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***Research article***

# On $k$–Airy convexity and applications to eigenvalue problems

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**Abstract:** Let Ai and Bi denote the Airy functions. For a fixed $k > 0$, we introduce the class $\mathrm{Airy}_k(\mathcal{I})$ of $k$–Airy convex functions on an interval $\mathcal{I} \subset [0, \infty)$. A function $f : \mathcal{I} \to \mathbb{R}$ is called $k$–Airy convex on $\mathcal{I}$ if, for every subinterval $[a, b] \subset \mathcal{I}$ and all $a < t < b$,

$$f(t) \le \frac{\mathrm{Ai}(kt)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(kt)}{\mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka)} f(a) + \frac{\mathrm{Ai}(ka)\ \mathrm{Bi}(kt) - \mathrm{Ai}(kt)\ \mathrm{Bi}(ka)}{\mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka)} f(b).$$

We establish fundamental properties of $\mathrm{Airy}_k(\mathcal{I})$. As an application, we derive lower bounds for eigenvalues in Airy-type boundary value problems.

**Keywords:** Airy functions; $k$–Airy convex functions; Airy differential equation; eigenvalue problems

## 1. Introduction

Convex functions arise throughout pure and applied mathematics, prominently in functional analysis, optimization, operations research, and economics. Given an interval $\mathcal{I} \subset \mathbb{R}$, a function $f : \mathcal{I} \to \mathbb{R}$ is said to be convex on $\mathcal{I}$ if

$$f(\lambda t_1 + (1 - \lambda)t_2) \le \lambda f(t_1) + (1 - \lambda) f(t_2) \qquad \text{for all } t_1, t_2 \in \mathcal{I} \text{ and all } \lambda \in [0, 1].$$

The literature on convexity is vast; see, for example, the monographs [1–3] for systematic treatments of foundational properties, equivalent characterizations, and applications.

On the other hand, for certain problems, the usual convexity assumptions are not satisfied, for instance, in optimization over curved domains (geodesic convexity on the symmetric positive definite (SPD) cone [4]) and in applied models—particularly in economics (see, e.g., [5]). For these reasons, a variety of generalized convexities have been introduced and analyzed.

de Finetti [6] introduced the class of functions $f$ satisfying

$$f(\lambda t_1 + (1 - \lambda)t_2) \le \max\{f(t_1), f(t_2)\} \qquad \text{for all } t_1, t_2 \in \mathcal{I} \text{ and all } \lambda \in [0, 1].$$

Functions obeying this condition are called quasi-convex on $\mathcal{I}$. It is immediate that every convex function on $\mathcal{I}$ is quasi-convex on $\mathcal{I}$. Further developments on quasi-convexity can be found in [7–9].

Orlicz [10] introduced the class of $s$-convex functions in the first sense. For a fixed $s \in (0,1]$, a function $f : [0,\infty) \to \mathbb{R}$ is said to be $s$-convex in the first sense if

$$f(\alpha x + \beta y) \ \le \ \alpha^s f(x) + \beta^s f(y) \quad \text{for all } x, y \ge 0 \text{ and } \alpha, \beta \ge 0 \text{ with } \alpha^s + \beta^s = 1.$$

When $s = 1$, the condition reduces to the usual convexity inequality.

Breckner [11] introduced the class of functions $f : [0,\infty) \to [0,\infty)$ satisfying

$$f(\lambda t_1 + (1-\lambda)t_2) \ \le \ \lambda^s f(t_1) + (1-\lambda)^s f(t_2) \qquad \text{for all } t_1, t_2 \ge 0 \text{ and all } \lambda \in [0,1],$$

where $s \in (0,1]$ is fixed. Any function $f$ obeying this inequality is called $s$-convex in the second sense. Some recent contributions related to this class of functions can be found in [12–14].

Other examples of generalized convexity include $m$-convex functions (Toader [15]; see also [16,17]), $h$-convex functions (Varošanec [18]; see also [19,20]), and multiplicatively $(P,m)$-convex functions (Zhang et al. [21]).

Many classes of differential equations are solved using special functions (see, e.g., Abramowitz and Stegun [22]). In particular, the Airy functions (see Section 2) have numerous applications across science and engineering, including quantum mechanics, optics, fluid dynamics, and electromagnetism. For background on Airy functions and their applications, see the monograph [23]. Motivated by the importance of the Airy functions, we introduce and analyze the class $\mathrm{Airy}_k(\mathcal{I})$ of $k$-Airy convex functions on an interval $\mathcal{I} \subset [0,\infty)$, for a fixed parameter $k > 0$, and develop properties and applications tailored to Airy-type operators.

We now outline the paper's structure. In Section 2, we recall a few basic properties of the Airy functions and provide some useful lemmas. In Section 3, we introduce the classes of $k$-Airy convex and concave functions. Some basic properties related to these classes of functions are studied in Section 4. Finally, Section 5 is devoted to an application to Airy eigenvalue problems.

Throughout the paper, $\mathcal{I} \subset [0,\infty)$ is an interval. By $CV(\mathcal{I})$, we mean the class of convex functions $f : \mathcal{I} \to \mathbb{R}$.

## 2. Preliminaries on Airy functions and auxiliary results

In this section, we recall a few basic properties of the Airy functions (see Abramowitz–Stegun [22] for details) and establish auxiliary results that will be used later.

Throughout, Ai and Bi denote the standard Airy functions, i.e., the linearly independent solutions of

$$y''(t) - t\,y(t) = 0, \qquad t \in \mathbb{R},$$

normalized so that their Wronskian satisfies

$$W(\mathrm{Ai},\mathrm{Bi})(t) := \mathrm{Ai}(t)\,\mathrm{Bi}'(t) - \mathrm{Ai}'(t)\,\mathrm{Bi}(t) = \frac{1}{\pi}, \qquad t \in \mathbb{R}. \tag{2.1}$$

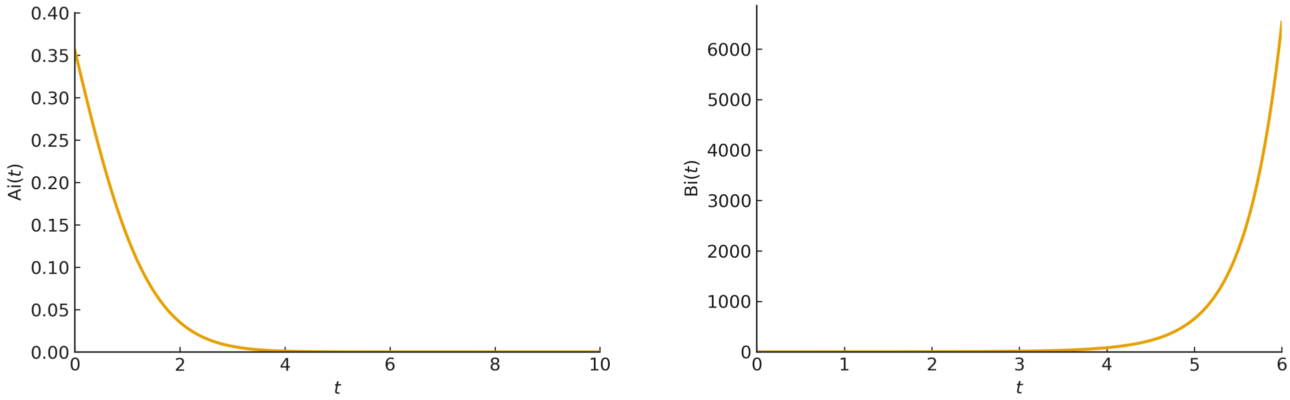
The Airy functions Ai and Bi satisfy the following properties (see Figure 1):

- Ai $> 0$ and strictly decreasing on $[0,\infty)$.

- Bi $> 0$ and strictly increasing on $[0, \infty)$.
- $\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}$, $\mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}$.
- $\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}$, $\mathrm{Bi}'(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{1}{3}\right)}$.
- As $t \to +\infty$,

$$\mathrm{Ai}(t) \sim \frac{1}{2\sqrt{\pi}}\, t^{-1/4}\, e^{-\frac{2}{3}t^{3/2}}, \qquad \mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}\, t^{-1/4}\, e^{\frac{2}{3}t^{3/2}}. \tag{2.2}$$

Here, $\Gamma(\cdot)$ denotes the Gamma function.



**(a)** Plot of $\mathrm{Ai}(t)$ on $[0, \infty)$ (generated with Python).

**(b)** Plot of $\mathrm{Bi}(t)$ on $[0, \infty)$ (generated with Python).

**Figure 1.** Airy functions Ai and Bi on the nonnegative real axis.

**Lemma 2.1.** *Let* $k > 0$ *and* $a, b \in [0, \infty)$ *with* $a < b$. *Then*

$$\mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka) > 0.$$

*Proof.* On $[0, \infty)$, Ai $> 0$ is strictly decreasing, and Bi $> 0$ is strictly increasing. Hence, $\mathrm{Ai}(ka) > \mathrm{Ai}(kb)$ and $\mathrm{Bi}(kb) > \mathrm{Bi}(ka)$. Therefore,

$$\mathrm{Ai}(ka)\,\mathrm{Bi}(kb) - \mathrm{Ai}(kb)\,\mathrm{Bi}(ka) > \mathrm{Ai}(kb)\,\mathrm{Bi}(kb) - \mathrm{Ai}(kb)\,\mathrm{Bi}(ka) = \mathrm{Ai}(kb)(\mathrm{Bi}(kb) - \mathrm{Bi}(ka)) > 0.$$

For $k > 0$ and a fixed subinterval $[a, b] \subset \mathcal{I}$, we introduce the coefficient functions

$$\alpha_{k,a,b}(t) := \frac{\mathrm{Ai}(kt)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(kt)}{\mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka)}, \quad \beta_{k,a,b}(t) := \frac{\mathrm{Ai}(ka)\ \mathrm{Bi}(kt) - \mathrm{Ai}(kt)\ \mathrm{Bi}(ka)}{\mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka)}, \quad t \in [a, b]. \tag{2.3}$$

Set

$$M(k, t) := \mathrm{Ai}(kt)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(kt), \qquad N(k, t) := \mathrm{Ai}(ka)\ \mathrm{Bi}(kt) - \mathrm{Ai}(kt)\ \mathrm{Bi}(ka). \tag{2.4}$$

**Lemma 2.2.** *Fix* $k > 0$ *and* $a, b \in \mathcal{I}$ *with* $a < b$. *The following properties hold:*

(i) $M(k, a) = N(k, b) = \mathrm{Ai}(ka)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(ka) > 0$.

(ii) *For every* $t \in [a,b]$,

$$\alpha_{k,a,b}(t) = \frac{M(k,t)}{M(k,a)}, \qquad \beta_{k,a,b}(t) = \frac{N(k,t)}{N(k,b)} = \frac{N(k,t)}{M(k,a)}.$$

(iii) *Endpoint values:*

$$\alpha_{k,a,b}(a) = \beta_{k,a,b}(b) = 1, \qquad \alpha_{k,a,b}(b) = \beta_{k,a,b}(a) = 0.$$

(iv) *For interior points* $a < t < b$,

$$0 < \alpha_{k,a,b}(t) < 1, \qquad 0 < \beta_{k,a,b}(t) < 1.$$

(v) *For every* $t \in (a,b)$,

$$\frac{\partial^2 M}{\partial t^2}(k,t) - k^3 t\, M(k,t) = 0, \qquad \frac{\partial^2 N}{\partial t^2}(k,t) - k^3 t\, N(k,t) = 0.$$

(vi) *Monotonicity in k: if* $0 < k_1 < k_2$, *then for every* $t \in [a,b]$,

$$\alpha_{k_1,a,b}(t) \ge \alpha_{k_2,a,b}(t), \qquad \beta_{k_1,a,b}(t) \ge \beta_{k_2,a,b}(t).$$

*Proof.* Assertions (i), (ii), and (iii) follow immediately from the definitions of $M(k,\cdot)$, $N(k,\cdot)$, $\alpha_{k,a,b}$, and $\beta_{k,a,b}$.

(iv) Let $a < t < b$. Since Ai is strictly decreasing and Bi is strictly increasing on $[0,\infty)$, with $\mathrm{Ai}, \mathrm{Bi} > 0$, we have

$$\mathrm{Ai}(kb) < \mathrm{Ai}(kt) < \mathrm{Ai}(ka), \qquad \mathrm{Bi}(ka) < \mathrm{Bi}(kt) < \mathrm{Bi}(kb).$$

Hence,

$$0 < M(k,t) < M(k,a), \qquad 0 < N(k,t) < N(k,b),$$

and therefore

$$0 < \alpha_{k,a,b}(t) = \frac{M(k,t)}{M(k,a)} < 1, \qquad 0 < \beta_{k,a,b}(t) = \frac{N(k,t)}{N(k,b)} < 1.$$

(v) Since Ai and Bi solve the Airy differential equation $y''(z) - z\,y(z) = 0$, the functions

$$y_1(t) := \mathrm{Ai}(kt), \qquad y_2(t) := \mathrm{Bi}(kt)$$

satisfy

$$y_1''(t) = k^2\,\mathrm{Ai}''(kt) = k^2(kt)\,\mathrm{Ai}(kt) = k^3 t\, y_1(t), \qquad y_2''(t) = k^2\,\mathrm{Bi}''(kt) = k^2(kt)\,\mathrm{Bi}(kt) = k^3 t\, y_2(t).$$

Hence, $y_j'' - k^3 t\, y_j = 0$ for $j = 1,2$. On the other hand,

$$M(k,t) = \mathrm{Ai}(kt)\ \mathrm{Bi}(kb) - \mathrm{Ai}(kb)\ \mathrm{Bi}(kt) = \mathrm{Bi}(kb)\, y_1(t) - \mathrm{Ai}(kb)\, y_2(t),$$

is a linear combination of $y_1$ and $y_2$ with $t$-independent coefficients. Therefore,

$$\frac{\partial^2 M}{\partial t^2}(k,t) - k^3 t\, M(k,t) = 0.$$

Similarly, $\dfrac{\partial^2 N}{\partial t^2}(k,t) - k^3 t\, N(k,t) = 0$.

(vi) Let $0 < k_1 < k_2$. From (iii) and (v), $\alpha_{k,a,b}$ satisfies

$$\alpha''_{k,a,b}(t) - k^3 t\, \alpha_{k,a,b}(t) = 0 \ \ (t \in (a,b)), \qquad \alpha_{k,a,b}(a) = 1,\ \alpha_{k,a,b}(b) = 0.$$

Define $w(t) := \alpha_{k_2,a,b}(t) - \alpha_{k_1,a,b}(t)$. Subtracting the two equations for $k = k_2$ and $k = k_1$, we get

$$w''(t) - k_2^3 t\, w(t) = (k_2^3 - k_1^3)\, t\, \alpha_{k_1,a,b}(t) \ \geq \ 0 \quad \text{on } (a,b),$$

because $k_2 > k_1$, $t > 0$, and $0 \leq \alpha_{k_1,a,b} \leq 1$ on $[a,b]$ (by (iv)). Moreover,

$$w(a) = \alpha_{k_2,a,b}(a) - \alpha_{k_1,a,b}(a) = 1 - 1 = 0, \qquad w(b) = \alpha_{k_2,a,b}(b) - \alpha_{k_1,a,b}(b) = 0 - 0 = 0.$$

We now apply the maximum principle for $L[y] := y'' - q(t)y$ with $q(t) := k_2^3 t \geq 0$ (see Protter and Weinberger [24]): If $L[w] \geq 0$ in $(a,b)$ and $w(a) = w(b) = 0$, then $w \leq 0$ on $[a,b]$. Therefore, $w \leq 0$, i.e.,

$$\alpha_{k_2,a,b}(t) \leq \alpha_{k_1,a,b}(t) \quad \text{for all } t \in [a,b].$$

The proof for $\beta_{k,a,b}(t)$ is identical.

**Lemma 2.3.** *Let $a, b \in \mathcal{I}$ with $a < b$. For every $t \in [a,b]$,*

$$\lim_{k \to 0^+} \alpha_{k,a,b}(t) = \frac{b-t}{b-a}, \qquad \lim_{k \to 0^+} \beta_{k,a,b}(t) = \frac{t-a}{b-a},$$

*and*

$$\lim_{k \to \infty} \alpha_{k,a,b}(t) = \lim_{k \to \infty} \beta_{k,a,b}(t) = 0.$$

*Proof.* Fix $t \in [a,b]$. At $k = 0$,

$$M(0,t) = M(0,a) = \mathrm{Ai}(0)\,\mathrm{Bi}(0) - \mathrm{Ai}(0)\,\mathrm{Bi}(0) = 0$$

and

$$N(0,t) = N(0,b) = \mathrm{Ai}(0)\,\mathrm{Bi}(0) - \mathrm{Ai}(0)\,\mathrm{Bi}(0) = 0,$$

so we can apply L'Hospital's rule. Differentiating, we obtain

$$\begin{aligned}
\frac{\partial M}{\partial k}(k,t) &= t\ \mathrm{Ai}'(kt)\ \mathrm{Bi}(kb) + b\ \mathrm{Ai}(kt)\ \mathrm{Bi}'(kb) - b\ \mathrm{Ai}'(kb)\ \mathrm{Bi}(kt) - t\ \mathrm{Ai}(kb)\ \mathrm{Bi}'(kt),\\
\frac{\partial M}{\partial k}(k,a) &= a\ \mathrm{Ai}'(ka)\ \mathrm{Bi}(kb) + b\ \mathrm{Ai}(ka)\ \mathrm{Bi}'(kb) - b\ \mathrm{Ai}'(kb)\ \mathrm{Bi}(ka) - a\ \mathrm{Ai}(kb)\ \mathrm{Bi}'(ka),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial N}{\partial k}(k,t) &= a\ \mathrm{Ai}'(ka)\ \mathrm{Bi}(kt) + t\ \mathrm{Ai}(ka)\ \mathrm{Bi}'(kt) - t\ \mathrm{Ai}'(kt)\ \mathrm{Bi}(ka) - a\ \mathrm{Ai}(kt)\ \mathrm{Bi}'(ka),\\
\frac{\partial N}{\partial k}(k,b) &= a\ \mathrm{Ai}'(ka)\ \mathrm{Bi}(kb) + b\ \mathrm{Ai}(ka)\ \mathrm{Bi}'(kb) - b\ \mathrm{Ai}'(kb)\ \mathrm{Bi}(ka) - a\ \mathrm{Ai}(kb)\ \mathrm{Bi}'(ka).
\end{aligned}$$

Evaluating at $k = 0$ and writing $A_0 := \mathrm{Ai}(0)$, $A_1 := \mathrm{Ai}'(0)$, $B_0 := \mathrm{Bi}(0)$, and $B_1 := \mathrm{Bi}'(0)$, we get

$$\frac{\partial M}{\partial k}(0,t) = tA_1B_0 + bA_0B_1 - bA_1B_0 - tA_0B_1 = (b-t)(A_0B_1 - A_1B_0),$$

$$\frac{\partial M}{\partial k}(0,a) = aA_1B_0 + bA_0B_1 - bA_1B_0 - aA_0B_1 = (b-a)(A_0B_1 - A_1B_0),$$

and

$$\frac{\partial N}{\partial k}(0,t) = aA_1B_0 + tA_0B_1 - tA_1B_0 - aA_0B_1 = (t-a)(A_0B_1 - A_1B_0),$$

$$\frac{\partial N}{\partial k}(0,b) = aA_1B_0 + bA_0B_1 - bA_1B_0 - aA_0B_1 = (b-a)(A_0B_1 - A_1B_0).$$

From the Wronskian identity (2.1), we have $A_0B_1 - A_1B_0 = 1/\pi \neq 0$. Hence, by L'Hospital's rule and Lemma 2.2 (ii), we obtain

$$\lim_{k\to 0^+} \alpha_k(t) = \frac{b-t}{b-a}, \qquad \lim_{k\to 0^+} \beta_k(t) = \frac{t-a}{b-a}.$$

The second property follows from the standard asymptotics of Airy functions as $k \to \infty$ (see Eq (2.2)).

## 3. *k*–Airy convex and concave functions

In this section, we introduce the classes of $k$–Airy convex and concave functions.

**Definition 3.1.** Let $k > 0$. A function $f : \mathcal{I} \to \mathbb{R}$ is $k$–Airy convex on $\mathcal{I}$ if, for every subinterval $[a,b] \subset \mathcal{I}$ and all $t \in (a,b)$,

$$f(t) \;\leq\; \alpha_{k,a,b}(t)\, f(a) \;+\; \beta_{k,a,b}(t)\, f(b), \tag{3.1}$$

where the coefficients $\alpha_{k,a,b}(t)$ and $\beta_{k,a,b}(t)$ are defined in Eq (2.3). We denote the class of such functions by $\mathrm{Airy}_k(\mathcal{I})$.

*Remark* 3.1. (i) For $t \in \{a,b\}$, the inequality (3.1) holds with equality (see Lemma 2.2 (iii)).

(ii) The weights $\alpha_k$ and $\beta_k$ generally do not satisfy $\alpha_k(t) + \beta_k(t) = 1$ (the Airy interpolation is not affine).

(iii) From Lemma 2.3, for every $t \in [a,b]$,

$$\lim_{k\to 0^+} \alpha_k(t) = \frac{b-t}{b-a}, \qquad \lim_{k\to 0^+} \beta_k(t) = \frac{t-a}{b-a}.$$

Therefore, as $k \to 0^+$, the $k$–Airy convexity reduces to ordinary convexity.

**Definition 3.2.** Let $k > 0$. A function $f : \mathcal{I} \to \mathbb{R}$ is $k$–Airy concave on $\mathcal{I}$ if, for every subinterval $[a,b] \subset \mathcal{I}$ and all $t \in (a,b)$,

$$f(t) \;\geq\; \alpha_{k,a,b}(t)\, f(a) \;+\; \beta_{k,a,b}(t)\, f(b). \tag{3.2}$$

We denote the class of such functions by $\mathrm{Airy}_k^*(\mathcal{I})$.

*Remark* 3.2. Clearly, $f \in \mathrm{Airy}_k^*(\mathcal{I})$ if and only if $-f \in \mathrm{Airy}_k(\mathcal{I})$.

**Example 3.1.** Consider

$$f : \mathcal{I} \to \mathbb{R}, \qquad f(t) := \operatorname{Ai}(t) - \tfrac{1}{2}\operatorname{Bi}(t),$$

with $\mathcal{I} \subset [0, \infty)$. We verify directly that, for every subinterval $[a, b] \subset \mathcal{I}$ and all $t \in (a, b)$,

$$f(t) \;=\; \alpha_{1,a,b}(t)\, f(a) \;+\; \beta_{1,a,b}(t)\, f(b).$$

Thus, $f \in \operatorname{Airy}_1(\mathcal{I}) \cap \operatorname{Airy}_1^*(\mathcal{I})$.

Moreover, since Ai and Bi solve $y''(t) - t\,y(t) = 0$, we have

$$f''(t) = \operatorname{Ai}''(t) - \tfrac{1}{2}\operatorname{Bi}''(t) = t\operatorname{Ai}(t) - \tfrac{1}{2}t\operatorname{Bi}(t) = t\,f(t).$$

Using the standard limits $\operatorname{Ai}(t) \to 0$ and $\operatorname{Bi}(t) \to +\infty$ as $t \to +\infty$, we obtain

$$\lim_{t\to+\infty} f(t) = -\infty, \qquad f(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} - \frac{1}{2}\,\frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} \approx 0.04756 > 0.$$

Hence, by continuity, there exists $t_0 > 0$ such that $f(t_0) = 0$; that is, $f$ changes sign on $\mathcal{I}$. Since $f''(t) = t\,f(t)$ and $t \geq 0$ on $\mathcal{I}$, the second derivative changes sign as $f$ does, so $f$ is neither convex nor concave on $\mathcal{I}$.

## 4. Basic properties

In this section, we develop basic properties of the classes $\operatorname{Airy}_k(\mathcal{I})$ and $\operatorname{Airy}_k^*(\mathcal{I})$.

**Proposition 4.1.** *Let $k > 0$ and $\lambda, \mu \in \mathbb{R}$, and set*

$$u(t) := \lambda\operatorname{Ai}(kt) + \mu\operatorname{Bi}(kt), \qquad t \in \mathcal{I}.$$

*Then $u \in \operatorname{Airy}_k(\mathcal{I}) \cap \operatorname{Airy}_k^*(\mathcal{I})$.*

*Proof.* Let $u_1(t) := \operatorname{Ai}(kt)$ and $u_2(t) := \operatorname{Bi}(kt)$ for $t \in \mathcal{I}$. Fix $[a, b] \subset \mathcal{I}$ and $t \in (a, b)$. Then

$$\begin{aligned}
&\alpha_{k,a,b}(t)\,u(a) \;+\; \beta_{k,a,b}(t)\,u(b)\\
&= \frac{\operatorname{Ai}(kt)\operatorname{Bi}(kb) - \operatorname{Ai}(kb)\operatorname{Bi}(kt)}{\operatorname{Ai}(ka)\operatorname{Bi}(kb) - \operatorname{Ai}(kb)\operatorname{Bi}(ka)}\,u(a) + \frac{\operatorname{Ai}(ka)\operatorname{Bi}(kt) - \operatorname{Ai}(kt)\operatorname{Bi}(ka)}{\operatorname{Ai}(ka)\operatorname{Bi}(kb) - \operatorname{Ai}(kb)\operatorname{Bi}(ka)}\,u(b)\\
&= \frac{u_1(t)u_2(b) - u_1(b)u_2(t)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u(a) + \frac{u_1(a)u_2(t) - u_1(t)u_2(a)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u(b)\\
&= \lambda\left[\frac{u_1(t)u_2(b) - u_1(b)u_2(t)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u_1(a) + \frac{u_1(a)u_2(t) - u_1(t)u_2(a)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u_1(b)\right]\\
&\quad + \mu\left[\frac{u_1(t)u_2(b) - u_1(b)u_2(t)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u_2(a) + \frac{u_1(a)u_2(t) - u_1(t)u_2(a)}{u_1(a)u_2(b) - u_1(b)u_2(a)}\,u_2(b)\right]\\
&= \lambda\,\frac{u_1(t)(u_2(b)u_1(a) - u_2(a)u_1(b))}{u_1(a)u_2(b) - u_1(b)u_2(a)} + \mu\,\frac{u_2(t)(u_1(a)u_2(b) - u_1(b)u_2(a))}{u_1(a)u_2(b) - u_1(b)u_2(a)}\\
&= \lambda\,u_1(t) + \mu\,u_2(t) = u(t).
\end{aligned}$$

Thus, the defining inequality (3.1) holds with equality, and the concave analogue (3.2) does as well. Hence, $u \in \operatorname{Airy}_k(\mathcal{I}) \cap \operatorname{Airy}_k^*(\mathcal{I})$.

The following proposition is immediate from Definitions 3.1 and 3.2.

**Proposition 4.2.** *Let* $k > 0$ *and* $\lambda, \mu \geq 0$.

(i) *If* $f, g \in \mathrm{Airy}_k(\mathcal{I})$*, then* $\lambda f + \mu g \in \mathrm{Airy}_k(\mathcal{I})$.
(ii) *If* $f, g \in \mathrm{Airy}^*_k(\mathcal{I})$*, then* $\lambda f + \mu g \in \mathrm{Airy}^*_k(\mathcal{I})$.

**Proposition 4.3.** *Let* $0 < k_1 \leq k_2$.

(i) *If* $f \geq 0$ *and* $f \in \mathrm{Airy}_{k_2}(\mathcal{I})$*, then* $f \in \mathrm{Airy}_{k_1}(\mathcal{I})$.
(ii) *If* $f \leq 0$ *and* $f \in \mathrm{Airy}_{k_1}(\mathcal{I})$*, then* $f \in \mathrm{Airy}_{k_2}(\mathcal{I})$.
(iii) *If* $f \geq 0$ *and* $f \in \mathrm{Airy}^*_{k_1}(\mathcal{I})$*, then* $f \in \mathrm{Airy}^*_{k_2}(\mathcal{I})$.
(iv) *If* $f \leq 0$ *and* $f \in \mathrm{Airy}^*_{k_2}(\mathcal{I})$*, then* $f \in \mathrm{Airy}^*_{k_1}(\mathcal{I})$.

*Proof.* Fix $a < b$ in $\mathcal{I}$ and $t \in (a, b)$. Recall that (see Lemma 2.2 (iii), (iv), and (vi)) the Airy weights satisfy $0 \leq \alpha_{k,a,b}(t), \beta_{k,a,b}(t) \leq 1$ and, for $0 < k_1 \leq k_2$,

$$\alpha_{k_2,a,b}(t) \leq \alpha_{k_1,a,b}(t), \qquad \beta_{k_2,a,b}(t) \leq \beta_{k_1,a,b}(t).$$

(i) If $f \geq 0$ and $f \in \mathrm{Airy}_{k_2}(\mathcal{I})$, then

$$f(t) \leq \alpha_{k_2,a,b} f(a) + \beta_{k_2,a,b} f(b) \leq \alpha_{k_1,a,b} f(a) + \beta_{k_1,a,b} f(b).$$

Thus, $f \in \mathrm{Airy}_{k_1}(\mathcal{I})$.

(ii) If $f \leq 0$ and $f \in \mathrm{Airy}_{k_1}(\mathcal{I})$, then

$$f(t) \leq \alpha_{k_1,a,b} f(a) + \beta_{k_1,a,b} f(b) \leq \alpha_{k_2,a,b} f(a) + \beta_{k_2,a,b} f(b).$$

Hence, $f \in \mathrm{Airy}_{k_2}(\mathcal{I})$.

(iii) If $f \geq 0$ and $f \in \mathrm{Airy}^*_{k_1}(\mathcal{I})$, then $-f \leq 0$ and $-f \in \mathrm{Airy}_{k_1}(\mathcal{I})$. By (ii) applied to $-f$, we get $-f \in \mathrm{Airy}_{k_2}(\mathcal{I})$, hence $f \in \mathrm{Airy}^*_{k_2}(\mathcal{I})$.

(iv) If $f \leq 0$ and $f \in \mathrm{Airy}^*_{k_2}(\mathcal{I})$, then $-f \geq 0$ and $-f \in \mathrm{Airy}_{k_2}(\mathcal{I})$. By (i) applied to $-f$, we get $-f \in \mathrm{Airy}_{k_1}(\mathcal{I})$, hence $f \in \mathrm{Airy}^*_{k_1}(\mathcal{I})$.

The next result characterizes the class $\bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I})$.

**Proposition 4.4.** *We have*

$$\bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I}) = \{ f \in \mathrm{CV}(\mathcal{I}) \ : \ f \leq 0 \text{ on } \mathcal{I} \}.$$

*Proof.* Let $f \in \bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I})$. Fix $a < b$ in $\mathcal{I}$ and $t \in [a, b]$. For each $k > 0$,

$$f(t) \leq \alpha_{k,a,b}(t)\, f(a) + \beta_{k,a,b}(t)\, f(b).$$

By Lemma 2.3, $\alpha_{k,a,b}(t) \to 0$ and $\beta_{k,a,b}(t) \to 0$ as $k \to \infty$ (for fixed $a < b$, $t \in [a, b]$). Letting $k \to \infty$ in the preceding inequality gives $f(t) \leq 0$. Since $a < b$ and $t \in [a, b]$ were arbitrary, we conclude $f \leq 0$ on $\mathcal{I}$.

Moreover, Lemma 2.3 also yields $\alpha_{k,a,b}(t) \to \frac{b-t}{b-a}$ and $\beta_{k,a,b}(t) \to \frac{t-a}{b-a}$ as $k \to 0^+$. Passing to the limit $k \to 0^+$ in the same inequality gives

$$f(t) \le \frac{b-t}{b-a} f(a) + \frac{t-a}{b-a} f(b).$$

Since $a < b$ and $t \in [a, b]$ were arbitrary, we conclude $f \in \mathrm{CV}(\mathcal{I})$. Consequently,

$$\bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I}) \subset \{\, f \in \mathrm{CV}(\mathcal{I}) : f \le 0 \text{ on } \mathcal{I} \,\}.$$

We now prove the reverse inclusion. Let $f \in \mathrm{CV}(\mathcal{I})$ with $f \le 0$ on $\mathcal{I}$. Fix $k > 0$, $a < b$ in $\mathcal{I}$, and $t \in (a, b)$. By Lemma 2.3 and the monotonicity of the Airy weights in $k$ (Lemma 2.2 (vi)) ,

$$\alpha_{k,a,b}(t) \le \frac{b-t}{b-a}, \qquad \beta_{k,a,b}(t) \le \frac{t-a}{b-a}.$$

Since $f(a), f(b) \le 0$, decreasing the nonnegative coefficients increases the linear combination; hence

$$\alpha_{k,a,b}(t)\, f(a) + \beta_{k,a,b}(t)\, f(b) \;\ge\; \frac{b-t}{b-a} f(a) + \frac{t-a}{b-a} f(b).$$

By convexity of $f$,

$$f(t) \;\le\; \frac{b-t}{b-a} f(a) + \frac{t-a}{b-a} f(b).$$

Combining the last two displays yields

$$f(t) \;\le\; \alpha_{k,a,b}(t)\, f(a) + \beta_{k,a,b}(t)\, f(b).$$

As $[a, b] \subset \mathcal{I}$ and $t \in (a, b)$ were arbitrary, $f \in \mathrm{Airy}_k(\mathcal{I})$. Since $k > 0$ was arbitrary, $f \in \bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I})$. Consequently,

$$\{\, f \in \mathrm{CV}(\mathcal{I}) : f \le 0 \text{ on } \mathcal{I} \,\} \subset \bigcap_{k>0} \mathrm{Airy}_k(\mathcal{I}).$$

This completes the proof.

In what follows, $\mathcal{I} \subset (0, \infty)$ denotes an open interval.

**Theorem 4.1** (Gradient-type inequality). *Let $k > 0$ and $f \in \mathrm{Airy}_k(\mathcal{I}) \cap C^1(\mathcal{I})$. Then, for every $x, y \in \mathcal{I}$,*

$$f(y) \;\ge\; \pi\big(\, \mathrm{Ai}(ky)\, \mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\, \mathrm{Bi}(ky)\big)\, f(x) \;+\; \frac{\pi}{k}\big(\, \mathrm{Ai}(kx)\, \mathrm{Bi}(ky) - \mathrm{Ai}(ky)\, \mathrm{Bi}(kx)\big)\, f'(x). \tag{4.1}$$

*Proof.* Let $x, y \in \mathcal{I}$. If $x = y$, then Eq (4.1) holds with equality, since

$$\pi\big(\, \mathrm{Ai}(kx)\, \mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\, \mathrm{Bi}(kx)\big) = \pi\, W(\mathrm{Ai}, \mathrm{Bi})(kx) = 1$$

and $\mathrm{Ai}(kx)\, \mathrm{Bi}(kx) - \mathrm{Ai}(kx)\, \mathrm{Bi}(kx) = 0$.

Assume now that $x < y$. Define, for $t \in [x, y]$,

$$g(t) := \frac{\mathrm{Ai}(kt)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kt)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; \frac{\mathrm{Ai}(kx)\,\mathrm{Bi}(kt) - \mathrm{Ai}(kt)\,\mathrm{Bi}(kx)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(y).$$

Since $f \in \mathrm{Airy}_k(\mathcal{I})$, the $k$-Airy convexity inequality (3.1) with $a = x$ and $b = y$ yields

$$f(t) \le g(t) \qquad \text{for all } t \in [x, y].$$

Note that $g(x) = f(x)$ and $g(y) = f(y)$ by construction. Hence, for $x < t \le y$,

$$\frac{f(t) - f(x)}{t - x} \;\le\; \frac{g(t) - g(x)}{t - x}.$$

Since $f \in C^1(\mathcal{I})$ and $g$ is $C^1$ on $[x, y]$, letting $t \to x^+$ gives

$$f'(x) \;\le\; g'(x). \tag{4.2}$$

On the other hand, for every $t \in [x, y]$,

$$g'(t) = \frac{k\,\mathrm{Ai}'(kt)\,\mathrm{Bi}(ky) - k\,\mathrm{Ai}(ky)\,\mathrm{Bi}'(kt)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; \frac{k\,\mathrm{Ai}(kx)\,\mathrm{Bi}'(kt) - k\,\mathrm{Ai}'(kt)\,\mathrm{Bi}(kx)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(y).$$

Evaluating at $t = x$ gives

$$g'(x) = \frac{k(\,\mathrm{Ai}'(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}'(kx))}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; \frac{k(\,\mathrm{Ai}(kx)\,\mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\,\mathrm{Bi}(kx))}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(y).$$

By the Wronskian identity (2.1),

$$\mathrm{Ai}(kx)\,\mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\,\mathrm{Bi}(kx) = W(\mathrm{Ai}, \mathrm{Bi})(kx) = \frac{1}{\pi},$$

hence

$$g'(x) = \frac{k(\,\mathrm{Ai}'(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}'(kx))}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; \frac{k}{\pi}\,\frac{f(y)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}.$$

Then, by Eq (4.2), we have

$$f'(x) \;\le\; \frac{k(\,\mathrm{Ai}'(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}'(kx))}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; \frac{k}{\pi}\,\frac{f(y)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}.$$

Equivalently,

$$\frac{k}{\pi}\,\frac{f(y)}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)} \;\ge\; \frac{k(\,\mathrm{Ai}(ky)\,\mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\,\mathrm{Bi}(ky))}{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}\,f(x) \;+\; f'(x).$$

Since the denominator $\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx) > 0$ for $0 < x < y$ (see Lemma 2.2 (iv)), multiplying both sides by $\frac{\pi}{k}(\,\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx))$ yields Eq (4.1).

The case $x > y$ is analogous. Indeed, applying the same argument on the interval $[y, x]$ (or, equivalently, interchanging the roles of $x$ and $y$ in the construction of $g$) yields (4.1). This completes the proof.

*Remark* 4.1. Let $x, y \in \mathcal{I}$. By Eq (2.1) we have

$$\lim_{k\to 0^+} \left( \mathrm{Ai}(kx)\,\mathrm{Bi}'(ky) - \mathrm{Ai}'(ky)\,\mathrm{Bi}(kx)\right) = \mathrm{Ai}(0)\,\mathrm{Bi}'(0) - \mathrm{Ai}'(0)\,\mathrm{Bi}(0) = W(\mathrm{Ai},\mathrm{Bi})(0) = \frac{1}{\pi}.$$

Moreover,

$$\begin{aligned}
&\lim_{k\to 0^+} \frac{\mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)}{k} \\
&= \lim_{k\to 0^+} \left[x\ \mathrm{Ai}'(kx)\,\mathrm{Bi}(ky) + y\ \mathrm{Ai}(kx)\,\mathrm{Bi}'(ky) - y\ \mathrm{Ai}'(ky)\,\mathrm{Bi}(kx) - x\ \mathrm{Ai}(ky)\,\mathrm{Bi}'(kx)\right] \\
&= x\left( \mathrm{Ai}'(0)\,\mathrm{Bi}(0) - \mathrm{Ai}(0)\,\mathrm{Bi}'(0)\right) + y\left( \mathrm{Ai}(0)\,\mathrm{Bi}'(0) - \mathrm{Ai}'(0)\,\mathrm{Bi}(0)\right) \\
&= (y-x)\, W(\mathrm{Ai},\mathrm{Bi})(0) = \frac{y-x}{\pi}.
\end{aligned}$$

Therefore, as $k \to 0^+$, the gradient-type inequality (4.1) reduces to the classical gradient inequality for convex functions:

$$f(y) \;\ge\; f(x) + (y-x)\, f'(x).$$

Similarly, if $f \in \mathrm{Airy}_k^*(\mathcal{I}) \cap C^1(\mathcal{I})$, we obtain the following gradient-type inequality.

**Theorem 4.2.** *Let $k > 0$ and $f \in \mathrm{Airy}_k^*(\mathcal{I}) \cap C^1(\mathcal{I})$. Then, for every $x, y \in \mathcal{I}$,*

$$f(y) \;\le\; \pi\left( \mathrm{Ai}(ky)\,\mathrm{Bi}'(kx) - \mathrm{Ai}'(kx)\,\mathrm{Bi}(ky)\right) f(x) \;+\; \frac{\pi}{k}\left( \mathrm{Ai}(kx)\,\mathrm{Bi}(ky) - \mathrm{Ai}(ky)\,\mathrm{Bi}(kx)\right) f'(x).$$

The following lemma will be invoked in the proof of the subsequent theorem.

**Lemma 4.1.** *Fix $k > 0$ and let $a < b$ be points of $\mathcal{I}$. For every $F \in C^2([a,b])$ and every $t \in [a,b]$, the following identity holds:*

$$\begin{aligned}
F(t) - \alpha_{k,a,b}(t)F(a) - \beta_{k,a,b}(t)F(b) = -\frac{\pi}{k\,M(k,a)}\Bigg[ & M(k,t)\int_a^t N(k,s)\left(F''(s) - k^3 s\,F(s)\right) ds \\
&+ N(k,t)\int_t^b M(k,s)\left(F''(s) - k^3 s\,F(s)\right) ds\Bigg],
\end{aligned} \tag{4.3}$$

*where $\alpha_{k,a,b}(t)$, $\beta_{k,a,b}(t)$ and $M(k,t)$, $N(k,t)$ are defined in Eqs (2.3) and (2.4), respectively.*

*Proof.* Set

$$y_1(x) := \mathrm{Ai}(kx), \qquad y_2(x) := \mathrm{Bi}(kx), \qquad M_k(x) := M(k,x), \qquad N_k(x) := N(k,x).$$

By Eq (2.1),

$$W(y_1,y_2)(x) = k\,W(\mathrm{Ai},\mathrm{Bi})(kx) = \frac{k}{\pi}.$$

Since

$$M_k(x) = y_1(x)\,y_2(b) - y_1(b)\,y_2(x), \qquad N_k(x) = y_1(a)\,y_2(x) - y_1(x)\,y_2(a),$$

the bilinearity of the Wronskian gives

$$\begin{aligned} W(M_k, N_k)(x) &= (y_1(a)y_2(b) - y_1(b)y_2(a))\, W(y_1, y_2)(x) \\ &= \frac{k}{\pi}\,(y_1(a)y_2(b) - y_1(b)y_2(a)), \end{aligned}$$

that is,

$$W(M_k, N_k)(x) = \frac{k\, M_k(a)}{\pi}. \tag{4.4}$$

Moreover, by Lemma 2.2 (v),

$$M_k''(x) - k^3 x\, M_k(x) = 0, \qquad N_k''(x) - k^3 x\, N_k(x) = 0. \tag{4.5}$$

Let $F \in C^2([a, b])$, and define

$$R(t) := -\frac{\pi}{k\, M_k(a)}\left[M_k(t)\int_a^t N_k(s)\,(F''(s) - k^3 s\, F(s))\, ds + N_k(t)\int_t^b M_k(s)\,(F''(s) - k^3 s\, F(s))\, ds\right],$$

for $t \in [a, b]$. Since $M_k(b) = 0$ and $N_k(a) = 0$, we have

$$R(a) = R(b) = 0. \tag{4.6}$$

Differentiating $R$ via Leibniz's rule, we obtain

$$R'(t) = -\frac{\pi}{k\, M_k(a)}\left[M_k'(t)\int_a^t N_k(s)\,(F''(s) - k^3 s\, F(s))\, ds + N_k'(t)\int_t^b M_k(s)\,(F''(s) - k^3 s\, F(s))\, ds\right]$$

and

$$\begin{aligned} R''(t) = -\frac{\pi}{k\, M_k(a)}\Bigg[ & M_k''(t)\int_a^t N_k(s)\,(F''(s) - k^3 s\, F(s))\, ds + N_k''(t)\int_t^b M_k(s)\,(F''(s) - k^3 s\, F(s))\, ds \\ & - \left(F''(t) - k^3 t F(t)\right)\, W(M_k, N_k)(t)\Bigg]. \end{aligned}$$

Using Eq (4.5) to replace $M_k''(t)$ and $N_k''(t)$ by $k^3 t M_k(t)$ and $k^3 t N_k(t)$, and substituting Eq (4.4), we obtain

$$R''(t) - k^3 t\, R(t) = F''(t) - k^3 t\, F(t). \tag{4.7}$$

Now define the interpolation error

$$H(t) := F(t) - \alpha_k(t)F(a) - \beta_k(t)F(b) = F(t) - \frac{M_k(t)}{M_k(a)}F(a) - \frac{N_k(t)}{N_k(b)}F(b).$$

Then

$$H(a) = H(b) = 0. \tag{4.8}$$

Differentiating twice and using Eq (4.5), we find

$$H''(t) - k^3 t\, H(t) = F''(t) - k^3 t\, F(t). \tag{4.9}$$

From Eqs (4.6)–(4.9), the functions $H$ and $R$ satisfy the same boundary value problem:

$$\begin{cases} y''(t) - k^3 t\, y(t) = F''(t) - k^3 t\, F(t), \quad a < t < b, \\ y(a) = y(b) = 0. \end{cases}$$

By the uniqueness of the solution (see Protter and Weinberger [24]), it follows that $H \equiv R$. Hence, Eq (4.3) holds.

The following result characterizes $k$-Airy convexity by the pointwise differential inequality $f'' - k^3 t f \geq 0$.

**Theorem 4.3.** *Let $k > 0$ and $f \in C^2(\mathcal{I})$. Then the following are equivalent:*

*(i)* $f \in \mathrm{Airy}_k(\mathcal{I})$.
*(ii)* *$f$ satisfies the second-order differential inequality*

$$f''(t) - k^3 t\, f(t) \;\geq\; 0, \qquad t \in \mathcal{I}. \tag{4.10}$$

*Proof.* (i) ⇒ (ii). Let $f \in \mathrm{Airy}_k(\mathcal{I})$. Assume, seeking a contradiction, that

$$f''(t_0) - k^3 t_0\, f(t_0) < 0$$

for some $t_0 \in \mathcal{I}$. By continuity, there exists $\delta > 0$ such that

$$f''(s) - k^3 s\, f(s) < 0 \qquad \text{for all } s \in (t_0 - \delta, t_0 + \delta) \subset \mathcal{I}.$$

Choose $a, b$ with $t_0 - \delta < a < t_0 < b < t_0 + \delta$. Since $f \in \mathrm{Airy}_k(\mathcal{I})$, for every $t \in [a, b]$ we have

$$f(t) \;\leq\; \alpha_{k,a,b}(t) f(a) + \beta_{k,a,b}(t) f(b). \tag{4.11}$$

Applying the identity Eq (4.3) (Lemma 4.1) with $F = f$ on $[a, b]$, we obtain

$$\begin{aligned} f(t) - \alpha_{k,a,b}(t) f(a) - \beta_{k,a,b}(t) f(b) = -\frac{\pi}{k\, M(k,a)} \bigg[ & M(k,t) \int_a^t N(k,s)\,(f''(s) - k^3 s\, f(s))\, ds \\ & + N(k,t) \int_t^b M(k,s)\,(f''(s) - k^3 s\, f(s))\, ds \bigg]. \end{aligned} \tag{4.12}$$

On $[a, b] \subset (0, \infty)$, we have (see Lemma 2.2 (ii)–(iv)) $M(k, \cdot) \geq 0$ and $N(k, \cdot) \geq 0$, with $M(k, a) > 0$. By the choice of $a, b$, the integrand $f''(s) - k^3 s\, f(s)$ in Eq (4.12) is strictly negative on $[a, b]$. Therefore, for any interior point $t \in (a, b)$, both integrals are strictly negative, so the bracket is $< 0$; since the prefactor $-\pi/(k\, M(k, a))$ is negative, we conclude that

$$f(t) - \alpha_{k,a,b}(t) f(a) - \beta_{k,a,b}(t) f(b) \;>\; 0 \qquad (t \in (a, b)),$$

which contradicts Eq (4.11). Consequently, Eq (4.10) holds on $\mathcal{I}$.

(ii) $\Rightarrow$ (i). Assume $f$ satisfies Eq (4.10). Fix $a < b$ in $\mathcal{I}$ and $t \in [a,b]$. Applying the identity Eq (4.3) with $F = f$ on $[a,b]$, we obtain

$$f(t) - \alpha_{k,a,b}(t)f(a) - \beta_{k,a,b}(t)f(b) = -\frac{\pi}{k\,M(k,a)}\Bigg[M(k,t)\int_a^t N(k,s)\,(f''(s) - k^3 s\,f(s))\,ds$$
$$+\; N(k,t)\int_t^b M(k,s)\,(f''(s) - k^3 s\,f(s))\,ds\Bigg].$$

Assumption (ii) gives $f''(s) - k^3 s\,f(s) \geq 0$ for all $s \in [a,b]$. Since $M(k,\cdot) \geq 0$ and $N(k,\cdot) \geq 0$ on $[a,b]$, and $M(k,a) > 0$, we deduce that

$$f(t) \leq \alpha_{k,a,b}(t)f(a) + \beta_{k,a,b}(t)f(b),$$

that is, $f \in \mathrm{Airy}_k(\mathcal{I})$.

Hence, both implications hold, and the proof of the theorem is complete.

Similarly, we obtain the following result.

**Theorem 4.4.** *Let $k > 0$ and $f \in C^2(\mathcal{I})$. Then the following statements are equivalent:*

*(i)* $f \in \mathrm{Airy}^*_k(\mathcal{I})$.
*(ii) $f$ satisfies the second-order differential inequality*

$$f''(t) - k^3 t\,f(t) \;\leq\; 0, \qquad t \in \mathcal{I}.$$

**Example 4.1.** Consider the concave function

$$f(t) = 1 - t^2, \qquad t > \sqrt{3}.$$

Then $f''(t) = -2$. Since $t > \sqrt{3}$ implies $1 - t^2 < -2 < 0$, and $t > 1$, we have

$$t(1 - t^2) < 1 - t^2 < -2 = f''(t).$$

Equivalently,

$$f''(t) - t\,f(t) > 0, \qquad t > \sqrt{3}.$$

By Theorem 4.3, it follows that $f \in \mathrm{Airy}_1((\sqrt{3},\infty))$.

**Example 4.2.** Let $k > 0$ and consider

$$f(t) = t\,\exp\left(\frac{1}{t}\right), \qquad 0 < t < k^{-3/5}.$$

A direct computation gives

$$f''(t) = \exp\left(\frac{1}{t}\right)\frac{1}{t^3} = k^{-3}t^{-5}\,(k^3 t\,f(t)).$$

Hence, if $0 < t < k^{-3/5}$ (equivalently, $t^{-5} > k^3$), we have $f''(t) > k^3 t\,f(t)$, i.e.,

$$f''(t) - k^3 t\,f(t) \;>\; 0, \qquad 0 < t < k^{-3/5}.$$

By Theorem 4.3, it follows that $f \in \mathrm{Airy}_k((0,k^{-3/5}))$.

**Proposition 4.5.** *Let* $k > 0$ *and* $f, g \in C^2(\mathcal{I})$*. Assume that*

$$f(t)\, g''(t) + 2\, f'(t)\, g'(t) \;\geq\; 0, \qquad t \in \mathcal{I}. \tag{4.13}$$

*Then:*

(i) *If* $g \geq 0$ *and* $f \in \mathrm{Airy}_k(\mathcal{I})$*, then* $fg \in \mathrm{Airy}_k(\mathcal{I})$*.*
(ii) *If* $g \leq 0$ *and* $f \in \mathrm{Airy}_k^*(\mathcal{I})$*, then* $fg \in \mathrm{Airy}_k(\mathcal{I})$*.*

*Proof.* We use the product rule

$$(fg)'' - k^3 t\,(fg) = g\,(f'' - k^3 t\, f) \;+\; (fg'' + 2f'g'). \tag{$*$}$$

(i) If $g \geq 0$ and $f \in \mathrm{Airy}_k(\mathcal{I})$, then $f'' - k^3 t\, f \geq 0$ on $\mathcal{I}$ (Theorem 4.3). By Eq (4.13), $fg'' + 2f'g' \geq 0$. Hence, the right-hand side of $(*)$ is $\geq 0$, so $(fg)'' - k^3 t(fg) \geq 0$, i.e., $fg \in \mathrm{Airy}_k(\mathcal{I})$.

(ii) If $g \leq 0$ and $f \in \mathrm{Airy}_k^*(\mathcal{I})$, then $f'' - k^3 t\, f \leq 0$ on $\mathcal{I}$ (Theorem 4.4). Since $g \leq 0$, the first term in $(*)$ satisfies $g\,(f'' - k^3 t\, f) \geq 0$. With Eq (4.13), the second term is $\geq 0$. Therefore, $(fg)'' - k^3 t(fg) \geq 0$, so $fg \in \mathrm{Airy}_k(\mathcal{I})$.

**Proposition 4.6.** *Let* $k > 0$ *and* $f, g \in C^2(\mathcal{I})$*. Assume that*

$$f(t)\, g''(t) + 2\, f'(t)\, g'(t) \;\leq\; 0, \qquad t \in \mathcal{I}. \tag{4.14}$$

*Then:*

(i) *If* $g \geq 0$ *and* $f \in \mathrm{Airy}_k^*(\mathcal{I})$*, then* $fg \in \mathrm{Airy}_k^*(\mathcal{I})$*.*
(ii) *If* $g \leq 0$ *and* $f \in \mathrm{Airy}_k(\mathcal{I})$*, then* $fg \in \mathrm{Airy}_k^*(\mathcal{I})$*.*

*Proof.* Set $\tilde{f} := -f$. By (4.14),

$$\tilde{f}\, g'' + 2\, \tilde{f}'\, g' = -(fg'' + 2f'g') \;\geq\; 0.$$

(i) If $g \geq 0$ and $f \in \mathrm{Airy}_k^*(\mathcal{I})$, then $\tilde{f} = -f \in \mathrm{Airy}_k(\mathcal{I})$. By Proposition 4.5 (i) (with $\tilde{f}$ in place of $f$), $\tilde{f}g \in \mathrm{Airy}_k(\mathcal{I})$. Hence, $-fg \in \mathrm{Airy}_k(\mathcal{I})$, that is, $fg \in \mathrm{Airy}_k^*(\mathcal{I})$.

(ii) If $g \leq 0$ and $f \in \mathrm{Airy}_k(\mathcal{I})$, then $\tilde{f} = -f \in \mathrm{Airy}_k^*(\mathcal{I})$. By Proposition 4.5 (ii) (with $\tilde{f}$ in place of $f$), $\tilde{f}g \in \mathrm{Airy}_k(\mathcal{I})$. Thus, $-fg \in \mathrm{Airy}_k(\mathcal{I})$, that is, $fg \in \mathrm{Airy}_k^*(\mathcal{I})$.

## 5. An application to Airy–eigenvalue problems

As an application of the preceding results, we consider the Airy–eigenvalue problem

$$-(f''(t) - k^3 t\, f(t)) = \lambda\, \phi(t)\, f(t), \qquad t \in \mathcal{I}, \tag{5.1}$$

where $k > 0$, $\lambda > 0$, and $\phi \in C(\mathcal{I})$ is a nonnegative, nontrivial function. Our goal is to derive a lower bound for the eigenvalue $\lambda$.

In the special case $k = 0$, Eq (5.1) reduces to

$$-f''(t) = \lambda\, \phi(t)\, f(t), \qquad t \in \mathcal{I}. \tag{5.2}$$

By a classical inequality of Lyapunov [25], if $f$ satisfies Eq (5.2) and $t_1, t_2 \in \mathcal{I}$ are two consecutive zeros of $f$, then

$$\lambda \;\geq\; \frac{4}{t_2 - t_1}\left(\int_{t_1}^{t_2} \phi(t)\,dt\right)^{-1}. \tag{5.3}$$

Our main result is the following.

**Theorem 5.1.** *Let $f \in C^2(\mathcal{I})$ be a solution of Eq* (5.1) *and let $t_1, t_2 \in \mathcal{I}$ be two consecutive zeros of $f$. Then*

$$\lambda \;\geq\; \frac{4k/\pi}{\operatorname{Ai}(kt_1)\operatorname{Bi}(kt_2) - \operatorname{Ai}(kt_2)\operatorname{Bi}(kt_1)}\left(\int_{t_1}^{t_2} \phi(t)\,dt\right)^{-1}. \tag{5.4}$$

*Proof.* Without loss of generality, we may assume that

$$f(t) > 0, \qquad t_1 < t < t_2.$$

Integrating Eq (5.1) over $(t_1, t_2)$, we obtain

$$f'(t_1) - f'(t_2) + k^3\int_{t_1}^{t_2} t\,f(t)\,dt = \lambda\int_{t_1}^{t_2} \phi(t)\,f(t)\,dt. \tag{5.5}$$

Let $c \in (t_1, t_2)$ be such that

$$f(c) = \max_{t_1 \leq t \leq t_2} f(t).$$

Then

$$\int_{t_1}^{t_2} \phi(t)\,f(t)\,dt \leq f(c)\int_{t_1}^{t_2} \phi(t)\,dt. \tag{5.6}$$

On the other hand, we have

$$k^3\int_{t_1}^{t_2} t\,f(t)\,dt \;\geq\; 0. \tag{5.7}$$

Hence, combining Eqs (5.5)–(5.7), we obtain

$$\lambda \;\geq\; \frac{f'(t_1) - f'(t_2)}{f(c)}\left(\int_{t_1}^{t_2} \phi(t)\,dt\right)^{-1}. \tag{5.8}$$

Furthermore, since $\lambda\,\phi\,f \geq 0$, it follows from Eq (5.1) that

$$f''(t) - k^3 t\,f(t) \leq 0, \qquad t \in \mathcal{I},$$

which implies, by Theorem 4.4, that $f \in \operatorname{Airy}_k^*(\mathcal{I})$. Then, applying Theorem 4.2 with $x = t_1$ and $y = c$, we obtain

$$f(c) \leq \pi(\operatorname{Ai}(kc)\operatorname{Bi}'(kt_1) - \operatorname{Ai}'(kt_1)\operatorname{Bi}(kc))f(t_1) + \frac{\pi}{k}(\operatorname{Ai}(kt_1)\operatorname{Bi}(kc) - \operatorname{Ai}(kc)\operatorname{Bi}(kt_1))f'(t_1).$$

Since $f(t_1) = 0$, this reduces to

$$f(c) \leq \frac{\pi}{k}(\operatorname{Ai}(kt_1)\operatorname{Bi}(kc) - \operatorname{Ai}(kc)\operatorname{Bi}(kt_1))f'(t_1).$$

On the other hand,

$$\mathrm{Ai}(kt_1)\,\mathrm{Bi}(kc) - \mathrm{Ai}(kc)\,\mathrm{Bi}(kt_1) > 0.$$

Consequently, it follows that

$$f'(t_1) \geq \frac{k}{\pi(\,\mathrm{Ai}(kt_1)\,\mathrm{Bi}(kc) - \mathrm{Ai}(kc)\,\mathrm{Bi}(kt_1))} f(c). \tag{5.9}$$

Similarly, applying Theorem 4.2 with $x = t_2$ and $y = c$ and taking into account that $f(t_2) = 0$, we obtain

$$f(c) \leq \frac{\pi}{k}(\,\mathrm{Ai}(kc)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kc))\,[-f'(t_2)],$$

which implies that

$$-f'(t_2) \geq \frac{k}{\pi(\,\mathrm{Ai}(kc)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kc))} f(c). \tag{5.10}$$

Summing Eqs (5.9) and (5.10), we obtain

$$\frac{f'(t_1) - f'(t_2)}{f(c)} \geq \frac{k}{\pi}\left[\frac{1}{\mathrm{Ai}(kt_1)\,\mathrm{Bi}(kc) - \mathrm{Ai}(kc)\,\mathrm{Bi}(kt_1)} + \frac{1}{\mathrm{Ai}(kc)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kc)}\right].$$

Furthermore, using the elementary inequality

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}, \qquad x, y > 0,$$

we obtain

$$\frac{f'(t_1) - f'(t_2)}{f(c)} \geq \frac{4k/\pi}{\mathrm{Bi}(kc)(\,\mathrm{Ai}(kt_1) - \mathrm{Ai}(kt_2)) + \mathrm{Ai}(kc)(\,\mathrm{Bi}(kt_2) - \mathrm{Bi}(kt_1))}. \tag{5.11}$$

Now, we introduce the function

$$\ell(t) = \mathrm{Bi}(kt)(\,\mathrm{Ai}(kt_1) - \mathrm{Ai}(kt_2)) + \mathrm{Ai}(kt)(\,\mathrm{Bi}(kt_2) - \mathrm{Bi}(kt_1)), \qquad t_1 \leq t \leq t_2.$$

Since on $[t_1, t_2]$, $\mathrm{Ai}(kt_1) - \mathrm{Ai}(kt_2) > 0$, $\mathrm{Bi}(kt_2) - \mathrm{Bi}(kt_1) > 0$, and $\mathrm{Ai}, \mathrm{Bi} > 0$, we have $\ell(t) > 0$. On the other hand, a straightforward computation gives

$$\ell''(t) = k^3 t\,\ell(t) > 0.$$

Therefore, $\ell$ is convex on $[t_1, t_2]$. Moreover, a direct evaluation shows that

$$\ell(t_1) = \ell(t_2) = \mathrm{Ai}(kt_1)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kt_1).$$

Hence,

$$\max_{t_1 \leq t \leq t_2} \ell(t) = \ell(t_1) = \ell(t_2) = \mathrm{Ai}(kt_1)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kt_1).$$

Then, by Eq (5.11), we obtain

$$\frac{f'(t_1) - f'(t_2)}{f(c)} \geq \frac{4k/\pi}{\mathrm{Ai}(kt_1)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kt_1)}.$$

Finally, combining the above inequality with Eq (5.8), we obtain Eq (5.4).

*Remark* 5.1. By L'Hospital's rule, one can show that

$$\lim_{k \to 0^+} \frac{k}{\mathrm{Ai}(kt_1)\,\mathrm{Bi}(kt_2) - \mathrm{Ai}(kt_2)\,\mathrm{Bi}(kt_1)} = \frac{\pi}{t_2 - t_1}.$$

Substituting this limit into Eq (5.4) yields the Lyapunov inequality (5.3).

## 6. Conclusions

Motivated by the many applications of Airy functions across science and engineering, we introduced and studied the class $\mathrm{Airy}_k(\mathcal{I})$ of $k$–Airy convex functions on $\mathcal{I}$. We also presented an application to Airy–eigenvalue problems. Further investigations of the class $\mathrm{Airy}_k(\mathcal{I})$ are possible; a few directions are listed below:

- If $f \in \mathrm{CV}(\mathcal{I})$, then by the Hermite–Hadamard inequality, for every subinterval $[t_1, t_2] \subset \mathcal{I}$,

  $$f\Big(\frac{t_1+t_2}{2}\Big) \le \frac{1}{t_2-t_1}\int_{t_1}^{t_2} f(t)\,dt \le \frac{f(t_1)+f(t_2)}{2}.$$

  A natural question is whether analogous bounds hold for the class $\mathrm{Airy}_k(\mathcal{I})$. For background on the Hermite–Hadamard inequality and its generalizations, see the monograph of Dragomir and Pearce [26] and the recent works [20, 27–29].
- One of the earliest generalizations of the Hermite–Hadamard inequality is Fejér's inequality (see, e.g., [26]). It can be stated as follows: Let $f \in \mathrm{CV}(\mathcal{I})$ and let $g \in C(\mathcal{I})$ be nonnegative and symmetric with respect to $\frac{t_1+t_2}{2}$, where $[t_1, t_2] \subset \mathcal{I}$. Then

  $$f\Big(\frac{t_1+t_2}{2}\Big)\int_{t_1}^{t_2} g(t)\,dt \le \int_{t_1}^{t_2} f(t)\,g(t)\,dt \le \frac{f(t_1)+f(t_2)}{2}\int_{t_1}^{t_2} g(t)\,dt.$$

  It would be natural to seek an extension of Fejér's inequality to the class $\mathrm{Airy}_k(\mathcal{I})$.
- Further integral inequalities for functions $f \in \mathrm{Airy}_k(\mathcal{I})$ merit investigation, for example, Jensen-type, Ostrowski-type, trapezoid-and midpoint-type, and Bullen-type inequalities.

### Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare there is no conflict of interest.

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