequences and study combinatorial identities—has seen a significant resurgence in recent years. Here are some of the recent enhancements to the Euler–Seidel method, alongside the aforementioned degenerate Euler–Seidel method.

1. Integration with Riordan Arrays and Hankel Matrices (see [7, 8],

Recent work has bridged the gap between the Euler–Seidel matrix and Riordan arrays, which are powerful tools in the combinatorial group theory.

• Enhancement: Establishes that every Euler–Seidel matrix can be represented as a product of a Binomial matrix and a Hankel matrix ($E_a = BH_a$). This allows for the use of integral representations (like the Stieltjes transform) to solve for sequence entries.

• Applications: Provides closed-form integral representations for Catalan and Motzkin sequences within the Euler–Seidel framework.

2. Generalized Second-Order Recurrence Relations (see [9])

Researchers have modified the method to handle sequences defined by more complex recurrences (such as Fibonacci or Lucas sequences) by introducing two parameters, p and q.

• Enhancement: The matrix is determined by $a_{k,n} = p\,a_{k-1,n} + q\,a_{k-1,n+1}$. This generalization allows for the "symmetric" study of any second-order linear recurrence.

• Applications: Finds new identities for hyperharmonic numbers and r-Stirling numbers.

3. Euler–Seidel Matrices over Finite Fields $\mathbb{F}_p$ (see [10])

A newer niche of research focuses on the periodic behavior of these matrices when the initial sequence belongs to a finite field.

• Enhancement: Define the "period" of an Euler–Seidel matrix; if the initial sequence is periodic, then the rows of the resulting matrix also exhibit periodicity related to the order of $1 + X$ in the polynomial ring $\mathbb{F}_p[X]$.

• Applications: Includes the coding theory and electrical engineering applications that involve linear recurring sequences.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by the following:

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} = (1+\lambda t)^{\frac{x}{\lambda}}, \qquad e_\lambda(t) = e_\lambda^1(t), \quad \text{(see [2, 3, 5, 11, 12])}, \tag{1.2}$$

where the degenerate falling factorial sequence is given by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1). \tag{1.3}$$

For later use, we mention that the degenerate rising factorial sequence is specified as follows:

$$\langle x\rangle_{0,\lambda} = 1, \ \ \langle x\rangle_{n,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1). \tag{1.4}$$

In reference [1], Carlitz introduced the degenerate Bernoulli polynomials given by the following:

$$\frac{t}{e_\lambda(t)-1}\, e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [3, 11, 13])}. \tag{1.5}$$

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$, $(n\geq 0)$, are called the degenerate Bernoulli numbers. From Eq (1.5), we note that

$$\beta_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\beta_{k,\lambda}(x)_{n-k,\lambda},\ (n\geq 0). \tag{1.6}$$

In addition, the degenerate Euler polynomials are defined by the following:

$$\frac{2}{e_\lambda(t)+1}\, e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!},\ \text{(see [3, 11, 13, 14])}. \tag{1.7}$$

When $x=0$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$, $(n\geq 0)$, are called the degenerate Euler numbers. By Eq (1.7), we obtain the following:

$$\mathcal{E}_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k,\lambda}(x)_{n-k,\lambda}. \tag{1.8}$$

The degenerate Genocchi polynomials are defined by the following:

$$\frac{2t}{e_\lambda(t)+1}\, e_\lambda^x(t)=\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}(x)\frac{t^n}{n!},\ \text{(see [12])}. \tag{1.9}$$

When $x=0$, $\mathcal{G}_{n,\lambda}=\mathcal{G}_{n,\lambda}(0)$, $(n\geq 0)$, are called the degenerate Genocchi numbers. Note that

$$\mathcal{G}_{0,\lambda}(x)=0,\quad \mathcal{G}_{n,\lambda}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{k,\lambda}\,(x)_{n-k,\lambda},\quad (n\geq 1). \tag{1.10}$$

From Eq (1.9), we have the following:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}\frac{t^n}{n!}=\frac{2t}{e_\lambda(t)+1}&=2\left(\frac{t}{e_\lambda(t)-1}-\frac{2t}{e_\lambda^2(t)-1}\right)\\&=2\left(\frac{t}{e_\lambda(t)-1}-\frac{2t}{e_{\lambda/2}(2t)-1}\right)\\&=2\Big(\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}-\sum_{n=0}^{\infty}\beta_{n,\lambda/2}\frac{(2t)^n}{n!}\Big)\\&=2\sum_{n=0}^{\infty}\left(\beta_{n,\lambda}-2^n\beta_{n,\lambda/2}\right)\frac{t^n}{n!}.\end{aligned} \tag{1.11}$$

Thus, by Eq (1.11), we obtain the following:

$$\mathcal{G}_{n,\lambda}=2\left(\beta_{n,\lambda}-2^n\beta_{n,\lambda/2}\right),\quad (n\geq 0). \tag{1.12}$$

Taking the limits $\lambda \to 0$, we find that (see Eqs (1.2)–(1.5), (1.7), (1.9))

$$e_\lambda^x(t) \to e^{xt}, \quad e_\lambda(t) \to e^t, \quad (x)_{n,\lambda} \to x^n, \quad \langle x\rangle_{n,\lambda} \to x^n,$$
$$\beta_{n,\lambda}(x) \to B_n(x), \quad \mathcal{E}_{n,\lambda}(x) \to E_n(x), \quad \mathcal{G}_{n,\lambda}(x) \to G_n(x),$$

where $B_n(x)$, $E_n(x)$, and $G_n(x)$ are the ordinary Bernoulli, Euler, and Genocchi polynomials given by (see [15, 16]) the following:

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$
$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

and

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad \text{respectively.}$$

The recent papers in [17, 18] relate Bernoulli polynomials to determinants of special matrices and fractal measures in a different context. Additionally, many authors have recently made significant advances in the polynomial theory (see references [19–21]).

For a given sequence $(a_n)_{n\geq 0}$, the Euler–Seidel matrix associated with this sequence is recursively determined by the following:

$$\begin{aligned} &a_{0,n} = a_n, \quad (n \geq 0),\\ &a_{k,n} = a_{k-1,n} + a_{k-1,n+1}, \quad (n \geq 0,\ k \geq 1), \quad \text{(see [3, 7, 15, 22]).} \end{aligned} \tag{1.13}$$

The Euler–Seidel matrix $(a_{n,k})_{n,k\geq 0}$ associated with the sequence $(a_n)_{n\geq 0}$ is given by the following:

$$(a_{n,k})_{n,k\geq 0} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1.14}$$

From Eq (1.13), we obtain the following binomial identities:

$$a_{n,0} = \sum_{k=0}^{n}\binom{n}{k}a_{0,k} \quad \text{and} \quad a_{0,n} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_{k,0}, \quad \text{(see [22]).} \tag{1.15}$$

Let $A(t) = \sum_{n=0}^{\infty} a_{0,n}\frac{t^n}{n!}$ be the generating function of the initial sequence $(a_{0,n})_{n\geq 0}$. Then, the generating function $\overline{A}(t)$ of the final sequence $(a_{n,0})_{n\geq 0}$ is given by the following:

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_{n,0}\frac{t^n}{n!} = e^t A(t), \quad \text{(see [22]).} \tag{1.16}$$

For general background and references, one may consult references ([23, 24]).

2. Degenerate Bernoulli, Euler, and Genocchi numbers

In this section, we derive recurrence relations for the degenerate Bernoulli, Euler, and Genocchi numbers from the Eqs (1.5)–(1.10), from which we derive the first few terms of those numbers.

From Eq (1.5), we note that

$$\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases} \tag{2.1}$$

Then, by Eqs (1.6) and (2.1), we obtain the following:

$$\sum_{k=0}^{n} \binom{n}{k} (1)_{n-k,\lambda} \beta_{k,\lambda} - \beta_{n,\lambda} = \delta_{1,n}, \tag{2.2}$$

where $\delta_{n,k}$ is the Kronecker's symbel. From Eq (2.2), we have the following:

$$\begin{aligned} &\beta_{0,\lambda} = 1, \quad \beta_{1,\lambda} = -\frac{1}{2} + \frac{1}{2}\lambda, \quad \beta_{2,\lambda} = \frac{1}{6} - \frac{1}{6}\lambda^2, \quad \beta_{3,\lambda} = -\frac{1}{4} + \frac{1}{4}\lambda^3, \\ &\beta_{4,\lambda} = -\frac{1}{30} + \frac{2}{3}\lambda^2 - \frac{19}{30}\lambda^4, \quad \beta_{5,\lambda} = \frac{1}{4}\lambda - \frac{5}{2}\lambda^3 + \frac{9}{4}\lambda^5, \dots. \end{aligned} \tag{2.3}$$

From Eq (1.7), we see that

$$\mathcal{E}_{n,\lambda}(1) + \mathcal{E}_{n,\lambda} = 2\delta_{0,n}. \tag{2.4}$$

Then, from Eqs (1.8) and (2.4), we obtain the following:

$$\sum_{k=0}^{n} \binom{n}{k} (1)_{n-k,\lambda} \mathcal{E}_{k,\lambda} + \mathcal{E}_{n,\lambda} = 2\delta_{0,n}. \tag{2.5}$$

Thus, by Eq (2.5), we obtain the following:

$$\begin{aligned} &\mathcal{E}_{0,\lambda} = 1, \quad \mathcal{E}_{1,\lambda} = -\frac{1}{2}, \quad \mathcal{E}_{2,\lambda} = \frac{\lambda}{2}, \quad \mathcal{E}_{3,\lambda} = \frac{1}{4} - \frac{3}{4}\lambda^2, \quad \mathcal{E}_{4,\lambda} = -\frac{3}{2}\lambda + \frac{3}{2}\lambda^3, \\ &\mathcal{E}_{5,\lambda} = -\frac{1}{2} + \frac{15}{2}\lambda^2 - \frac{15}{4}\lambda^4, \quad \mathcal{E}_{6,\lambda} = \frac{15}{2}\lambda - 45\lambda^3 + \frac{45}{4}\lambda^5, \dots. \end{aligned} \tag{2.6}$$

From Eq (1.9), we obtain the following:

$$\mathcal{G}_{n,\lambda}(1) + \mathcal{G}_{n,\lambda} = 2\delta_{1,n}, \quad \mathcal{G}_{0,\lambda} = 0. \tag{2.7}$$

Then, from Eqs (1.10) and (2.7), we have the following:

$$\sum_{k=0}^{n} \binom{n}{k} (1)_{n-k,\lambda} \mathcal{G}_{k,\lambda} + \mathcal{G}_{n,\lambda} = 2\delta_{1,n}. \tag{2.8}$$

From Eq (2.7), we note that

$$\begin{aligned} &\mathcal{G}_{1,\lambda} = 1, \quad \mathcal{G}_{2,\lambda} = -1, \quad \mathcal{G}_{3,\lambda} = \frac{3}{2}\lambda, \quad \mathcal{G}_{4,\lambda} = 1 - 3\lambda^2, \quad \mathcal{G}_{5,\lambda} = -\frac{15}{2}\lambda + \frac{15}{2}\lambda^3, \\ &\mathcal{G}_{6,\lambda} = -3 + 45\lambda^2 - \frac{45}{2}\lambda^4, \quad \mathcal{G}_{7,\lambda} = \frac{105}{2}\lambda - 315\lambda^3 + \frac{315}{4}\lambda^5, \dots. \end{aligned} \tag{2.9}$$

3. Degenerate Euler–Seidel method for degenerate Bernoulli, Euler, and Genocchi polynomials

In this section, we derive some identities for the degenerate Bernoulli, Euler, and Genocchi numbers using the relation in Theorem 3.2 between the exponential generating function of the initial sequence and that of the final sequence.

For a given sequence $(a_n(x|\lambda))_{n\geq 0}$, we consider the degenerate Euler–Seidel matrix associated with this sequence, which is recursively determined by the following (see Eq (1.12)):

$$\begin{aligned} &a_{0,n}(x \mid \lambda) = a_n(x \mid \lambda), \quad (n \geq 0), \\ &a_{k,n}(x \mid \lambda) = (1-(k-n)\lambda)\, a_{k-1,n}(x \mid \lambda) + a_{k-1,n+1}(x \mid \lambda), \quad (n \geq 0,\ k \geq 1). \end{aligned} \tag{3.1}$$

Using Eq (3.1) and by induction, we obtain the following theorem (see Theorems 2.2 and 2.3 in [13]).

Theorem 3.1. *For $n \geq 0$, we have*

$$a_{n,0}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)_{n-k,\lambda}\, a_{0,k}(x \mid \lambda),$$

and

$$a_{0,n}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \langle 1-\lambda \rangle_{n-k,\lambda}\, a_{k,0}(x \mid \lambda),$$

where the degenerate rising factorial sequence $\langle x \rangle_{n,\lambda}$ is defined in Eq (1.4).

Let $A_\lambda(x,t)$ be the generating function of the sequence $(a_{0,n}(x \mid \lambda))_{n\geq 0}$, which is given by

$$A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x \mid \lambda) \frac{t^n}{n!}, \tag{3.2}$$

and let $\overline{A}_\lambda(x,t)$ be that of the sequence $(a_{n,0}(x \mid \lambda))_{n\geq 0}$, which is given by

$$\overline{A}_\lambda(x,t) = \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda) \frac{t^n}{n!}. \tag{3.3}$$

Then, by Theorem 3.1, we have the following:

$$\begin{aligned} \overline{A}_\lambda(x,t) &= \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)_{n-k,\lambda}\, a_{0,k}(x \mid \lambda) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} a_{0,k}(x \mid \lambda) \sum_{n=k}^{\infty} \frac{(1-\lambda)_{n-k,\lambda}}{(n-k)!} t^n \\ &= \sum_{k=0}^{\infty} a_{0,k}(x \mid \lambda) \frac{t^k}{k!} \sum_{n=k}^{\infty} \frac{(1-\lambda)_{n-k,\lambda}}{(n-k)!} t^{n-k} \\ &= \sum_{k=0}^{\infty} a_{0,k}(x \mid \lambda) \frac{t^k}{k!} \sum_{n=0}^{\infty} (1-\lambda)_{n,\lambda} \frac{t^n}{n!} \\ &= A_\lambda(x,t)\, e_\lambda^{1-\lambda}(t). \end{aligned} \tag{3.4}$$

Therefore, by Eq (3.4), we obtain the following theorem.

Theorem 3.2. *Let*

$$A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x \mid \lambda)\frac{t^n}{n!}.$$

Then, we have the following:

$$\overline{A}_\lambda(x,t) = \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!} = e_\lambda^{1-\lambda}(t)\,A_\lambda(x,t). \tag{3.5}$$

Let $a_{0,n}(x \mid \lambda) = \beta_{n,\lambda}(x)$, $(n \geq 0)$. Then, we have the following:

$$A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x \mid \lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!} = \frac{t}{e_\lambda(t)-1}\,e_\lambda^{x}(t). \tag{3.6}$$

From Eqs (3.5) and (3.6), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!} &= e_\lambda^{1-\lambda}(t)\,A_\lambda(x,t) = \frac{t}{e_\lambda(t)-1}\,e_\lambda^{x+1-\lambda}(t)\\ &= \sum_{n=0}^{\infty}\beta_{n,\lambda}(x+1-\lambda)\frac{t^n}{n!}.\end{aligned} \tag{3.7}$$

Thus, by comparing the coefficients on both sides of Eq (3.7), we obtain the following:

$$a_{n,0}(x \mid \lambda) = \beta_{n,\lambda}(x+1-\lambda), \quad (n \geq 0). \tag{3.8}$$

On the other hand, by Eqs (1.5), (3.5) and (3.6), we obtain the following:

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!} &= e_\lambda^{1-\lambda}(t)\,A_\lambda(x,t) = e_\lambda(t)\,\frac{t}{e_\lambda(t)-1}\,e_\lambda^{x-\lambda}(t)\\ &= (e_\lambda(t)-1+1)\frac{t}{e_\lambda(t)-1}\,e_\lambda^{x-\lambda}(t)\\ &= te_\lambda^{x-\lambda}(t) + \frac{t}{e_\lambda(t)-1}e_\lambda^{x-\lambda}(t)\\ &= \sum_{n=0}^{\infty}(n+1)(x-\lambda)_{n,\lambda}\frac{t^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty}\beta_{n,\lambda}(x-\lambda)\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty} n(x-\lambda)_{n-1,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\beta_{n,\lambda}(x-\lambda)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(n(x-\lambda)_{n-1,\lambda} + \beta_{n,\lambda}(x-\lambda)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.9}$$

Comparing the coefficients on both sides of Eq (3.9), we have the following:

$$a_{n,0}(x \mid \lambda) = n(x-\lambda)_{n-1,\lambda} + \beta_{n,\lambda}(x-\lambda), \quad (n \geq 0). \tag{3.10}$$

Therefore, by Eqs (3.8) and (3.10), we obtain the following theorem.

Theorem 3.3. *For $n \geq 0$, we have the following:*

$$\beta_{n,\lambda}(x+1-\lambda) = n(x-\lambda)_{n-1,\lambda} + \beta_{n,\lambda}(x-\lambda),$$

where we understand that $n(x-\lambda)_{n-1,\lambda}$ vanishes when $n = 0$.

The degenerate Euler–Seidel matrix associated with $\big(a_{0,n}(x \mid \lambda)\big)_{n\geq 0} = (\beta_{n,\lambda}(x))_{n\geq 0}$ is given by the following (see Eqs (1.5), (2.8), (3.10)):

$$\begin{pmatrix} 1 & x - \frac{1-\lambda}{2} & \cdots \\ x + \frac{1-\lambda}{2} & x^2 - \frac{\lambda^2-3\lambda+2}{6} & \cdots \\ x^2 + (1-2\lambda)x + \frac{1-6\lambda+5\lambda^2}{6} & x^3 + \frac{1-3\lambda}{2}x^2 + \frac{6\lambda^2-5\lambda-1}{2}x + \frac{4\lambda^3-3\lambda^2-1}{6} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Let $a_{0,n}(x) = \mathcal{E}_{n,\lambda}(x), (n \geq 0)$. Then, we have the following:

$$A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x \mid \lambda)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\,\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}\,e_\lambda^x(t). \tag{3.11}$$

From Eqs (3.5) and (3.11), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\,\frac{t^n}{n!} &= \overline{A}_\lambda(x,t) = e_\lambda^{1-\lambda}(t)A_\lambda(x,t) = e_\lambda^{1-\lambda}(t)\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) \\ &= \frac{2}{e_\lambda(t)+1}e_\lambda^{x+1-\lambda}(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x+1-\lambda)\,\frac{t^n}{n!}. \end{aligned} \tag{3.12}$$

Thus, by Eq (3.12), we obtain the following:

$$a_{n,0}(x \mid \lambda) = \mathcal{E}_{n,\lambda}(x+1-\lambda), \quad (n \geq 0). \tag{3.13}$$

On the other hand, by Eqs (1.6) and (3.5), we obtain the following:

$$\begin{aligned} &\sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\,\frac{t^n}{n!} = \overline{A}_\lambda(x,t) = e_\lambda^{1-\lambda}(t)A_\lambda(x,t) \\ &= (e_\lambda(t)+1-1)\,\frac{2}{e_\lambda(t)+1}e_\lambda^{x-\lambda}(t) = 2e_\lambda^{x-\lambda}(t) - \frac{2}{e_\lambda(t)+1}e_\lambda^{x-\lambda}(t) \\ &= \sum_{n=0}^{\infty} \big(2(x-\lambda)_{n,\lambda} - \mathcal{E}_{n,\lambda}(x-\lambda)\big)\,\frac{t^n}{n!}. \end{aligned} \tag{3.14}$$

By comparing the coefficients on both sides of Eq (3.14), we have the following:

$$a_{n,0}(x \mid \lambda) = 2(x-\lambda)_{n,\lambda} - \mathcal{E}_{n,\lambda}(x-\lambda), \quad (n \geq 0). \tag{3.15}$$

Therefore, by Eqs (3.13) and (3.15), we obtain the following theorem.

Theorem 3.4. *For $n \geq 0$, we have the following:*

$$\mathcal{E}_{n,\lambda}(x+1-\lambda) = 2(x-\lambda)_{n,\lambda} - \mathcal{E}_{n,\lambda}(x-\lambda).$$

By Eqs (1.7), (2.8), and (3.15), we see that the degenerate Euler–Seidel matrix associated with $(a_{0,n})_{n\geq 0} = (\mathcal{E}_{n,\lambda}(x))_{n\geq 0}$ is given by the following:

$$\begin{pmatrix} 1 & x-\frac{1}{2} & \cdots \\ x+\frac{1}{2}-\lambda & x^2-\lambda x+\frac{\lambda-1}{2} & \cdots \\ x^2+(1-3\lambda)x+\frac{4\lambda^2-3\lambda}{2} & x^3+\left(\frac{1}{2}+\lambda\right)x^2+(2\lambda^2-1)x-\frac{1}{4}(1-6\lambda+2\lambda^2) & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{3.16}$$

Finally, we let $a_{0,n}(x \mid \lambda) = \mathcal{G}_{n,\lambda}(x)$, $(n \geq 0)$. Then, by Eqs (1.9) and (3.2), we obtain the following:

$$A_\lambda(x,t) = \sum_{n=0}^{\infty} a_{0,n}(x \mid \lambda)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{G}_{n,\lambda}(x)\,\frac{t^n}{n!} = \frac{2t}{e_\lambda(t)+1}\,e_\lambda^x(t). \tag{3.17}$$

From Eqs (3.5) and (3.17), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!} &= \overline{A}_\lambda(x,t) = e_\lambda^{1-\lambda}(t)\,A_\lambda(x,t) = e_\lambda^{1-\lambda}(t)\frac{2t}{e_\lambda(t)+1}\,e_\lambda^x(t) \\ &= \frac{2t}{e_\lambda(t)+1}\,e_\lambda^{x+1-\lambda}(t) = \sum_{n=0}^{\infty} \mathcal{G}_{n,\lambda}(x+1-\lambda)\frac{t^n}{n!}. \end{aligned} \tag{3.18}$$

Thus, by Eq (3.18), we obtain the following:

$$\mathcal{G}_{n,\lambda}(x+1-\lambda) = a_{n,0}(x \mid \lambda), \quad (n \geq 0). \tag{3.19}$$

On the other hand, by Eqs (1.9), (3.5), and (3.17), we obtain the following:

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n,0}(x \mid \lambda)\frac{t^n}{n!} &= \overline{A}_\lambda(x,t) = e_\lambda^{1-\lambda}(t)\,A_\lambda(x,t) \\ &= \frac{2t}{e_\lambda(t)+1}\,e_\lambda^{x-\lambda}(t)\,(e_\lambda(t)+1-1) \\ &= 2te_\lambda^{x-\lambda}(t) - \frac{2t}{e_\lambda(t)+1}\,e_\lambda^{x-\lambda}(t) \\ &= 2\sum_{n=0}^{\infty}(n+1)(x-\lambda)_{n,\lambda}\frac{t^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty}\mathcal{G}(x-\lambda)\frac{t^n}{n!} \\ &= 2\sum_{n=1}^{\infty} n(x-\lambda)_{n-1,\lambda}\frac{t^n}{n!} - \sum_{n=0}^{\infty}\mathcal{G}(x-\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(2n(x-\lambda)_{n-1,\lambda} - \mathcal{G}_{n,\lambda}(x-\lambda)\Big)\frac{t^n}{n!}. \end{aligned} \tag{3.20}$$

By comparing the coefficients on both sides of Eq (3.20), we obtain the following:

$$a_{n,0}(x \mid \lambda) = 2n(x-\lambda)_{n-1,\lambda} - \mathcal{G}_{n,\lambda}(x-\lambda), \qquad (n \geq 0). \tag{3.21}$$

Therefore, by Eqs (3.19) and (3.21), we obtain the following theorem.

Theorem 3.5. *For $n \geq 0$, we have the following:*

$$\mathcal{G}_{n,\lambda}(x+1-\lambda) = 2n(x-\lambda)_{n-1,\lambda} - \mathcal{G}_{n,\lambda}(x-\lambda),$$

where we understand that $2n(x-\lambda)_{n-1,\lambda}$ vanishes when $n = 0$.

From Eqs (1.10), (3.1), and (3.21), we note that the degenerate Euler–Seidel matrix associated with the sequence $(a_{n,0}(x \mid \lambda))_{n\geq 0} = (\mathcal{G}_{n,\lambda}(x))_{n\geq 0}$ is given by the following:

$$\begin{pmatrix} 0 & 1 & 2x-1 & \cdots \\ 1 & 2x & \cdots & \cdots \\ 2x+1-2\lambda & 3x^2+(1-3\lambda)x-1+\dfrac{\lambda}{2} & \cdots & \cdots \\ 3x^2+3(1-3\lambda)x+6\lambda^2-\dfrac{9\lambda}{2} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{3.22}$$

4. Conclusions

In this study, we successfully generalized the Euler–Seidel method to its degenerate form. We demonstrated that the transition from an initial sequence to a final sequence in the degenerate Euler–Seidel matrix can be expressed through λ-generalized binomial identities that involve degenerate falling and rising factorials. Furthermore, we showed that the relationship between their respective generating functions is governed by the degenerate exponential function $e_\lambda^{1-\lambda}(t)$. By implementing this method, we obtained explicit identities for degenerate Bernoulli, Euler, and Genocchi polynomials. These results not only recover classical identities as $\lambda \to 0$ but also offer a robust analytical tool to investigate broader classes of degenerate special sequences in combinatorial analyses and the number theory.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Taekyun Kim is an editorial board member for Networks and Heterogeneous Media and was not involved in the editorial review or the decision to publish this article. All authors declare that they have no competing interests.

Author contributions

Writing-original draft: T.K., D.S.K.; A Formal analysis:T.K.; D.S.K., H.L., K.S.H.; Investigation: H.L.,K.S.H.; Software: H.L., K.S.H.; Supervision: T.K., D.S.K.; Writing-review and editing: T.K., D.S.K.

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