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Research article

Refined stability analysis of complex-valued neural networks with time-varying delays

N. Mohamed Thoiyab[1], Mostafa Fazly[2,*], R. Vadivel[3] and Nallappan Gunasekaran[4,*]

[1] Department of Mathematics, Jamal Mohamed College, Affiliated to Bharathidasan University, Tiruchirappalli 620020, Tamilnadu, India
[2] Department of Mathematics, University of Texas at San Antonio, San Antonio, TX 78249, United States of America
[3] Department of Mathematics, Faculty of Science and Technology, Phuket Rajabhat University, Phuket 83000, Thailand
[4] Department of Natural Sciences, Eastern Michigan Joint College of Engineering, Beibu Gulf University, Qinzhou 535011, China

* **Correspondence:** Emial: mostafa.fazly@utsa.edu, gunasmaths@gmail.com.

Abstract: In this paper, we investigate the global asymptotic stability of complex-valued neural networks (CVNNs) subject to time-varying delays and parameter uncertainties. We establish novel stability conditions that guarantee both the existence and uniqueness of equilibrium states, as well as the global convergence of the network trajectories. By constructing a suitable Lyapunov-Krasovskii functional, the approach inherently accounts for the stability of CVNNs subject to time-varying delays. Finally, numerical examples are presented to verify the theoretical findings, illustrating both the effectiveness and the practical applicability of the proposed approach.

Keywords: Complex-valued neural networks; Lyapunov-Krasovskii functional; Global asymptotic stability; Time-varying delays
Mathematics subject classification: 93D05, 34D23, 34D08, 68T07

1. Introduction

Since the 1980s, neural networks (NNs) have attracted extensive attention owing to their wide applicability in various domains, including image analysis, pattern recognition, associative memory systems, and the resolution of challenging optimization problems [1–4]. In most NN models, the basic computational unit, commonly referred to as a neuron or node, receives inputs either from other nodes within the network or from external sources. These inputs are then processed to yield an output.

Each input is typically scaled by a weight that indicates its importance relative to other inputs, contributing to the node's final output. Consequently, the node operates on a weighted sum of its inputs. Moreover, a key element in altering the analysis of NNs is the existence of a distinct equilibrium point. Indeed, dynamic NN models are significantly influenced by various methods of equilibrium point stability analysis. Numerous stability analysis approaches have been developed for dynamic systems with time delays, including complete stability, exponential stability, and global asymptotic stability, as investigated in studies such as [5–11] and the references therein. The stability and instability of NNs with time delays have been widely examined in previous publications, with the primary focus being on the utilization of Lyapunov-based methodologies and tools from non-smooth analysis. As a result, one of the key challenges in this area is the comprehensive analysis of global asymptotic stability and the formulation of effective control strategies for these networks. Recently, this topic has garnered increasing attention within the research community (see [12–16] and the references therein). Considerable progress has been made in exploring different facets such as stochastic stability, global convergence, state estimation, and system stabilization. It is expected that the study of NNs will address not only equilibrium properties but also other dynamical features, such as bifurcations, chaotic phenomena, and periodic oscillations. It should be noted that, in the existing literature, most results pertain to real-valued NNs rather than complex-valued NNs (CVNNs), such as in [17–19]. Therefore, the primary aim of this paper is to examine CVNNs.

Recently, CVNNs, characterized by their use of complex-valued states, connection weights, outputs, and activation functions, have gained prominence as effective tools for solving sophisticated real-world problems. Instead of processing data in the real-valued domain (real-valued NNs or RVNNs), CVNNs are made to process data in the complex domain, which includes both real and imaginary components. The requirement to handle data that naturally contains amplitude and phase information—such as radar, sonar, optical, and wireless communication signals—more precisely and effectively is the main driving force behind the use of CVNNs. A signal's energy (amplitude) and timing (phase) are represented by complex numbers in the frequency domain, which is used in many practical applications. While RVNNs frequently have trouble handling the phase relationships, CVNNs can directly represent them. CVNNs circumvent the phase distortion that may arise from splitting inputs into separate real-valued channels by processing complex numbers directly. Current developments are based on their activation functions and learning algorithms [20]. The influence of exogenous perturbations on the synchronization performance is constrained at a prescribed level. In the meantime, the intermittent linear state feedback controller can be derived by solving a set of linear matrix inequalities [21, 22]. Because these NNs offer a wide range of applications, including signal filtering, machine learning, phase-shift keying modulation, and multi valued associative memory systems, there has been growing interest in them (see [23–26] and the references therein). Unlike RVNNs, CVNNs offer significant advantages in complex signal processing, particularly in their ability to represent phase shifts and delays using complex numbers. Additionally, they support more efficient data encoding and can simplify processing structures. It is well established that the effectiveness of delayed NNs in practical scenarios largely depends on their dynamic characteristics, making the study of CVNN dynamics an essential area of research [27, 28]. On the other hand, complex-valued NNs exhibit distinct and more intricate characteristics compared with their real-valued counterparts. It is crucial to analyze the dynamic behavior of CVNNs because of their unique structure [29–33]. Liouville's theorem states that any complete function that stays bounded throughout the complex plane must be constant in the context of complex analysis. According to this conclusion, unless a function is simple, it cannot be both analytic and limited throughout the whole complex domain. As a result, many widely used activation functions fall short of meeting both requirements at the same time. Because it is typically not possible to achieve both boundedness and analyticity, this constraint presents a fundamental issue for constructing appropriate activation functions for CVNNs. There are a variety of activation functions within the complex domain. When activation functions can be delineated into their real and imaginary components, significant advancements have been made regarding the diverse dynamic behaviors of CVNNs (see [27, 34–36] and the references therein).

The Lyapunov functional method serves as a fundamental tool for establishing the asymptotic stability of NNs. Recent advancements have introduced novel stability criteria by leveraging complex-valued Lyapunov functionals, as detailed in [28, 37]. Numerous studies have explored the stability characteristics of complex-valued recurrent neural networks (CVRNNs) influenced by time delays (see [38–41] and the references therein). Moreover, the global exponential stability of CVRNNs subject to asynchronous time delays is examined in [41]. Further, the work in [42] analyzes the exponential stability of memristor-based CVNNs under time-delay conditions. Additionally, the stability properties and occurrence of Hopf bifurcation in fractional-order, time-delayed, complex-valued single-neuron models are explored in [43].

This paper's primary contributions, driven by the aforementioned issues and anchored in an established theoretical framework, are summarized as follows.

1. A comprehensive investigation is performed on the global asymptotic stability of CVNNs that include time-varying delays.
2. New and verifiable requirements are established to ensure the presence and uniqueness of equilibrium states, along with the global asymptotic stability of the proposed CVNN model.
3. The stability study is meticulously formulated via the establishment of a suitable Lyapunov-Krasovskii functional (LKF).
4. Multiple numerical simulations are provided for validation of the theoretical results and to demonstrate the practical significance and efficacy of the suggested methodology.

Notations:

This work uses the following notations. The imaginary unit is represented by $i = \sqrt{-1}$. The collection of real numbers is denoted $\mathbb{R}$, while $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ signify the spaces of n-dimensional real vectors and $n \times m$ real matrices, respectively. Similarly, $\mathbb{C}$, $\mathbb{C}^n$, and $\mathbb{C}^{n\times m}$ denote the collections of complex numbers, n-dimensional complex vectors, and $n \times m$ complex matrices, respectively. For a complex matrix $B = (b_{ij})$, the *(Re)* and *(Im)* components are represented by b^R_{ij} and b^I_{ij}, respectively. The absolute value of a matrix $B \in \mathbb{R}^{n\times m}$ is computed element-wise as $|B| = (|b_{ij}|)$. A matrix $D \in \mathbb{C}^{n\times n}$ is classified as Hermitian positive semi-definite if $u^*Du \geq 0$ for every vector $u \in \mathbb{C}^n$, and as Hermitian positive definite if $u^*Du > 0$ for all non-zero vectors u. *(Re)* and *(Im)* denote the real and imaginary parts of the number. In this context, u^* represents the conjugate transpose of u.

2. Preliminaries

Consider a hybrid CVNNs that has delays that change over time, which is represented by the equation that follows:

$$\dot{z}_j(t) = -\check{a}_{jj}z_j(t) + \sum_{k=1}^{n} \check{b}_{jk}\psi_k(z_j(t)) + \sum_{k=1}^{n} \check{c}_{jk}\psi_k(z_j(t-\hat{\sigma}(t))) + K_j(t), \tag{2.1}$$

where $j = 1, 2, \cdots, n$. The matrix form of Eq (2.1) is given by

$$\dot{z}(t) = -\check{A}z(t) + \check{B}\Psi(z(t)) + \check{C}\Psi(z(t-\hat{\sigma}(t))) + K(t), \tag{2.2}$$

where $z(t) = (z_1(t), z_2(t), \cdots, z_n(t)) \in \mathbb{C}^n$ represents the state of the neurons at time t. The activation function vectors are defined as $\Psi(z(t)) = (\psi_1(z_1(t)), \psi_2(z_2(t)), \cdots, \psi_n(z_n(t)))^T \in \mathbb{C}^n$ without time delay and $\Psi(z(t-\hat{\sigma}(t))) = (\psi_1(z_1(t-\hat{\sigma}(t))), \psi_2(z_2(t-\hat{\sigma}(t))), \cdots, \psi_n(z_n(t-\hat{\sigma}(t))))^T \in \mathbb{C}^n$ with time-varying delays. The connection weight matrices are defined as follows: $\check{A} = \text{diag}[\check{a}_{11}, \check{a}_{22}, \cdots, \check{a}_{nn}] \in \mathbb{R}^{n\times n}$ is the diagonal matrix of connection weights, $\check{B} = (\check{b}_{jk})_{n\times n} \in \mathbb{C}^{n\times n}$ is the complex-valued connection weight matrix without time delays, and $\check{C} = (\check{c}_{jk})_{n\times n} \in \mathbb{C}^{n\times n}$ is the complex-valued connection weight matrix with time delays. The external input vector is denoted $K(t) = (K_1(t), K_2(t), \cdots, K_n(t)) \in \mathbb{C}^n$. It is assumed that the time-varying delay, denoted by the symbol $\hat{\sigma}(t)$, will fulfill the following conditions:

$$0 \le \hat{\sigma}(t) \le \hat{\sigma}_1, \ \dot{\hat{\sigma}}(t) \le \hat{\sigma}_2 < 1. \tag{2.3}$$

There are a number of assumptions that are taken into consideration in order to ensure that the NN model (2.1) remains stable. Throughout the entirety of this investigation, the activation functions will be subjected to the circumstances listed below. Suppose $z_j = \check{x}_j + i\check{y}_j$. Moreover, $\psi_j(.)$ can be separated into the *(Re)* and *(Im)* parts as follows:

$$\psi_j(z_j) = \psi_j^R(\check{x}_j, \check{y}_j) + i\psi_j^I(\check{x}_j, \check{y}_j),$$

where $\psi_j^R : \mathbb{R}^2 \to \mathbb{R}$, $\psi_j^I : \mathbb{R}^2 \to \mathbb{R}$.

Assumption $(A1)$**.** Let us assume that there are positive constants η_j^{KR} and η_j^{KI} that exist in such a way that the partial derivatives of ψ_k^K with respect to $\check{x}_j$ and $\check{y}_j$, respectively, are continuous and continue to be bounded, i.e.,

$$|\frac{\partial\psi_k^R}{\partial\check{y}_j}| \le \eta_j^{RR}, \ |\frac{\partial\psi_k^R}{\partial\check{x}_j}| \le \eta_j^{RI}, \ |\frac{\partial\psi_k^I}{\partial\check{y}_j}| \le \eta_j^{IR}, \ |\frac{\partial\psi_k^I}{\partial\check{x}_j}| \le \eta_j^{II},$$

for all $\check{x}_j, \check{y}_j \in \mathbb{R}$, where $j, k = 1, 2, \cdots, n$.

Assumption $(A2)$**.** For every $\check{x}_j, \check{x}'_j, \check{y}_j, \check{y}'_j \in \mathbb{R}$, and $\check{x}_j \neq \check{x}'_j, \check{y}_j \neq \check{y}'_j$, in a manner that ensures the following requirements are addressed:

$|\psi_k^R(\check{x}_j - \check{x}'_j) - \psi_k^R(\check{y}_j - \check{y}'_j)| \le \eta_j^{RR}|\check{x}_j - \check{x}'_j| + \eta_j^{RI}|\check{y}_j - \check{y}'_j|$;

$|\psi_k^I(\check{x}_j - \check{x}'_j) - \psi_k^I(\check{y}_j - \check{y}'_j)| \le \eta_j^{IR}|\check{x}_j - \check{x}'_j| + \eta_j^{II}|\check{y}_j - \check{y}'_j|$.

Assumption $(A3)$**.** The external input in Eq (2.1) may be separated into its *(Re)* and *(Im)* components in the following manner:

$K_j(t) = K_j^R(t) + iK_j^I(t)$.

The input functions are limited in such a way that the value of $|K_j^R(t)| \leq K_j^R$ and $|K_j^I(t)| \leq K_j^I$.

The following computation is used to transform the equilibrium point of Eq (2.1) to the point of origin. To achieve this, the following subsequent alteration will be used: $\check{v}_j(\cdot) = z_j(\cdot) - z_j^*$, $j = 1, 2, \cdots, n$.. By applying the aforementioned transformation, the system (2.1) is converted into the following form:

$$\dot{\check{v}}_j(t) = \quad -\check{a}_{jj}\check{v}_j(t) + \textstyle\sum_{k=1}^{n} \check{b}_{jk}\chi_{1j}(\check{v}_k(t)) + \sum_{k=1}^{n} \check{c}_{jk}\chi_{1j}(\check{v}_k(t - \hat{\sigma}_1)), \tag{2.4}$$

where $j, k = 1, 2, \cdots, n$, $\chi_{1j}(0) = 0$, and $\chi_{1j}(\check{v}_j(\cdot)) = \psi_j(\check{v}_j(\cdot) + z_j^*) - \psi_j(z_j^*)$, $\forall\ j$. At this point, it is evident that the function χ_{1j} satisfies the necessary conditions for ψ_j, specifically meeting the requirements outlined in Assumptions ($A1$)–($A3$). The next step is to express the connection weight matrices by splitting them into their *(Re)* and *(Im)* components, as seen in the following illustration:

$$\check{B} = (\check{b})_{n\times n} = (\check{b}_{jk}^R)_{n\times n} + i(\check{b}_{jk}^I)_{n\times n},\ \check{C} = (\check{c})_{n\times n} = (\check{c}_{jk}^R)_{n\times n} + i(\check{c}_{jk}^I)_{n\times n}.$$

According to the subsequent expression, the matrix form of Eq (2.4) is demonstrated as follows:

$$\dot{\check{v}}(t) = -\check{A}\check{v}(t) + \check{B}\chi_{1j}(\check{v}(t)) + \check{C}\chi_{1j}(\check{v}(t - \hat{\sigma}_1)), \tag{2.5}$$

where

$$\begin{aligned}
\check{v}(t) =&\check{x}_1(t) + i\check{y}_1(t),\\
\chi_{1j}(\check{v}(t)) =&\chi_{1j}^R(\check{x}_1(t), \check{y}_1(t)) + i\chi_{1j}^I(\check{x}_1(t), \check{y}_1(t)),\\
\chi_{1j}(\check{v}(t - \hat{\sigma}_1)) =&\chi_{1j}^R(\check{v}(t - \hat{\sigma}_1)) + i\chi_{1j}^I(\check{v}(t - \hat{\sigma}_1)).
\end{aligned}$$

Currently, the matrix form of the *(Re)* and *(Im)* portions of Eq (2.5) that have been separated may be stated as follows:

$$\begin{aligned}
\dot{\check{x}}_1(t) =& -\check{A}\check{x}_1(t) + \check{B}^R\chi_{1j}^R(\check{v}(t)) - \check{B}^I\chi_{1j}^I(\check{v}(t)) + \check{C}^R\chi_{1j}^R(\check{v}(t - \hat{\sigma}_1))\\
&- \check{C}^I\chi_{1j}^I(\check{v}(t - \hat{\sigma}_1)), && (2.6)\\
\dot{\check{y}}_1(t) =& -\check{A}\check{y}_1(t) + \check{B}^R\chi_{1j}^I(\check{v}(t)) + \check{B}^I\chi_{1j}^R(\check{v}(t)) + \check{C}^R\chi_{1j}^I(\check{v}(t - \hat{\sigma}_1))\\
&- \check{C}^I\chi_{1j}^R(\check{v}(t - \hat{\sigma}_1)). && (2.7)
\end{aligned}$$

Furthermore, it is possible to express Eqs (2.6) and (2.7) in the following form:

$$\begin{aligned}
\dot{\check{x}}_{1j}(t) =& -\check{a}_{jj}\check{x}_{1j}(t) + \sum_{k=1}^{n}\check{b}_{jk}^R\chi_{1j}^R(\check{x}_{1j}(t), \check{y}_{1j}(t)) - \sum_{k=1}^{n}\check{b}_{jk}^I\chi_{1j}^I(\check{x}_{1j}(t), \check{y}_{1j}(t))\\
&+ \sum_{k=1}^{n}\check{c}_{jk}^R\chi_{1j}^R(\check{x}_{1j}(t - \hat{\sigma}_1), \check{y}_{1j}(t - \hat{\sigma}_1)) - \sum_{k=1}^{n}\check{c}_{jk}^I\chi_{1j}^I(\check{x}_{1j}(t - \hat{\sigma}_1), \check{y}_{1j}(t - \hat{\sigma}_1)), && (2.8)\\
\dot{\check{y}}_{1j}(t) =& -\check{a}_{jj}\check{y}_{1j}(t) + \sum_{k=1}^{n}\check{b}_{jk}^R\chi_{1j}^I(\check{x}_{1j}(t), \check{y}_{1j}(t)) + \sum_{k=1}^{n}\check{b}_{jk}^I\chi_{1j}^R(\check{x}_{1j}(t), \check{y}_{1j}(t))\\
&+ \sum_{k=1}^{n}\check{c}_{jk}^R\chi_{1j}^I(\check{x}_{1j}(t - \hat{\sigma}_1), \check{y}_{1j}(t - \hat{\sigma}_1)) - \sum_{k=1}^{n}\check{c}_{jk}^I\chi_{1j}^R(\check{x}_{1j}(t - \hat{\sigma}_1), \check{y}_{1j}(t - \hat{\sigma}_1)), && (2.9)
\end{aligned}$$

where $\forall\ j = 1, 2, \cdots, n$.

Lemma 2.1. *[27] The following inequality is valid for all vectors* $u = (u_1, u_2, \dots, u_n)^T \in \mathbb{C}^n$ *and* $v = (v_1, v_2, \dots, v_n)^T \in \mathbb{C}^n$*, as well as for any Hermitian positive definite matrix* $D \in \mathbb{C}^{n\times n}$*:*

$$2u^*v = 2v^*u \leq \beta u^* D u + \frac{1}{\beta} v^* D^{-1} v, \quad \forall \beta > 0.$$

3. Main results

For the purpose of ensuring the global asymptotic stability of the system described by Eq (2.4), we develop unique sufficient conditions in this section. These requirements are derived under the constraint that is presented in which Eq (2.3). Taking into consideration the assumptions that have been made, the system in Eq (2.4), that is in accordance with Eq (2.3), has at least one equilibrium point. Consequently, it is of utmost importance to demonstrate not only the singularity but also the global asymptotic stability of this equilibrium point for the system (2.4), or, to put it another way, for the initial system (2.1).

Theorem 3.1. Assume that the activation functions χ_{1j} are in accordance with the criteria (A1)–(A3). Furthermore, suppose that there are positive constants $\check{\alpha}_j$ and $\check{\beta}_j$ that ensure that the following inequalities are valid:

$$\begin{aligned}\hat{\Xi}_1 =& min_{1\leq j\leq n}\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{IR}_j \\ &+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RR}_j]\} > 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\hat{\Xi}_2 =& min_{1\leq j\leq n}\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RI}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{II}_j] - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{II}_j \\ &+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RI}_j]\} > 0. \end{aligned} \tag{3.2}$$

Then the origin is the unique equilibrium point of the system (2.4).

Proof. This theorem aims to demonstrate that the origin is the unique equilibrium point of the system (2.4). Suppose, for the sake of contradiction, that the system (2.3) possesses an equilibrium point denoted $\check{v}^* = (\check{v}^*_1, \check{v}^*_2, \dots, \check{v}^*_m)^T \neq 0$. The set of points that satisfy the following equations is considered to be the equilibrium states of the system (2.4):

$$-\check{a}_{jj}\check{v}^*_j + \sum_{k=1}^{n} \check{b}_{jk}\chi_{1j}(\check{v}^*_k) + \sum_{k=1}^{n} \check{c}_{jk}\chi_{1j}(\check{v}^*_k) = 0, \ \forall j. \tag{3.3}$$

Equation (3.3) above may be split into its *(Re)* and *(Im)* components in the following manner:

$$\begin{aligned}&- \check{a}_{jj}\check{x}^*_{1j} + \sum_{k=1}^{n} \check{b}^R_{jk}\chi^R_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) - \sum_{k=1}^{n} \check{b}^I_{jk}\chi^I_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) + \sum_{k=1}^{n} \check{c}^R_{jk}\chi^R_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) \\ &- \sum_{k=1}^{n} \check{c}^I_{jk}\chi^I_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) = 0, \forall j, \end{aligned} \tag{3.4}$$

$$- \check{a}_{jj}\check{y}^*_{1j} + \sum_{k=1}^{n} \check{b}^R_{jk}\chi^I_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) + \sum_{k=1}^{n} \check{b}^I_{jk}\chi^R_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) + \sum_{k=1}^{n} \check{c}^R_{jk}\chi^I_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j})$$
$$+ \sum_{k=1}^{n} \check{c}^I_{jk}\chi^R_{1j}(\check{x}^*_{1j}, \check{y}^*_{1j}) = 0, \forall j. \tag{3.5}$$

Applying Assumptions (A1) through (A3) to Eqs (3.4) and (3.5) results in the following equations being produced:

$$\check{a}_{jj}|\check{x}^*_{1j}| \leq \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{b}^I_{jk}|\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}|$$
$$+\eta^{RI}_j|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{c}^I_{jk}|\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}_j|\check{y}_{1j}|), \forall j, \tag{3.6}$$

$$\check{a}_{jj}|\check{y}^*_{1j}| \leq \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}_j|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{b}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}|$$
$$+\eta^{II}_j|\check{y}^*_{1j}|) + \sum_{k=1}^{n} |\check{c}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|), \forall j. \tag{3.7}$$

Multiplying Eq (3.6) by $\sum_{j=1}^{n} \check{\alpha}_j$ and Eq (3.7) by $\sum_{j=1}^{n} \check{\beta}_j$, leads to

$$0 \leq -\sum_{j=1}^{n} \check{\alpha}_j\{\check{a}_{jj}|\check{x}^*_{1j}| - \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|) - \sum_{k=1}^{n} |\check{b}^I_{jk}|\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}|\check{y}^*_{1j}|)$$
$$- \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|) - \sum_{k=1}^{n} |\check{c}^I_{jk}|\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}_j|\check{y}_{1j}|)\}, \tag{3.8}$$

$$0 \leq -\sum_{j=1}^{n} \check{\beta}_j\{\check{a}_{jj}|\check{y}^*_{1j}| - \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}_j|\check{y}^*_{1j}|) - \sum_{k=1}^{n} |\check{b}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|)$$
$$- \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}| + \eta^{II}_j|\check{y}^*_{1j}|) - \sum_{k=1}^{n} |\check{c}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}| + \eta^{RI}_j|\check{y}^*_{1j}|)\}. \tag{3.9}$$

By adding Eqs (3.8) and (3.9), and simplifying the equation, the equation becomes

$$0 \leq -\sum_{j=1}^{n} \check{\alpha}_j\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{IR}_j$$
$$+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RR}_j]\}|\check{x}^*_{1j}| - \sum_{j=1}^{n} \check{\beta}_j\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RI}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{II}_j]$$
$$- \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{II}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RI}_j]\}|\check{y}^*_{1j}|, \tag{3.10}$$

$$0 \leq -\sum_{j=1}^{n} \check{\alpha}_j\hat{\Xi}_1|\check{x}^*_{1j}| - \sum_{j=1}^{n} \check{\beta}_j\hat{\Xi}_2|\check{y}^*_{1j}|, \tag{3.11}$$

where

$$\hat{\Xi}_1 = min_{1\le j\le n}\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{IR}_j$$
$$+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RR}_j]\} > 0,$$
$$\hat{\Xi}_2 = min_{1\le j\le n}\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RI}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{II}_j] - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{II}_j$$
$$+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RI}_j]\} > 0,$$

and $0 \le -\hat{\Xi}\Big(\sum_{j=1}^{n}\check{\alpha}_j|\check{x}^*_{1j}| + \sum_{j=1}^{n}\check{\beta}_j|\check{y}^*_{1j}|\Big)$, where $\hat{\Xi} = min\{\hat{\Xi}_1, \hat{\Xi}_2\}$, since $\hat{\Xi} > 0$, $\sum_{j=1}^{n}\check{\alpha}_j|\check{x}^*_{1j}| + \sum_{j=1}^{n}\check{\beta}_j|\check{y}_{1j}| > 0$, and $\check{x}^*_{1j}, \check{y}^*_{1j} \ne 0$. But $-\hat{\Xi}\Big(\sum_{j=1}^{n}\check{\alpha}_j|\check{x}^*_{2j}| + \sum_{j=1}^{n}\check{\beta}_j|\check{y}^*_{1j}|\Big) < 0$. The preceding conclusion is contradicted by this, suggesting that the only solution is $\check{x}^*_{1j} = \check{y}^*_{1j} = 0$. Therefore, in conjunction with the equation $\check{v}^* = 0$, the singular equilibrium point of the system (2.4) is established or established. The origin is therefore the only place where the system is in a state of equilibrium.

Remark 3.2. *Theorem 3.1 indicates that the feasibility of Conditions* (3.3) *and* (3.2) *is fundamentally influenced by the delay signal $\hat{\sigma}(t)$. These conditions, however, also incorporate the finer details of the time-varying delay, potentially relaxing the constraints and allowing for a larger upper limit on the delay. This expansion consequently improves the robustness and applicability of the derived criteria for global asymptotic stability.*

Remark 3.3. *In this paper, the delay in the hybrid CVNNs is considered to be a constant delay satisfying* $0 \le \hat{\sigma}(t) \le \hat{\sigma}_1$. *In contrast, the work in [44] evaluates the time-varying delay $\hat{\sigma}(t)$ directly by its upper bound $\hat{\sigma}_1$ during the stability analysis, which may introduce additional conservativeness. By explicitly incorporating the delay term into the analysis, the proposed criteria provide a more accurate estimation of the delay effect and consequently lead to less conservative stability conditions.*

Remark 3.4. *By omitting the discrete component of the system* (2.1)*, we can impose a simplified Lipschitz continuity condition on the system's nonlinear terms. This assumption is crucial, as it facilitates the application of classical fixed-point theorems and Lyapunov stability theory to rigorously establish the existence, uniqueness, and global asymptotic stability of the equilibrium point. The simplification also reduces the mathematical complexity associated with hybrid dynamic systems that combine continuous and discrete dynamics. Moreover, in our proposed framework, the external input vector is not constant or static but evolves continuously over time.*

Theorem 3.5. Assume that the activation functions χ_{1j} satisfy Conditions $(A1)$–$(A3)$. Furthermore, suppose that positive constants $\check{\alpha}_j$ and $\check{\beta}_j$ exist such that Eqs (3.2) and (3.3) hold. Then the CVNNs (2.4) are globally asymptotically stable at the origin.

Proof. We analyze the LKF presented below:

$$V(t) = \sum_{j=1}^{n}\check{\alpha}_j|\check{x}_{1j}(t)| + \sum_{j=1}^{n}\check{\beta}_j|\check{y}_{1j}(t)| + \sum_{j=1}^{n}\check{\gamma}_j\int_{t-\hat{\sigma}_1}^{t}|\check{x}_{1j}(\xi)|d\xi + \sum_{j=1}^{n}\check{\delta}_j\int_{t-\hat{\sigma}_1}^{t}|\check{x}_{1j}(\xi)|d\xi, \tag{3.12}$$

where

$$\check{\gamma}_j = \sum_{i=1}^{n} \check{\alpha}_j\Big(|\check{b}^R_{jk}|\eta^{RR}_j + |\check{b}^I_{jk}|\eta^{IR}_j\Big) + \sum_{k=1}^{n} \check{\beta}_j\Big(|\check{b}^R_{jk}|\eta^{IR}_j + |\check{b}^I_{jk}|\eta^{RR}_j\Big),$$
$$\check{\delta}_j = \sum_{i=1}^{n} \check{\alpha}_j\Big(|\check{b}^R_{jk}|\eta^{RI}_j + |\check{b}^I_{jk}|\eta^{II}_j\Big) + \sum_{k=1}^{n} \check{\beta}_j\Big(|\check{b}^R_{jk}|\eta^{II}_j + |\check{b}^I_{jk}|\eta^{RI}_j\Big).$$

The upper Dini- derivative of $V(t)$ along the trajectories of Eq (2.3) can be derived as follows:

$$\begin{aligned}
D^+V(t) =& \sum_{j=1}^{n} \check{\alpha}_j(sgn\ x_{1j})\dot{\check{x}}_{1j}(t) + \sum_{j=1}^{n} \check{\beta}_j(sgn\ y_{1j})\dot{\check{y}}_{1j}(t) + \sum_{j=1}^{n} \check{\gamma}_j[\check{x}_{1j}(t) - \check{x}_{1j}(t-\hat{\sigma}_1)] \\
&+ \sum_{j=1}^{n} \check{\delta}_j[\check{x}_{1j}(t) - \check{x}_{1j}(t-\hat{\sigma}_1)], \\
D^+V(t) \leq& \sum_{j=1}^{n} \check{\alpha}_j\Big(- \check{a}_{jj}|\check{x}_{1j}(t)| + \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{RR}_j|\check{x}_{1j}(t)| + \eta^{RI}_j|\check{y}_{1j}|(t)) + \sum_{k=1}^{n} |\check{b}^I_{jk}|(\eta^{IR}_j|\check{x}_{1j}(t)| + \eta^{II} \\
&|\check{y}_{1j}(t)|) + \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{RR}_j|\check{x}_{1j}(t-\hat{\sigma}_1)| + \eta^{RI}_j|\check{y}_{1j}(t-\hat{\sigma}_1)|) + \sum_{k=1}^{n} |\check{c}^I_{jk}|(\eta^{IR}_j|\check{x}_{1j}(t-\hat{\sigma}_1)| \\
&+ \eta^{II}_j|\check{y}_{1j}(t-\hat{\sigma}_1)|)\Big) + \sum_{j=1}^{n} \check{\beta}_j\Big(- \check{a}_{jj}|\check{y}_{1j}(t)| + \sum_{k=1}^{n} |\check{b}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}(t)| + \eta^{II}_j|\check{y}^*_{1j}(t)|) \\
&+ \sum_{k=1}^{n} |\check{b}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}(t)| + \eta^{RI}_j|\check{y}^*_{1j}(t)|) + \sum_{k=1}^{n} |\check{c}^R_{jk}|(\eta^{IR}_j|\check{x}^*_{1j}(t-\hat{\sigma}_1)| + \eta^{II}_j|\check{y}^*_{1j}(t-\hat{\sigma}_1)|) \\
&+ \sum_{k=1}^{n} |\check{c}^I_{jk}|(\eta^{RR}_j|\check{x}^*_{1j}(t-\hat{\sigma}_1)| + \eta^{RI}_j|\check{y}^*_{1j}(t-\hat{\sigma}_1)|)\Big) + \sum_{j=1}^{n}\Big[\sum_{k=1}^{n} \check{\alpha}_j\Big(|\check{b}^R_{jk}|\eta^{RR}_j + |\check{b}^I_{jk}|\eta^{IR}_j\Big) \\
&+ \sum_{k=1}^{n} \check{\beta}_j\Big(|\check{b}^R_{jk}|\eta^{IR}_j + |\check{b}^I_{jk}|\eta^{RR}_j\Big)\Big][\check{x}_{1j}(t) - \check{x}_{1j}(t-\hat{\sigma}_1)] + \sum_{j=1}^{n}\Big[\sum_{k=1}^{n} \check{\alpha}_j\Big(|\check{b}^R_{jk}|\eta^{RI}_j \\
&+ |\check{b}^I_{jk}|\eta^{II}_j\Big) + \sum_{k=1}^{n} \beta_j\Big(|\check{b}^R_{jk}|\eta^{II}_j + |\check{b}^I_{jk}|\eta^{RI}_j\Big)\Big[\check{y}_{1j}(t) - \check{y}_{1j}(t-\hat{\sigma}_1)\Big], \\
D^+V(t) \leq& - \sum_{j=1}^{n} \check{\alpha}_j\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{n} \frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| \\
&+ |\check{c}^R_{jk}|)\eta^{IR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RR}_j]\}|\check{x}_{2j}(t)| \\
&- \sum_{j=1}^{n} \check{\beta}_j\{\check{a}_{jj} - \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RI}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{II}_j] \\
&- \sum_{k=1}^{n} \frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{II}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RI}_j]\}|\check{y}_{1j}(t)|,
\end{aligned} \tag{3.13}$$

$$D^+V(t) \leq - \sum_{j=1}^{n} \check{\alpha}_j\hat{\Xi}_1|\check{x}_{1j}(t)| - \sum_{j=1}^{n} \check{\beta}_j\hat{\Xi}_2|\check{y}_{1j}(t)|, \tag{3.14}$$

where

$$\hat{\Xi}_1 = min_{1\le j\le n}\{\check{a}_{jj} - \sum_{k=1}^{n}\frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{n}\frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{IR}_j$$
$$+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RR}_j]\} > 0,$$
$$\hat{\Xi}_2 = min_{1\le j\le n}\{\check{a}_{jj} - \sum_{k=1}^{n}\frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RI}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{II}_j] - \sum_{k=1}^{n}\frac{1}{\check{\beta}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{II}_j$$
$$+(|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{RI}_j]\} > 0,$$

and

$$D^+V(t) \le -\hat{\Xi}\Big(\sum_{j=1}^{n}\check{\alpha}_j|\check{x}_{1j}(t)| + \sum_{j=1}^{n}\check{\beta}_j|\check{y}_{1j}(t)|\Big).$$

The expression $\hat{\Xi} = \min(\hat{\Xi}_1, \hat{\Xi}_2) > 0$ is required. We have the condition that $\dot{V}(t) < 0$ holds true for all nonzero values of $\check{x}_{1j}(t), \check{y}_{1j}(t)$ and positive values of $\check{\alpha}_j, \check{\beta}_j$. Therefore, according to the Lyapunov stability theory, the origin of the system (2.4) under condition (2.3) is globally asymptotically stable. Furthermore, we can conclude that any system (2.1) that meets condition (2.3) is globally asymptotically robust stable (GARS). This completes the proof.

Remark 3.6. *In contrast to much of the existing literature in this research field (e.g., [45]), our approach does not require decomposing the complex-valued activation functions into their (Re) and (Im) components, nor does it require us to construct an equivalent RVNN. Using the linear inequality method allows us to directly analyze the original complex-valued system while rigorously establishing the existence, uniqueness, and global exponential stability of the equilibrium point. This not only simplifies the analytical process but also preserves the inherent structure and dynamics of the proposed CVNNs.*

4. Numerical simulations

In this section, numerical examples validate the theoretical findings, such as the existence, uniqueness, and global asymptotic stability, which are given to show the usefulness of the suggested method. The proposed results are also investigated using numerical simulations, which are shown below.

Example 4.1. Consider the following matrix representation of Eq (2.5):

$$\check{A} = \begin{bmatrix}\check{a} & 0\\ 0 & \check{a}\end{bmatrix},\ \check{B} = \begin{bmatrix}-i & 0.2+0.3i\\ -0.5+i & i\end{bmatrix},\ \check{C} = \begin{bmatrix}0.2i & 0.1-i\\ -0.1+i & -0.2i\end{bmatrix},$$

and $\eta^{RR}_j = \eta^{IR}_j = \eta^{RI}_j = \eta^{II}_j = \frac{1}{2}, \check{\alpha}_j = \check{\beta}_j = 3, \chi_{1j}(\check{v}_j)(t) = 0.5(|\check{x}_{1j}(t) + 1| - |\check{y}_{1j}(t) - 1|) + i0.5(|\check{x}_{1j}(t) + 1| - |\check{y}_{1j}(t) - 1|)$, $j = 1, 2, \hat{\sigma}(t) = 0.2(sin^2(t)) + 0.3 \le 0.5 = \hat{\sigma}_1$, and $\dot{\hat{\sigma}}(t) = 0.2(sin2(t)) \le 0.2 = \hat{\sigma}_2, \check{a} > 0$.

Since

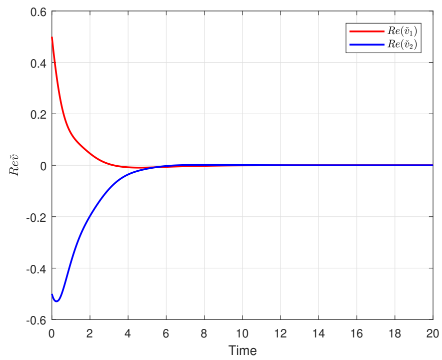
$$\hat{\Xi}_1 = min_{1\le j\le 2}\{\check{a}_{jj} - \sum_{k=1}^{2}\frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{RR}_j + (|\check{b}^I_{jk}| + |\check{c}^I_{jk}|)\eta^{IR}_j] - \sum_{k=1}^{2}\frac{1}{\check{\alpha}_j}[(|\check{b}^R_{jk}| + |\check{c}^R_{jk}|)\eta^{IR}_j$$

$$+(|\breve{b}^I_{jk}| + |\breve{c}^I_{jk}|)\eta^{RR}_j]\} > 0,$$

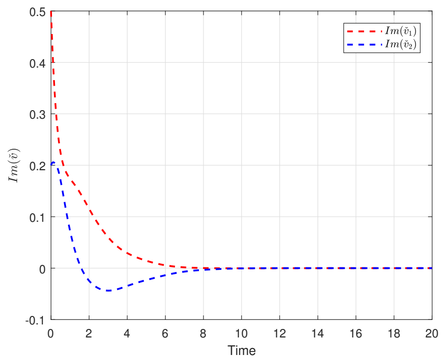
only if $\breve{a} - 1.53 > 0$ and $\breve{a} - 1.17 > 0$. Since $\hat{\Xi}_1 = min_{(}\breve{a} - 1.53, \breve{a} - 1.17) > 0$. Therefore, $\breve{a} > 1.53$.

$$\hat{\Xi}_2 = min_{1\leq j\leq 2}\{\breve{a}_{jj} - \sum_{k=1}^{2}\frac{1}{\breve{\beta}_j}[(|\breve{b}^R_{jk}| + |\breve{c}^R_{jk}|)\eta^{RI}_j + (|\breve{b}^I_{jk}| + |\breve{c}^I_{jk}|)\eta^{II}_j] - \sum_{k=1}^{2}\frac{1}{\breve{\beta}_j}[(|\breve{b}^R_{jk}| + |\breve{c}^R_{jk}|)\eta^{II}_j$$
$$+(|\breve{b}^I_{jk}| + |\breve{c}^I_{jk}|)\eta^{RI}_j]\} > 0,$$

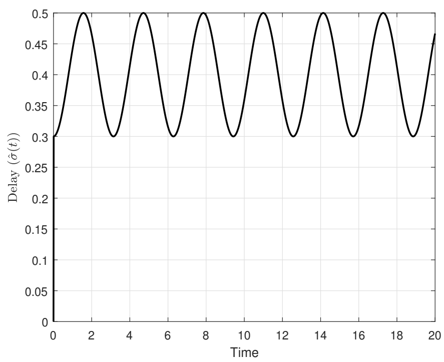
only if $\breve{a} - 1.53 > 0$ and $\breve{a} - 1.17 > 0$. Since $\hat{\Xi}_2 = min_{(}\breve{a} - 1.53, \breve{a} - 1.17) > 0$. Therefore, $\breve{a} > 1.53$. Since $\hat{\Xi}_1, \hat{\Xi}_2 > 0$ is valid only if $\breve{a} > 1.53$.



(a) Responses of $Re(\check{v})$



(b) Responses of $Im(\check{v})$



(c) Responses of Delay ($\hat{\sigma}(t)$)

Figure 1. *(Re)* and *(Im)* parts of the states and the responses of the delay $\hat{\sigma}(t)$ in Example 4.1.

The dynamics of a two-neuron CVNNs with time-varying delays were investigated using the Euler method with a time step of $\Delta t = 0.01$ over a total simulation time of $T = 20$. The initial condition was set as $\check{v}_0 = [0.5 + 0.5i,\ -0.5 + 0.2i]^T$, and the delay function was defined as $\hat{\sigma}(t) = 0.2\sin^2(t) + 0.3$, introducing a periodic time delay. The system matrices were $\breve{A} = 2I$, $\breve{B} = \begin{bmatrix} i & 0.2 + 0.3i \\ -0.5 + i & i \end{bmatrix}$, and $\breve{C} = \begin{bmatrix} 0.2i & 0.1 - i \\ -0.1 + i & -0.2i \end{bmatrix}$, and the nonlinear activation function used was the element-wise complex hyperbolic tangent, $\chi(\check{v}) = \tanh(\check{v})$. The simulation results, including the evolution of the *(Re)* and *(Im)* parts of the state variables and their trajectories in the complex plane, reveal the influence of the time-varying delay on the system's stability and convergence toward equilibrium. Figure 1 demonstrates the evolution of both the *(Re)* and *(Im)* parts of $\check{v}_1(t), \check{v}_2(t)$, showing convergence to the origin, thereby indicating global asymptotic stability. Figure 1 displays the periodic profile of the delay $\hat{\sigma}(t)$, while Figure 1 shows the state trajectories in the complex plane, revealing the interdependence and convergence of neuron states. Lastly, Figure 2 illustrates the final steady-state values of the *(Re)* and *(Im)* components, providing insight into the system's equilibrium. Collectively, these figures confirm the system's stability and highlight the influence of time delays and nonlinear dynamics.

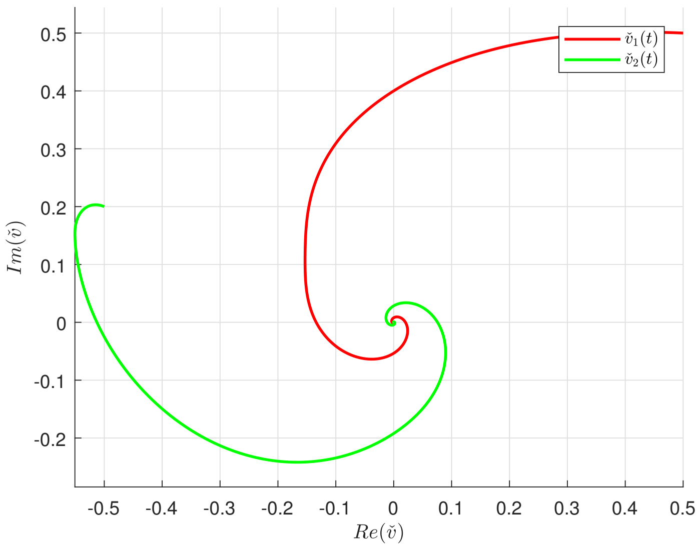


Figure 2. Phase portraits of the states in Example 4.1.

Example 4.2. Consider the CVNNs defined by Eq (2.4), subject to the conditions in Eq (2.3). The corresponding matrix form of the system is given by Eq (2.5), which will be analyzed in the subsequent discussion.

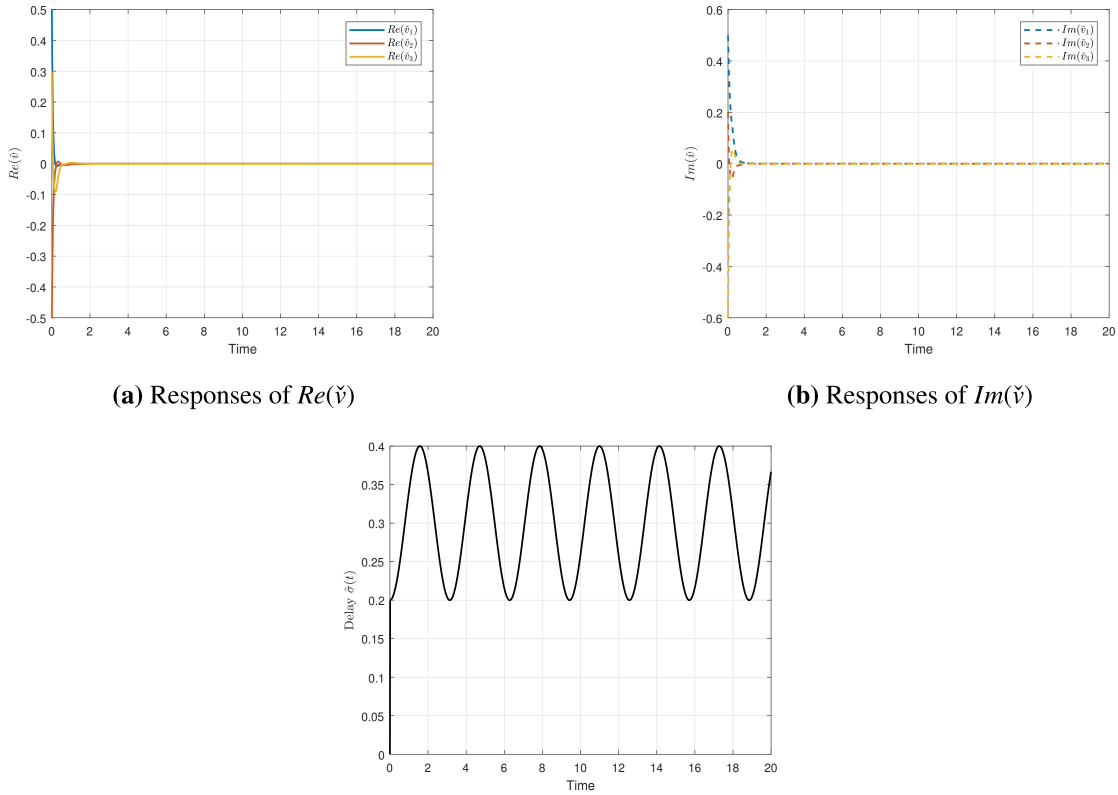
$$\check{A} = \begin{bmatrix} \check{a} & 0 & 0 \\ 0 & \check{a} & 0 \\ 0 & 0 & \check{a} \end{bmatrix}, \ \check{B} = \begin{bmatrix} -2i & 1+i & 2+i \\ 1+i & 1+2i & 0.1i \\ 2+i & 0.1i & 5i \end{bmatrix}, \ \check{C} = \begin{bmatrix} -i & 2+i & 1-i \\ 2+i & 1+3i & 4-2i \\ 1-i & 4-2i & 4i \end{bmatrix},$$

and $\eta_j^{RR} = \eta_j^{IR} = \eta_j^{RI} = \eta_j^{II} = \frac{3}{4}$, $\check{\alpha}_j = \check{\beta}_j = 3$, $\chi_{1j}(\check{v}_j)(t) = 0.75(|\hat{x}_{1j}(t) - 1| - |\check{y}_{1j}(t) + 1|) + i0.75(|\hat{x}_{1j}(t) - 1| - |\check{y}_{1j}(t) + 1|)$, $j = 1, 2, 3$, $\hat{\sigma}(t) = 0.2(1 + sin^2(t)) \leq 0.4 = \hat{\sigma}_1$, and
$\dot{\hat{\sigma}}(t) = 0.2(2sin(t)cos(t)) = 0.2sin2(t) \leq 0.2 = \hat{\sigma}_2$, $\check{a} > 0$. Since

$$\hat{\Xi}_1 = min_{1\leq j\leq 3}\{\check{a}_{jj} - \sum_{i=i}^{3} \frac{1}{\check{\alpha}_j}[(|b_{ij}^R| + |c_{ij}^R|)\eta_j^{RR} + (|b_{ij}^I| + |c_{ij}^I|)\eta_j^{IR}] - \sum_{i=i}^{3} \frac{1}{\check{\alpha}_j}[(|b_{ij}^R| + |c_{ij}^R|)\eta_j^{IR}$$

$$+ (|b_{ij}^I| + |c_{ij}^I|)\eta_j^{RR}]\} > 0,$$

only if $\check{a} - 9.5 > 0$, $\check{a} - 5.1 > 0$ and $\check{a} - 9.35$. Since $\hat{\Xi}_1 = min(\check{a} - 14.95,\ \check{a} - 5.1,\ \check{a} - 9.35) > 0$. Therefore, $\check{a} > 14.95$.



(a) Responses of $Re(\check{v})$

(b) Responses of $Im(\check{v})$

(c) Responses of Delay ($\hat{\sigma}(t)$)

Figure 3. The *(Re)* and *(Im)* parts of the states and the responses of the delay $\hat{\sigma}(t)$ in Example 4.2.

$$\hat{\Xi}_2 = min_{1\le j\le 3}\{\check{a}_{jj} - \sum_{i=i}^{3} \frac{1}{\check{\beta}_j}[(|b_{ij}^R| + |c_{ij}^R|)\eta_j^{RI} + (|b_{ij}^I| + |c_{ij}^I|)\eta_j^{II}] - \sum_{i=i}^{3} \frac{1}{\check{\beta}_j}[(|b_{ij}^R| + |c_{ij}^R|)\eta_j^{II}$$
$$+ (|b_{ij}^I| + |c_{ij}^I|)\eta_j^{RI}]\} > 0,$$

only if $\check{a} - 9.5 > 0$, $\check{a} - 5.1 > 0$ and $\check{a} - 9.35$. Since $\hat{\Xi}_2 = min(\check{a} - 14.95,\ \check{a} - 5.1,\ \check{a} - 9.35) > 0$. Therefore, $\check{a} > 14.95$. Since $\hat{\Xi}_1, \hat{\Xi}_2 > 0$ is valid only if $\check{a} > 14.95$. The figures in this study illustrate the dynamic behavior of a CVNN composed of three neurons subject to time-varying delays and nonlinear activation functions. The simulation is performed over a time interval of $T = 20$ seconds with a fixed time step $\Delta t = 0.01$ using the Euler method. The initial condition is set as $\check{v}(0) = [0.5 + 0.5i,\ -0.5 + 0.2i,\ 0.3 - 0.6i]^T$. The time delay is defined as a smooth periodic function $\hat{\sigma}(t) = 0.2(1 + \sin^2(t))$, introducing bounded time-dependent shifts in the system's response. The activation function used is $\chi(\check{v}) = 0.75\,(|\mathrm{Re}(\check{v}) - 1| - |\mathrm{Im}(\check{v}) + 1|) + 0.75i(|\mathrm{Re}(\check{v}) - 1| - |\mathrm{Im}(\check{v}) + 1|)$, which nonlinearly couples the *(Re)* and *(Im)* parts of the complex-valued states. Figure 3 demonstrates the evolution of both the *(Re)* and *(Im)* parts of $\check{v}_1(t)$, $\check{v}_2(t)$, $\check{v}_3(t)$, showing convergence to the origin, thereby indicating global asymptotic stability. Figure 3 displays the periodic profile of the delay $\hat{\sigma}(t)$, while Figure 3 shows the state trajectories in the complex plane, revealing the interdependence and convergence of the neuron states. Lastly, Figure 4 illustrates the final steady-state values of the *(Re)* and *(Im)* components, providing insight into the system's equilibrium. Collectively, these figures confirm the system's stability and highlight the influence of time delays and nonlinear dynamics.

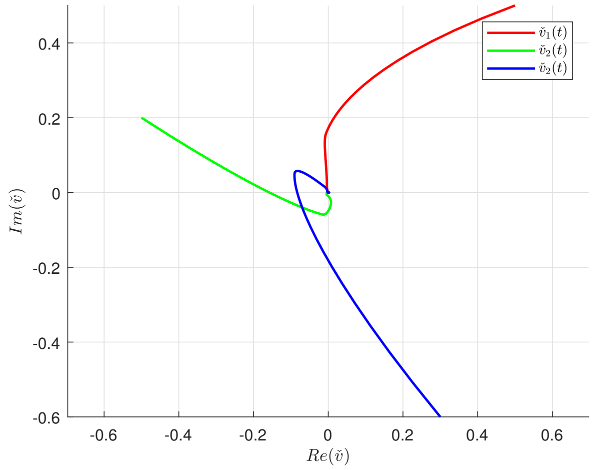


Figure 4. Phase portraits of the states in Example 4.2.

Remark 4.1. *This study primarily advances the field by establishing novel stability criteria for CVNNs experiencing time-varying delays, achieved via a LKF methodology. The approach distinguishes itself from prior research by circumventing restrictive assumptions and linear approximations, instead embracing a wider range of nonlinear activation functions by decomposing them into the (Re) and (Im) elements. This advancement not only bolsters the theoretical solidity of the model but also aids in practical execution. Additionally, the tractability of the derived conditions significantly enhances their application potential in real-world contexts, such as those involving communication delays, signal processing, and complex-valued data systems.*

5. Conclusions

In this paper, the global asymptotic stability of hybrid CVNNs with time-varying delays is thoroughly investigated. Together with proving the system's global asymptotic robust stability as in Theorem 3.5, we have also developed a number of novel, readily verifiable premises. Both the existence and uniqueness of equilibrium points are guaranteed by these conditions. The investigation was conducted by implementing the concept of a suitable LKF. Because the stability criteria are derived by breaking down the activation functions into their real and imaginary components and using the Lipschitz continuity properties of these components, they may be computed. The stability analysis of CVNNs serves as the theoretical foundation for numerical comparisons by verifying the numerical simulation's results, robustness, convergence, and physical meaning. It is used to ensure that a suggested numerical approach appropriately reflects the desired behavior of the theoretical model. Numerical examples are shown to illustrate the applicability of the suggested method and validate the theoretical findings. Our focus in the future will be on applying these results to fractional-order CVNNs. The purpose of this study is to make the stability analysis framework more useful and broaden its application.

Author contributions

N. Mohamed Thoiyab: Writing-review and editing, writing-original draft, conceptualization.
Mostafa Fazly: Writing-review and editing, visualization, investigation, funding acquisition.
R. Vadivel: Writing-review and editing, methodology, validation.
Nallappan Gunasekaran: Methodology, formal analysis, software, supervision.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors claim that they have no conflicts of interest in publishing this work.

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