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Research article

A novel numerical method on a posteriori mesh for a singularly perturbed convection-diffusion problem with mixed type boundary conditions

Jian Huang[1], **Zhongdi Cen**[1] **and Xianbin Wu**[2,*]

[1] Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, P.R. China
[2] School of Data Science, Zhuhai College of Science and Technology, Zhuhai 519041, China

* **Correspondence:** Email: wxb3210@zcst.edu.cn.

Abstract: This article introduces a novel numerical method for solving a singularly-perturbed convection-diffusion problem with mixed type boundary conditions. The approach begins by transforming the second-order boundary value problem into a system of first-order differential equations. The stability result of its corresponding differential operator vector is obtained although the system of equations is in a special form. The transformed system is approximated through a hybrid difference method, followed by a posteriori error analysis that relies on the stability result of the differential operator vector. A posteriori mesh and approximation solution are derived by designing a solution-adaptive algorithm derived from the a posteriori error bound. Finally, the second-order uniform convergence of the novel numerical method is corroborated by numerical experiments.

Keywords: singularly perturbed; convection-diffusion; difference method; a posteriori error analysis; mesh equidistribution

1. Introduction

The present article deals with the following singularly perturbed convection-diffusion problem having mixed-type boundary conditions [1]:

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \omega = (0,1), \\ \varphi_0 u(0) - \varphi_1 u'(0) = \gamma_1, \quad u(1) = \gamma_2, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter, $a(x), b(x)$, and $f(x)$ are sufficiently smooth functions satisfying $a(x) \geq \alpha > 0$ and $\beta^* \geq b(x) \geq \beta > 0$ for $x \in \bar{\omega}$, $\varphi_0, \varphi_1, \gamma_1$, and γ_2 are constants, and $\varphi_1 \neq 0$. Under the given assumptions, the singularly-perturbed boundary value problem (1.1) possesses a unique solution $u(x) \in C^2(\omega) \cap C^1(\bar{\omega})$ that exhibits a weak boundary layer at $x = 0$ (see [1] for details). Singularly-perturbed problems featuring mixed-type boundary conditions were also studied in [2,3]. This class of equations is well-suited for modeling the diffusion and migration of chemicals, pollutants, and other substances in fluids (see, e.g. [4–6]).

Owing to the presence of boundary layers or inner layers in singular perturbation problems, classical numerical methods typically encounter difficulties in obtaining satisfactory numerical solutions. One effective method for solving the singular perturbation problems is to make use of the special meshes which are very fine in the boundary layers or internal layers [6, 7]. This type of mesh can be divided into two categories: a priori meshes and a posteriori meshes. The construction of a priori meshes have to use a priori information of the exact solutions [8]. However, in many cases, a priori information of the exact solutions is often unattainable or difficult to achieve. The construction of a posteriori meshes is achieved through a solution-adaptive algorithm that utilizes an a posteriori error bound, enabling automatic adaptation to the singularity of the exact solution in the absence of a priori information. Hence, the a posteriori mesh method has more extensive application (see, e.g. [9–14] and references therein). However, most a posteriori mesh methods for solving second-order singularly perturbed boundary value problems achieve only first-order convergence. Kopteva [15] put forward a four-point difference method on an a posteriori mesh for second-order singularly perturbed convection-diffusion equations, demonstrating its second-order accuracy. Linß and Kopteva [16] formulated a second-order defect-correction method for such equations on an a posteriori mesh. Linß and Radojev [17] enhanced a first-order upwind difference scheme to second-order accuracy through Richardson extrapolation on an a posteriori mesh. More recently, Cen et al. [18] introduced a three-point difference scheme of second-order accuracy on an a posteriori mesh for a singularly perturbed convection-diffusion equation.

In this article, we first transform the second-order boundary value problem (1.1) into a system of first-order differential equations. The stability result of its corresponding differential operator vector is derived out, although the system of equations is in a special form. The transformed system is approximated through a hybrid difference method, followed by the deduction of a posteriori error analysis for the discretization method on an arbitrary mesh by using the stability result of the differential operator vector. Subsequently, an a posteriori mesh and approximation solution are derived by designing a solution-adaptive algorithm derived from the a posteriori error bound. Finally, the second-order uniform convergence of the method is corroborated by numerical experiments.

Our method has two main advantages. First, our method can obtain the approximation solutions of the exact solution and its derivative simultaneously. Second, our method is to derive a posteriori error analysis of the second-order discretization method for the system of first-order equations, which makes a posteriori error analysis easier compared to the methods given in [15, 18], which directly perform a posteriori error analysis for the second-order discretization method of the original second-order differential equation (1.1).

This article is structured as follows. It begins with an exposition of some properties of the continuous problem in Section 2. This is followed by a description of the discretization method and the accompanying a posteriori error analysis in Section 3. The presentation concludes with numerical experiments in Section 4, which serve to validate the theoretical findings.

Notation. We introduce a generic positive constant C, which is independent of the parameter ε and the computational mesh. Note that C may assume distinct positive values at different instances. Furthermore, for a function g defined on $\bar{\omega}$, we use the notation g_i to stand for the function value $g(x_i)$.

2. The continuous problem

The second-order singularly-perturbed convection-diffusion problem with mixed-type boundary conditions, given in Eq (1.1), admits an equivalent reformulation as the following system of first-order equations:

$$\begin{cases} L_1\mathbf{u}(x) \equiv u_1'(x) - u_2(x) = 0, & x \in \Omega, \\ L_2\mathbf{u}(x) \equiv \varepsilon u_2'(x) + a(x)u_2(x) - b(x)u_1(x) = f(x), & x \in \Omega, \\ u_1(0) = \lambda, \quad u_2(0) = (\varphi_0\lambda - \gamma_1)/\varphi_1, \end{cases} \tag{2.1}$$

where $\mathbf{u}(x) = (u_1(x), u_2(x))^T, \Omega = (0, 1]$, and λ is a constant determined by the right boundary condition in (1.1) that will be obtained by using a shooting method based on the secant iterative method. For details on the equivalence and the shooting technique, we refer the reader to Osborne [19] and Keller [20]. In the following, system (2.1) instead of the second-order differential equation (1.1) will be discretized.

The system of Eq (2.1) is in a special form, which brings certain difficulties to the derivation of the stability result. First, we apply the method of proof by contradiction to prove the differential operator vector $\mathbf{L} = (L_1, L_2)^T$ admits a maximum principle.

Lemma 2.1 If $u_1(0) \geq 0$, $u_2(0) \geq 0$, $L_1\mathbf{u}(x) \geq 0$, and $L_2\mathbf{u}(x) \geq 0$ for $x \in \Omega$, then we have $\mathbf{u}(x) \geq 0$ for $x \in \bar{\Omega}$.

Proof. We prove this lemma by using a similar technique given in [21, 22]. Let $\mathbf{s}(x) = (s_1(x), s_2(x))^T$ be the vector function defined by

$$s_1(x) = \frac{2\varepsilon}{\beta^*} e^{-\beta^*(1-x)/\varepsilon}, \qquad s_2(x) = e^{-\beta^*(1-x)/\varepsilon}.$$

It is straightforward to observe that $L_1\mathbf{s}(x) > 0$ and $L_2\mathbf{s}(x) > 0$ for the small parameter ε.

Assume the lemma is false, and let

$$\xi = \max\left\{\max_{x\in\bar{\Omega}} (-u_1/s_1)(x), \max_{x\in\bar{\Omega}} (-u_2/s_2)(x)\right\}.$$

Then, we easily know $\xi > 0$ and

$$u_1(x) + \xi s_1(x) \geq 0, \qquad u_2(x) + \xi s_2(x) \geq 0.$$

Since the functions $u_1(x)+\xi s_1(x)$ and $u_2(x)+\xi s_2(x)$ are continuous on $\bar{\Omega}$, there exists a point $x^* \in \bar{\Omega}$ at which the function $u_1(x) + \xi s_1(x)$ or $u_2(x) + \xi s_2(x)$ reaches its minimum value 0. Also, since the functions $u_k(0) \geq 0$ and $s_k(x) > 0$ for $k = 1, 2$, x^* is not equal to 0.

Case I: $(u_1 + \xi s_1)(x^*) = 0$ for $x^* \in \Omega$. This means $(u_1 + \xi s_1)(x)$ reaches its minimum at x^*. Hence, we have

$$0 < L_1(\mathbf{u} + \xi\mathbf{s})(x^*) = (u_1 + \xi s_1)'(x^*) - (u_2 + \xi s_2)(x^*) \leq 0,$$

which leads to a contradiction.

Case II: $(u_2 + \xi s_2)(x^*) = 0$ for $x^* \in \Omega$. This means $(u_2 + \xi s_2)(x)$ reaches its minimum at x^*. Hence, we have the following result:

$$0 < L_2(\mathbf{u} + \xi \mathbf{s})(x^*) = \varepsilon (u_2 + \xi s_2)'(x^*) + a(x^*)(u_2 + \xi s_2)(x^*) - b(x^*)(u_1 + \xi s_1)(x^*) \le 0,$$

which also leads to a contradiction.

From the preceding analysis, it follows that $\mathbf{u}(x) \ge 0$ for all $x \in \bar{\Omega}$.

By using the above lemma with a barrier function vector $\mathbf{W}(x) = (W_1(x), W_2(x))^T$ defined as

$$W_1(x) = C\frac{\alpha}{\beta^*}e^{-\beta^*(1-x)/\alpha} \max\left\{|u_1(0)|, |u_2(0)|, \max_{x\in\Omega}|L_1\mathbf{u}(x)|, \max_{x\in\Omega}|L_2\mathbf{u}(x)|\right\},$$
$$W_2(x) = Ce^{-\beta^*(1-x)/\alpha} \max\left\{|u_1(0)|, |u_2(0)|, \max_{x\in\Omega}|L_1\mathbf{u}(x)|, \max_{x\in\Omega}|L_2\mathbf{u}(x)|\right\},$$

we obtain the following stability result concerning the differential operator vector.

Lemma 2.2 (Stability). A positive constant C exists such that

$$\|\mathbf{u}(x)\| \le C \max\left\{|u_1(0)|, |u_2(0)|, \max_{x\in\Omega}|L_1\mathbf{u}(x)|, \max_{x\in\Omega}|L_2\mathbf{u}(x)|\right\},$$

where $\|\mathbf{u}(x)\| = \max_{x\in\bar{\Omega}}\{|u_1(x)|, |u_2(x)|\}$.

3. Discretization scheme

In this section, we present a hybrid difference method on a posteriori mesh for system (2.1). First, we construct an arbitrary mesh $\Omega^N = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ with mesh sizes $h_i = x_i - x_{i-1}$ for $1 \le i \le N$.

In this section, we introduce a hybrid difference method on an a posteriori mesh for system (2.1), beginning with the construction of an arbitrary mesh with the corresponding mesh sizes given by $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. On this mesh Ω^N, we propose the following discretization scheme

$$\begin{cases} L_1^N\mathbf{U}_i = 0, \quad L_2^N\mathbf{U}_i = \tilde{f}_i, & i = 1, 2, \cdots, N, \\ U_{1,0} = \lambda, \ U_{2,0} = (\varphi_0\lambda - \gamma_1)/\varphi_1, \end{cases} \tag{3.1}$$

where $\mathbf{U} = (U_1, U_2)^T$ represents the approximate solution of the exact solution $\mathbf{u} = (u_1, u_2)^T$,

$$L_1^N\mathbf{U}_i = D^-U_{1,i} - \frac{U_{2,i-1} + U_{2,i}}{2}, \qquad i = 1, 2, \cdots, N, \tag{3.2}$$

$$L_2^N\mathbf{U}_i = \begin{cases} \varepsilon D^-U_{2,i} + \frac{1}{2}(a_{i-1}U_{2,i-1} + a_iU_{2,i}) - \frac{1}{2}(b_{i-1}U_{1,i-1} + b_iU_{1,i}), & \frac{\varepsilon}{h_i} \ge \frac{a_{i-1}}{2}, \\ \varepsilon D^-U_{2,i} + a_iU_{2,i} - b_iU_{1,i}, & \text{otherwise}, \end{cases} \tag{3.3}$$

$$D^-U_{1,i} = \frac{U_{1,i} - U_{1,i-1}}{h_i}, \quad \tilde{f}_i = \begin{cases} \frac{1}{2}(f_{i-1} + f_i), & \frac{\varepsilon}{h_i} \ge \frac{a_{i-1}}{2}, \\ f_i, & \text{otherwise}. \end{cases} \tag{3.4}$$

The system of difference equations (3.1) is also in a special form. By applying the method of contradiction as that in Lemma 2.1, we can prove that the difference operator vector $\mathbf{L}^N = \left(L_1^N, L_2^N\right)^T$ admits a discrete maximum principle.

Lemma 3.1 (Discrete maximum principle). If $U_{1,0} \ge 0$, $U_{2,0} \ge 0$, $L_1^N\mathbf{U}_i \ge 0$, and $L_2^N\mathbf{U}_i \ge 0$, then $\mathbf{U}_i \ge 0$ for all $0 \le i \le N$.

Proof. Let $\mathbf{s}_i = (s_{1,i}, s_{2,i})^T > 0$ be the vector function defined by

$$s_{1,i} = \frac{2\varepsilon}{\beta^*} e^{-\beta^*(1-x_i)/\varepsilon}, \quad s_{2,i} = e^{-\beta^*(1-x_i)/\varepsilon}, \qquad 0 \le i \le N.$$

It is straightforward to observe that $L_1^N \mathbf{s}_i > 0$ and $L_2^N \mathbf{s}_i > 0$ for $1 \le i \le N$.

Assume the lemma is false, and define

$$\eta = \max\left\{ \max_{0\le i\le N} \left(-U_{1,i}/s_{1,i}\right), \max_{0\le i\le N} \left(-U_{2,i}/s_{2,i}\right) \right\}.$$

Then, we easily know $\eta > 0$ and

$$U_{1,i} + \eta s_{1,i} \ge 0, \quad U_{2,i} + \eta s_{2,i} \ge 0, \qquad 0 \le i \le N.$$

It is immediate from the definition of η that there exists an i^* such that

$$U_{1,i^*} + \eta s_{1,i^*} = 0, \qquad 0 \le i^* \le N,$$

or

$$U_{2,i^*} + \eta s_{2,i^*} = 0, \qquad 0 \le i^* \le N.$$

Since $U_{k,0} \ge 0$ and $s_{k,0} > 0$ for $k = 1, 2$, i^* is not equal to 0.

Case I: $U_{1,i^*} + \eta s_{1,i^*} = 0$ for $1 \le i^* \le N$. This means $(U_{1,i} + \eta s_{1,i})$ reaches its minimum at i^*. Hence, we have the following result

$$0 < L_1^N \left(\mathbf{U}_{i^*} + \eta \mathbf{s}_{i^*}\right) = D^- \left(U_{1,i^*} + \eta s_{1,i^*}\right) - \frac{\left(U_{2,i^*-1} + \eta s_{2,i^*-1}\right) + \left(U_{2,i^*} + \eta s_{2,i^*}\right)}{2} \le 0,$$

which leads to a contradiction.

Case II: $U_{2,i^*} + \eta s_{2,i^*} = 0$ for $1 \le i^* \le N$. This means $(U_{2,i} + \eta s_{2,i})$ reaches its minimum at i^*. If $\frac{\varepsilon}{h_{i^*}} \ge \frac{a_{i^*-1}}{2}$, then we have

$$\begin{aligned} 0 < L_2^N \left(\mathbf{U}_{i^*} + \eta \mathbf{s}_{i^*}\right) &= \varepsilon D^- \left(U_{2,i^*} + \eta s_{2,i^*}\right) + \frac{1}{2}\left[a_{i^*-1}\left(U_{2,i^*-1} + \eta s_{2,i^*-1}\right) + a_{i^*}\left(U_{2,i^*} + \eta s_{2,i^*}\right)\right] \\ &\quad -\frac{1}{2}\left[b_{i^*-1}\left(U_{1,i^*-1} + \eta s_{1,i^*-1}\right) + b_{i^*}\left(U_{1,i^*} + \eta s_{1,i^*}\right)\right] \\ &= \left(\frac{\varepsilon}{h_{i^*}} - \frac{a_{i^*-1}}{2}\right)\left(U_{2,i^*-1} + \eta s_{2,i^*-1}\right) - \frac{1}{2}\left[b_{i^*-1}\left(U_{1,i^*-1} + \eta s_{1,i^*-1}\right)\right. \\ &\quad \left. + b_{i^*}\left(U_{1,i^*} + \eta s_{1,i^*}\right)\right] \le 0, \end{aligned}$$

which is a contradiction. If $\frac{\varepsilon}{h_{i^*}} < \frac{a_{i^*-1}}{2}$, then we have

$$0 < L_2^N \left(\mathbf{U}_{i^*} + \eta \mathbf{s}_{i^*}\right) = \varepsilon D^- \left(U_{2,i^*} + \eta s_{2,i^*}\right) + a_{i^*}\left(U_{2,i^*} + \eta s_{2,i^*}\right) - b_{i^*}\left(U_{1,i^*} + \eta s_{1,i^*}\right) \le 0,$$

which also leads to a contradiction.

From the preceding analysis, it follows that $\mathbf{U}_i \ge 0$ for all $0 \le i \le N$.

Following the approach employed in Lemma 2.2, we establish the following stability result of the discretization scheme (3.1) by applying the above discrete maximum principle.

Lemma 3.2. A positive constant C exists such that

$$\|\mathbf{U}\|_{\Omega^N} \le C \max\left\{\left|U_{1,0}\right|, \left|U_{2,0}\right|, \max_{1\le i\le N}\left|L_1^N \mathbf{U}_i\right|, \max_{1\le i\le N}\left|L_2^N \mathbf{U}_i\right|\right\},$$

where $\|\mathbf{U}\|_{\Omega^N} = \max_{0\le i\le N}\left\{\left|U_{1,i}\right|, \left|U_{2,i}\right|\right\}$.

We now proceed to perform an a posteriori error analysis pertaining to the discretization scheme (3.1), conducted on an arbitrary mesh Ω^N. As that in [12, 18, 23], we define a piecewise quadratic interpolation function vector $\bar{\mathbf{U}}(x) = \left(\bar{U}_1(x), \bar{U}_2(x)\right)^T$ as

$$\begin{cases} \bar{U}_1(x) = \frac{1}{2}(x - x_{i-1})(x - x_i) D^- D^- U_{1,i} + (x - x_i) D^- U_{1,i} + U_{1,i}, \\ \bar{U}_2(x) = \frac{1}{2}(x - x_{i-1})(x - x_i) D^- D^- U_{2,i} + (x - x_i) D^- U_{2,i} + U_{2,i} \end{cases} \tag{3.5}$$

for $x \in [x_{i-1}, x_i]$ and $i = 1, \ldots, N$, with

$$D^- U_{k,0} = 0, \qquad D^- D^- U_{k,i} = \frac{D^- U_{k,i} - D^- U_{k,i-1}}{h_i}, \qquad k = 1, 2.$$

We then derive the following results without difficulty:

$$\bar{U}_1(x_i) = U_{1,i}, \qquad \bar{U}_2(x_i) = U_{2,i}, \tag{3.6}$$

for $x \in [x_{i-1}, x_i]$ and

$$\left[\bar{U}_1(x)\right]' = (x - x_{i-1/2}) D^- D^- U_{1,i} + D^- U_{1,i}, \quad \left[\bar{U}_2(x)\right]' = (x - x_{i-1/2}) D^- D^- U_{2,i} + D^- U_{2,i}, \tag{3.7}$$

for $x \in [x_{i-1}, x_i]$ and $1 \le i \le N$, with $x_{i-1/2} = (x_{i-1} + x_i)/2$. We also define

$$p(x) = a(x)\bar{U}_2(x) - b(x)\bar{U}_1(x), \quad p^I(x) = p_i + (x - x_i) D^- p_i, \tag{3.8}$$

$$f^I(x) = f_i + (x - x_i) D^- f_i, \qquad x \in [x_{i-1}, x_i] \tag{3.9}$$

for $1 \le i \le N$.

Theorem 3.3 Denote by $\mathbf{u}(x)$, $\mathbf{U}$, and $\bar{\mathbf{U}}(x)$ the exact solution of Eq (2.1), the approximate solution of Eq (3.1) on Ω^N, and its piecewise quadratic interpolation defined in Eq (3.5), respectively. Then, the bound

$$\begin{aligned} \left\|\bar{\mathbf{U}}(x) - \mathbf{u}(x)\right\|_{\bar{\Omega}} \le\; & C \max_{1\le i\le N} h_i^2 \Big[1 + \left|D^- U_{1,i}\right| + \left|D^- U_{2,i}\right| + \left|D^- D^- U_{1,i}\right| + \left|D^- D^- U_{2,i}\right| \\ & + \left|D^- D^- (aU_2)_i\right| + \left|D^- D^- (bU_1)_i\right| + \left|D^- D^- f_i\right|\Big] \end{aligned}$$

holds, where $\left\|\bar{\mathbf{U}}(x) - \mathbf{u}(x)\right\|_{\bar{\Omega}} = \max_{x\in\bar{\Omega}}\{|U_1(x) - u_1(x)|, |U_2(x) - u_2(x)|\}$.

Proof. Given $x \in (x_{i-1}, x_i)$, the following deduction can be derived:

$$\begin{aligned} \left|L_1\bar{\mathbf{U}}(x) - L_1\mathbf{u}(x)\right| &= \left|\left[\bar{U}_1(x)\right]' - \bar{U}_2(x)\right| \\ &= \Big|(x - x_{i-1/2}) D^- D^- U_{1,i} + D^- U_{1,i} \end{aligned}$$

$$
\begin{aligned}
& -\left[\frac{1}{2}(x-x_{i-1})(x-x_i)D^-D^-U_{2,i}+(x-x_i)D^-U_{2,i}+U_{2,i}\right]\Big| \\
= \ & \Big|(x-x_{i-1/2})D^-D^-U_{1,i}-\frac{U_{2,i}-U_{2,i-1}}{2} \\
& -\left[\frac{1}{2}(x-x_{i-1})(x-x_i)D^-D^-U_{2,i}+(x-x_i)D^-U_{2,i}\right]\Big| \\
= \ & \left|-\frac{1}{2}(x-x_{i-1})(x-x_i)D^-D^-U_{2,i}+(x-x_{i-1/2})D^-\left(D^-U_{1,i}-U_{2,i}\right)\right| \\
= \ & \left|-\frac{1}{2}(x-x_{i-1})(x-x_i)D^-D^-U_{2,i}-\frac{1}{2}(x-x_{i-1/2})h_iD^-D^-U_{2,i}\right| \\
\leq \ & Ch_i^2\left|D^-D^-U_{2,i}\right|,
\end{aligned}
\tag{3.10}
$$

where we have used the first equation in Eq (3.1) and Eqs (3.5)–(3.7).

For the case where $x\in(x_{i-1},x_i)$ and $\frac{\varepsilon}{h_i}>\frac{a_{i-1}}{2}$, the following deduction can be derived:

$$
\begin{aligned}
\left|L_2\bar{\mathbf{U}}(x)-L_2\mathbf{u}(x)\right| = \ & \left|\varepsilon\bar{U}_2'(x)+p(x)-f(x)\right| \\
= \ & \left|\varepsilon\left[(x-x_{i-1/2})D^-D^-U_{2,i}+D^-U_{2,i}\right]+p(x)-f(x)\right| \\
= \ & \Big|\varepsilon\left[(x-x_{i-1/2})D^-D^-U_{2,i}+D^-U_{2,i}\right]+p^I(x)-f^I(x) \\
& +\left(p(x)-p^I(x)\right)-\left(f(x)-f^I(x)\right)\Big| \\
= \ & \Big|\varepsilon\left[(x-x_{i-1/2})D^-D^-U_{2,i}+D^-U_{2,i}\right]+a_iU_{2,i}-b_iU_{1,i}-f_i \\
& +(x-x_i)D^-(aU_2)_i-(x-x_i)D^-(bU_1)_i-(x-x_i)D^-f_i \\
& +\left(p(x)-p^I(x)\right)-\left(f(x)-f^I(x)\right)\Big| \\
= \ & \Big|\varepsilon(x-x_{i-1/2})D^-D^-U_{2,i}+\frac{h_i}{2}\left[D^-(aU_2)_i-D^-(bU_1)_i-D^-f_i\right] \\
& +(x-x_i)\left[D^-(aU_2)_i-D^-(bU_1)_i-D^-f_i\right] \\
& +\left(p(x)-p^I(x)\right)-\left(f(x)-f^I(x)\right)\Big| \\
= \ & \Big|(x-x_{i-1/2})D^-\left(\varepsilon D^-U_{2,i}+a_iU_{2,i}-b_iU_{1,i}-f_i\right) \\
& +\left(p(x)-p^I(x)\right)-\left(f(x)-f^I(x)\right)\Big| \\
= \ & \Big|\frac{1}{2}(x-x_{i-1/2})h_i\left[D^-D^-(aU_2)_i-D^-D^-(bU_1)_i-D^-D^-f_i\right] \\
& +\left(p(x)-p^I(x)\right)-\left(f(x)-f^I(x)\right)\Big| \\
\leq \ & Ch_i^2\left[\left|D^-D^-(aU_2)_i\right|+\left|D^-D^-(bU_1)_i\right|+\left|D^-D^-f_i\right|\right] \\
& +\left|p(x)-p^I(x)\right|+\left|f(x)-f^I(x)\right| \\
\leq \ & Ch_i^2\Big[1+\left|D^-U_{1,i}\right|+\left|D^-U_{2,i}\right|+\left|D^-D^-U_{1,i}\right|+\left|D^-D^-U_{2,i}\right| \\
& +\left|D^-D^-(aU_2)_i\right|+\left|D^-D^-(bU_1)_i\right|+\left|D^-D^-f_i\right|\Big],
\end{aligned}
\tag{3.11}
$$

where we have used the second equation in Eqs (3.1), (3.5)–(3.9), and the following linear interpolation remainder formula:

$$\left\|g(x) - g^I(x)\right\|_{[x_{i-1},x_i]} \;\le\; \frac{1}{2}h_i^2 \sup_{x\in(x_{i-1},x_i)} |g''(x)|\,. \tag{3.12}$$

For the case where $x \in (x_{i-1}, x_i)$ and $\frac{\varepsilon}{h_i} \le \frac{a_{i-1}}{2}$, the following deduction also can be derived:

$$\begin{aligned}
\left|L_2\bar{\mathbf{U}}(x) - L_2\mathbf{u}(x)\right| &= \left|\varepsilon \bar{U}_2'(x) + p(x) - f(x)\right| \\
&= \left|\varepsilon\left[(x - x_{i-1/2})\, D^-D^-U_{2,i} + D^-U_{2,i}\right] + p(x) - f(x)\right| \\
&= \Big|\varepsilon\left[(x - x_{i-1/2})\, D^-D^-U_{2,i} + D^-U_{2,i}\right] + p^I(x) - f^I(x) \\
&\quad + \left(p(x) - p^I(x)\right) - \left(f(x) - f^I(x)\right)\Big| \\
&= \Big|\varepsilon\left[(x - x_{i-1/2})\, D^-D^-U_{2,i} + D^-U_{2,i}\right] + a_iU_{2,i} - b_iU_{1,i} - f_i \\
&\quad + (x - x_i)\, D^-\left(aU_2\right)_i - (x - x_i)\, D^-\left(bU_1\right)_i - (x - x_i)\, D^-f_i \\
&\quad + \left(p(x) - p^I(x)\right) - \left(f(x) - f^I(x)\right)\Big| \\
&= \Big|\varepsilon\,(x - x_{i-1/2})\, D^-D^-U_{2,i} + (x - x_i)\, D^-\left(aU_2 - bU_1 - f\right)_i \\
&\quad + \left(p(x) - p^I(x)\right) - \left(f(x) - f^I(x)\right)\Big| \\
&= \Big|\varepsilon\,(x - x_{i-1/2})\, D^-D^-U_{2,i} + \varepsilon\,(x - x_i)\, D^-D^-U_{2,i} \\
&\quad + \left(p(x) - p^I(x)\right) - \left(f(x) - f^I(x)\right)\Big| \\
&\le C h_i^2 \Big(1 + \left|D^-U_{1,i}\right| + \left|D^-U_{2,i}\right| + \left|D^-D^-U_{1,i}\right| \\
&\quad + \left|D^-D^-U_{2,i}\right| + \left|D^-D^-f_i\right|\Big),
\end{aligned} \tag{3.13}$$

where we also have used the second equation in Eqs (3.1), (3.5)–(3.9), and (3.12).

Therefore, by combining Eqs (3.10) and (3.11) with Eq (3.13), the desired result follows from Lemma 3.2.

Theorem 3.3 implies the following corollary.

Corollary 3.4 Suppose that the condition

$$\begin{aligned}
&h_i^2\Big[1 + \left|D^-U_{1,i}\right| + \left|D^-U_{2,i}\right| + \left|D^-D^-U_{1,i}\right| + \left|D^-D^-U_{2,i}\right| \\
&\quad + \left|D^-D^-\left(aU_2\right)_i\right| + \left|D^-D^-\left(bU_1\right)_i\right| + \left|D^-D^-f_i\right|\Big] \le CN^{-2}
\end{aligned} \tag{3.14}$$

for $1 \le i \le N$ is met. Then, the bound

$$\left\|\bar{\mathbf{U}}(x) - \mathbf{u}(x)\right\|_{\bar{\Omega}} \le CN^{-2}$$

can be established, where C is a sufficiently large constant independent of ε and N.

By implementing the adapted de-Boor algorithm [11, 23–26], the a posteriori adapted mesh can be generated through the solution of the equidistribution problem

$$M_ih_i = \frac{1}{N}\sum_{j=1}^{N} M_jh_j, \qquad i = 1, \ldots, N,$$

where the discrete monitoring function M_i is specified as follows:

$$\begin{aligned} M_i &= 1 + \left|D^- U_{1,i}\right|^{1/2} + \left|D^- U_{2,i}\right|^{1/2} + \left|D^- D^- U_{1,i}\right|^{1/2} + \left|D^- D^- U_{2,i}\right|^{1/2} \\ &\quad + \left|D^- D^- (aU_2)_i\right|^{1/2} + \left|D^- D^- (bU_1)_i\right|^{1/2} + \left|D^- D^- f_i\right|^{1/2}. \end{aligned}$$

This means that if $\max_{1\le i\le N} h_i^2 M_i^2 \le CN^{-2}$ is satisfied, then the condition (3.14) of Corollary 3.4 can also be satisfied. Therefore, when the termination condition (3.15) is met, it can be inferred from Corollary 3.4 that the error of the approximation solution on an a posteriori adapted mesh is second-order and ε-uniformly efficient.

Algorithm:

Step 1. Initialization mesh: The uniform mesh $\Omega^{N,(0)} = \left\{x_i^{(0)} \mid x_i^{(0)} = i/N, 0 \le i \le N\right\}$ serves as an initial iteration mesh for $k = 0$.

Step 2. Numerical solution: The solution $\left\{\mathbf{U}_i^{(k)}\right\}_{i=0}^N$ is computed by addressing the discretization problem (3.1) on the kth iterative mesh $\Omega^{N,(k)} = \left\{x_i^{(k)} \mid 0 \le i \le N\right\}$, where the mesh size is defined as $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$. Meanwhile, the notations $Q_i^{(k)} = h_i^{(k)} M_i^{(k)}$ and $I_i^{(k)} = \sum_{j=1}^{i} Q_j^{(k)}$ are introduced, where $M_i^{(k)}$ is the discrete monitor function's value at $x_i^{(k)}$ and $M_0^{(k)} = M_1^{(k)}$.

Step 3. Test mesh: If the user-defined constant $\bar{C} \ge 1$ meets the termination condition

$$\max_{1\le i\le N} \left\{Q_i^{(k)}\right\} \le \bar{C} I_N^{(k)}/N, \tag{3.15}$$

proceed to Step 5; otherwise, continue to Step 4.

Step 4. New mesh: Denote $Z_i^{(k)} = iI_N^{(k)}/N$; a new adapted mesh $\Omega^{N,(k+1)} = \left\{x_i^{(k+1)} | 0 \le i \le N\right\}$ is obtained by linearly interpolating $\left(Z_i^{(k)}, x_i^{(k+1)}\right)$ to $\left(I_i^{(k)}, x_i^{(k)}\right)$; proceed to Step 2 by incrementing k to $k+1$.

Step 5. Final adapted mesh and approximation solution: $\Omega^{N,*} = \Omega^{N,(k)}$ is designated as the final a posteriori mesh, and $\left\{\mathbf{U}_i^*\right\}_{i=0}^N = \left\{\mathbf{U}_i^{(k)}\right\}_{i=0}^N$ is designated as the final approximation solution.

4. Numerical experiments

This section presents numerical experiments designed to verify the accuracy and efficiency of the proposed method, with error estimates and convergence rates illustrated through the following two examples.

Example 4.1. We examine the following second-order singularly perturbed convection-diffusion problem with mixed type boundary conditions:

$$\begin{cases} \varepsilon u''(x) + u'(x) - u(x) = f(x), & x \in (0,1), \\ u(0) - u'(0) = 5 - 2\varepsilon\left(1 + e^{-1/\varepsilon}\right), \quad u(1) = 0, & \end{cases}$$

where the selection of $f(x)$ should ensure that the exact solution satisfies $u(x) = 2(1-\varepsilon) - x(1 + x - 2\varepsilon) + 2\varepsilon\left(e^{-x/\varepsilon} - e^{-1/\varepsilon}\right)$.

Example 4.2. We examine the following second-order singularly perturbed convection-diffusion problem with mixed type boundary conditions:

$$\begin{cases} \varepsilon u''(x) + (1+x)\,u'(x) - (2 - \sin x)\,u(x) = 1 + 2\sin x + x(1-x) + e^{-x/\varepsilon}, & x \in (0,1), \\ 2u(0) - u'(0) = 2(1+\varepsilon), \quad u(1) = 1 + \varepsilon e^{-1/\varepsilon}. & \end{cases}$$

Table 1. Numerical results of Eq (3.1) on a posteriori and Shishkin meshes for Example 4.1.

ε			The mesh discretization parameter N					
			64	128	256	512	1024	2048
2^0	a posteriori	e^N	2.1697e-5	5.3485e-6	1.3306e-6	3.3105e-7	8.2567e-8	2.0618e-8
		r^N	2.020	2.007	2.007	2.003	2.002	-
		C^N	0.089	0.088	0.087	0.087	0.087	0.087
		K^N	2	3	3	3	3	4
	Shishkin	e^N	1.9333e-5	4.8331e-6	1.2083e-6	3.0207e-7	7.5517e-8	1.8879e-8
		r^N	2.000	2.000	2.000	2.000	2.000	-
2^{-4}	a posteriori	e^N	4.6447e-4	1.1681e-4	2.9209e-5	7.2864e-6	1.8201e-6	4.5500e-7
		r^N	1.991	2.000	2.003	2.001	2.000	-
		C^N	1.903	1.914	1.914	1.910	1.909	1.908
		K^N	3	3	4	4	4	4
	Shishkin	e^N	3.7720e-3	9.3786e-4	2.3415e-4	5.8547e-5	1.4636e-5	3.6588e-6
		r^N	2.008	2.002	2.000	2.000	2.000	-
2^{-8}	a posteriori	e^N	7.7863e-4	1.9437e-4	5.0072e-5	1.2774e-5	3.2218e-6	8.0900e-7
		r^N	2.002	1.957	1.971	1.987	1.994	-
		C^N	3.189	3.185	3.282	3.349	3.378	3.393
		K^N	4	4	4	4	4	4
	Shishkin	e^N	3.8602e-3	1.4959e-3	5.3807e-4	1.3448e-4	4.5755e-5	1.4941e-5
		r^N	1.368	1.475	2.000	1.555	1.615	-
2^{-12}	a posteriori	e^N	8.9863e-4	2.2462e-4	5.6362e-5	1.4150e-5	3.5495e-6	8.8978e-7
		r^N	2.000	1.995	1.994	1.995	1.996	-
		C^N	3.681	3.680	3.694	3.709	3.722	3.732
		K^N	5	5	5	5	5	5
	Shishkin	e^N	3.8677e-3	1.4984e-3	5.3912e-4	1.3475e-4	4.5839e-5	1.4968e-5
		r^N	1.368	1.475	2.000	1.556	1.615	-
2^{-16}	a posteriori	e^N	9.5252e-4	2.3340e-4	5.8478e-5	1.4635e-5	3.6639e-6	9.1920e-7
		r^N	2.029	1.997	1.998	1.998	1.995	-
		C^N	3.902	3.824	3.832	3.837	3.842	3.855
		K^N	8	6	8	6	5	5
	Shishkin	e^N	3.8681e-3	1.4986e-3	5.3918e-4	1.3476e-4	4.5845e-5	1.4969e-5
		r^N	1.368	1.475	2.000	1.556	1.615	-
2^{-20}	a posteriori	e^N	9.6008e-4	2.3837e-4	5.9503e-5	1.4756e-5	3.6968e-6	9.2603e-7
		r^N	2.010	2.002	2.012	1.997	1.997	-
		C^N	3.933	3.905	3.900	3.868	3.876	3.884
		K^N	12	12	7	6	7	9
	Shishkin	e^N	3.8682e-3	1.4986e-3	5.3918e-4	1.3476e-4	4.5845e-5	1.4970e-5
		r^N	1.368	1.475	2.000	1.556	1.615	-
2^{-24}	a posteriori	e^N	9.6552e-4	2.3978e-4	5.9293e-5	1.4809e-5	3.7120e-6	9.2800e-7
		r^N	2.010	2.016	2.001	1.996	2.000	-
		C^N	3.955	3.929	3.886	3.882	3.892	3.892
		K^N	33	42	36	22	7	8
	Shishkin	e^N	3.8682e-3	1.4986e-3	5.3918e-4	1.3476e-4	4.5845e-5	1.4970e-5
		r^N	1.368	1.475	2.000	1.556	1.615	-

Table 2. Numerical results of Eq (3.1) on a posteriori and Shishkin meshes for Example 4.2.

ε			The mesh discretization parameter N					
			64	128	256	512	1024	2048
2^{0}	a posteriori	e^N	5.9141e-5	1.4560e-5	3.6048e-6	8.9316e-7	2.2283e-7	5.5646e-8
		r^N	2.022	2.014	2.013	2.003	2.002	-
		C^N	0.242	0.239	0.236	0.234	0.234	0.233
		K^N	3	3	3	4	4	4
	Shishkin	e^N	6.0185e-5	1.5047e-5	3.7617e-6	9.4042e-7	2.3510e-7	5.8776e-8
		r^N	2.000	2.000	2.000	2.000	2.000	-
2^{-4}	a posteriori	e^N	6.2304e-4	1.5168e-4	3.7671e-5	9.3843e-6	2.3417e-6	5.8482e-7
		r^N	2.038	2.010	2.005	2.003	2.001	-
		C^N	2.552	2.485	2.469	2.460	2.455	2.453
		K^N	3	4	4	4	4	4
	Shishkin	e^N	7.6326e-3	1.9086e-3	4.7618e-4	1.1899e-4	2.9748e-5	7.4367e-6
		r^N	2.000	2.003	2.001	2.000	2.000	-
2^{-8}	a posteriori	e^N	1.7419e-3	4.2604e-4	1.1583e-4	2.5751e-5	6.2402e-6	1.6096e-6
		r^N	2.032	1.879	2.169	2.045	1.955	-
		C^N	7.135	6.980	7.591	6.750	6.543	6.751
		K^N	4	4	4	4	4	4
	Shishkin	e^N	8.2048e-3	3.1819e-3	1.1617e-3	2.8065e-4	9.5187e-5	3.1015e-5
		r^N	1.367	1.454	2.049	1.560	1.618	-
2^{-12}	a posteriori	e^N	2.6240e-3	4.1211e-4	1.0360e-4	2.8719e-5	8.1251e-6	2.1572e-6
		r^N	2.671	1.992	1.851	1.822	1.913	-
		C^N	10.748	6.752	6.790	7.529	8.520	9.048
		K^N	5	5	5	5	5	5
	Shishkin	e^N	8.1951e-3	3.1695e-3	1.1328e-3	2.8322e-4	9.6163e-5	3.1390e-5
		r^N	1.371	1.484	2.000	1.558	1.615	-
2^{-16}	a posteriori	e^N	1.9247e-3	4.7339e-4	1.1050e-4	3.0052e-5	7.7404e-6	1.9820e-6
		r^N	2.024	2.099	1.878	1.957	1.965	-
		C^N	7.884	7.756	7.242	7.878	8.116	8.313
		K^N	13	11	9	7	7	7
	Shishkin	e^N	8.1942e-3	3.1684e-3	1.1320e-3	2.8270e-4	9.5881e-5	3.1241e-5
		r^N	1.371	1.485	2.002	1.560	1.618	-
2^{-20}	a posteriori	e^N	2.1716e-3	3.8853e-4	1.0549e-4	3.0087e-5	7.7749e-6	2.1036e-6
		r^N	2.483	1.881	1.810	1.952	1.886	-
		C^N	8.895	6.366	6.914	7.887	8.153	8.823
		K^N	19	11	13	9	8	8
	Shishkin	e^N	8.1941e-3	3.1684e-3	1.1320e-3	2.8267e-4	9.5863e-5	3.1232e-5
		r^N	1.371	1.485	2.002	1.560	1.618	-
2^{-24}	a posteriori	e^N	1.8256e-3	3.9757e-4	1.1247e-4	2.9904e-5	8.0020e-6	1.9894e-6
		r^N	2.199	1.822	1.911	1.902	2.008	-
		C^N	7.478	6.514	7.371	7.839	8.391	8.344
		K^N	33	15	24	37	8	18
	Shishkin	e^N	8.1941e-3	3.1684e-3	1.1320e-3	2.8267e-4	9.5862e-5	3.1232e-5
		r^N	1.371	1.485	2.002	1.560	1.618	-

The shooting method based on the secant iterative method is employed to solve the boundary value problems in Examples 4.1 and 4.2. γ_1 and γ_2 are selected as two initial guesses for λ, then λ^j is iteratively computed such that the solution meets the specified right boundary condition $U_{1,N} = \gamma_2$ until $\left|\lambda^j - \lambda^{j-1}\right| < 10^{-10}$. When the tolerance value for terminating the iteration is small enough, the iteration error will not influence the numerical results of the discretization scheme. Due to the fact that the problem under consideration is a linear problem, the iteration termination condition for obtaining λ can be reached with a small number of iterations. The details of the convergence for the shooting method can be found in Osborne [19] and Keller [20]. For each given λ, the solution-adaptive algorithm given in Section 3 is utilized to solve $\mathbf{U}$.

For Example 4.1, the maximum error is given by

$$e^N = \max_{0\le i\le N}\left\{\left|U_{1,i} - u_{1,i}\right|, \left|U_{2,i} - u_{2,i}\right|\right\},$$

and for Example 4.2, the maximum error is given by

$$e^N = \max_{0\le i\le N}\left\{\left|U^N_{1,i} - U^{2N}_{1,i}\right|, \left|U^N_{2,i} - U^{2N}_{2,i}\right|\right\}.$$

Their associated convergence rates are given by

$$r^N = \log_2\left(e^N/e^{2N}\right),$$

and the error constants are calculated by

$$C^N = e^N/N^{-2}.$$

Tables 1 and 2 present the error estimates, convergence rates, and error constants of scheme (3.1) on a posteriori meshes for Examples 4.1 and 4.2 under different ε values. According to the numerical results, scheme (3.1) converges with second-order accuracy and realizes the optimal convergence order under the current discretization scheme. In addition, it is evident that the error constant C^N does not grow substantially with diminishing ε or increasing N, demonstrating the scheme's uniform convergence in ε. The number of iterations K^N for generating a posteriori adapted mesh is also listed in Tables 1 and 2, which indicates that a posteriori adapted meshes can be quickly obtained.

For the purposes of comparison, a Shishkin mesh is adopted in conjunction with the hybrid difference method (3.1) to solve problem (2.1). The Shishkin mesh is chosen as

$$\bar{\omega}^N = \left\{x_i = ih^{(1)}, 0 \le i \le N/2; x_i = \tau + (i - N/2)\, h^{(2)}, N/2 < i \le N\right\}$$

with $h^{(1)} = 2\sigma/N, h^{(2)} = 2\,(1-\sigma)\,/N$, and $\sigma = \min\left\{\frac{1}{2}, \frac{2}{\alpha}\varepsilon \ln N\right\}$. Numerical results obtained with the hybrid difference scheme on the Shishkin mesh $\bar{\omega}^N$ are also reported in Tables 1 and 2, respectively. The numerical results presented in Tables 1 and 2 clearly demonstrate that the a posteriori mesh yields significantly better accuracy than the Shishkin mesh. Furthermore, our method can obtain the approximation solutions for both the exact solution and its derivative.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Authors contribution

J. Huang designed the numerical algorithm and conducted numerical experiments. Z. Cen carried out the literature review, wrote the main manuscript text and participated in designing the numerical algorithm and analyzing the error. X. Wu participated in analyzing the error and designing the numerical algorithm. All authors reviewed and approved the final manuscript.

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