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Research article

Some energy-preserving relaxation-type schemes for two-dimensional space fractional nonlinear Schrödinger equations

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Abstract: In this paper, we develop some energy-preserving relaxation-type schemes to solve two-dimensional space fractional nonlinear Schrödinger equations with periodic boundary conditions. First, we change the original system into an equivalent relaxation form by introducing some new variables, which transforms the energy conservation law of the original system into quadratic invariants, and satisfies the mass conservation law of the original system. Then an implicit relaxation scheme is applied to deal with the time derivative, and the resulting semi discrete system can exactly preserve the mass and energy conservation laws. However, the obtained semi discrete system is nonlinear. Next a linear implicit relaxation scheme is directly used for the modified system to arrive at a semi discrete scheme, and the conservation of the semi discrete system is analyzed. Second, the resulting semi discrete systems are discretized by the Fourier spectral method with periodic boundary conditions, and the efficient iterative algorithms of the fully-discrete systems are given. Finally, numerical experiments of some space fractional nonlinear Schrödinger equations are given to verify the correctness of the theoretical results.

Keywords: fractional nonlinear Schrödinger equation; conservation law; Fourier spectral method; relaxation scheme; energy-preserving scheme

1. Introduction

In this paper, we consider the following two-dimensional space fractional nonlinear Schrödinger (NLS) equation ($1 < \alpha \leq 2$) [1–3]:

$$i\frac{\partial u(x,y,t)}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}}u(x,y,t) + \mu h(|u(x,y,t)|^2)u(x,y,t) = 0, \quad (1.1)$$
$$u(x,y,0) = u_0(x,y), (x,y) \in R^2.$$

The fractional operator [4–6] $-(-\Delta^*)^{\frac{\alpha}{2}}u(x,y)$ is defined by a sequential Riesz fractional-order derivative and can be expressed by

$$-(-\Delta^*)^{\frac{\alpha}{2}}u(x,y) = \frac{\partial^\alpha}{\partial|x|^\alpha}u(x,y) + \frac{\partial^\alpha}{\partial|y|^\alpha}u(x,y). \tag{1.2}$$

Moreover, the fractional NLS system (1.1) has the following mass and energy conservation laws:

$$\begin{aligned} &Mass: M(t) = \int_{R^2} |u(x,y,t)|^2 dxdy = M(0), \\ &Energy: E(t) = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|_{L^2}^2 - \mu\int_{R^2} H(|u|^2)dxdy = E(0), \end{aligned}$$

where $H(\zeta) = \int_0^\zeta h(\varsigma)d\varsigma$.

In the literature, a great deal of mathematical studies have been carried out on space fractional NLS equations [7,8], strongly coupled fractional NLS equations [9], and coupled fractional NLS equations [10–12]. The space fractional NLS equation was first introduced in [13] by generalizing the Feynman path integral over Levy trajectories, and its well-posedness has been investigated. Moreover, several numerical studies such as fourth-order implicit explicit schemes, Runge Kutta schemes and exponential operator splitting schemes have been carried out to solve the space fractional NLS equation in the past decades. However, most of these numerical methods can not maintain the laws of conservation of the space fractional NLS equation. From the numerical point of view, conservative numerical schemes perform better than the non conservative ones, and the non conservative numerical scheme may easily show nonlinear blow-up. In particular, some conservative numerical methods such as the energy-preserving finite difference scheme [14, 15], the Fourier spectral method [16], the Galerkin finite element method [17], symplectic scheme [18], and the multi symplectic scheme [19] have been used with Dirichlet or periodic boundary conditions, and strict theoretical analyses, such as the existence, uniqueness, convergence and stability of the numerical schemes, have been carried out. Recently, a relaxation scheme has been designed and investigated for solving some class and fractional NLS equations with different boundary conditions. In [20], a relaxation scheme was proposed for the class NLS equation. It is more efficient than the Crank Nicolson scheme, and can effectively avoid costly numerical treatment of the nonlinearity. In [21], a relaxation scheme was proposed for the fractional NLS equation, which was proven to conserve both mass and energy. However, these schemes apply only to special cases ($h(|u|^2) = |u|^2$). However, in [22], a new relaxation scheme was proposed to solve the fractional NLS equation ($h(|u|^2) \neq |u|^2$), and the study proved that the scheme conserves mass and energy conservation laws.

The main difficulties to deal with the fractional NLS (1.1) conclude that (a) the fractional operator $-(-\Delta^*)^{\frac{\alpha}{2}}u(x,y)$ is nonlocal, and (b) $h(|u|^2) = |u|^2$ is strongly nonlinear, and an iterative algorithm needs to be constructed. The main goal of this paper is to construct an energy-preserving relaxation-type scheme [23, 24] for solving the two-dimensional space fractional NLS equation (1.1) with periodic boundary conditions. However, all energy-preserving relaxation-type schemes for the fractional NLS equation (1.1) ($h(|u|^2) \neq |u|^2$) face the problem of how to efficiently solve a large nonlinear system at each time step, and the accuracy is often lower. In order to efficiently handle the nonlinearity of these problems, various numerical schemes have been developed, such as the linear implicit scheme [25, 26], scalar auxiliary variable (SAV) scheme [27–29], and the split-step scheme [16, 30]. As far as we know, there have been few reports on a linear implicit energy-preserving relaxation-type scheme for the two-dimensional fractional NLS equation (1.1). On the other hand, when $\alpha = 2$, the Riesz fractional operator reduces to the Laplace operator, and the fractional NLS equation correspondingly reduces to the classical NLS equation. Numerous numerical schemes for the Riesz fractional operator, such as the finite difference method [31–33], finite element method [34–35], the Fourier spectral method etc., have been developed to solve both classical and fractional NLS systems while preserving certain conservation properties. Nevertheless, to the extent of the author's knowledge, the Fourier spectral method, which is an important approach to solve fractional NLS equations with periodic boundary conditions has seldom been explored.

In this paper, we first transform the original system into an equivalent form that converts the energy conservation law of the original system into quadratic invariants while still satisfying the original mass conservation law. For the two-dimensional fractional NLS equation (1.1), we then present several semi-discrete time-stepping schemes based on the implicit relaxation method and analyze their conservation properties. In addition, we introduce a linear implicit relaxation scheme, which is designed to preserve the energy conservation law of the system. Next, we apply the Fourier spectral method to discretize the Riesz fractional operator in space and present key properties of the associated spectral differentiation matrix. By combining the Fourier spectral method in space with the relaxation scheme in time, we construct a fully discrete numerical scheme for the fractional NLS equation (1.1). Efficient iterative algorithms for solving the resulting fully discrete systems are also provided. Finally, we conduct numerical experiments on several cases of the fractional NLS equation (1.1) to verify the theoretical analysis. For these experiments, we report the numerical accuracy, central processing unit (CPU) computation times, the preservation of invariants, and graphical representations of solitary wave solutions.

The remainder of this paper is arranged as follows. In Section 2, the semi-discrete scheme is shown by using an implicit relaxation scheme for the two-dimensional space fractional NLS equation (1.1), and the resulting semi-discrete system preserves the mass and energy analytical properties of the fractional equation. In addition, the linear implicit relaxation scheme is presented for the two-dimensional space fractional NLS equation (1.1) to arrive at a semi discrete scheme, which only needs to solve decoupled linear system at each time step, and can preserve the energy conservation law of the modified system. In Section 3, the Fourier spectral method is applied to the resulted semi discrete systems with periodic boundary conditions, and the efficient iterative algorithms of the fully discrete systems are given. In Section 4, the numerical experiments are presented, and the results verify the efficiency of the numerical schemes. Finally, a conclusion and a discussion are presented in Section 5.

2. Semi-discrete scheme for the two-dimensional space fractional NLS equation

Recently, relaxation schemes have been designed and investigated for solving some space fractional NLS systems, but little attention was paid to the space fractional NLS equation (1.1) with $h(|u|^2) \neq |u|^2$; in particular, $h(|u|^2)$ is not in the form of a power function. In this section, we show an equivalent relaxation form of the space fractional NLS equation (1.1) by using some auxiliary variables. The new system is discretized by the relaxation-type scheme in time, which can conserve some conservation laws.

Here, we introduce the following notation:

$$h(|u|^2) = f'(|u|^2)g(\varphi),\ F_1(|u|^2) = f'(|u|^2),\ F_2(\varphi) = g'(\varphi), \psi_1 = g(\varphi), \psi_2 = f(|u|^2),$$
$$u_t^n = \frac{1}{\tau}(u^{n+1} - u^n), u^{n+\frac{1}{2}} = \frac{1}{2}(u^n + u^{n+1}), \varphi^{n+\frac{1}{2}} = \frac{1}{2}(\varphi^n + \varphi^{n+1}),$$
$$\varphi^{n-\frac{1}{2}} = \frac{1}{2}(\varphi^n + \varphi^{n-1}), \widetilde{u}^{n+\frac{1}{2}} = \frac{3u^n - u^{n-1}}{2},$$

and τ denotes the time step.

Over the past decade, some attention has been paid to the following two-dimensional space fractional NLS equation with generalized nonlinear power:

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}} u + \mu|u|^{2\lambda}u = 0. \tag{2.1}$$

In order to obtain the relaxation scheme, the fractional NLS equation (2.1) has been rewritten as the systems of two equations presented in previous literature [16, 20, 22]

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}} u + \mu\varphi u = 0,\ \varphi = |u|^{2\lambda}, \tag{2.2}$$

and

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}} u + \mu\varphi^\lambda u = 0,\ \Big(\frac{\varphi^{\lambda+1}}{\lambda+1}\Big)' = \Big(\frac{\varphi^\lambda}{\lambda}\Big)'|u|^2. \tag{2.3}$$

Moreover, Eqs (2.2) and (2.3) are discretized by the following numerical schemes:

$$iu_t^n - (-\Delta^*)^{\frac{\alpha}{2}} u^{n+\frac{1}{2}} + \mu\varphi^{n+\frac{1}{2}}u^{n+\frac{1}{2}} = 0,\ \frac{1}{2}(\varphi^{n+\frac{1}{2}} + \varphi^{n-\frac{1}{2}}) = |u^n|^{2\lambda}, \tag{2.4}$$

and

$$iu_t^n - (-\Delta^*)^{\frac{\alpha}{2}} u^{n+\frac{1}{2}} + \mu(\varphi^{n+\frac{1}{2}})^\lambda u^{n+\frac{1}{2}} = 0,\ \frac{(\varphi^{n+\frac{1}{2}})^{\lambda+1} - (\varphi^{n-\frac{1}{2}})^{\lambda+1}}{\lambda+1} = \frac{(\varphi^{n+\frac{1}{2}})^{\lambda} - (\varphi^{n-\frac{1}{2}})^{\lambda}}{\lambda}|u^n|^2, \tag{2.5}$$

The scheme in Eq (2.5) is a conservation scheme; however, the scheme in Eq (2.4) is conserved only for $\lambda = 1$. Through this analysis, we find that the treatment of the conservation-related numerical scheme considers the numerical method unexpectedly, and it is also necessary to consider the equivalent relaxation form. The resulting equivalent relaxation system can be expressed in the following form:

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}} u + \frac{\mu}{\lambda}\frac{\partial}{\partial|u|^2}\Big(\frac{\varphi^\lambda}{\lambda}|u|^2\Big)u = 0,\ \Big(\frac{\varphi^{\lambda+1}}{\lambda+1}\Big)' = \frac{\partial}{\partial\varphi}\Big(\frac{\varphi^\lambda}{\lambda}|u|^2\Big). \tag{2.6}$$

In the paper, we generalize the relaxation form above to the following form:

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}} u + \mu\frac{\partial F(|u|^2, \varphi)}{\partial|u|^2}u = 0, \tag{2.7}$$

$$\widetilde{g}'(\varphi) = \frac{\partial F(|u|^2, \varphi)}{\partial\varphi}, \tag{2.8}$$

where $\widetilde{g}'(\varphi)$ is a linear or nonlinear function of φ.

Theorem 1. *The equivalent relaxation form in Eqs (2.7) and (2.8) satisfies the laws of conservation of mass and energy*

$$M(t) = \int_{R^2} |u|^2 dxdy = M(0),$$
$$E(t) = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|_{L^2}^2 - \mu \int_{R^2} F(|u|^2, \varphi) - \widetilde{g}(\varphi)dxdy = E(0).$$

Proof: Computing the product of Eq (2.7) with u, and taking the imaginary part yields $M(t) = \int_{R^2} |u|^2 dxdy = M(0)$. Computing the product of Eq (2.7) with $\frac{\partial u}{\partial t}$ yields

$$\Big\langle i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}}u + \mu\frac{\partial F(|u|^2, \varphi)}{\partial|u|^2}u, \frac{\partial u}{\partial t}\Big\rangle = 0.$$

Taking the real part of the equation above , and noting that

$$Re\Big\langle\mu\frac{\partial F(|u|^2, \varphi)}{\partial|u|^2}u, \frac{\partial u}{\partial t}\Big\rangle = \Big\langle\mu\frac{\partial F(|u|^2, \varphi)}{\partial|u|^2}, \frac{\partial|u|^2}{\partial t}\Big\rangle,$$

we can see that

$$\frac{d}{dt}\Big(\|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|_{L^2}^2\Big) - \Big\langle\mu\frac{\partial F(|u|^2, \varphi)}{\partial|u|^2}, \frac{\partial|u|^2}{\partial t}\Big\rangle = 0. \tag{2.9}$$

Computing the product of Eq (2.8) with $\frac{\partial\varphi}{\partial t}$ yields

$$\Big\langle\widetilde{g}'(\varphi), \frac{\partial\varphi}{\partial t}\Big\rangle = \Big\langle\frac{\partial F(|u|^2, \varphi)}{\partial\varphi}, \frac{\partial\varphi}{\partial t}\Big\rangle. \tag{2.10}$$

It follows from Eqs (2.9) and (2.10) that

$$E(t) = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|_{L^2}^2 - \mu \int_{R^2} F(|u|^2, \varphi) - \widetilde{g}(\varphi)dxdy = E(0).$$

This ends the proof.

In this paper, we mainly consider the case $F(|u|^2, \varphi) = f(|u|^2)g(\varphi)$. Moreover, the fractional NLS equation (1.1) can be expressed in the following form:

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}}u + \mu f'(|u|^2)g(\varphi)u = 0, \tag{2.11}$$
$$\widetilde{g}'(\varphi) = f(|u|^2)g'(\varphi). \tag{2.12}$$

In order to obtain the energy-preserving relaxation-type scheme, we introduce some auxiliary variables

$$F_1(|u|^2) = f'(|u|^2), F_2(\varphi) = g'(\varphi), \widetilde{g}'(\varphi) = G'(\varphi)G(\varphi),$$
$$\psi_1 = g(\varphi), \psi_2 = f(|u|^2), \rho = G'(\varphi) = G_1(\varphi).$$

Then, Eqs (2.11) and (2.12) can then be expressed the following form:

$$i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}}u + \mu F_1(|u|^2)\psi_1 u = 0, \tag{2.13}$$

$$G_1(\varphi)\rho = \psi_2 F_2(\varphi), \tag{2.14}$$

$$\frac{\partial \psi_1}{\partial t} = F_2(\varphi)\frac{\partial \varphi}{\partial t}, \tag{2.15}$$

$$\frac{\partial \psi_2}{\partial t} = 2F_1(|u|^2)Re(\overline{u}\frac{\partial u}{\partial t}), \tag{2.16}$$

$$\frac{\partial \rho}{\partial t} = G_1(\varphi)\frac{\partial \varphi}{\partial t}, \tag{2.17}$$

and $\bar{u}$ represents the conjugate of u.

The numerical methods described above can also be used to solve the following fractional logarithmic NLS equation:

$$iu_t - (-\Delta^*)^{\frac{\alpha}{2}}u + \mu \ln |u|^4 u = 0, \tag{2.18}$$

and can be expressed as the equivalent form in Eqs (2.13)–(2.17), where

$$\psi_1 = \ln \varphi, \psi_2 = |u|^2, \rho = \sqrt{4\varphi^{\frac{1}{2}} + c}, F_1(|u|^2) = 1, F_2(\varphi) = \frac{1}{\varphi}, G_1(\varphi) = \frac{\varphi^{-\frac{1}{2}}}{\sqrt{4\varphi^{\frac{1}{2}} + c}}.$$

Theorem 2. *The equivalent systems (2.13)–(2.17) of the space fractional NLS equation (1.1) satisfies laws of conservation of mass and modified energy*

$$M(t) = \int_{R^2} |u|^2 dxdy = \int_{R^2} |u_0|^2 dxdy = M(0),$$
$$\widetilde{E}(t) = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|_{L^2}^2 - \mu\int_{R^2} 2\psi_1\psi_2 - \rho^2 dxdy = \widetilde{E}(0),$$

which transforms the energy conservation law of the original system into quadratic invariants.

Proof: Computing the product of Eq (2.13) with $\frac{\partial u}{\partial t}$ yields

$$\left\langle i\frac{\partial u}{\partial t} - (-\Delta^*)^{\frac{\alpha}{2}}u + \mu F_1(|u|^2)\psi_1 u, \frac{\partial u}{\partial t}\right\rangle = 0.$$

Taking the real part of the equation above yields

$$Re\left\langle \mu F_1(|u|^2)\psi_1 u, \frac{\partial u}{\partial t}\right\rangle = \left\langle \mu F_1(|u|^2)\psi_1 u, 2Re(\overline{u}\frac{\partial u}{\partial t})\right\rangle.$$

Computing the product of Eqs (2.15) and (2.16) with ψ_2, ψ_1, respectively, yields

$$\left\langle \frac{\partial \psi_1}{\partial t}, \psi_2\right\rangle = \left\langle F_2(\varphi)\frac{\partial \varphi}{\partial t}, \psi_2\right\rangle, \left\langle \frac{\partial \psi_2}{\partial t}, \psi_1\right\rangle = \left\langle 2F_1(|u|^2)Re(\overline{u}\frac{\partial u}{\partial t}), \psi_1\right\rangle.$$

Adding above two equation can yield

$$\frac{d}{dt}\int_{R^2}\psi_1\psi_2 dxdy = \left\langle F_2(\varphi)\frac{\partial\varphi}{\partial t},\psi_2\right\rangle + \left\langle 2F_1(|u|^2)Re(\overline{u}\frac{\partial u}{\partial t}),\psi_1\right\rangle.$$

Computing the product of Eqs (2.14) and (2.17) with $\frac{\partial\varphi}{\partial t}$ and ρ, respectively, yields

$$\left\langle G_1(\varphi)\rho,\frac{\partial\varphi}{\partial t}\right\rangle = \left\langle \psi_2 F_2(\varphi),\frac{\partial\varphi}{\partial t}\right\rangle,\ \left\langle\frac{\partial\rho}{\partial t},\rho\right\rangle = \left\langle G_1(\varphi)\frac{\partial\varphi}{\partial t},\rho\right\rangle.$$

It is easy to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}\rho^2 dxdy = \left\langle \psi_2 F_2(\varphi),\frac{\partial\varphi}{\partial t}\right\rangle.$$

The above mentioned analysis yields

$$\widetilde{E}(t) = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u\|^2_{L^2} + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u\|^2_{L^2} - \mu\int_{R^2}2\psi_1\psi_2 - \rho^2 dxdy = \widetilde{E}(0).$$

This ends the proof.

Applying a relaxation-type scheme to the modified fractional NLS system in Eqs (2.13)–(2.17) in time yields the following semi discrete system:

$$iu^n_t - (-\Delta^*)^{\frac{\alpha}{2}}u^{n+\frac{1}{2}} + \mu F_1(|u^{n+\frac{1}{2}}|^2)\psi_1^{n+\frac{1}{2}}u^{n+\frac{1}{2}} = 0, \tag{2.19}$$

$$G_1(\varphi^{n+\frac{1}{2}})\rho^{n+\frac{1}{2}} = \psi_2^{n+\frac{1}{2}}F_2(\varphi^{n+\frac{1}{2}}), \tag{2.20}$$

$$\psi^n_{1t} = F_2(\varphi^{n+\frac{1}{2}})\varphi^n_t, \tag{2.21}$$

$$\psi^n_{2t} = 2F_1(|u^{n+\frac{1}{2}}|^2)Re(u^{n+\frac{1}{2}}\overline{u}^n_t), \tag{2.22}$$

$$\rho^n_t = G_1(\varphi^{n+\frac{1}{2}})\varphi^n_t. \tag{2.23}$$

Theorem 3. *The relaxation-type scheme in Eqs (2.19)–(2.23) of the modified fractional NLS system in Eqs (2.13)–(2.17) satisfies the laws of conservation of mass and modified energy*

$$M^n = \int_{R^2}|u^n|^2 dxdy = M(0),$$
$$\widetilde{E}^n = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^n\|^2_{L^2} + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^n\|^2_{L^2} - \mu(2\langle\psi^n_1,\psi^n_2\rangle - \|\rho^n\|^2) = \widetilde{E}(0).$$

Proof: Computing the product of Eq (2.19) with $u^{n+\frac{1}{2}}$ and taking the imaginary part yields

$$M^n = \int_{R^2}|u^n|^2 dxdy = M^0.$$

Computing the product of Eq (2.19) with u^n_t yields

$$\left\langle iu^n_t - (-\Delta^*)^{\frac{\alpha}{2}}u^{n+\frac{1}{2}} + \mu F_1(|u^{n+\frac{1}{2}}|^2)\psi_1^{n+\frac{1}{2}}u^{n+\frac{1}{2}},u^n_t\right\rangle = 0.$$

Taking the real part of the first equation above yields

$$\frac{1}{2\tau}\Big(\|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^{n+1}\|_{L^2}^2+\|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^{n+1}\|_{L^2}^2-\|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^{n}\|_{L^2}^2-\|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^{n}\|_{L^2}^2\Big)$$
$$-\Big\langle\mu F_1(|u^{n+\frac{1}{2}}|^2)\psi_1^{n+\frac{1}{2}},2Re(u^{n+\frac{1}{2}}\overline{u}_t^n)\Big\rangle=0. \tag{2.24}$$

Computing the product of Eqs (2.21) and (2.22) with $\psi_2^{n+\frac{1}{2}}$, $\psi_1^{n+\frac{1}{2}}$, respectively, yields

$$\Big\langle\psi_{1t}^n,\psi_2^{n+\frac{1}{2}}\Big\rangle=\Big\langle F_2(\varphi^{n+\frac{1}{2}})\varphi_t^n,\psi_2^{n+\frac{1}{2}}\Big\rangle,\ \Big\langle\psi_{2t}^n,\psi_1^{n+\frac{1}{2}}\Big\rangle=\Big\langle 2F_1(|u^{n+\frac{1}{2}}|^2)Re(u^{n+\frac{1}{2}}\overline{u}_t^n),\psi_1^{n+\frac{1}{2}}\Big\rangle.$$

Adding the equation above and noting that

$$\begin{aligned}\langle\psi_{1t}^n,\psi_2^{n+\frac{1}{2}}\rangle+\langle\psi_{2t}^n,\psi_1^{n+\frac{1}{2}}\rangle&=\frac{1}{2\tau}\langle\psi_1^{n+1}-\psi_1^n,\psi_2^{n+1}+\psi_2^n\rangle+\frac{1}{2\tau}\langle\psi_2^{n+1}-\psi_2^n,\psi_1^{n+1}+\psi_1^n\rangle\\&=\frac{\langle\psi_1^{n+1},\psi_2^{n+1}\rangle-\langle\psi_1^n,\psi_2^n\rangle}{\tau},\end{aligned}$$

we can obtain

$$\frac{\langle\psi_1^{n+1},\psi_2^{n+1}\rangle-\langle\psi_1^n,\psi_2^n\rangle}{\tau}=\langle F_2(\varphi^{n+\frac{1}{2}})\varphi_t^n,\psi_2^{n+\frac{1}{2}}\rangle+\langle 2F_1(|u^{n+\frac{1}{2}}|^2)Re(u^{n+\frac{1}{2}}\overline{u}_t^n),\psi_1^{n+\frac{1}{2}}\rangle. \tag{2.25}$$

Computing the product of Eqs (2.20) and (2.23) with φ_t^n and $\rho^{n+\frac{1}{2}}$, respectively, yields

$$\begin{aligned}&\langle G_1(\varphi^{n+\frac{1}{2}})\rho^{n+\frac{1}{2}},\varphi_t^n\rangle=\langle\psi_2^{n+\frac{1}{2}}F_2(\varphi^{n+\frac{1}{2}}),\varphi_t^n\rangle,\\&\langle\rho_t^n,\rho^{n+\frac{1}{2}}\rangle=\langle G_1(\varphi^{n+\frac{1}{2}})\varphi_t^n,\rho^{n+\frac{1}{2}}\rangle,\end{aligned}$$

and

$$\langle\rho_t^n,\rho^{n+\frac{1}{2}}\rangle=\frac{\|\rho^{n+1}\|^2-\|\rho^n\|^2}{2\tau}=\langle\psi_2^{n+\frac{1}{2}}F_2(\varphi^{n+\frac{1}{2}}),\varphi_t^n\rangle. \tag{2.26}$$

It follows from Eqs (2.24)–(2.26) that

$$\begin{aligned}&\|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^{n+1}\|_{L^2}^2+\|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^{n+1}\|_{L^2}^2-\mu(2\langle\psi_1^{n+1},\psi_2^{n+1}\rangle-\|\rho^{n+1}\|^2)\\&=\|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^{n}\|_{L^2}^2+\|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^{n}\|_{L^2}^2-\mu(2\langle\psi_1^{n},\psi_2^{n}\rangle-\|\rho^{n}\|^2).\end{aligned}$$

This ends the proof.

The relaxation scheme above is nonlinear, and leads to nonlinear algebra systems at each time step for $\lambda\neq 1$. In order to handle the nonlinearity of the problems efficiently, many numerical schemes such as the linearized scheme and the split-step scheme etc., have been developed. However, little attention has been paid to linear implicit relaxation schemes to solve the two-dimensional fractional NLS equation (1.1) with ($h(|u|^2)\neq|u|^2$). We apply the linear implicit relaxation scheme in time to arrive at the following semi discrete system:

$$iu_t^n-(-\Delta^*)^{\frac{\alpha}{2}}u^{n+\frac{1}{2}}+\mu\widetilde{u}^{n+\frac{1}{2}}F_1(|\widetilde{u}^{n+\frac{1}{2}}|^2)\psi_1^{n+\frac{1}{2}}=0, \tag{2.27}$$

$$G_1(\widetilde{\varphi}^{n+\frac{1}{2}})\rho^{n+\frac{1}{2}} = \psi_2^{n+\frac{1}{2}}F_2(\widetilde{\varphi}^{n+\frac{1}{2}}), \tag{2.28}$$

$$\psi_{1t}^n = F_2(\widetilde{\varphi}^{n+\frac{1}{2}})\varphi_t^n, \tag{2.29}$$

$$\psi_{2t}^n = 2F_1(|\widetilde{u}^{n+\frac{1}{2}}|^2)Re(\widetilde{u}^{n+\frac{1}{2}}\overline{u}_t^n), \tag{2.30}$$

$$\rho_t^n = G_1(\widetilde{\varphi}^{n+\frac{1}{2}})\varphi_t^n. \tag{2.31}$$

Theorem 4. *The linear implicit relaxation-type scheme in Eqs (2.27)–(2.31) of the modified fractional NLS system in Eqs (2.13)–(2.17) satisfies the law of conservation of modified energy*

$$\widetilde{E}^n = \widetilde{E}^{n-1}\cdots = \widetilde{E}^0,\ \widetilde{E}^n = \|\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}u^n\|_{L^2}^2 + \|\frac{\partial^{\alpha/2}}{\partial|y|^{\alpha/2}}u^n\|_{L^2}^2 - \mu(2\langle\psi_1^n,\psi_2^n\rangle - \|\rho^n\|^2).$$

The proof of Theorem 4 can be easily shown. Therefore, the details are omitted.

3. A fully discrete scheme for the two-dimensional space fractional NLS equation

Recently, some numerical schemes such as finite difference schemes [4], the finite element method [5], and high-order numerical schemes [36, 37] have been designed for the discrete fractional Riesz derivative in line with the following lemma.

Lemma 1 ([4])**.** *Suppose that $u \in L_1(R)$ and*

$$u \in \mathcal{L}^{2+\alpha}(R) := \{u| \int_{-\infty}^{+\infty}(1+|\xi|)^{2+\alpha}|\mathcal{F}(u(\xi))|d\xi < \infty\}.$$

For a fixed h, we can obtain

$$-\Delta_h^\alpha u(x) = \frac{\partial^\alpha}{\partial|x|^\alpha}u + O(h^2),$$

where $1 < \alpha < 2$, and

$$\Delta_h^\alpha u(x) = h^{-\alpha}\sum_{k=-\infty}^{\infty} g_k^{(\alpha)}u(x-kh),\ g_k^{(\alpha)} = \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)}.$$

Some high-order numerical scheme of the Reisz derivative [4] is given by compact difference scheme

$$\begin{aligned}
&-\Delta_h^\alpha u(x) = \mathcal{A}_x^\alpha[\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)] + O(h^4),\\
&-\frac{\alpha}{24}\Delta_h^\alpha u(x+h) + (1+\frac{\alpha}{12})\Delta_h^\alpha u(x) - \frac{\alpha}{24}\Delta_h^\alpha u(x-h) = \frac{\partial^\alpha}{\partial|x|^\alpha}u(x) + O(h^4),
\end{aligned}$$

and the extrapolating difference scheme [18]

$$\delta_h^\alpha u(x) = \frac{\partial^\alpha}{\partial|x|^\alpha}u(x) + O(h^4),$$

where

$$\mathcal{A}_x^{\alpha}u(x) = \frac{\alpha}{24}u(x+h) + (1-\frac{\alpha}{12})u(x) + \frac{\alpha}{24}u(x-h),\ \delta_h^{\alpha}u(x) = \frac{4}{3}\Delta_h^{\alpha}u(x) - \frac{1}{3}\Delta_{2h}^{\alpha}u(x).$$

In addition, other numerical schemes of the Riesz fractional operator, such as the finite element method, the Fourier spectral method, the local discontinuous Galerkin method, and the collocation method, have been developed to solve some fractional difference equations. It is well known that the Fourier spectral method is highly accurate and efficiency for periodic problems. In this section, we propose the Fourier spectral method in space to solve the resulting semi discrete systems with periodic boundary conditions, and the iterative algorithms of the fully discrete systems are given.

3.1. Fourier spectral method

Let the domain $\Omega = (a,b)\times(a,b)$ be uniform partition with $h = h_x = h_y = (b-a)/J$. Then $(x_j, y_k) = (a+jh, a+kh),\ j,k = 0,\ 1,\ 2,\cdots,J,\ u_{j,k}^n = u(x_j,y_k,t_n),\ U_{j,k}^n \approx u(x_j,y_k,t_n)$,

$$\Omega_h = \{(x_j,y_k)|0\le j,k\le J-1\},\ \Omega_\tau = \{t_n|1\le n\le N-1\}.$$

For any grid function $U = \{U_{j,k}\},\ V = \{V_{j,k}\}$, we define the discrete inner product, l_h^2-norm, l_h^∞-norm as

$$\langle U,V\rangle_h = h^2\sum_{j=0}^{J-1}\sum_{k=0}^{J-1}U_{j,k}\overline{V}_{j,k},\ \|U\|_h^2 = \langle U,U\rangle_h,\ \|U\|_{h,\infty}^2 = \max_{0\le j,k\le J-1}|U_{jk}|.$$

Let

$$V_J = span\{g_j(x)g_k(y),\ j,k = 0,1,\cdots,J-1\},$$

be the interpolation space, where $g_j(x)$ and $g_k(x)$ are trigonometric polynomials of degree $J/2$ given by

$$g_j(x) = \sum_{l=-J/2}^{J/2}\frac{1}{Jc_l}e^{il\mu(x-x_j)},\ g_k(y) = \sum_{m=-J/2}^{J/2}\frac{1}{Jc_m}e^{im\mu(y-y_k)},$$

where $c_l = 1(|l|\ne J/2),\ c_{J/2} = c_{-J/2} = 2$. Denote the interpolation operator $I_J : L^2(\Omega)\to V_J$ as

$$I_JU(x,y) = \sum_{j=0}^{J-1}\sum_{k=0}^{J-1}U_{j,k}g_j(x)g_k(y).$$

The values for the derivatives $I_JU(x,y,t)$ at the collocation points (x_i,y_j) are obtained as

$$\frac{\partial^k I_JU(x_i,y_j)}{\partial x^k} = \sum_{n=0}^{J-1}U_{n,j}\frac{d^k g_n(x_i)}{dx^k} = \sum_{n=0}^{J-1}(D_k^x)_{i,n}U_{n,j} = (D_k^xU)_{i,j},$$
$$\frac{\partial^k I_JU(x_i,y_j)}{\partial y^k} = \sum_{n=0}^{J-1}U_{i,n}\frac{d^k g_n(y_j)}{dy^k} = \sum_{n=0}^{J-1}U_{i,m}(D_k^y)_{j,n} = (UD_k^y)_{i,j},$$

where D_k^x and D_k^y represents Fourier spectral differential matrices in the x and y directions, with the elements given by

$$(D_k^x)_{i,n} = \frac{d^k g_n(x_i)}{dx^k},\ (D_k^y)_{j,n} = \frac{d^k g_n(y_j)}{dy^k}.$$

In this paper, we mainly consider Fourier spectral differential matrices of fractional-order operators (1.2). First, the values for one-dimensional fractional order operators (1.2) of $U(x)$ at the collocation points x_j are obtained by

$$-(-\Delta^*)^{\frac{\alpha}{2}} I_J U(x_j) = -\sum_{k=-J/2}^{J/2} |k\mu|^\alpha \left(\frac{1}{Jc_k} \sum_{l=0}^{J-1} U(x_l) e^{-ik\mu(x_l-a)} \right) e^{ik\mu(x_j-a)} = -(D_{2,\alpha}U)_j, \tag{3.1}$$

where $D_{2,\alpha}$ represents a fractional-order Fourier pseudo-spectral differential matrix with the elements

$$(D_{2,\alpha})_{jl} = \sum_{k=-J/2}^{J/2} \frac{1}{Jc_k} |k\mu|^\alpha e^{ik\mu(x_j-x_l)}.$$

Second, the values for two-dimensional fractional-order operators (1.2) of $U(x, y)$ at the collocation points (x_j, y_m) are obtained by

$$-(-\Delta^*)^{\frac{\alpha}{2}} I_J U(x_j, y_m) = -(D_{2,\alpha}^x U)_{j,m} - (U D_{2,\alpha}^y)_{j,m}, \tag{3.2}$$

where $D_{2,\alpha}^x$ and $D_{2,\alpha}^y$ represents fractional-order Fourier pseudo spectral differential matrices in the x and y directions, and $D_{2,\alpha}^x = D_{2,\alpha}^y = D_{2,\alpha}$.

Lemma 2. *For any grid functions U^n, we can obtain*

$$Im\langle D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y, U^{n+\frac{1}{2}} \rangle_h = 0,$$
$$Re\langle D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y, U_t^n \rangle_h = -\frac{1}{2\tau}(|U^{n+1}|_h^2 - |U^n|_h^2),$$

where $|U^n|_h^2 = \langle D_{2,\alpha}^x U^n + U^n D_{2,\alpha}^y, U^n \rangle_h$, and $Im(s)$ and $Re(s)$ represent the imaginary part and the real part of a complex number s, respectively.

On the other hand, we can also consider the following fractional Laplacian:

$$-(-\Delta)^{\alpha/2} u(\boldsymbol{x}) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha}{2} + \frac{n}{2})}{\pi^{n/2}\Gamma(1 - \frac{\alpha}{2})} P.V. \int_{R^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\alpha}} d\boldsymbol{y}.$$

For the periodic boundary condition, the values for the two-dimensional fractional Laplacian of $U(x, y)$ at the collocation points (x_j, y_m) are obtained by

$$-(-\Delta)^{\frac{\alpha}{2}} I_J U(x_j, y_m) = -\sum_{k=-J/2}^{J/2} \sum_{m=-J/2}^{J/2} |(k\mu)^2 + (m\mu)^2|^{\alpha/2}$$
$$\left(\frac{1}{Jc_k} \frac{1}{Jc_m} \sum_{l=0}^{J-1} \sum_{p=0}^{J-1} U(x_l, y_p) e^{-ik\mu(x_l-a)} e^{-ik\mu(x_p-a)} \right) e^{ik\mu(x_j-a)} e^{ik\mu(x_m-a)} = -(D_{2,\alpha}^{x,y} U)_{j,m},$$

where, $D^{x,y}_{2,\alpha}$ represents the Fourier pseudospectral differential matrix with the elements

$$D^{x,y}_{2,\alpha} = \sum_{k=-J/2}^{J/2} \sum_{m=-J/2}^{J/2} \frac{1}{Jc_k} \frac{1}{Jc_m} |(k\mu)^2 + (m\mu)^2|^{\alpha/2} e^{ik\mu(x_j - x_l)} e^{ik\mu(x_m - x_p)}.$$

In order to analyze the conservation of the numerical scheme, we define the following semi norm [35]:

$$|u|_h = \sqrt{\langle -D^x_2 u, u\rangle_h + \langle -u D^y_2, u\rangle_h}.$$

Lemma 3. *For any grid function u^n, we can obtain*

$$Im\langle D^x_2 u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} D^y_2, u^{n+\frac{1}{2}}\rangle_h = 0,$$
$$Re\langle D^x_2 u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} D^y_2, u^n_t\rangle_h = Re\langle D^x_2 u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} D^y_2, u^n_t\rangle_h = -\frac{1}{2\tau}(|u^{n+1}|^2_h - |u^n|^2_h),$$

where $Im(s)$ and $Re(s)$ represent the imaginary part and the real part of a complex number s, respectively.

3.2. Fully discrete scheme

First, applying the relaxation-type scheme in time and the Fourier spectral method in space to the two-dimensional modified fractional NLS system (2.13)–(2.17), we can obtain the following numerical scheme:

$$iU^n_t + D^x_{2,\alpha} U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D^y_{2,\alpha} + \mu F_1(|U^{n+\frac{1}{2}}|^2)\Psi^{n+\frac{1}{2}}_1 U^{n+\frac{1}{2}} = 0, \tag{3.3}$$
$$G_1(\Phi^{n+\frac{1}{2}})\Upsilon^{n+\frac{1}{2}} = \Psi^{n+\frac{1}{2}}_2 F_2(\Phi^{n+\frac{1}{2}}), \tag{3.4}$$
$$\Psi^n_{1t} = F_2(\Phi^{n+\frac{1}{2}})\Phi^n_t, \tag{3.5}$$
$$\Psi^n_{2t} = 2F_1(|U^{n+\frac{1}{2}}|^2)Re(U^{n+\frac{1}{2}}\overline{U}^n_t), \tag{3.6}$$
$$\Upsilon^n_t = G_1(\Phi^{n+\frac{1}{2}})\Phi^n_t, \tag{3.7}$$

where $\Phi^n_{j,k} \approx \varphi(x_j, y_k, t_n), \Psi^n_{1j,k} \approx \psi_1(x_j, y_k, t_n), \Psi^n_{2j,k} \approx \psi_2(x_j, y_k, t_n), \Upsilon^n_{j,k} \approx \rho(x_j, y_k, t_n)$. It follows from Lemma 2 that we can obtain following theorem.

Theorem 5. *The fully discrete scheme (3.3)–(3.7) of the modified fractional NLS system (2.13)–(2.17) can preserve the mass and modified energy conservation laws*

$$M^n = M^{n-1} \cdots = M^0,\ \widetilde{E}^n = \widetilde{E}^{n-1} \cdots = \widetilde{E}^0,$$

where

$$M^n = \|U^n\|^2_h,\ \widetilde{E}^n = |U^n|^2_h - 2\mu\langle\Psi^n_1, \Psi^n_2\rangle_h + \mu\|\Upsilon^n\|^2_h.$$

Theorem 5 can be easily shown. Therefore, the details are omitted. The fully discrete schemes in Eqs (3.3)–(3.7) show how to efficiently solve a large nonlinear system at each time step. The numerical scheme in Eqs (3.3)–(3.7) can be rewritten as follows:

$$i\frac{U^{n+1}-U^n}{\tau}+D^x_{2,\alpha}\frac{U^{n+1}+U^n}{2}+\frac{U^{n+1}+U^n}{2}D^y_{2,\alpha}+\mu F_1\Big(\Big|\frac{U^{n+1}+U^n}{2}\Big|^2\Big)\frac{\Psi_1^{n+1}+\Psi_1^n}{2}\frac{U^{n+1}+U^n}{2}=0, \tag{3.8}$$

$$G_1\Big(\frac{\Phi^{n+1}+\Phi^n}{2}\Big)\frac{\Upsilon^{n+1}+\Upsilon^n}{2}=\frac{\Psi_2^{n+1}+\Psi_2^n}{2}F_2\Big(\frac{\Phi^{n+1}+\Phi^n}{2}\Big), \tag{3.9}$$

$$\Psi_1^{n+1}-\Psi_1^n=F_2\Big(\frac{\Phi^{n+1}+\Phi^n}{2}\Big)(\Phi^{n+1}-\Phi^n), \tag{3.10}$$

$$\Psi_2^{n+1}-\Psi_2^n=2F_1\Big(\Big|\frac{U^{n+1}+U^n}{2}\Big|^2\Big)Re\Big(\frac{U^{n+1}+U^n}{2}\overline{U^{n+1}-U^n}\Big), \tag{3.11}$$

$$\Phi^{n+1}-\Phi^n=(\Upsilon^{n+1}-\Upsilon^n)/G_1\Big(\frac{\Phi^{n+1}+\Phi^n}{2}\Big), \tag{3.12}$$

and given the following iterative algorithm to solve the numerical scheme in Eqs (3.8)–(3.12):

$$\begin{aligned}i\frac{U^{n+1,s+1}-U^n}{\tau}&+D^x_{2,\alpha}\frac{U^{n+1,s+1}+U^n}{2}+\frac{U^{n+1,s+1}+U^n}{2}D^y_{2,\alpha}\\&+\mu F_1\Big(\Big|\frac{U^{n+1,s}+U^n}{2}\Big|^2\Big)\frac{\Psi_1^{n+1,s}+\Psi_1^n}{2}\frac{U^{n+1,s}+U^n}{2}=0,\end{aligned} \tag{3.13}$$

$$\frac{\Upsilon^{n+1,s+1}+\Upsilon^n}{2}=\frac{\Psi_2^{n+1,s}+\Psi_2^n}{2}F_2\Big(\frac{\Phi^{n+1,s}+\Phi^n}{2}\Big)/G_1\Big(\frac{\Phi^{n+1,s}+\Phi^n}{2}\Big), \tag{3.14}$$

$$\Psi_1^{n+1,s+1}-\Psi_1^n=F_2\Big(\frac{\Phi^{n+1,s}+\Phi^n}{2}\Big)(\Phi^{n+1,s}-\Phi^n), \tag{3.15}$$

$$\Psi_2^{n+1,s+1}-\Psi_2^n=2F_1\Big(\Big|\frac{U^{n+1,s}+U^n}{2}\Big|^2\Big)Re\Big(\frac{U^{n+1,s}+U^n}{2}\overline{U^{n+1,s}-U^n}\Big), \tag{3.16}$$

$$\Phi^{n+1,s+1}-\Phi^n=(\Upsilon^{n+1,s}-\Upsilon^n)/G_1\Big(\frac{\Phi^{n+1,s}+\Phi^n}{2}\Big), \tag{3.17}$$

where $U^{n+1,0}=U^n, \Upsilon^{n+1,0}=\Upsilon^n, \Psi_1^{n+1,0}=\Psi_1^n, \Psi_2^{n+1,0}=\Psi_2^n, \Phi^{n+1,0}=\Phi^n$.

If $(U^n,\ \Upsilon^n,\ \Psi_1^n,\ \Psi_2^n,\ \Phi^n), n=1,2,3,\cdots$ are known, then $(U^{n+1},\ \Upsilon^{n+1},\ \Psi_1^{n+1},\ \Psi_2^{n+1},\ \Phi^{n+1})$ is derived by the numerical scheme in Eqs (3.13)–(3.17). Each iteration requires solving the system of linear equations, and we apply the fast Fourier transform to solve the linear equations.

Second, applying the linear implicit relaxation-type scheme in time and the Fourier spectral method in space to the modified fractional NLS system in Eqs (2.13)–(2.17), we can obtain the following numerical scheme:

$$i\frac{U^{n+1}-U^n}{\tau}+D^x_{2,\alpha}\frac{U^{n+1}+U^n}{2}+\frac{U^{n+1}+U^n}{2}D^y_{2,\alpha}+\rho\widetilde{U}^{n+\frac12}F_1(|\widetilde{U}^{n+\frac12}|^2)\frac{\Psi_1^{n+1}+\Psi_1^n}{2}=0, \tag{3.18}$$

$$\frac{\Upsilon^{n+1}+\Upsilon^n}{2}=\frac{\Psi_2^{n+1}+\Psi_2^n}{2}F_2(\widetilde{\Phi}^{n+\frac12})/G_1(\widetilde{\Phi}^{n+\frac12}), \tag{3.19}$$

$$\Psi_1^{n+1}-\Psi_1^n=F_2(\widetilde{\Phi}^{n+\frac12})(\Phi^{n+1}-\Phi^n), \tag{3.20}$$

$$\Psi_2^{n+1} - \Psi_2^n = 2F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)Re(\widetilde{U}^{n+\frac{1}{2}}\overline{U^{n+1} - U^n}), \tag{3.21}$$

$$\Phi^{n+1} - \Phi^n = (\Upsilon^{n+1} - \Upsilon^n)/G_1(\widetilde{\Phi}^{n+\frac{1}{2}}), \tag{3.22}$$

where $\widetilde{U}^{n+\frac{1}{2}} = \dfrac{3U^{n+1} - U^n}{2}$. The first step can obtain a numerical scheme with the following form:

$$i\frac{U^1 - U^0}{\tau} + D_{2,\alpha}^x\frac{U^1 + U^0}{2} + \frac{U^1 + U^0}{2}D_{2,\alpha}^y + \rho F_1\Big(\Big|\frac{U^1 + U^0}{2}\Big|^2\Big)\frac{\Psi_1^1 + \Psi_1^0}{2}\frac{U^1 + U^0}{2} = 0, \tag{3.23}$$

$$\frac{\Upsilon^1 + \Upsilon^0}{2} = \frac{\Psi_2^1 + \Psi_2^0}{2}F_2\Big(\frac{\Phi^1 + \Phi^0}{2}\Big)/G_1\Big(\frac{\Phi^1 + \Phi^0}{2}\Big), \tag{3.24}$$

$$\Psi_1^1 - \Psi_1^0 = F_2\Big(\frac{\Phi^1 + \Phi^0}{2}\Big)(\Phi^1 - \Phi^0), \tag{3.25}$$

$$\Psi_2^1 - \Psi_2^0 = 2F_1\Big(\Big|\frac{U^1 + U^0}{2}\Big|^2\Big)Re\Big(\frac{U^1 + U^0}{2}\overline{U^1 - U^0}\Big), \tag{3.26}$$

$$\Phi^1 - \Phi^0 = (\Upsilon^1 - \Upsilon^0)/G_1\Big(\frac{\Phi^1 + \Phi^0}{2}\Big). \tag{3.27}$$

Theorem 6. *The fully discrete schemes in Eqs (3.18)–(3.27) of the modified fractional NLS system in Eqs (2.13)–(2.17) can preserve the modified energy conservation law*

$$\widetilde{E}^n = \widetilde{E}^{n-1} \cdots = \widetilde{E}^0, \ \widetilde{E}^n = |U^n|_h^2 - 2\mu\langle\Psi_1^n, \Psi_2^n\rangle_h + \mu\|\Upsilon^n\|_h^2.$$

Theorem 6 can be easily shown. Therefore, the details are omitted. The first step of the numerical scheme in Eqs (3.23)–(3.27) can be solved by the iterative algorithm above. To solve the numerical scheme in Eqs (3.18)–(3.22), we introduce $U^{n+1} = P^{n+1} + iQ^{n+1}$. The numerical scheme in Eqs (3.18)–(3.22) can be obtained by solving the following linear equations:

$$\begin{aligned}
&-2Q^{n+1} + \tau D_{2,\alpha}^x P^{n+1} + \tau P^{n+1}D_{2,\alpha}^y + \tau\rho\widetilde{U}^{n+\frac{1}{2}}F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)\Psi_1^{n+1}\\
&\quad = -2Q^n - \tau D_{2,\alpha}^x P^n + \tau P^n D_{2,\alpha}^y - \tau\rho\widetilde{U}^{n+\frac{1}{2}}F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)\Psi_1^n,\\
&-2P^{n+1} + \tau D_{2,\alpha}^x Q^{n+1} + \tau Q^{n+1}D_{2,\alpha}^y = -2P^n - \tau D_{2,\alpha}^x Q^n + \tau Q^n D_{2,\alpha}^y,\\
&\Upsilon^{n+1} - F_2(\widetilde{\Phi}^{n+\frac{1}{2}})/G_1(\widetilde{\Phi}^{n+\frac{1}{2}})\Psi_2^{n+1} = \Psi_2^n F_2(\widetilde{\Phi}^{n+\frac{1}{2}})/G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) - \Upsilon^n,\\
&\Psi_1^{n+1} - F_2(\widetilde{\Phi}^{n+\frac{1}{2}})\Phi^{n+1} = \Psi_1^n - F_2(\widetilde{\Phi}^{n+\frac{1}{2}})\Phi^n,\\
&\Psi_2^{n+1} - 2F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)(\widetilde{P}^{n+\frac{1}{2}}P^{n+1} + \widetilde{Q}^{n+\frac{1}{2}}Q^{n+1}) = \Psi_2^n + 2F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)(\widetilde{P}^{n+\frac{1}{2}}P^n + \widetilde{Q}^{n+\frac{1}{2}}Q^n),\\
&\Phi^{n+1} - \Upsilon^{n+1}/G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) = \Phi^n - \Upsilon^n/G_1(\widetilde{\Phi}^{n+\frac{1}{2}}).
\end{aligned}$$

In Ref [22], a theoretical analysis is given for $h(|u|^2) = |u|^2$. Moreover, in a similar analysis of the literature [22], we can obtain the theoretical analysis of $h(|u|^2) = |u|^2$ for our method. In this paper, we only show the unique solvability analysis for the fully discrete scheme with $h(|u|^2) \neq |u|^2$. Due to the nonlinear terms, there are still some difficulties in the convergence analysis of the fully discrete scheme with $h(|u|^2) \neq |u|^2$.

Theorem 7. *When $\rho > 0$, the fully discrete scheme in Eqs (3.18)–(3.22) of the modified fractional NLS system in Eqs (2.13)–(2.17) has a unique solution.*

Proof: Assume that for a fixed n, the pairs $(U^n, \Upsilon^n, \Psi_1^n, \Psi_2^n, \Phi^n)$ and $(U^{n-1}, \Upsilon^{n-1}, \Psi_1^{n-1}, \Psi_2^{n-1}, \Phi^{n-1})$ are already determined. The corresponding homogeneous system of the fully discrete scheme in Eqs (3.18)–(3.22) has the form

$$2iU^{n+1} + \tau D_{2,\alpha}^x U^{n+1} + \tau U^{n+1} D_{2,\alpha}^y + \tau\rho \widetilde{U}^{n+\frac{1}{2}} F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)\Psi_1^{n+1} = 0, \tag{3.28}$$

$$\Upsilon^{n+1} G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) - \Psi_2^{n+1} F_2(\widetilde{\Phi}^{n+\frac{1}{2}}) = 0, \tag{3.29}$$

$$\Psi_1^{n+1} - F_2(\widetilde{\Phi}^{n+\frac{1}{2}})\Phi^{n+1} = 0, \tag{3.30}$$

$$\Psi_2^{n+1} - 2F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2) Re(\widetilde{U}^{n+\frac{1}{2}} \overline{U^{n+1}}) = 0, \tag{3.31}$$

$$\Phi^{n+1} G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) - \Upsilon^{n+1} = 0. \tag{3.32}$$

Computing the discrete inner product of Eq (3.28) with U^{n+1}, and taking the real part yields

$$Re\langle \tau D_{2,\alpha}^x U^{n+1} + \tau U^{n+1} D_{2,\alpha}^y + \tau\rho \widetilde{U}^{n+\frac{1}{2}} F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2)\Psi_1^{n+1}, U^{n+1}\rangle_h = 0. \tag{3.33}$$

Computing the discrete inner product of Eqs (3.29)–(3.32) with Φ^{n+1}, Ψ_2^{n+1}, Ψ_1^{n+1} and $Upsilon^{n+1}$ yields

$$\langle \Upsilon^{n+1} G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) - \Psi_2^{n+1} F_2(\widetilde{\Phi}^{n+\frac{1}{2}}), \Phi^{n+1}\rangle = 0, \tag{3.34}$$

$$\langle \Psi_1^{n+1} - F_2(\widetilde{\Phi}^{n+\frac{1}{2}})\Phi^{n+1}, \Psi_2^{n+1}\rangle = 0, \tag{3.35}$$

$$\langle \Psi_2^{n+1} - 2F_1(|\widetilde{U}^{n+\frac{1}{2}}|^2) Re(\widetilde{U}^{n+\frac{1}{2}} \overline{U^{n+1}}), \Psi_1^{n+1}\rangle = 0, \tag{3.36}$$

$$\langle \Phi^{n+1} G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) - \Upsilon^{n+1}, \Upsilon^{n+1}\rangle = 0. \tag{3.37}$$

From Eqs (3.34)–(3.37) can yield

$$Re\langle D_{2,\alpha}^x U^{n+1} + \tau U^{n+1} D_{2,\alpha}^y, U^{n+1}\rangle_h + \frac{\rho}{2}\langle \Upsilon^{n+1}, \Upsilon^{n+1}\rangle_h = 0. \tag{3.38}$$

Noting that $\rho > 0$, we can obtain $Re\langle D_{2,\alpha}^x U^{n+1} + \tau U^{n+1} D_{2,\alpha}^y, U^{n+1}\rangle_h = 0, \langle \Upsilon^{n+1}, \Upsilon^{n+1}\rangle_h = 0$. Thus the homogeneous system has only a trivial solution, and hence the function $U^{n+1}, \Upsilon^{n+1}, \Psi_1^{n+1}, \Psi_2^{n+1}, \Phi^{n+1}$ is uniquely determined.

This ends the proof.

4. Numerical experiments

In this section, we use the Fourier spectral method in space, the implicit relaxation scheme, and the linear implicit relaxation scheme in time to solve the space fractional NLS equation (1.1). First, we check the numerical errors and convergence rates of the numerical schemes with periodic boundary conditions. To test the convergence orders and error in the temporal direction and spatial direction, we assume the following form for sufficient small τ:

$$\|F_u(h)\|_h = \|u_i^n(h,\tau) - u_{2i}^n(h/2,\tau)\|_h,\ Order_{l_h^2}^h = \log_2 \frac{\|F_u(h)\|_h}{\|F_u(h/2)\|_h}, \tag{4.1}$$

$$\|F_u(h)\|_{h,\infty} = \max_{1\le i\le J, 1\le n\le N} |u_i^n(h,\tau) - u_{2i}^n(h/2,\tau)|,\ Order_\infty^h = \log_2 \frac{\|F_u(h)\|_{h,\infty}}{\|F_u(h/2)\|_{h,\infty}}, \tag{4.2}$$

and for a sufficient small h, because of

$$\|G_u(\tau)\|_h = \|u_i^n(h,\tau) - u_i^{2n}(h,\tau/2)\|, \ Order^{\tau}_{l_h^2} = \log_2 \frac{\|G_u(\tau)\|_h}{\|G_u(\tau/2)\|_h}, \tag{4.3}$$

$$\|G_u(\tau)\|_{h,\infty} = \max_{1\le i\le J, 1\le n\le N} |u_i^n(h,\tau) - u_i^{2n}(h,\tau/2)|, \ Order^{\tau}_{\infty} = \log_2 \frac{\|G_u(\tau)\|_{h,\infty}}{\|G_u(\tau/2)\|_{h,\infty}}. \tag{4.4}$$

Second, we check the conservation properties of the numerical schemes including the one-dimensional case and two-dimensional case with periodic boundary conditions. Third, we show the CPU times and spatial evolution diagrams of the soliton for the space fractional NLS equation with periodic boundary conditions.

4.1. Example A

Example 1. Consider the space fractional NLS equation

$$iu_t - (-\Delta^*)^{\frac{\alpha}{2}} u + 2|u|^2 u = 0,$$

and show the following three relaxation scheme:
Relaxation scheme I

$$\begin{aligned} &iU_t^n + D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y + \mu \Phi^{n+\frac{1}{2}} U^{n+\frac{1}{2}} = 0, \\ &\frac{1}{2}(\Phi^{n+\frac{1}{2}} + \Phi^{n-\frac{1}{2}}) = |U^n|^2, \end{aligned}$$

Relaxation scheme II

$$\begin{aligned} &iU_t^n + D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y + \mu \Phi^{n+\frac{1}{2}} U^{n+\frac{1}{2}} = 0, \\ &\Phi^{n+1} - \Phi^n = 2Re(U^{n+\frac{1}{2}} \cdot \overline{U^{n+1} - U^n}), \end{aligned}$$

Relaxation scheme III

$$\begin{aligned} &iU_t^n + D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y + \mu \Phi^{n+\frac{1}{2}} \widetilde{U}^{n+\frac{1}{2}} = 0, \\ &\Phi^{n+1} - \Phi^n = 2Re(\widetilde{U}^{n+\frac{1}{2}} \cdot \overline{U^{n+1} - U^n}). \end{aligned}$$

Moreover, Relaxation schemes II and III are special cases of the relaxation schemes in Eqs (3.3)–(3.7) and Eqs (3.18)–(3.22), and Relaxation scheme I is a special case of the relaxation scheme in Eq (2.4) with $\lambda = 2$.

First, consider one-dimensional space fractional NLS equation

$$iu_t - (-\Delta^*)^{\frac{\alpha}{2}} u + 2|u|^2 u = 0, \tag{4.5}$$

$$u(x,0) = \mathrm{sech}(x) \cdot \exp(2ix), x \in [-20, 20]. \tag{4.6}$$

In this example, we solve the one-dimensional fractional NLS equation in Eqs (4.5)–(4.6) with the Fourier spectral method in space, and the relaxation scheme in time with periodic boundary conditions. We first verify the numerical errors and convergence orders by Eqs (4.3)–(4.4). Figure 1 shows the time errors and convergence orders for different values of α by Relaxation schemes I, II, and III.

From Figure 1, we find that Relaxation schemes I, II, and III are of order 2 in time. We then fix the time step $\tau = 10^{-7}$ to test the space numerical errors for different values of α by Eqs (4.1) and (4.2) and the fourth-order finite difference method with periodic boundary conditions. The results are listed in Figures 2 and 3. From Figures 2 and 3, we can see that the Fourier spectral method has a significant advantage over the fourth-order finite difference method in terms of numerical precision. Second, we examine the conservation of Relaxation schemes I, II, and III with different values of α, $t \in [0, 500], \tau = 0.001$, and $J = 256$. The results are listed in Figures 4–9. From Figures 4–9, it is seen that Relaxation schemes I and II preserves the mass and energy conservation simultaneously very well, and the energy varies with the α values. Moreover, Relaxation scheme III also preserves energy conservation very well.

Now, consider two-dimensional space fractional NLS equation

$$iu_t - (-\Delta^*)^{\frac{\alpha}{2}} u + |u|^2 u = 0, \tag{4.7}$$

$$u(x, y, 0) = sech(x)sech(y), (x, y) \in [-2\pi, 2\pi] \times [-2\pi, 2\pi]. \tag{4.8}$$

In this example, we first examine the conservation of Relaxation scheme II with the periodic boundary conditions with different values of α, $t \in [0, 500], \tau = 0.001$, and $N = 256$. The results are listed in Figures 10 and 11. From Figures 10 and 11, it is seen Relaxation scheme II preserves the mass and energy conservation simultaneously very well, and the energy varies with the α values. Finally, we show the CPU times by Relaxation schemes I, II, and III, respectively. The results are listed in Table 1. From Table 1, it is seen that Relaxation scheme I takes the least time and has the highest computational efficiency, followed by the Relaxation scheme III and the Crank–Nicolson scheme, and Relaxation scheme II takes the longest time.

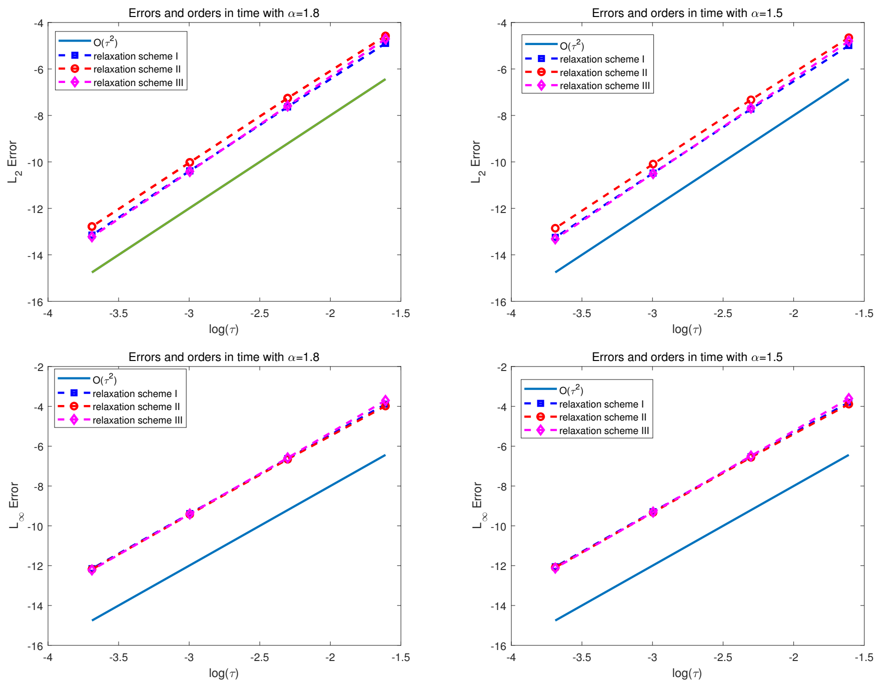

Figure 1. Time convergence orders and errors with $J = 256$.

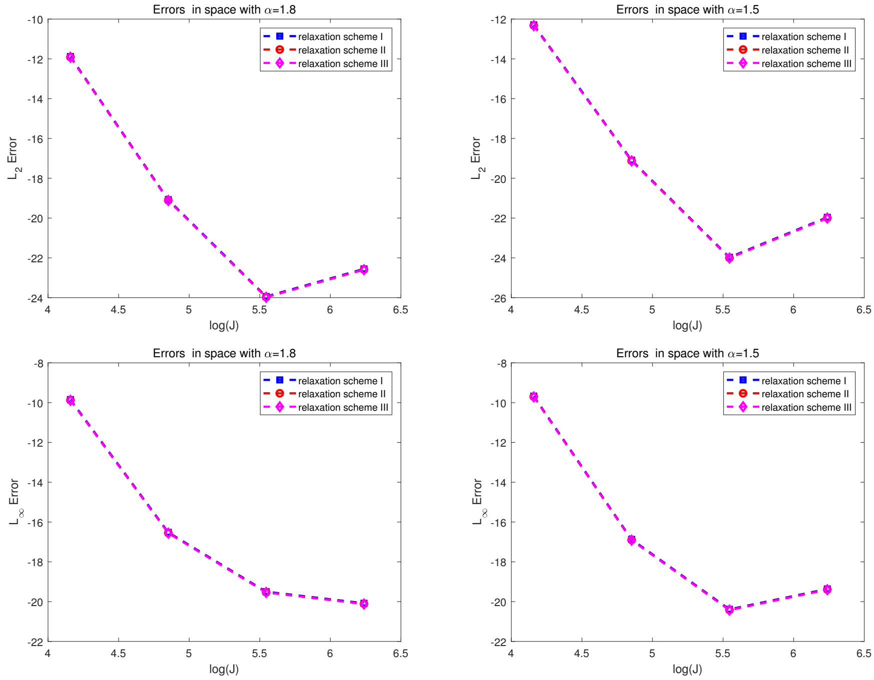

Figure 2. Space numerical errors (Fourier spectral method) with $\tau = 10^{-7}$.

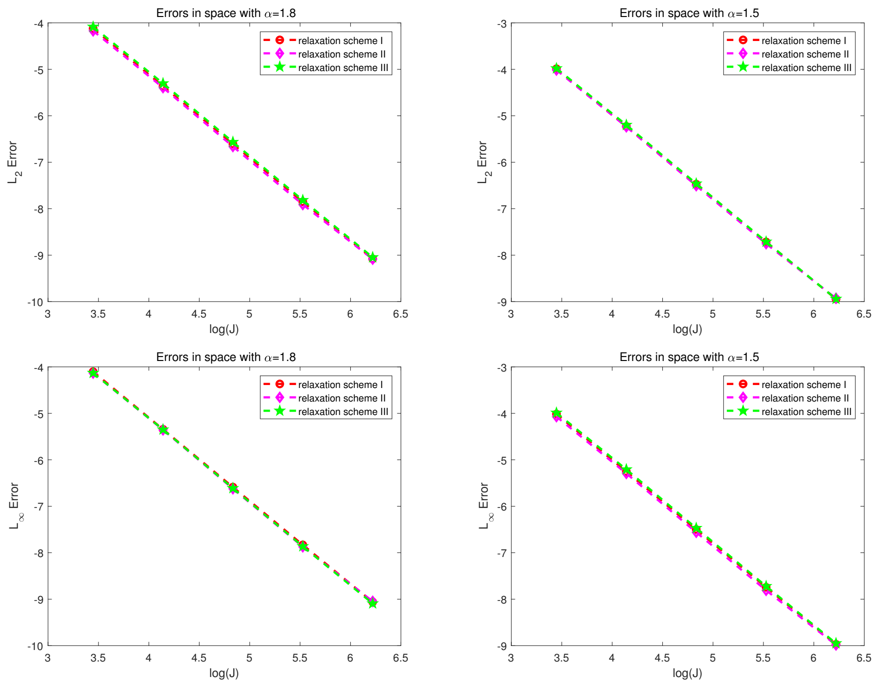

Figure 3. Space numerical errors (fourth-order finite difference method) with $\tau = 10^{-7}$.

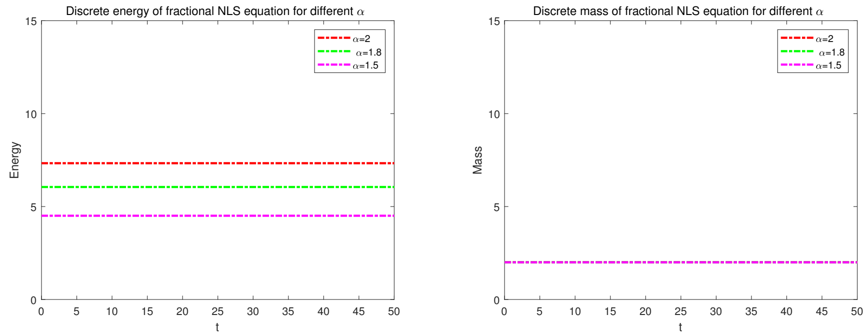

Figure 4. Discrete mass and energy of the one-dimensional fractional NLS equation in Eqs (4.5) and (4.6) with Relaxation scheme I.

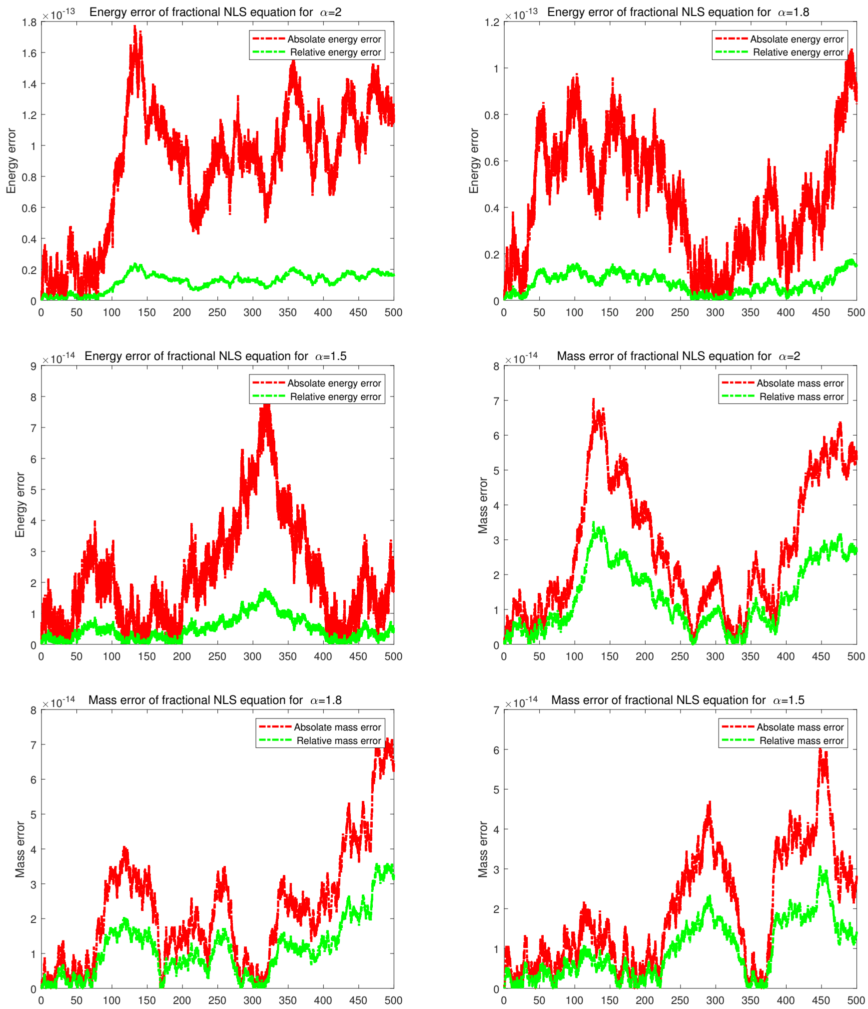

Figure 5. Energy and mass errors of the one-dimensional fractional NLS equation in Eqs (4.5) and (4.6) with Relaxation scheme I.

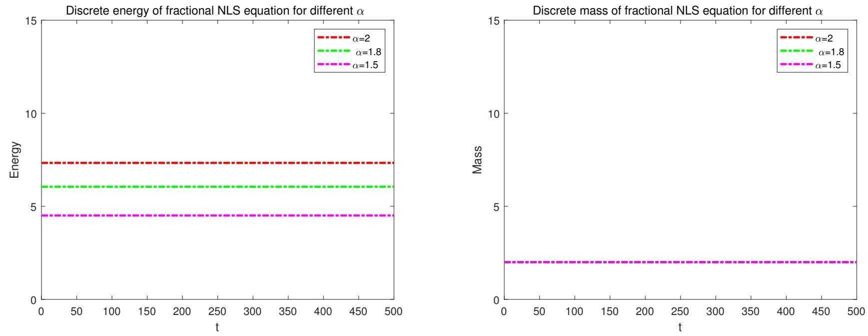

Figure 6. Discrete mass and energy of the one-dimensional fractional NLS equation in Eqs (4.5) and (4.6) with Relaxation scheme II.

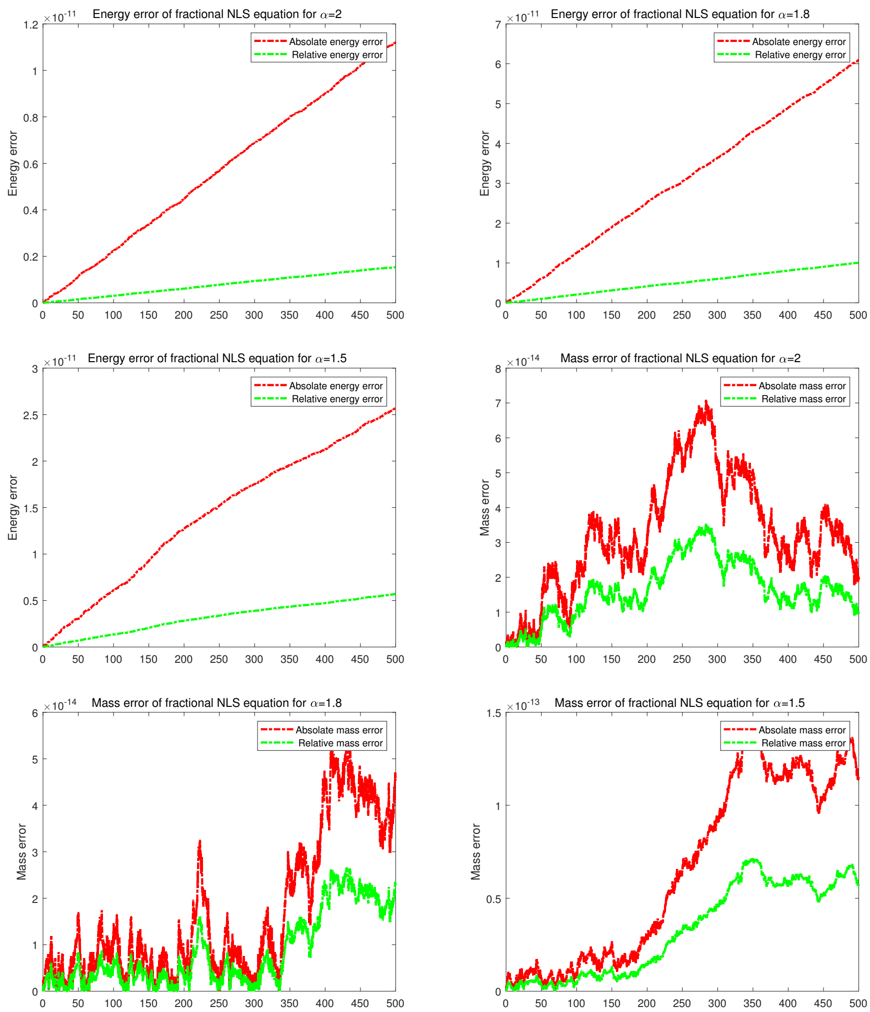

Figure 7. Energy and mass errors of the one-dimensional fractional equation in Eqs (4.5) and (4.6) with Relaxation scheme II.

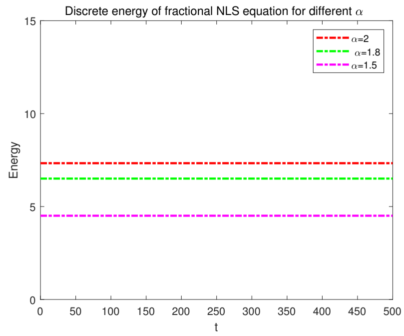

Figure 8. Discrete energy of the one-dimensional fractional NLS equation in Eqs (4.5) and (4.6) with Relaxation scheme III.

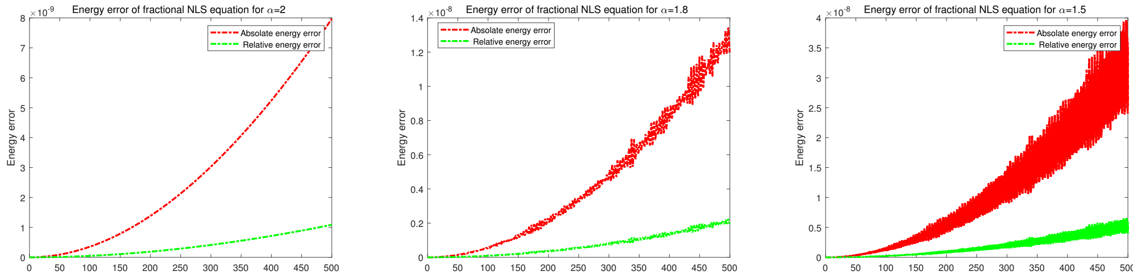

Figure 9. Energy errors of the one-dimensional fractional NLS equation in Eqs (4.5) and (4.6) with Relaxation scheme III.

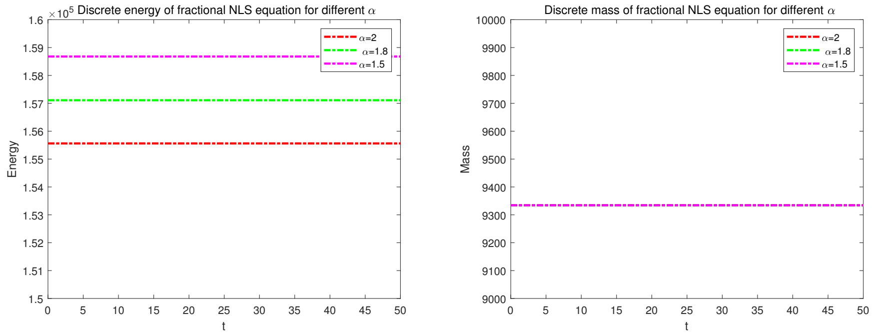

Figure 10. Discrete mass and energy of the two-dimensional fractional NLS equation in Eqs (4.7) and (4.8) with Relaxation scheme II.

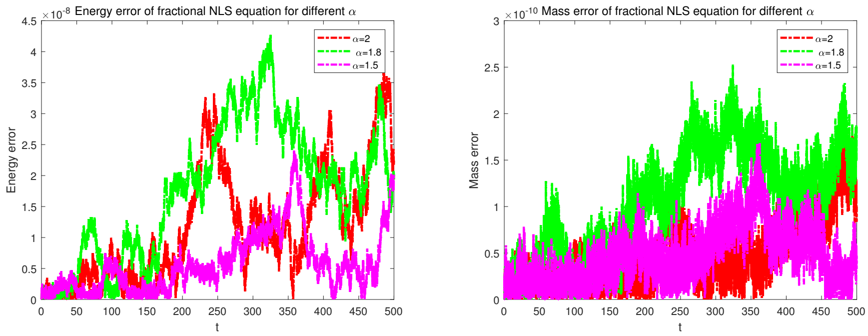

Figure 11. Mass and energy errors of the two-dimensional fractional NLS equation in Eqs (4.7) and (4.8) with Relaxation scheme II.

Table 1. CPU time for difference numerical schemes with $J = 128, \tau = 0.01$.

		Relaxation scheme I	Relaxation scheme II	Relaxation scheme III	Crank–Nicolson scheme
	$T = 10$	31.96	56.32	32.64	54.25
$\alpha = 1.4$	$T = 50$	151.32	261.87	153.65	259.35
	$T = 80$	245.32	485.24	256.24	478.36
	$T = 10$	33.96	60.45	34.25	56.38
$\alpha = 1.6$	$T = 50$	156.42	270.96	158.12	268.23
	$T = 80$	248.19	490.68	265.32	485.28
	$T = 10$	35.24	60.27	36.58	60.85
$\alpha = 1.8$	$T = 50$	160.85	267.95	160.87	278.64
	$T = 80$	251.83	490.35	265.54	490.65

4.2. *Example B*

Example 2. Consider the following fractional NLS equation:

$$iu_t - (-\Delta^*)^{\frac{\alpha}{2}} u + \mu |u|^{\lambda} u = 0, \tag{4.9}$$

and show the following relaxation scheme:

$$iU_t^n + D_{2,\alpha}^x U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} D_{2,\alpha}^y + \mu \Psi_1^{n+\frac{1}{2}} \widetilde{U}^{n+\frac{1}{2}} = 0, \tag{4.10}$$

$$G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) \Upsilon^{n+\frac{1}{2}} = \Psi_2^{n+\frac{1}{2}} (\widetilde{\Phi}^{n+\frac{1}{2}})^{\lambda-1}, \tag{4.11}$$

$$\Psi_{1t}^n = (\widetilde{\Phi}^{n+\frac{1}{2}})^{\lambda-1} \Phi_t^n, \tag{4.12}$$

$$\Psi_{2t}^n = 2Re(\widetilde{U}^{n+\frac{1}{2}} \overline{U}_t^n), \tag{4.13}$$

$$\Upsilon_t^n = G_1(\widetilde{\Phi}^{n+\frac{1}{2}}) \Phi_t^n, \tag{4.14}$$

where $G_1(\varphi) = \sqrt{\frac{2}{\lambda+1} \varphi^{\lambda+1} + C}$. In the numerical test, we select the parameter $\mu = 2$. For the one-dimensional and two-dimensional space fractional NLS equations, we consider the following initial conditions:

$$u(x, 0) = \text{sech}(x) \cdot \exp(2ix), x \in [-20, 20], \tag{4.15}$$

$$u(x, y, 0) = \exp(-x^2 - y^2), (x, y) \in [-3\pi, 3\pi] \times [-3\pi, 3\pi]. \tag{4.16}$$

In this example, we first simulate the solitary waves of the one-dimensional fractional NLS equation for different values of α and λ by the relaxation scheme in Eqs (4.10)–(4.14) with $t \in [0, 50]$, $\tau = 0.001$ and $J = 256$. The results are listed in Figures 12–15. Second, we simulate the solitary wave solution of the two-dimensional fractional NLS equation by the relaxation scheme with $(x, y) \in [-3\pi, 3\pi]^2$, $\tau = 0.001$ and $J = 256$. Figures 16–21 display the wave forms of the numerical solution for different time. It follows from Figures 12–21 that α, λ affect the propagation velocity of the solitary wave. For smaller values of α and larger values of λ, the propagation of the soliton gets slower, thus indicating the presence of the quantum sub-diffusion. The numerical results indicate that the maximum of the solution increases faster with smaller values of α, and the solution will eventually blow up.

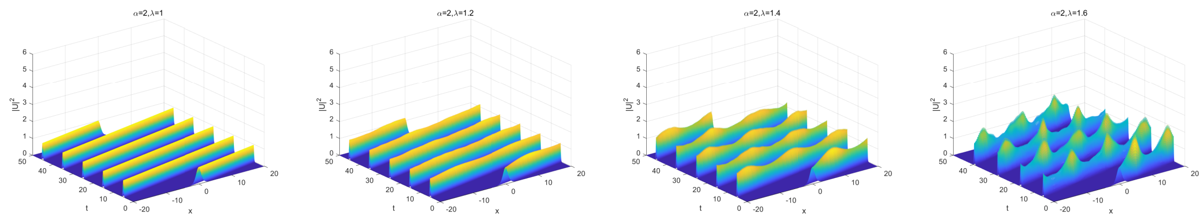

Figure 12. The wave forms of the numerical solution for the one-dimensional fractional NLS equation in Eq (4.9) with $\alpha = 2,\ \lambda = 1,\ 1.2,\ 1.4,\ 1.6$.

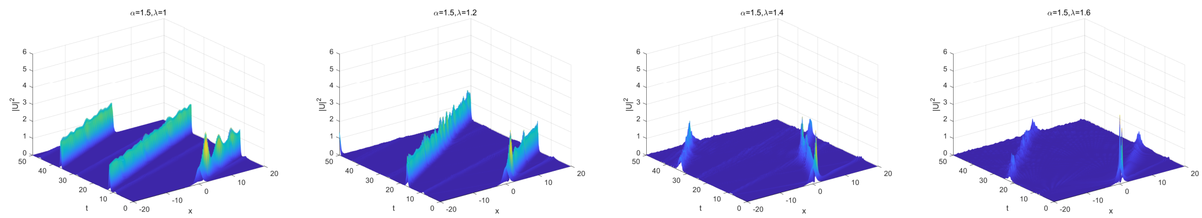

Figure 13. The wave forms of the numerical solution for the one-dimensional fractional NLS equation in Eq (4.9) with $\alpha = 1.5,\ \lambda = 1,\ 1.2,\ 1.4,\ 1.6$.

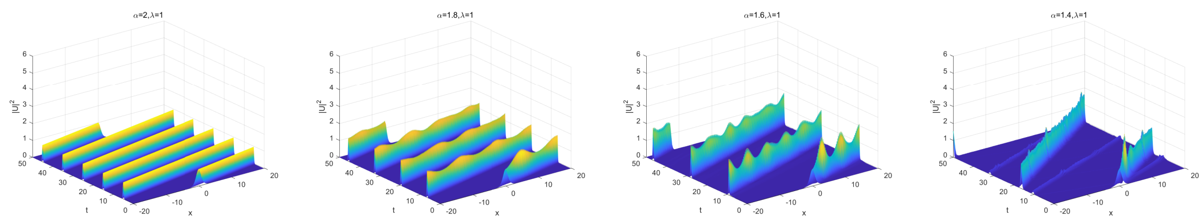

Figure 14. The wave forms of the numerical solution for the one-dimensional fractional NLS equation in Eq (4.9) with $\alpha = 2,\ 1.8,\ 1.6,\ 1.4, \lambda = 1$.

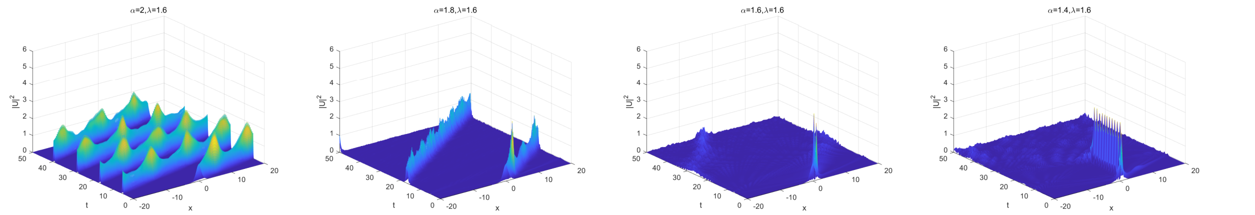

Figure 15. The wave forms of the numerical solution for the one-dimensional fractional NLS equation in Eq (4.9) $\alpha = 2,\ 1.8,\ 1.6,\ 1.4, \lambda = 1.6$.

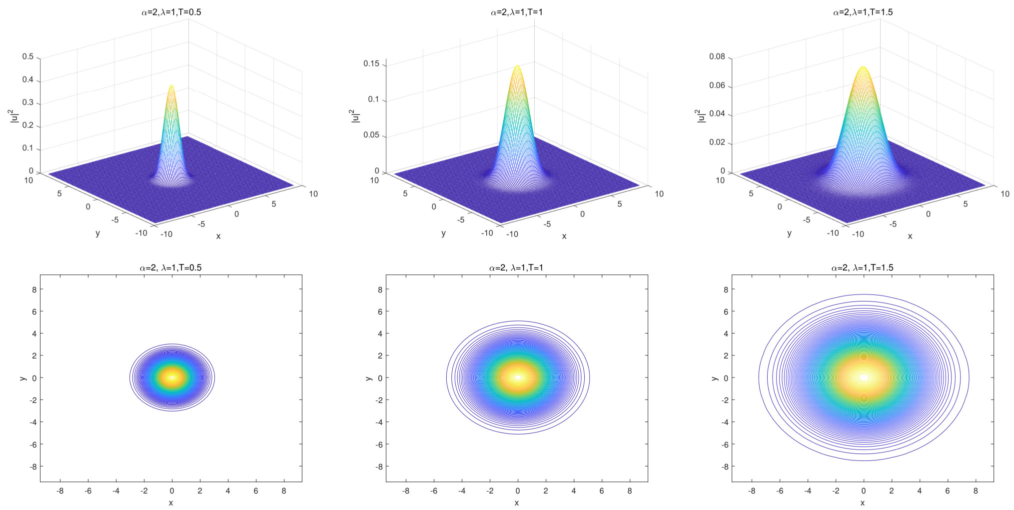

Figure 16. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 2, \lambda = 1, T = 0.5, 1, 1.5$.

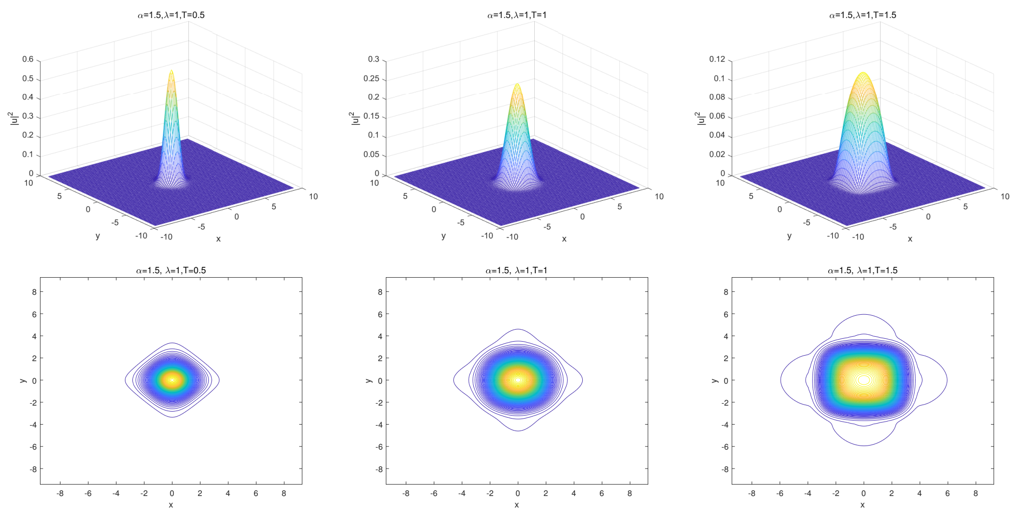

Figure 17. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 1.5, \lambda = 1, T = 0.5, 1, 1.5$.

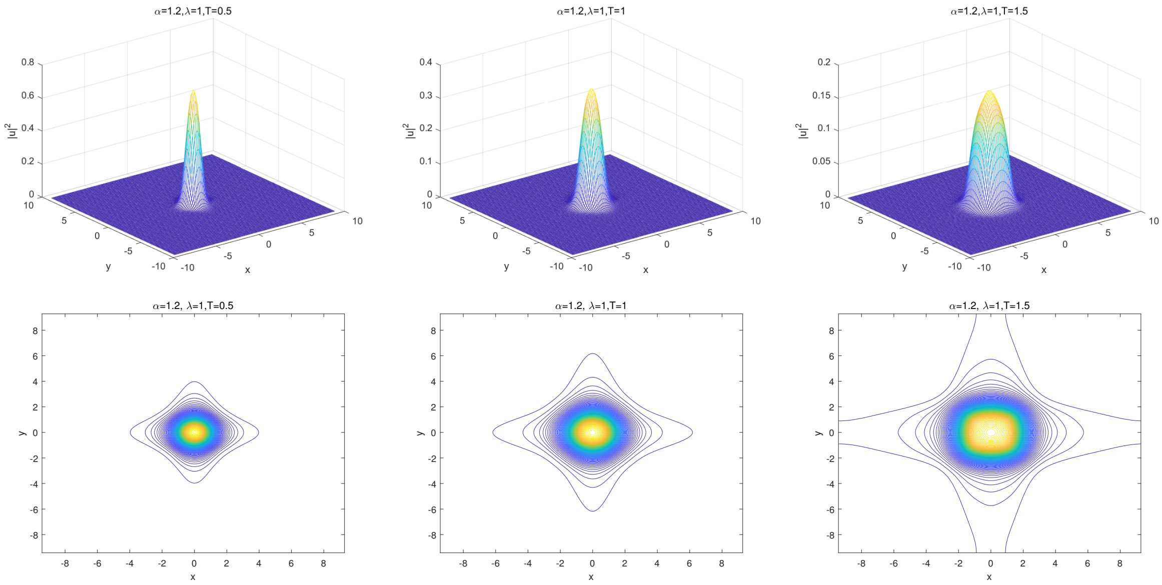

Figure 18. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 1.2, \lambda = 1, T = 0.5, 1, 1.5$.

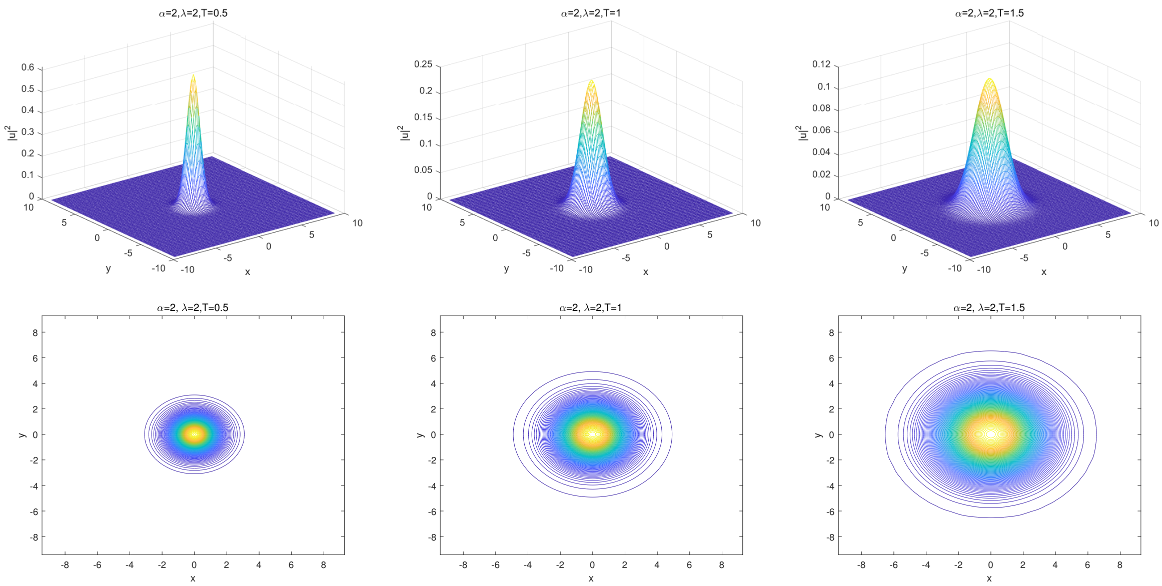

Figure 19. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 2, \lambda = 2, T = 0.5, 1, 1.5$.

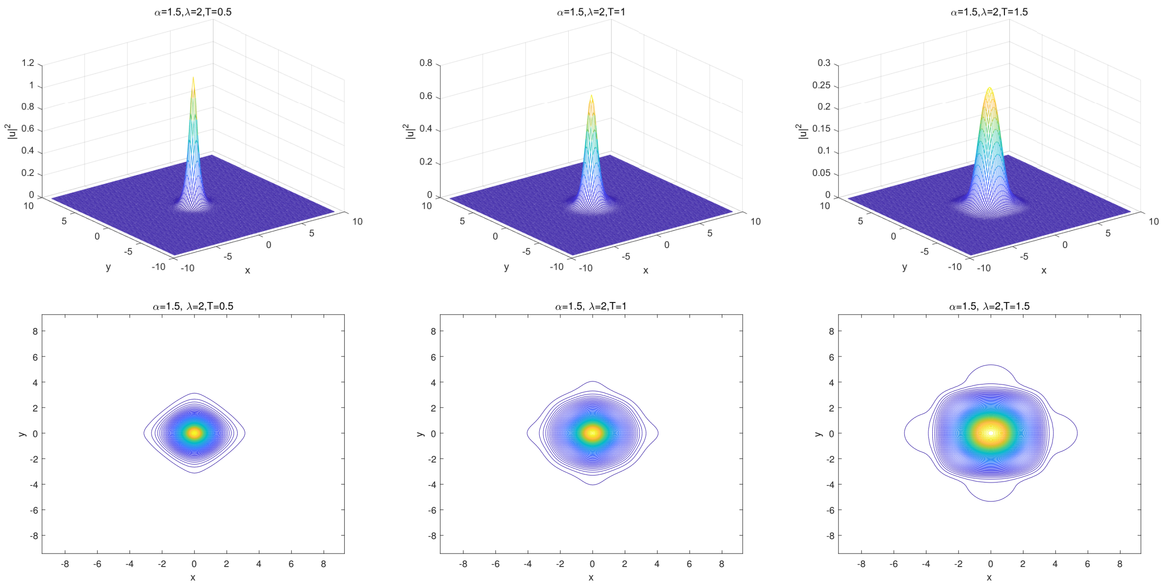

Figure 20. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 1.5, \lambda = 2, T = 0.5, 1, 1.5$.

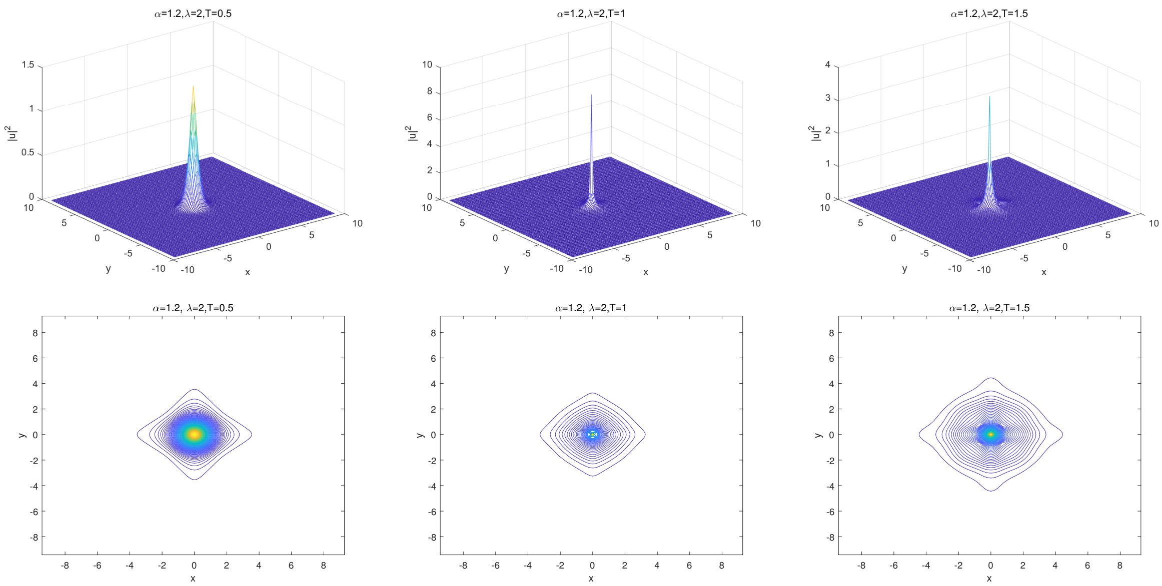

Figure 21. The wave forms of the numerical solution for the two-dimensional fractional NLS equation in Eqs (4.15) and (4.16) with $\alpha = 1.2, \lambda = 2, T = 0.5, 1, 1.5$.

5. Conclusions

In this paper, we show some energy-preserving relaxation-type schemes to solve the two-dimensional space fractional NLS equation with periodic boundary conditions. First, we change original system into an equivalent form by introducing some new variables, which transforms the energy conservation law of the original system into quadratic invariants. We then show the semi discrete schemes of the fractional NLS equation by the two relaxation schemes including an implicit relaxation scheme and a linear implicit relaxation scheme, which preserve the energy conservation laws of the modified fractional NLS system. Second, we show the spectral differentiation matrix of the one-dimensional and two-dimensional fractional operators , and give fully discrete scheme of the fractional NLS equation based on the above mentioned spectral differentiation matrix for the discrete space fractional derivative with periodic boundary conditions. Finally, numerical experiments of some fractional NLS equations are given to verify the correctness of the theoretical results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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