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*Research article*

## **Global stability of delayed HIV-1 models with saturated viral and cellular infection rates and two types of target cells**

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**Abstract:** In this study, we investigate two mathematical models, formulated using delay differential equations, to capture HIV-1 transmission dynamics. Both models incorporate CD4<sup>+</sup>T cells and macrophages as target cells and saturated non-linear infection transmission terms. The first model introduces a discrete time delay, while the second employs a distributed delay to reflect more realistic biological insights. Our theoretical analysis explores key properties such as the basic reproduction number, non-negativity, boundedness, and the existence of equilibria. The global stability analysis of the disease-free and endemic equilibrium are analysed using the LaSalle invariance principle by proposing Lyapunov functions. We conducted sensitivity analysis to find the significant parameters related to the infection dynamics. Numerical simulations are performed to validate our theoretical results and visualize the behavior of the system under different parametric conditions. This study shows the critical role of time delays in shaping HIV-1 infection dynamics. Incorporation of such delays into mathematical models are essential for accurately capturing the progression of the disease and proposing effective intervention strategies for HIV-1.

**Keywords:** HIV-1 infection; macrophages; sensitivity analysis; discrete and distributed delays; global stability; LaSalle invariance principle; numerical simulations

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### **1. Introduction**

A key challenge in the transmission dynamics of the human immunodeficiency virus (HIV) is that it leads to the acquisition of the acquired immune deficiency syndrome (AIDS). This is due to the viral infections that cause the reduction or destruction of the immune system [1]. In the past, there has been significant modeling of the dynamics of the infections caused by the human immunodeficiency virus type 1. This has emerged as a core key area in mathematical biology [2]. On the other hand, there

has also been a focus on models involving delays, which has proven valuable in accurately modeling aspects of the infection dynamics [3, 4].

A number of studies have examined delay-induced models of HIV-1 dynamics in relation to CD4<sup>+</sup>T cells [5, 6]. Such models make allowance for biological aspects like inflammatory cytokines, cell infection by HIV, immune reactions, and time delays for a deeper understanding and treatment approach for HIV [7, 8]. For example, a particular study stresses the importance of inflammatory cytokines and cell infection in modeling the dynamics of HIV-1. It puts particular emphasis on the significance of time delay for viral reproduction as well as for treatment [9]. Another study examines the global stability and growth of a Hopf bifurcation for delayed models for HIV. It helps in understanding the effect of a hidden reservoir and immunities [10].

Studies on the role of the CD4<sup>+</sup>T cell homeostatic proliferation include the effect of intracellular delays on stability at equilibrium [11]. To expand this, mathematical models are formulated for studying HIV-1 dynamics with both CD4<sup>+</sup>T cells and macrophages [12, 13]. Such models help in gaining insights into viral persistence, immune reactions, and possible treatments. In [14], the authors investigated the proliferation of CD4<sup>+</sup>T cells and macrophages, discovering intricate behaviors including bistability, tristability, and bifurcations. Raezah et al. [15] formulated a more realistic model including viral and cellular transmissions, immune reactions, and time delays to analyze stability, treatment via drugs, and viral parameters. Some studies [16, 17] investigated the global stability for an HIV infection model with cytotoxic T lymphocyte (CTL) immune response and distributed time delays, explaining how immune reactions determine disease progression. In [14], the authors assumed both virus-to-cell and cell-to-cell transmission mechanisms in HIV-1 infections, especially with macrophage contributions. Chen and Zhao [18] presented a model with a discrimination mechanism for intercellular versus viral infections. They determined the global behavior of the system and studied the role of intercellular infection.

Saturated infection terms are often introduced in epidemic models to account for situations where the number of infected individuals becomes very large. Instead of using the standard incidence term  $\beta SI$ , the modified form  $\frac{\beta SI}{1+cI}$  is employed to capture saturation effects. This adjustment reflects the biological reality that contact rates cannot increase indefinitely as infections grow, making the model more realistic during large-scale outbreaks [19, 20]. In the context of HIV modeling, incorporating saturated infection rates acknowledges that viral replication and target cell infection do not rise without bound as viral load or target cell availability increases [21]. These processes naturally approach a biological maximum due to limitations such as receptor availability and immune regulation [22]. Ahmed et al. [23] studied an HIV-1 model that included saturated virus–target and infected–target incidences, together with CTL immune response and two types of intracellular delays. They analyzed the global stability of equilibria using Lyapunov functionals, focusing on the basic reproduction number and CTL activation. In this paper, we also consider the role of saturated infection terms. Specifically, our work examines delayed HIV-1 dynamic models that incorporate both saturated viral and cellular infection rates, while distinguishing between CD4<sup>+</sup> T lymphocytes and macrophages as key target cells.

Distributed delay models assume a continuum of time lags instead of fixed time lags, providing further detail on the dynamics of HIV transmission [24]. Herein, these delays reflect both biological and behavioral processes, representing, for example, the time taken by infected cells to produce new virions and an individual cell's time to change behavior in response to HIV prevalence [25]. Distributed

delay has commonly been used to model incubation periods or lagging immune responses [26], as is suitably applied to capture time-lagged effects in viral replication, immune activation, and treatment response [27]. In this work, we incorporate both discrete and distributed delays to model the main biological time lags, such as between viral entry into target cells, replication, and production of virions, and lagged immune activation, essential to accurately simulate the course of HIV-1 infection and, thus, treatment effects.

The model introduced in [28] describes five interacting populations: uninfected CD4<sup>+</sup> T cells, uninfected macrophages, infected CD4<sup>+</sup> T cells, infected macrophages, and free HIV-1 virions. Notably, this model did not incorporate time delays. Raedah et al. [12] extended the model of [28] by including cell-to-cell transmission mechanisms. In this paper, we extend the model proposed in [12] by introducing four discrete and distributed time delays, along with saturated infection terms. Accordingly, we formulate two mathematical models: One incorporating discrete delays and another incorporating distributed delays. We derive the basic reproduction number and apply stability theory to establish conditions ensuring the global stability of the disease-free and endemic equilibrium. Furthermore, we conduct sensitivity analysis to identify key parameters that significantly influence HIV-1 transmission dynamics. Finally, numerical simulations are performed using dde23 solver in Matlab (R2016a) to illustrate the dynamic behavior of the models and to validate the analytical results.

The remaining sections of this article are organized as follows. Section 2 introduces the formulation of the HIV-1 dynamic model, incorporating discrete time delays and saturated infection rates. We also present a comprehensive mathematical analysis of the delayed model including the identification of equilibria and the investigation of their stability under both delayed and non-delayed conditions. Section 3 contains the formulation and analysis of the model with distributed delay. Numerical simulations are provided in Section 4 to illustrate and support the analytical findings. Finally, Section 5 concludes the research with a discussion of the key results and their implications for future research and therapeutic strategies.

## 2. Mathematical model formulation

In our model, there are five populations, namely the uninfected CD4<sup>+</sup>T cells  $T_1(t)$  (target cells), uninfected macrophages  $T_2(t)$  (another target cell type), infected CD4<sup>+</sup>T cells  $I_1(t)$ , infected macrophages  $I_2(t)$  and free HIV-1 virions,  $V(t)$ . We propose the following system of ordinary differential equations (ODE) without delay:

$$\dot{T}_1 = \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1}, \quad (2.1)$$

$$\dot{I}_1 = \left[ \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right] - \delta_1 I_1, \quad (2.2)$$

$$\dot{T}_2 = \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2}, \quad (2.3)$$

$$\dot{I}_2 = \left[ \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right] - \delta_2 I_2, \quad (2.4)$$

$$\dot{V} = b_1 \delta_1 I_1 + b_2 \delta_2 I_2 - \varphi V. \quad (2.5)$$

The short descriptions of the model parameters are presented in Table 1 and the detailed description of the interactions among the components in the HIV-1 dynamics model (2.5) are provided below:

(i) The HIV-1 virus primarily targets uninfected CD4<sup>+</sup>T cells and macrophages, which are represented in this model as  $T_1(t)$  and  $T_2(t)$ , respectively. These uninfected cells are produced at constant rates  $\lambda_1$  and  $\lambda_2$  and die naturally at rates  $\gamma_1$  and  $\gamma_2$ , respectively. However, these populations are also decreased through infection by free virus particles  $V$  and also by the infected cells  $I_1$  and  $I_2$ .

(ii) The infection transmissions are captured using saturated terms with  $\beta_1$  and  $\beta_2$  as the maximum infection rates of CD4<sup>+</sup>T cells and macrophages, respectively, by the virus that caused the infected cells  $I_1$  and  $I_2$ . The infection transmission is modeled using saturated terms with  $\mu_i, \bar{\mu}_i, i = 1, 2$  as the saturation constants.

(iii) After getting infected, the uninfected cells transition into the infected compartments  $I_1(t)$  and  $I_2(t)$ . These infected cells die at rates  $\delta_1$  and  $\delta_2$ , but before dying, they can produce new virions. The number of virions produced per infected cell is denoted by  $b_1$  for CD4<sup>+</sup>T cells and  $b_2$  for macrophages. The free virus  $V$  decay at a rate  $\varphi$ .

**Table 1.** Parameter descriptions for HIV-1 dynamics model (2.1–2.5).

Parameter	Description
$\lambda_1$	recruitment rate of uninfected CD4 <sup>+</sup> T cells ( $T_1$ )
$\lambda_2$	recruitment rate of uninfected macrophages ( $T_2$ )
$\gamma_1$	natural mortality rate of CD4 <sup>+</sup> T cells
$\gamma_2$	natural mortality rate of macrophages
$\beta_1$	infection rate of CD4 <sup>+</sup> T cells by free-virus
$\bar{\beta}_1$	infection rate of CD4 <sup>+</sup> T cells by infected cells
$\beta_2$	infection rate of macrophages by virus
$\bar{\beta}_2$	infection rate of uninfected macrophages by infected macrophages
$\delta_1$	mortality rate of infected CD4 <sup>+</sup> T cells ( $I_1$ )
$\delta_2$	mortality rate of infected macrophages ( $I_2$ )
$b_1$	rate of virions production per infected CD4 <sup>+</sup> T cell
$b_2$	rate of virions produced per infected macrophage
$\varphi$	rate of clearance of free virus particles ( $V$ )
$\mu_i, \bar{\mu}_i, i = 1, 2$	Saturation constants

### 2.1. Model with discrete delays

We incorporate into system (2.1)–(2.5) the discrete time delays due to the latent periods of the uninfected cells. We assume that,  $\sigma_1$  and  $\sigma_2$  represent the amount of time that has passed since both HIV-1 particles and the infected cells had made contact with uninfected cells before those cells became infected cells.  $\omega_1$  and  $\omega_2$  are the maturation times of new virions.

The survival probability of the cells during the latent period  $[t - \sigma_1, t]$  is governed by  $e^{-\epsilon_1 \sigma_1}$ , where  $\epsilon_1 > 0$ ; similarly, in the latent period  $[t - \sigma_2, t]$ , the survival probability is  $e^{-\epsilon_2 \sigma_2}$ , where  $\epsilon_2 > 0$ . The factor  $e^{-\epsilon_1 \omega_1}$  represents the survival of the infected cells during the delay period  $[t - \omega_1, t]$ , where  $\epsilon_1 > 0$ . The factor  $e^{-\epsilon_2 \omega_2}$  represents the loss of the infected cells during the latent time  $[t - \omega_2, t]$ , where

$\varepsilon_2 > 0$ . The model takes the following form:

$$\dot{T}_1 = \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1}, \quad (2.6)$$

$$\dot{I}_1 = e^{-\varepsilon_1 \sigma_1} \left[ \frac{\beta_1 T_1 (t - \sigma_1) V (t - \sigma_1)}{1 + \mu_1 V (t - \sigma_1)} + \frac{\bar{\beta}_1 T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{1 + \bar{\mu}_1 I_1 (t - \sigma_1)} \right] - \delta_1 I_1, \quad (2.7)$$

$$\dot{T}_2 = \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2}, \quad (2.8)$$

$$\dot{I}_2 = e^{-\varepsilon_2 \sigma_2} \left[ \frac{\beta_2 T_2 (t - \sigma_2) V (t - \sigma_2)}{1 + \mu_2 V (t - \sigma_2)} + \frac{\bar{\beta}_2 T_2 (t - \sigma_2) I_2 (t - \sigma_2)}{1 + \bar{\mu}_2 I_2 (t - \sigma_2)} \right] - \delta_2 I_2, \quad (2.9)$$

$$\dot{V} = b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1 (t - \omega_1) + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2 (t - \omega_2) - \varphi V. \quad (2.10)$$

We assume the initial conditions as follows. Let  $\check{I} = \max\{\sigma_1, \omega_1, \sigma_2, \omega_2\}$  and  $\mathbb{C}$  be the Banach space of continuous functions from the interval  $[-\check{I}, 0]$  into  $\mathbb{R}_{\geq 0}^5$ . For the system given by Eqs (2.6)–(2.10), we consider the following initial conditions:

$$\begin{aligned} T_1(\theta) &= \chi_1(\theta), \quad T_2(\theta) = \chi_2(\theta), \quad I_1(\theta) = \chi_3(\theta), \quad I_2(\theta) = \chi_4(\theta), \quad V(\theta) = \chi_5(\theta), \\ \chi_j(\theta) &\geq 0, \quad j = 1, \dots, 5, \quad \theta \in [-\check{I}, 0], \end{aligned} \quad (2.11)$$

where  $(\chi_1(\theta), \dots, \chi_5(\theta)) \in \mathbb{C}([-\check{I}, 0], \mathbb{R}_{\geq 0}^5)$ , and the norm  $\|\chi_j\| = \sup_{-\check{I} \leq \theta \leq 0} |\chi_j(\theta)|$  for  $\chi_j \in \mathbb{C}$ ,  $j = 1, 2, 3, 4, 5$ .

We consider the delay differential system (2.6)–(2.10) with state variables:

$$X(t) = (T_1(t), I_1(t), T_2(t), I_2(t), V(t)).$$

Then the system can be written in the abstract form:

$$\dot{X}(t) = F(X(t), X(t - \sigma_1), X(t - \sigma_2), X(t - \omega_1), X(t - \omega_2)),$$

where  $F : \mathbb{R}^{5(m+1)} \rightarrow \mathbb{R}^5$  is continuous and locally Lipschitz in its arguments. By the standard existence and uniqueness theorem for functional differential equations [29, 30], for any continuous initial history function  $\phi \in C([-\tau, 0], \mathbb{R}^5)$  with  $\tau = \max\{\sigma_1, \sigma_2, \omega_1, \omega_2\}$ , there exists a unique local solution  $X(t)$  defined on  $[0, T)$  for some  $T > 0$ .

## 2.2. Nonnegativity of the solution

**Lemma 1.** Any solution to system (2.6)–(2.10) that adheres to the initial conditions specified in Eq (2.11) remains nonnegative and is eventually bounded.

*Proof.* We have  $\dot{T}_1|_{T_1=0} = \lambda_1 > 0$  and  $\dot{T}_2|_{T_2=0} = \lambda_2 > 0$ ; hence,  $T_1(t) > 0$ ,  $T_2(t) > 0$  for all  $t \geq 0$  (see Proposition B.7 of [31]). From Eqs (2.6)–(2.10), we have

$$I_1(t) = \chi_3(0)e^{-\delta_1 t} + e^{-\varepsilon_1 \sigma_1} \int_0^t e^{-\delta_1(t-z)} \left[ \frac{\beta_1 T_1(z - \sigma_1) V(z - \sigma_1)}{1 + \mu_1 V(z - \sigma_1)} + \frac{\bar{\beta}_1 T_1(z - \sigma_1) I_1(z - \sigma_1)}{1 + \bar{\mu}_1 I_1(z - \sigma_1)} \right] dz,$$

$$I_2(t) = \chi_4(0)e^{-\delta_2 t} + e^{-\varepsilon_2 \sigma_2} \int_0^t e^{-\delta_2(t-z)} \left[ \frac{\beta_2 T_2(z - \sigma_2) V(z - \sigma_2)}{1 + \mu_2 V(z - \sigma_2)} + \frac{\bar{\beta}_2 T_2(z - \sigma_2) I_2(z - \sigma_2)}{1 + \bar{\mu} I_2(z - \sigma_2)} \right] dz,$$

$$V(t) = \chi_5(0)e^{-\varphi t} + \int_0^t e^{-\varphi(t-z)} b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(z - \omega_1) dz + \int_0^t e^{-\varphi(t-z)} b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(z - \omega_2) dz.$$

These show that  $I_1(t) \geq 0$ ,  $I_2(t) \geq 0$  and  $V(t) \geq 0$  for all  $t \in [0, \check{J}]$  [32]. Hence, by recursive argumentation, we obtain that  $I_1(t) \geq 0$ ,  $I_2(t) \geq 0$  and  $V(t) \geq 0$  for all  $t \geq 0$ .

We now proceed to show that all components  $T_1, T_2, I_1, I_2$ , and  $V$  are ultimately bounded. From Eqs (2.6) and (2.8), we have  $\limsup_{t \rightarrow \infty} T_1(t) \leq \frac{\lambda_1}{\gamma_1}$ , and  $\limsup_{t \rightarrow \infty} T_2(t) \leq \frac{\lambda_2}{\gamma_2}$ .

Next, let us denote  $\check{F}_1(t) = e^{-\varepsilon_1 \sigma_1} T(t - \sigma_1) + I_1(t)$  and  $\check{F}_2(t) = e^{-\varepsilon_2 \sigma_2} T(t - \sigma_2) + I_2(t)$ , then,

$$\begin{aligned} \frac{d\check{F}_1(t)}{dt} &= \lambda_1 e^{-\varepsilon_1 \sigma_1} - \gamma_1 e^{-\varepsilon_1 \sigma_1} T_1(t - \sigma_1) - \delta_1 I_1(t) \\ &\leq \lambda_1 e^{-\varepsilon_1 \sigma_1} - o_1 (e^{-\varepsilon_1 \sigma_1} T_1(t - \sigma_1) + I_1(t)) \leq \lambda_1 - o_1 \check{F}_1(t), \end{aligned}$$

where  $o_1 = \min\{\gamma_1, \delta_1\}$ .

Similarly,

$$\begin{aligned} \frac{d\check{F}_2(t)}{dt} &= \lambda_2 e^{-\varepsilon_2 \sigma_2} - \gamma_2 e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) - \delta_2 I_2(t) \\ &\leq \lambda_2 e^{-\varepsilon_2 \sigma_2} - o_2 (e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) + I_2(t)) \leq \lambda_2 - o_2 \check{F}_2(t), \end{aligned}$$

where  $o_2 = \min\{\gamma_2, \delta_2\}$ . Hence,  $\limsup_{t \rightarrow \infty} \check{F}_1(t) \leq j_1$ ,  $\limsup_{t \rightarrow \infty} \check{F}_2(t) \leq j_2$ , where  $j_1 = \frac{\lambda_1}{o_1}$  and  $j_2 = \frac{\lambda_2}{o_2}$ .

Thus, we get  $\limsup_{t \rightarrow \infty} I_1(t) \leq j_1$  and  $\limsup_{t \rightarrow \infty} I_2(t) \leq j_2$  for all  $t \geq 0$ . On the other hand,

$$\begin{aligned} \dot{V}(t) &= b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(t - \omega_1) + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(t - \omega_2) - \varphi V(t) \\ &\leq b_1 \delta_1 e^{-\varepsilon_1 \omega_1} j_1 + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} j_2 - \varphi V(t) \leq b_1 \delta_1 j_1 + b_2 \delta_2 j_2 - \varphi V(t). \end{aligned}$$

Thus,  $\limsup_{t \rightarrow \infty} V(t) \leq j_3$ , where  $j_3 = \frac{b_1 \delta_1 j_1}{\varphi} + \frac{b_2 \delta_2 j_2}{\varphi}$ .

From the above analysis, we can conclude that all solutions of the system (2.6)–(2.10) are non-negative and ultimately bounded.

According to Lemma 1, we demonstrate that the domain

$$\Phi = \{(T_1, I_1, T_2, I_2, V) \in \mathbb{C}_{\geq 0}^5 : \|T_1\| \leq \frac{\lambda_1}{\gamma_1}, \|I_1\| \leq j_1, \|T_2\| \leq \frac{\lambda_2}{\gamma_2}, \|I_2\| \leq j_2, \|V\| \leq j_3\}$$

is positively invariant for system (2.6)–(2.10).

### 2.3. Equilibria

In order to determine the equilibria of the system (2.6)–(2.10), we solve the following system of equations:

$$0 = \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1}, \quad (2.12)$$

$$0 = e^{-\epsilon_1 \sigma_1} \left[ \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right] - \delta_1 I_1, \quad (2.13)$$

$$0 = \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2}, \quad (2.14)$$

$$0 = e^{-\epsilon_2 \sigma_2} \left[ \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right] - \delta_2 I_2, \quad (2.15)$$

$$0 = b_1 \delta_1 e^{-\epsilon_1 \omega_1} I_1 + b_2 \delta_2 e^{-\epsilon_2 \omega_2} I_2 - \varphi V. \quad (2.16)$$

Solving the system (2.12)–(2.16), we found two equilibria for the mathematical model in Eqs (2.6)–(2.10) listed below:

(i) the infection-free equilibrium (IFE)  $\Pi_0 = (T_1^0, 0, T_2^0, 0, 0)$ , where  $T_1^0 = \frac{\lambda_1}{\gamma_1}$ ,  $T_2^0 = \frac{\lambda_2}{\gamma_2}$ , and

(ii) the infection-present equilibrium (IPE)  $\check{\Pi} = (\check{T}_1, \check{I}_1, \check{T}_2, \check{I}_2, \check{V})$ ,

where

$$\begin{aligned} \check{T}_1 &= \frac{\lambda_1(1 + \mu_1 \check{V})(1 + \bar{\mu}_1 \check{I}_1)}{\gamma_1(1 + \mu_1 \check{V})(1 + \bar{\mu}_1 \check{I}_1) + \beta_1 \check{V}(1 + \bar{\mu}_1 \check{I}_1) + \bar{\beta}_1 \check{I}_1(1 + \mu_1 \check{V})}, \\ \check{I}_1 &= \frac{-\check{B}_1 + \sqrt{\check{B}_1^2 - 4\check{A}_1 \check{C}_1}}{2\check{A}_1}, \\ \check{T}_2 &= \frac{\lambda_2(1 + \mu_2 \check{V})(1 + \bar{\mu}_2 \check{I}_2)}{\gamma_2(1 + \mu_2 \check{V})(1 + \bar{\mu}_2 \check{I}_2) + \beta_2 \check{V}(1 + \bar{\mu}_2 \check{I}_2) + \bar{\beta}_2 \check{I}_2(1 + \mu_2 \check{V})}, \\ \check{I}_2 &= \frac{-\check{B}_2 + \sqrt{\check{B}_2^2 - 4\check{A}_2 \check{C}_2}}{2\check{A}_2}, \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} \check{A}_1 &= \delta_1 \left( (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1) (1 + \mu_1 \check{V}) + \check{V} \beta_1 \bar{\mu}_1 \right), \\ \check{B}_1 &= - \left( (e^{-\epsilon_1 \sigma_1} \bar{\beta}_1 \lambda_1 - \delta_1 \gamma_1) (1 + \mu_1 \check{V}) - \check{V} \beta_1 (\delta_1 - \lambda_1 \bar{\mu}_1 e^{-\epsilon_1 \sigma_1}) \right), \\ \check{C}_1 &= -e^{-\epsilon_1 \sigma_1} \check{V} \beta_1 \lambda_1, \\ \check{A}_2 &= \delta_2 \left( (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2) (1 + \mu_2 \check{V}) + \check{V} \beta_2 \bar{\mu}_2 \right), \\ \check{B}_2 &= - \left( (e^{-\epsilon_2 \sigma_2} \bar{\beta}_2 \lambda_2 - \delta_2 \gamma_2) (1 + \mu_2 \check{V}) - \check{V} \beta_2 (\delta_2 - \lambda_2 \bar{\mu}_2 e^{-\epsilon_2 \sigma_2}) \right), \\ \check{C}_2 &= -e^{-\epsilon_2 \sigma_2} \check{V} \beta_2 \lambda_2, \end{aligned} \quad (2.18)$$

and  $\check{V}$  satisfies the following equation:

$$\varphi \check{V} = b_1 \delta_1 e^{-\epsilon_1 \omega_1} \check{I}_1 + b_2 \delta_2 e^{-\epsilon_2 \omega_2} \check{I}_2. \quad (2.19)$$

### 2.3.1. The basic reproduction number $\mathcal{R}_0^*$

We calculate the basic reproduction number  $\mathcal{R}_0^*$  of system (2.6)–(2.10) by implementing the next-generation-matrix method.

We consider the infected compartments  $X(t) = [I_1(t), I_2(t), V(t)]^\top$ , where  $I_1$  and  $I_2$  denote infected cells, and  $V$  denotes the virus population. At the disease-free equilibrium (DFE), we have

$$I_1 = I_2 = V = 0, \quad T_1 = \frac{\lambda_1}{\gamma_1}, \quad T_2 = \frac{\lambda_2}{\gamma_2}. \quad (2.20)$$

The system can be decomposed as

$$\dot{X} = F(X) - V(X),$$

where  $F(X) = [F_{I_1}, F_{I_2}, F_V]^\top$  represents new infections and  $V(X) = [V_{I_1}, V_{I_2}, V_V]^\top$  represents transitions out of infectious compartments. For our model and considering the equation for  $I_1$ ,  $I_2$ , and  $V$ , we get

$$\begin{aligned} F_{I_1} &= e^{-\epsilon_1 \sigma_1} \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right), & V_{I_1} &= \delta_1 I_1, \\ F_{I_2} &= e^{-\epsilon_2 \sigma_2} \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right), & V_{I_2} &= \delta_2 I_2, \end{aligned}$$

and

$$F_V = b_1 \delta_1 e^{-\epsilon_1 \omega_1} I_1 + b_2 \delta_2 e^{-\epsilon_2 \omega_2} I_2, \quad V_V = \varphi V.$$

Let us denote the Jacobian of  $F(X)$  at the DFE  $\mathcal{F}$  and the Jacobian of  $V(X)$  at the DFE by  $\mathcal{V}$ . Then the next generation matrix is given by

$$K = \mathcal{F} \mathcal{V}^{-1}.$$

Considering Eq (2.20),  $\mathcal{F}$  and  $\mathcal{V}$  are derived at the DFE as below:

$$\mathcal{F} = \begin{bmatrix} \bar{\beta}_1 e^{-\epsilon_1 \sigma_1} T_1^0 & 0 & \beta_1 e^{-\epsilon_1 \sigma_1} T_1^0 \\ 0 & \bar{\beta}_2 e^{-\epsilon_2 \sigma_2} T_2^0 & \beta_2 e^{-\epsilon_2 \sigma_2} T_2^0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ b_1 \delta_1 e^{-\epsilon_1 \omega_1} & b_2 \delta_2 e^{-\epsilon_2 \omega_2} & \varphi \end{bmatrix}.$$

The basic reproduction number is given by

$$\mathcal{R}_0^* = \rho(K),$$

where  $\rho(K)$  denotes the spectral radius (largest eigenvalue) of the matrix  $K$ . Thus, we obtain the basic reproduction number as

$$\mathcal{R}_0^* = \frac{1}{2} \left( \frac{\bar{\beta}_1 T_1^0 e^{-\epsilon_1 \sigma_1}}{\delta_1} - \frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi} + \frac{\bar{\beta}_2 T_2^0 e^{-\epsilon_2 \sigma_2}}{\delta_2} - \frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi} \right)$$

$$\begin{aligned}
& + \frac{1}{2} \left[ \left( \frac{\bar{\beta}_1 T_1^0 e^{-\epsilon_1 \sigma_1}}{\delta_1} - \frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi} + \frac{\bar{\beta}_2 T_2^0 e^{-\epsilon_2 \sigma_2}}{\delta_2} - \frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi} \right)^2 \right. \\
& \left. + 4 \frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi} \frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi} \right]^{\frac{1}{2}}. \tag{2.21}
\end{aligned}$$

We have derived the basic reproduction number for the system with discrete delay. Now, we discuss the existence of feasible endemic equilibrium  $\check{\Pi}$  using the following propositions by finding the condition for a positive value for  $\check{V}$  using the following proposition.

**Proposition 1.** Assume that  $\mathcal{R}_0^* > 1$ . If  $\frac{\bar{\beta}_1 T_1^0 e^{-\epsilon_1 \sigma_1}}{\delta_1} + \frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi} < 1$ ,  $\frac{\bar{\beta}_2 T_2^0 e^{-\epsilon_2 \sigma_2}}{\delta_2} + \frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi} < 1$ , then  $\check{M} = \frac{\frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi}}{1 - \frac{\bar{\beta}_1 T_1^0 e^{-\epsilon_1 \sigma_1}}{\delta_1}} + \frac{\frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi}}{1 - \frac{\bar{\beta}_2 T_2^0 e^{-\epsilon_2 \sigma_2}}{\delta_2}} > 1$ .

*Proof.* Let

$$\begin{aligned}
\check{\mathcal{U}}_1 &= \frac{\bar{\beta}_1 T_1^0 e^{-\epsilon_1 \sigma_1}}{\delta_1}, \quad \check{\mathcal{U}}_2 = \frac{b_1 \beta_1 T_1^0 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)}}{\varphi}, \\
\check{\mathcal{W}}_1 &= \frac{\bar{\beta}_2 T_2^0 e^{-\epsilon_2 \sigma_2}}{\delta_2}, \quad \check{\mathcal{W}}_2 = \frac{b_2 \beta_2 T_2^0 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)}}{\varphi}, \\
\check{\mathcal{U}} &= \check{\mathcal{U}}_1 + \check{\mathcal{U}}_2, \quad \check{\mathcal{W}} = \check{\mathcal{W}}_1 + \check{\mathcal{W}}_2.
\end{aligned} \tag{2.22}$$

Hence,  $\check{\mathcal{U}} < 1$  and  $\check{\mathcal{W}} < 1$ , and  $\mathcal{R}_0^*$  takes the form

$$\mathcal{R}_0^* = \frac{1}{2} (\check{\mathcal{U}} + \check{\mathcal{W}}) + \frac{1}{2} \left( [\check{\mathcal{U}} - \check{\mathcal{W}}]^2 + 4\check{\mathcal{U}}_2\check{\mathcal{W}}_2 \right)^{\frac{1}{2}}. \tag{2.23}$$

Following the same technique of Lemma 2 [12], and Since  $\mathcal{R}_0^* > 1$ , we have  $[\check{\mathcal{U}} - \check{\mathcal{W}}]^2 + 4\check{\mathcal{U}}_2\check{\mathcal{W}}_2 > (2 - (\check{\mathcal{U}} + \check{\mathcal{W}}))^2$ , then  $\check{\mathcal{U}} + \check{\mathcal{W}} - \check{\mathcal{U}}\check{\mathcal{W}} + \check{\mathcal{U}}_2\check{\mathcal{W}}_2 > 1$ . Since  $1 - \check{\mathcal{U}}_1 > 0$  and  $1 - \check{\mathcal{W}}_1 > 0$ , we get

$$\frac{\check{\mathcal{U}}_2 + \check{\mathcal{W}}_2 - \check{\mathcal{U}}_1\check{\mathcal{W}}_2 - \check{\mathcal{U}}_2\check{\mathcal{W}}_1}{(1 - \check{\mathcal{U}}_1)(1 - \check{\mathcal{W}}_1)} > 1,$$

which leads to

$$\frac{\check{\mathcal{U}}_2(1 - \check{\mathcal{W}}_1) + \check{\mathcal{W}}_2(1 - \check{\mathcal{U}}_1)}{(1 - \check{\mathcal{U}}_1)(1 - \check{\mathcal{W}}_1)} > 1.$$

Therefore,  $\check{M} > 1$ .

**Proposition 2.** Suppose that  $\mathcal{R}_0^* > 1$ , then an endemic equilibrium  $\check{\Pi}$  exists for the system (2.6)–(2.10).

*Proof.* Any equilibrium must satisfy Eqs (2.12)–(2.15). For the equilibrium denoted by  $\check{\Pi}$ , we assume  $V \neq 0$ . Then, using Eq (2.16), we obtain

$$\frac{b_1 \delta_1 e^{-\epsilon_1 \omega_1} I_1}{\varphi} + \frac{b_2 \delta_2 e^{-\epsilon_2 \omega_2} I_2}{\varphi} - V = 0.$$

By substituting the expressions from Eqs (2.12)–(2.15) into the equation above, we derive

$$e^{-(\epsilon_1\sigma_1+\epsilon_1\omega_1)}\left(\frac{b_1\lambda_1}{\varphi}-\frac{b_1\gamma_1T_1}{\varphi}\right)+e^{-(\epsilon_2\sigma_2+\epsilon_2\omega_2)}\left(\frac{b_2\lambda_2}{\varphi}-\frac{b_2\gamma_2T_2}{\varphi}\right)-V=0.$$

Since  $T_1, I_1, T_2, I_2$  depend on  $V$  through Eqs (2.17) and (2.18), we can define the function

$$\check{G}(V)=e^{-(\epsilon_1\sigma_1+\epsilon_1\omega_1)}\left(\frac{b_1\lambda_1}{\varphi}-\frac{b_1\gamma_1T_1(V)}{\varphi}\right)+e^{-(\epsilon_2\sigma_2+\epsilon_2\omega_2)}\left(\frac{b_2\lambda_2}{\varphi}-\frac{b_2\gamma_2T_2(V)}{\varphi}\right)-V,$$

which is continuous on  $[0, \infty)$  and differentiable on  $(0, \infty)$ . Now consider

$$\check{V}^*=\frac{b_1\lambda_1e^{-(\epsilon_1\sigma_1+\epsilon_1\omega_1)}}{\varphi}+\frac{b_2\lambda_2e^{-(\epsilon_2\sigma_2+\epsilon_2\omega_2)}}{\varphi}>0.$$

Substituting  $V = \check{V}^*$  into Eqs (2.17) and (2.18) implies that  $T_1(\check{V}^*), I_1(\check{V}^*), T_2(\check{V}^*), I_2(\check{V}^*)$  are all strictly positive, and

$$\check{G}(\check{V}^*)=-\frac{b_1\gamma_1T_1(\check{V}^*)e^{-(\epsilon_1\sigma_1+\epsilon_1\omega_1)}}{\varphi}-\frac{b_2\gamma_2T_2(\check{V}^*)e^{-(\epsilon_2\sigma_2+\epsilon_2\omega_2)}}{\varphi}<0.$$

Hence  $\check{G}$  attains negative values. Furthermore we obtain

$$\begin{aligned} \check{G}(0) &= \frac{\check{\mathcal{U}}_2\gamma_1\bar{\beta}_1}{\beta_1(\gamma_1\bar{\mu}_1+\bar{\beta}_1)}\left(1-\frac{2}{2-(1-\check{\mathcal{U}}_1)+|1-\check{\mathcal{U}}_1|}\right) \\ &\quad + \frac{\check{\mathcal{W}}_2\gamma_2\bar{\beta}_2}{\beta_2(\gamma_2\bar{\mu}_2+\bar{\beta}_2)}\left(1-\frac{2}{2-(1-\check{\mathcal{W}}_1)+|1-\check{\mathcal{W}}_1|}\right), \end{aligned}$$

and

$$\check{G}'(0)=-e^{-(\epsilon_1\sigma_1+\epsilon_1\omega_1)}\frac{b_1\gamma_1}{\varphi}T_1'(0)-e^{-(\epsilon_2\sigma_2+\epsilon_2\omega_2)}\frac{b_2\gamma_2}{\varphi}T_2'(0)-1.$$

As the behavior of  $\check{G}(0)$  depends sensitively on the values of  $\check{\mathcal{U}}_1$  and  $\check{\mathcal{W}}_1$ , we classify the analysis into six cases according to their values:

- (1)  $\check{\mathcal{U}}_1 = 1$  and  $\check{\mathcal{W}}_1 \leq 1$ :  $\check{G}(0) = 0$  and  $\lim_{\check{\mathcal{U}}_1 \rightarrow 1} \check{G}'(0) = +\infty$ .
- (2)  $\check{\mathcal{U}}_1 \leq 1$  and  $\check{\mathcal{W}}_1 = 1$ :  $\check{G}(0) = 0$  and  $\lim_{\check{\mathcal{W}}_1 \rightarrow 1} \check{G}'(0) = +\infty$ .
- (3)  $\check{\mathcal{U}}_1 < 1$  and  $\check{\mathcal{W}}_1 < 1$ :  $\check{G}(0) = 0$  and  $\check{G}'(0) = \check{M} - 1 > 0$  (Proposition 1).
- (4)  $\check{\mathcal{U}}_1 \leq 1$  and  $\check{\mathcal{W}}_1 > 1$ :

$$\check{G}(0)=\frac{\check{\mathcal{W}}_2\gamma_2\bar{\beta}_2}{\beta_2(\gamma_2\bar{\mu}_2+\bar{\beta}_2)}\left(\frac{\check{\mathcal{W}}_1-1}{\check{\mathcal{W}}_1}\right)>0.$$

- (5)  $\check{\mathcal{U}}_1 > 1$  and  $\check{\mathcal{W}}_1 \leq 1$ :

$$\check{G}(0)=\frac{\check{\mathcal{U}}_2\gamma_1\bar{\beta}_1}{\beta_1(\gamma_1\bar{\mu}_1+\bar{\beta}_1)}\left(\frac{\check{\mathcal{U}}_1-1}{\check{\mathcal{U}}_1}\right)>0.$$

(6)  $\check{\mathcal{U}}_1 > 1$  and  $\check{\mathcal{W}}_1 > 1$ :

$$\check{G}(0) = \frac{\check{\mathcal{U}}_2 \gamma_1 \bar{\beta}_1}{\beta_1 (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1)} \left( \frac{\check{\mathcal{U}}_1 - 1}{\check{\mathcal{U}}_1} \right) + \frac{\check{\mathcal{W}}_2 \gamma_2 \bar{\beta}_2}{\beta_2 (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2)} \left( \frac{\check{\mathcal{W}}_1 - 1}{\check{\mathcal{W}}_1} \right) > 0.$$

Thus,

$$\check{G}(0) \geq 0, \quad \text{and in cases (1)–(3) with } \mathcal{R}_0^* > 1: \quad \check{G}'(0) > 0.$$

Define  $H(V) = |\check{G}(V)|$  on  $[0, \check{V}^*]$ . Since  $H$  is continuous, it achieves a minimum at some  $V_0 \in [0, \check{V}^*]$ .

Because  $\check{G}(0) \geq 0$  and  $\check{G}(\check{V}^*) < 0$ , the function  $\check{G}$  takes values of opposite sign on  $[0, \check{V}^*]$ . Hence there must exist  $\hat{V} \in (0, \check{V}^*)$  with  $\check{G}(\hat{V}) = 0$ ; otherwise  $H$  would be bounded below by a strictly positive value, contradicting the sign change. Therefore,

$$\check{G}(\check{V}) = 0 \quad \text{for some } 0 < \check{V} < \check{V}^*.$$

From Eqs (2.17) and the condition  $\mathcal{R}_0^* > 1$ , the corresponding values satisfy

$$\check{T}_1 > 0, \quad \check{I}_1 > 0, \quad \check{T}_2 > 0, \quad \check{I}_2 > 0, \quad \check{V} > 0.$$

Thus the endemic equilibrium

$$\check{\Pi} = (\check{T}_1, \check{I}_1, \check{T}_2, \check{I}_2, \check{V})$$

is well defined and strictly positive whenever  $\mathcal{R}_0^* > 1$ .

Based on these results:

- (i) If  $\mathcal{R}_0^* \leq 1$ , then the system admits only the disease-free equilibrium  $\Pi_0$ ;
- (ii) If  $\mathcal{R}_0^* > 1$ , then both the disease-free equilibrium  $\Pi_0$  and the endemic equilibrium  $\check{\Pi}$  exist.

#### 2.4. Global stability

For the system described by Eqs (2.6)–(2.10), we aim to establish the global asymptotic stability of both equilibria  $\Pi_0$  and  $\check{\Pi}$ . Our analysis proceeds according to the approach outlined in [33, 34].

Let  $\check{\Gamma}'_i$  denote the largest invariant subset of

$$\check{\Gamma}'_i = \left\{ (T_1, I_1, T_2, I_2, V) : \frac{d\check{\Xi}_i}{dt} = 0 \right\}, \quad i = 1, 2,$$

where  $\check{\Xi}_i$  is a Lyapunov function candidate. Furthermore, let

$$\Theta(y) = y - 1 - \ln(y). \tag{2.24}$$

Clearly  $\Theta(1) = 0$  and  $\Theta(y) \geq 0$  for  $y > 0$ .

To demonstrate global stability, we employ the following lemma:

**Lemma 2.** *Assume that  $\mathcal{R}_0^* \leq 1$ . Then:*

- (i)  $\check{\mathcal{U}}_1 \leq 1$ ,  $\check{\mathcal{U}}_2 \leq 1$ ,  $\check{\mathcal{W}}_1 \leq 1$ ,  $\check{\mathcal{W}}_2 \leq 1$ ;
- (ii)  $0 < M \leq 1$  where  $M = \check{\mathcal{U}} + \check{\mathcal{W}} - \check{\mathcal{U}}\check{\mathcal{W}} + \check{\mathcal{U}}_2\check{\mathcal{W}}_2$ .

*Proof.* (i) Let  $\mathcal{R}_0^* \leq 1$ , then

$$(\check{U} + \check{W}) + |\check{U} - \check{W}| \leq 2,$$

which leads to

$$2 \geq \begin{cases} 2\check{U} & \text{if } \check{U} \geq \check{W} \\ 2\check{W} & \text{if } \check{W} \geq \check{U} \end{cases},$$

hence  $\max\{\check{U}, \check{W}\} \leq 1$ . Therefore  $\check{\mathcal{U}}_1 \leq 1$ ,  $\check{\mathcal{U}}_2 \leq 1$ ,  $\check{\mathcal{W}}_1 \leq 1$ ,  $\check{\mathcal{W}}_2 \leq 1$ .

(ii) The inequality  $\mathcal{R}_0^* \leq 1$  is equivalent to

$$\check{U} + \check{W} + \sqrt{(\check{U} - \check{W})^2 + 4\check{\mathcal{U}}_2\check{\mathcal{W}}_2} \leq 2,$$

hence

$$\check{U} + \check{W} \leq 2.$$

Furthermore,

$$\sqrt{(\check{U} - \check{W})^2 + 4\check{\mathcal{U}}_2\check{\mathcal{W}}_2} \leq 2 - (\check{U} + \check{W}).$$

Squaring both sides and using  $2 - (\check{U} + \check{W}) \geq 0$  yields

$$\begin{aligned} (\check{U} - \check{W})^2 + 4\check{\mathcal{U}}_2\check{\mathcal{W}}_2 &\leq (2 - (\check{U} + \check{W}))^2 \\ \check{\mathcal{U}}_2\check{\mathcal{W}}_2 &\leq 1 - (\check{U} + \check{W} - \check{U}\check{W}). \end{aligned}$$

Now substitute

$$\check{U} = \check{\mathcal{U}}_1 + \check{\mathcal{U}}_2, \quad \check{W} = \check{\mathcal{W}}_1 + \check{\mathcal{W}}_2,$$

and we get

$$\check{U} + \check{W} - \check{U}\check{W} = \check{\mathcal{U}}_1 + \check{\mathcal{W}}_1 - \check{\mathcal{U}}_1\check{\mathcal{W}}_1 + \check{\mathcal{U}}_2(1 - \check{\mathcal{W}}_1 - \check{\mathcal{W}}_2) + \check{\mathcal{W}}_2(1 - \check{\mathcal{U}}_1).$$

From (i), we have

$$1 - \check{\mathcal{W}}_1 - \check{\mathcal{W}}_2 \geq 0, \quad 1 - \check{\mathcal{U}}_1 \geq 0.$$

Because all terms are positive,

$$\check{U} + \check{W} - \check{U}\check{W} \geq \check{\mathcal{U}}_1 + \check{\mathcal{W}}_1 - \check{\mathcal{U}}_1\check{\mathcal{W}}_1,$$

and therefore,

$$\check{\mathcal{U}}_2\check{\mathcal{W}}_2 \leq 1 - (\check{\mathcal{U}}_1 + \check{\mathcal{W}}_1 - \check{\mathcal{U}}_1\check{\mathcal{W}}_1).$$

Hence,

$$M = \check{\mathcal{U}}_1 + \check{\mathcal{W}}_1 - \check{\mathcal{U}}_1\check{\mathcal{W}}_1 + \check{\mathcal{U}}_2\check{\mathcal{W}}_2 \leq 1.$$

Because all quantities  $\check{\mathcal{U}}_1$ ,  $\check{\mathcal{U}}_2$ ,  $\check{\mathcal{W}}_1$ , and  $\check{\mathcal{W}}_2$  are strictly positive, each term in  $M$  is positive and therefore  $M > 0$ . Hence,  $0 < M \leq 1$ .

**Theorem 1.** For the system defined by Eqs (2.6)–(2.10), if  $\mathcal{R}_0^* \leq 1$ , then the disease-free equilibrium  $\Pi_0$  is globally asymptotically stable.

*Proof.* Suppose that  $\mathcal{R}_0^* \leq 1$ . Now, we formulate a function  $\check{\Xi}_0(T_1, I_1, T_2, I_2, V)$  as follows:

$$\begin{aligned} \check{\Xi}_0 = & \check{\eta}_1 \left[ T_1^0 \Theta \left( \frac{T_1}{T_1^0} \right) + e^{\epsilon_1 \sigma_1} I_1 \right] + \check{\eta}_1 \int_0^{\sigma_1} \left( \frac{\beta_1 T_1(t-\xi) V(t-\xi)}{1 + \mu_1 V(t-\xi)} + \frac{\bar{\beta}_1 T_1(t-\xi) I_1(t-\xi)}{1 + \bar{\mu}_1 I_1(t-\xi)} \right) d\xi \\ & + \check{\eta}_2 \left[ T_2^0 \Theta \left( \frac{T_2}{T_2^0} \right) + e^{\epsilon_2 \sigma_2} I_2 \right] + \check{\eta}_2 \int_0^{\sigma_2} \left( \frac{\beta_2 T_2(t-\xi) V(t-\xi)}{1 + \mu_2 V(t-\xi)} + \frac{\bar{\beta}_2 T_2(t-\xi) I_2(t-\xi)}{1 + \bar{\mu}_2 I_2(t-\xi)} \right) d\xi \\ & + \check{\eta}_3 \left( b_1 \delta_1 e^{-\epsilon_1 \omega_1} \int_0^{\omega_1} I_1(t-\xi) d\xi + b_2 \delta_2 e^{-\epsilon_2 \omega_2} \int_0^{\omega_2} I_2(t-\xi) d\xi \right) + \check{\eta}_3 V, \end{aligned}$$

where

$$\begin{aligned} \check{\eta}_1 &= \delta_1 \delta_2 b_1 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)} (1 - \check{\mathcal{W}}_1), \check{\eta}_2 = \delta_1 \delta_2 b_2 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)} (1 - \check{\mathcal{U}}_1), \\ \check{\eta}_3 &= \delta_1 \delta_2 (1 - \check{\mathcal{U}}_1) (1 - \check{\mathcal{W}}_1). \end{aligned} \quad (2.25)$$

Clearly,  $\check{\Xi}_0(T_1, I_1, T_2, I_2, V) > 0$  for any  $T_1, I_1, T_2, I_2, V > 0$ , and  $\check{\Xi}_0(T_1^0, 0, T_2^0, 0, 0) = 0$ . We calculate  $\frac{d\check{\Xi}_0}{dt}$  as

$$\begin{aligned} \frac{d\check{\Xi}_0}{dt} = & \check{\eta}_1 \left( 1 - \frac{T_1^0}{T_1} \right) \dot{T}_1 + \check{\eta}_1 e^{\epsilon_1 \sigma_1} \dot{I}_1 \\ & + \check{\eta}_1 \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\beta_1 T_1(t-\sigma_1) V(t-\sigma_1)}{1 + \mu_1 V(t-\sigma_1)} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\bar{\beta}_1 T_1(t-\sigma_1) I_1(t-\sigma_1)}{1 + \bar{\mu}_1 I_1(t-\sigma_1)} \right) \\ & + \check{\eta}_2 \left( 1 - \frac{T_2^0}{T_2} \right) \dot{T}_2 + \check{\eta}_2 e^{\epsilon_2 \sigma_2} \dot{I}_2 \\ & + \check{\eta}_2 \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\beta_2 T_2(t-\sigma_2) V(t-\sigma_2)}{1 + \mu_2 V(t-\sigma_2)} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\bar{\beta}_2 T_2(t-\sigma_2) I_2(t-\sigma_2)}{1 + \bar{\mu}_2 I_2(t-\sigma_2)} \right) \\ & + \check{\eta}_3 (b_1 \delta_1 e^{-\epsilon_1 \omega_1} (I_1 - I_1(t-\omega_1)) + b_2 \delta_2 e^{-\epsilon_2 \omega_2} (I_2 - I_2(t-\omega_2))) + \check{\eta}_3 \dot{V}. \end{aligned}$$

Using Eqs (2.6)–(2.10), we find

$$\begin{aligned} \frac{d\check{\Xi}_0}{dt} = & \check{\eta}_1 \left( 1 - \frac{T_1^0}{T_1} \right) \left( \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right) \\ & + \check{\eta}_1 e^{\epsilon_1 \sigma_1} \left( e^{-\epsilon_1 \sigma_1} \left[ \frac{\beta_1 T_1(t-\sigma_1) V(t-\sigma_1)}{1 + \mu_1 V(t-\sigma_1)} + \frac{\bar{\beta}_1 T_1(t-\sigma_1) I_1(t-\sigma_1)}{1 + \bar{\mu}_1 I_1(t-\sigma_1)} \right] - \delta_1 I_1 \right) \\ & + \check{\eta}_1 \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\beta_1 T_1(t-\sigma_1) V(t-\sigma_1)}{1 + \mu_1 V(t-\sigma_1)} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\bar{\beta}_1 T_1(t-\sigma_1) I_1(t-\sigma_1)}{1 + \bar{\mu}_1 I_1(t-\sigma_1)} \right) \\ & + \check{\eta}_2 \left( 1 - \frac{T_2^0}{T_2} \right) \left( \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\ & + \check{\eta}_2 e^{\epsilon_2 \sigma_2} \left( e^{-\epsilon_2 \sigma_2} \left[ \frac{\beta_2 T_2(t-\sigma_2) V(t-\sigma_2)}{1 + \mu_2 V(t-\sigma_2)} + \frac{\bar{\beta}_2 T_2(t-\sigma_2) I_2(t-\sigma_2)}{1 + \bar{\mu}_2 I_2(t-\sigma_2)} \right] - \delta_2 I_2 \right) \\ & + \check{\eta}_2 \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\beta_2 T_2(t-\sigma_2) V(t-\sigma_2)}{1 + \mu_2 V(t-\sigma_2)} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\bar{\beta}_2 T_2(t-\sigma_2) I_2(t-\sigma_2)}{1 + \bar{\mu}_2 I_2(t-\sigma_2)} \right) \end{aligned}$$

$$\begin{aligned}
 &+ \check{\eta}_3 (b_1 \delta_1 e^{-\varepsilon_1 \omega_1} (I_1 - I_1(t - \omega_1)) + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} (I_2 - I_2(t - \omega_2))) \\
 &+ \check{\eta}_3 (b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(t - \omega_1) + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(t - \omega_2) - \varphi V).
 \end{aligned} \tag{2.26}$$

Using Eq (2.25), we get

$$\begin{aligned}
 \frac{d\check{\Xi}_0}{dt} &= -\check{\eta}_1 \gamma_1 \frac{(T_1 - T_1^0)^2}{T_1} - \check{\eta}_2 \gamma_2 \frac{(T_2 - T_2^0)^2}{T_2} + \frac{\delta_1 \bar{\mu}_1}{1 + \bar{\mu}_1 I_1} (\check{\eta}_3 b_1 e^{-\varepsilon_1 \omega_1} - \check{\eta}_1 e^{\varepsilon_1 \sigma_1}) I_1^2 \\
 &+ \frac{\delta_2 \bar{\mu}_2}{1 + \bar{\mu}_2 I_2} (\check{\eta}_3 b_2 e^{-\varepsilon_2 \omega_2} - \check{\eta}_2 e^{\varepsilon_2 \sigma_2}) I_2^2 - \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (1 - M) V \\
 &+ \frac{V^2}{(1 + \mu_1 V)(1 + \mu_2 V)} ((\check{\eta}_1 \beta_1 \mu_2 T_1^0 - \check{\eta}_3 \varphi \mu_2) + (\check{\eta}_2 \beta_2 \mu_1 T_2^0 - \check{\eta}_3 \varphi \mu_1) - \check{\eta}_3 \varphi \mu_1 \mu_2 V) \\
 &= -\check{\eta}_1 \gamma_1 \frac{(T_1 - T_1^0)^2}{T_1} - \check{\eta}_2 \gamma_2 \frac{(T_2 - T_2^0)^2}{T_2} - \frac{b_1 \delta_1^2 \delta_2 \bar{\mu}_1 e^{-\varepsilon_1 \omega_1}}{1 + \bar{\mu}_1 I_1} (1 - \check{\mathcal{W}}_1) \check{\mathcal{W}}_1 I_1^2 \\
 &- \frac{b_2 \delta_2 \delta_1^2 \bar{\mu}_2 e^{-\varepsilon_2 \omega_2}}{1 + \bar{\mu}_2 I_2} (1 - \check{\mathcal{W}}_2) \check{\mathcal{W}}_2 I_2^2 - \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (1 - M) V \\
 &- \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (\mu_1 (1 - \check{\mathcal{W}}) (1 - \check{\mathcal{W}}_1) + \mu_2 (1 - \check{\mathcal{U}}) (1 - \check{\mathcal{W}}_1)) V^2 \\
 &- \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (\mu_1 \mu_2 (1 - \check{\mathcal{W}}_1) (1 - \check{\mathcal{W}}_1)) V^3.
 \end{aligned} \tag{2.27}$$

As a consequence, we have  $\frac{d\check{\Xi}_0}{dt} \leq 0$  whenever  $\mathcal{R}_0^* \leq 1$ , for all  $T_\ell, I_\ell, V \in (0, \infty)$ , with  $\ell = 1, 2$ . Furthermore, equality holds, i.e.,  $\frac{d\check{\Xi}_0}{dt} = 0$ , when  $T_\ell(t) = T_\ell^0, I_\ell(t) = 0$ , and  $V(t) = 0$  for all  $t$ , and for both  $\ell = 1, 2$ . Therefore, the trajectories of system (2.6)–(2.10) converge to the largest invariant subset  $\check{\Gamma}'_0$ , which is given by  $\check{\Gamma}'_0 = \{\Pi_0\}$ . By invoking LaSalle’s invariance principle (LIP) (see [35–38]), we conclude that  $\Pi_0$  is globally asymptotically stable.

**Theorem 2.** For the system defined by Eqs (2.6)–(2.10), if  $\mathcal{R}_0^* > 1$ , then the endemic equilibrium  $\check{\Pi}$  is globally asymptotically stable.

*Proof.* Formulate a function  $\check{\Xi}_1(T_1, I_1, T_2, I_2, V)$  as

$$\begin{aligned}
 \check{\Xi}_1 &= \bar{\eta}_1 \left[ \check{T}_1 \Theta \left( \frac{T_1}{\check{T}_1} \right) + e^{\varepsilon_1 \sigma_1} \check{I}_1 \Theta \left( \frac{I_1}{\check{I}_1} \right) \right] + \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \int_0^{\sigma_1} \Theta \left( \frac{T_1(t - \xi) V(t - \xi) (1 + \mu_1 \check{V})}{\check{T}_1 \check{V} (1 + \mu_1 V(t - \xi))} \right) d\xi \\
 &+ \frac{\bar{\eta}_1 \bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \int_0^{\sigma_1} \Theta \left( \frac{T_1(t - \xi) I_1(t - \xi) (1 + \bar{\mu}_1 \check{I}_1)}{\check{T}_1 \check{I}_1 (1 + \bar{\mu}_1 I_1(t - \xi))} \right) d\xi \\
 &+ \bar{\eta}_2 \left[ \check{T}_2 \Theta \left( \frac{T_2}{\check{T}_2} \right) + e^{\varepsilon_2 \sigma_2} \check{I}_2 \Theta \left( \frac{I_2}{\check{I}_2} \right) \right] + \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \int_0^{\sigma_2} \Theta \left( \frac{T_2(t - \xi) V(t - \xi) (1 + \mu_2 \check{V})}{\check{T}_2 \check{V} (1 + \mu_2 V(t - \xi))} \right) d\xi \\
 &+ \frac{\bar{\eta}_2 \bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \int_0^{\sigma_2} \Theta \left( \frac{T_2(t - \xi) I_2(t - \xi) (1 + \bar{\mu}_2 \check{I}_2)}{\check{T}_2 \check{I}_2 (1 + \bar{\mu}_2 I_2(t - \xi))} \right) d\xi \\
 &+ b_1 \delta_1 \check{I}_1 e^{-\varepsilon_1 \omega_1} \int_0^{\omega_1} \Theta \left( \frac{I_1(t - \xi)}{\check{I}_1} \right) d\xi + b_2 \delta_2 \check{I}_2 e^{-\varepsilon_2 \omega_2} \int_0^{\omega_2} \Theta \left( \frac{I_2(t - \xi)}{\check{I}_2} \right) d\xi + \check{V} \Theta \left( \frac{V}{\check{V}} \right),
 \end{aligned}$$

where

$$\begin{aligned}\bar{\eta}_1 &= b_1 e^{-(\epsilon_1 \sigma_1 + \epsilon_1 \omega_1)} \left( 1 + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1 (1 + \mu_1 \check{V})}{\beta_1 \check{T}_1 \check{V} (1 + \bar{\mu}_1 \check{I}_1)} \right), \\ \bar{\eta}_2 &= b_2 e^{-(\epsilon_2 \sigma_2 + \epsilon_2 \omega_2)} \left( 1 + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2 (1 + \mu_2 \check{V})}{\beta_2 \check{T}_2 \check{V} (1 + \bar{\mu}_2 \check{I}_2)} \right).\end{aligned}\quad (2.28)$$

It is clear that  $\check{\Xi}_1(T_1, I_1, T_2, I_2, V) > 0$  for all  $T_1, I_1, T_2, I_2, V > 0$ , and  $\check{\Xi}_1(\check{T}_1, \check{I}_1, \check{T}_2, \check{I}_2, \check{V}) = 0$ . Calculating  $\frac{d\check{\Xi}_1}{dt}$  along system, we get

$$\begin{aligned}\frac{d\check{\Xi}_1}{dt} &= \bar{\eta}_1 \left( 1 - \frac{\check{T}_1}{T_1} \right) \left( \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right) \\ &+ \bar{\eta}_1 e^{\epsilon_1 \sigma_1} \left( 1 - \frac{\check{I}_1}{I_1} \right) \left( \frac{\beta_1 e^{-\epsilon_1 \sigma_1} T_1 (t - \sigma_1) V (t - \sigma_1)}{1 + \mu_1 V (t - \sigma_1)} + \frac{\bar{\beta}_1 e^{-\epsilon_1 \sigma_1} T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{1 + \bar{\mu}_1 I_1 (t - \sigma_1)} - \delta_1 I_1 \right) \\ &+ \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \left[ \frac{(1 + \mu_1 \check{V}) T_1 V}{(1 + \mu_1 V) \check{T}_1 \check{V}} - \frac{(1 + \mu_1 \check{V}) T_1 (t - \sigma_1) V (t - \sigma_1)}{(1 + \mu_1 V (t - \sigma_1)) \check{T}_1 \check{V}} \right. \\ &+ \ln \left( \frac{(1 + \mu_1 V) T_1 (t - \sigma_1) V (t - \sigma_1)}{(1 + \mu_1 V (t - \sigma_1)) T_1 V} \right) \left. + \frac{\bar{\eta}_1 \bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \left[ \frac{(1 + \bar{\mu}_1 \check{I}_1) T_1 I_1}{(1 + \bar{\mu}_1 I_1) \check{T}_1 \check{I}_1} \right. \right. \\ &- \left. \frac{(1 + \bar{\mu}_1 \check{I}_1) T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{(1 + \bar{\mu}_1 I_1 (t - \sigma_1)) \check{T}_1 \check{I}_1} + \ln \left( \frac{(1 + \bar{\mu}_1 I_1) T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{(1 + \bar{\mu}_1 I_1 (t - \sigma_1)) T_1 I_1} \right) \right] \\ &+ \bar{\eta}_2 \left( 1 - \frac{\check{T}_2}{T_2} \right) \left( \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\ &+ \bar{\eta}_2 e^{\epsilon_2 \sigma_2} \left( 1 - \frac{\check{I}_2}{I_2} \right) \left( \frac{\beta_2 e^{-\epsilon_2 \sigma_2} T_2 (t - \sigma_2) V (t - \sigma_2)}{1 + \mu_2 V (t - \sigma_2)} + \frac{\bar{\beta}_2 e^{-\epsilon_2 \sigma_2} T_2 (t - \sigma_2) I_2 (t - \sigma_2)}{1 + \bar{\mu}_2 I_2 (t - \sigma_2)} - \delta_2 I_2 \right) \\ &+ \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \left[ \frac{(1 + \mu_2 \check{V}) T_2 V}{(1 + \mu_2 V) \check{T}_2 \check{V}} - \frac{(1 + \mu_2 \check{V}) T_2 (t - \sigma_2) V (t - \sigma_2)}{(1 + \mu_2 V (t - \sigma_2)) \check{T}_2 \check{V}} \right. \\ &+ \ln \left( \frac{(1 + \mu_2 V) T_2 (t - \sigma_2) V (t - \sigma_2)}{(1 + \mu_2 V (t - \sigma_2)) T_2 V} \right) \left. + \frac{\bar{\eta}_2 \bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \left[ \frac{(1 + \bar{\mu}_2 \check{I}_2) T_2 I_2}{(1 + \bar{\mu}_2 I_2) \check{T}_2 \check{I}_2} \right. \right. \\ &- \left. \frac{(1 + \bar{\mu}_2 \check{I}_2) T_2 (t - \sigma_2) I_2 (t - \sigma_2)}{(1 + \bar{\mu}_2 I_2 (t - \sigma_2)) \check{T}_2 \check{I}_2} + \ln \left( \frac{(1 + \bar{\mu}_2 I_2) T_2 (t - \sigma_2) I_2 (t - \sigma_2)}{(1 + \bar{\mu}_2 I_2 (t - \sigma_2)) T_2 I_2} \right) \right] \\ &+ b_1 \delta_1 \check{I}_1 e^{-\epsilon_1 \omega_1} \left[ \frac{I_1}{\check{I}_1} - \frac{I_1 (t - \omega_1)}{\check{I}_1} + \ln \left( \frac{I_1 (t - \omega_1)}{I_1} \right) \right] + b_2 \delta_2 \check{I}_2 e^{-\epsilon_2 \omega_2} \left[ \frac{I_2}{\check{I}_2} - \frac{I_2 (t - \omega_2)}{\check{I}_2} \right. \\ &+ \left. \ln \left( \frac{I_2 (t - \omega_2)}{I_2} \right) \right] + \left( 1 - \frac{\check{V}}{V} \right) (b_1 \delta_1 e^{-\epsilon_1 \omega_1} I_1 (t - \omega_1) + b_2 \delta_2 e^{-\epsilon_2 \omega_2} I_2 (t - \omega_2) - \varphi V).\end{aligned}$$

Thus,

$$\frac{d\check{\Xi}_1}{dt} = \bar{\eta}_1 \left( 1 - \frac{\check{T}_1}{T_1} \right) (\lambda_1 - \gamma_1 T_1) - \bar{\eta}_1 \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\beta_1 \check{T}_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 \check{T}_1 I_1}{1 + \bar{\mu}_1 I_1} \right)$$

$$\begin{aligned}
& + \bar{\eta}_1 \left( \frac{\beta_1 T_1(t - \sigma_1) V(t - \sigma_1)}{1 + \mu_1 V(t - \sigma_1)} + \frac{\bar{\beta}_1 T_1(t - \sigma_1) I_1(t - \sigma_1)}{1 + \bar{\mu}_1 I_1(t - \sigma_1)} - e^{\varepsilon_1 \sigma_1} \delta_1 I_1 \right. \\
& \left. - \frac{\beta_1 T_1(t - \sigma_1) V(t - \sigma_1) \check{I}_1}{1 + \mu_1 V(t - \sigma_1) I_1} - \frac{\bar{\beta}_1 T_1(t - \sigma_1) I_1(t - \sigma_1) \check{I}_1}{1 + \bar{\mu}_1 I_1(t - \sigma_1) I_1} + e^{\varepsilon_1 \sigma_1} \delta_1 \check{I}_1 \right) \\
& + \bar{\eta}_1 \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\beta_1 T_1(t - \sigma_1) V(t - \sigma_1)}{1 + \mu_1 V(t - \sigma_1)} + \frac{\beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \ln \left( \frac{(1 + \mu_1 V) T_1(t - \sigma_1) V(t - \sigma_1)}{(1 + \mu_1 V(t - \sigma_1)) T_1 V} \right) \right) \\
& + \bar{\eta}_1 \left( \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\bar{\beta}_1 T_1(t - \sigma_1) I_1(t - \sigma_1)}{1 + \bar{\mu}_1 I_1(t - \sigma_1)} + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \ln \left( \frac{(1 + \bar{\mu}_1 I_1) T_1(t - \sigma_1) I_1(t - \sigma_1)}{(1 + \bar{\mu}_1 I_1(t - \sigma_1)) T_1 I_1} \right) \right) \\
& + \bar{\eta}_2 \left( 1 - \frac{\check{T}_2}{T_2} \right) (\lambda_2 - \gamma_2 T_2) - \bar{\eta}_2 \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\beta_2 \check{T}_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 \check{T}_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\
& + \bar{\eta}_2 e^{\varepsilon_2 \sigma_2} \left( \frac{\beta_2 e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) V(t - \sigma_2)}{1 + \mu_2 V(t - \sigma_2)} + \frac{\bar{\beta}_2 e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) I_2(t - \sigma_2)}{1 + \bar{\mu}_2 I_2(t - \sigma_2)} - \delta_2 I_2 \right. \\
& \left. - \frac{\beta_2 e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) V(t - \sigma_2) \check{I}_2}{1 + \mu_2 V(t - \sigma_2) I_2} - \frac{\bar{\beta}_2 e^{-\varepsilon_2 \sigma_2} T_2(t - \sigma_2) I_2(t - \sigma_2) \check{I}_2}{1 + \bar{\mu}_2 I_2(t - \sigma_2) I_2} + \delta_2 \check{I}_2 \right) \\
& + \bar{\eta}_2 \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\beta_2 T_2(t - \sigma_2) V(t - \sigma_2)}{1 + \mu_2 V(t - \sigma_2)} + \frac{\beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \ln \left( \frac{(1 + \mu_2 V) T_2(t - \sigma_2) V(t - \sigma_2)}{(1 + \mu_2 V(t - \sigma_2)) T_2 V} \right) \right) \\
& + \bar{\eta}_2 \left( \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\bar{\beta}_2 T_2(t - \sigma_2) I_2(t - \sigma_2)}{1 + \bar{\mu}_2 I_2(t - \sigma_2)} + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \ln \left( \frac{T_2(t - \sigma_2) I_2(t - \sigma_2) (1 + \bar{\mu}_2 I_2)}{(1 + \bar{\mu}_2 I_2(t - \sigma_2)) T_2 I_2} \right) \right) \\
& + \left( b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1 - b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(t - \omega_1) + b_1 \delta_1 \check{I}_1 e^{-\varepsilon_1 \omega_1} \ln \left( \frac{I_1(t - \omega_1)}{I_1} \right) \right) \\
& + \left( b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2 - b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(t - \omega_2) + b_2 \delta_2 \check{I}_2 e^{-\varepsilon_2 \omega_2} \ln \left( \frac{I_2(t - \omega_2)}{I_2} \right) \right) \\
& + b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(t - \omega_1) + b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(t - \omega_2) - \varphi V \\
& - b_1 \delta_1 e^{-\varepsilon_1 \omega_1} I_1(t - \omega_1) \frac{\check{V}}{V} - b_2 \delta_2 e^{-\varepsilon_2 \omega_2} I_2(t - \omega_2) \frac{\check{V}}{V} + \varphi \check{V}.
\end{aligned} \tag{2.29}$$

Simplify Eq (2.29). We then have

$$\begin{aligned}
\frac{d\check{\Xi}_1}{dt} & = \bar{\eta}_1 \left( 1 - \frac{\check{T}_1}{T_1} \right) (\lambda_1 - \gamma_1 T_1) + \bar{\eta}_1 \left( \frac{\beta_1 \check{T}_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 \check{T}_1 I_1}{1 + \bar{\mu}_1 I_1} \right) - \bar{\eta}_1 e^{\varepsilon_1 \sigma_1} \delta_1 I_1 \\
& - \bar{\eta}_1 \left( \frac{\beta_1 T_1(t - \sigma_1) V(t - \sigma_1) \check{I}_1}{1 + \mu_1 V(t - \sigma_1) I_1} + \frac{\bar{\beta}_1 T_1(t - \sigma_1) I_1(t - \sigma_1) \check{I}_1}{1 + \bar{\mu}_1 I_1(t - \sigma_1) I_1} - e^{\varepsilon_1 \sigma_1} \delta_1 \check{I}_1 \right) \\
& + \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \ln \left( \frac{(1 + \mu_1 V) T_1(t - \sigma_1) V(t - \sigma_1)}{(1 + \mu_1 V(t - \sigma_1)) T_1 V} \right) + \frac{\bar{\eta}_1 \bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \ln \left( \frac{(1 + \bar{\mu}_1 I_1) T_1(t - \sigma_1) I_1(t - \sigma_1)}{(1 + \bar{\mu}_1 I_1(t - \sigma_1)) T_1 I_1} \right) \\
& + \bar{\eta}_2 \left( 1 - \frac{\check{T}_2}{T_2} \right) (\lambda_2 - \gamma_2 T_2) + \bar{\eta}_2 \left( \frac{\beta_2 \check{T}_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 \check{T}_2 I_2}{1 + \bar{\mu}_2 I_2} \right) - \bar{\eta}_2 e^{\varepsilon_2 \sigma_2} \delta_2 I_2 \\
& - \bar{\eta}_2 \left( \frac{\beta_2 T_2(t - \sigma_2) V(t - \sigma_2) \check{I}_2}{1 + \mu_2 V(t - \sigma_2) I_2} + \frac{\bar{\beta}_2 T_2(t - \sigma_2) I_2(t - \sigma_2) \check{I}_2}{1 + \bar{\mu}_2 I_2(t - \sigma_2) I_2} - e^{\varepsilon_2 \sigma_2} \delta_2 \check{I}_2 \right) \\
& + \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \ln \left( \frac{(1 + \mu_2 V) T_2(t - \sigma_2) V(t - \sigma_2)}{(1 + \mu_2 V(t - \sigma_2)) T_2 V} \right) + \frac{\bar{\eta}_2 \bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \ln \left( \frac{(1 + \bar{\mu}_2 I_2) T_2(t - \sigma_2) I_2(t - \sigma_2)}{(1 + \bar{\mu}_2 I_2(t - \sigma_2)) T_2 I_2} \right)
\end{aligned}$$

$$\begin{aligned}
& + b_1\delta_1 e^{-\varepsilon_1\omega_1} I_1 + b_1\delta_1 \check{I}_1 e^{-\varepsilon_1\omega_1} \ln\left(\frac{I_1(t-\omega_1)}{I_1}\right) + b_2\delta_2 e^{-\varepsilon_2\omega_2} I_2 + b_2\delta_2 \check{I}_2 e^{-\varepsilon_2\omega_2} \ln\left(\frac{I_2(t-\omega_2)}{I_2}\right) \\
& - \varphi V - b_1\delta_1 e^{-\varepsilon_1\omega_1} I_1(t-\omega_1) \frac{\check{V}}{V} - b_2\delta_2 e^{-\varepsilon_2\omega_2} I_2(t-\omega_2) \frac{\check{V}}{V} + \varphi \check{V}.
\end{aligned} \tag{2.30}$$

Applying the equilibrium conditions for  $\check{\Pi}$ :

$$\begin{aligned}
\lambda_1 &= \gamma_1 \check{T}_1 + \frac{\beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1}, \\
\lambda_2 &= \gamma_2 \check{T}_2 + \frac{\beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2}, \\
\delta_1 \check{I}_1 &= e^{-\varepsilon_1\sigma_1} \left( \frac{\beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \right), \\
\delta_2 \check{I}_2 &= e^{-\varepsilon_2\sigma_2} \left( \frac{\beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \right), \\
\varphi \check{V} &= b_1\delta_1 \check{I}_1 e^{-\varepsilon_1\omega_1} + b_2\delta_2 \check{I}_2 e^{-\varepsilon_2\omega_2} = \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} + \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \\
&= b_1 e^{-(\varepsilon_1\omega_1 + \varepsilon_1\sigma_1)} \left( \frac{\beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \right) + b_2 e^{-(\varepsilon_2\omega_2 + \varepsilon_2\sigma_2)} \left( \frac{\beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \right),
\end{aligned}$$

and reducing Eq (2.30), we get

$$\begin{aligned}
\frac{d\check{\Xi}_1}{dt} &= -\bar{\eta}_1 \gamma_1 \left( \frac{(T_1 - \check{T}_1)^2}{T_1} \right) + \bar{\eta}_1 \left( \frac{\beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} + \frac{\bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \right) \left( 1 - \frac{\check{T}_1}{T_1} \right) + \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \left( \frac{(1 + \mu_1 \check{V}) V}{(1 + \mu_1 V) \check{V}} + 1 \right. \\
&\quad \left. - \frac{V}{\check{V}} - \frac{(1 + \mu_1 \check{V}) T_1 (t - \sigma_1) V (t - \sigma_1) \check{I}_1}{(1 + \mu_1 V (t - \sigma_1)) \check{T}_1 \check{V} I_1} + \ln \left( \frac{(1 + \mu_1 V) T_1 (t - \sigma_1) V (t - \sigma_1)}{(1 + \mu_1 V (t - \sigma_1)) T_1 V} \right) \right) \\
&\quad + \frac{\bar{\eta}_1 \bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \left( \frac{(1 + \bar{\mu}_1 \check{I}_1) I_1}{(1 + \bar{\mu}_1 I_1) \check{I}_1} + 1 - \frac{I_1}{\check{I}_1} - \frac{(1 + \bar{\mu}_1 \check{I}_1) T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{(1 + \bar{\mu}_1 I_1 (t - \sigma_1)) \check{T}_1 I_1} \right. \\
&\quad \left. + \ln \left( \frac{(1 + \bar{\mu}_1 I_1) T_1 (t - \sigma_1) I_1 (t - \sigma_1)}{(1 + \bar{\mu}_1 I_1 (t - \sigma_1)) T_1 I_1} \right) \right) + \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \left( 1 - \frac{\check{V} I_1 (t - \omega_1)}{V \check{I}_1} + \ln \left( \frac{I_1 (t - \omega_1)}{I_1} \right) \right) \\
&\quad - \bar{\eta}_2 \gamma_2 \left( \frac{(T_2 - \check{T}_2)^2}{T_2} \right) + \bar{\eta}_2 \left( \frac{\beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} + \frac{\bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \right) \left( 1 - \frac{\check{T}_2}{T_2} \right) + \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \left( \frac{(1 + \mu_2 \check{V}) V}{(1 + \mu_2 V) \check{V}} + 1 \right. \\
&\quad \left. - \frac{V}{\check{V}} - \frac{(1 + \mu_2 \check{V}) T_2 (t - \sigma_2) V (t - \sigma_2) \check{I}_2}{(1 + \mu_2 V (t - \sigma_2)) \check{T}_2 \check{V} I_2} + \ln \left( \frac{(1 + \mu_2 V) T_2 (t - \sigma_2) V (t - \sigma_2)}{(1 + \mu_2 V (t - \sigma_2)) T_2 V} \right) \right) \\
&\quad + \frac{\bar{\eta}_2 \bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \left( \frac{(1 + \bar{\mu}_2 \check{I}_2) I_2}{(1 + \bar{\mu}_2 I_2) \check{I}_2} + 1 - \frac{I_2}{\check{I}_2} - \frac{(1 + \bar{\mu}_2 \check{I}_2) T_2 (t - \sigma_2) I_2 (t - \sigma_2)}{(1 + \bar{\mu}_2 I_2 (t - \sigma_2)) \check{T}_2 I_2} \right)
\end{aligned}$$

$$+ \ln \left( \frac{(1 + \bar{\mu}_2 I_2) T_2(t - \sigma_2) I_2(t - \sigma_2)}{(1 + \bar{\mu}_2 I_2(t - \sigma_2)) T_2 I_2} \right) + \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \left( 1 - \frac{\check{V} I_2(t - \omega_2)}{V \check{I}_2} + \ln \left( \frac{I_2(t - \omega_2)}{I_2} \right) \right). \quad (2.31)$$

Use the following equalities for all  $\ell = 1, 2$ :

$$\begin{aligned} \ln \left( \frac{(1 + \mu_\ell V) T_\ell(t - \sigma_\ell) V(t - \sigma_\ell)}{(1 + \mu_\ell V(t - \sigma_\ell)) T_\ell V} \right) &= \ln \left( \frac{\check{T}_\ell}{T_\ell} \right) + \ln \left( \frac{(1 + \mu_\ell \check{V}) T_\ell(t - \sigma_\ell) V(t - \sigma_\ell) \check{I}_\ell}{(1 + \mu_\ell V(t - \sigma_\ell)) \check{T}_\ell \check{V} I_\ell} \right) \\ &\quad + \ln \left( \frac{I_\ell \check{V}}{\check{I}_\ell V} \right) + \ln \left( \frac{1 + \mu_\ell V}{1 + \mu_\ell \check{V}} \right), \\ \ln \left( \frac{(1 + \bar{\mu}_\ell I_\ell) T_\ell(t - \sigma_\ell) I_\ell(t - \sigma_\ell)}{(1 + \bar{\mu}_\ell I_\ell(t - \sigma_\ell)) T_\ell I_\ell} \right) &= \ln \left( \frac{\check{T}_\ell}{T_\ell} \right) + \ln \left( \frac{(1 + \bar{\mu}_\ell \check{I}_\ell) T_\ell(t - \sigma_\ell) I_\ell(t - \sigma_\ell)}{(1 + \bar{\mu}_\ell I_\ell(t - \sigma_\ell)) \check{T}_\ell \check{I}_\ell} \right) \\ &\quad + \ln \left( \frac{1 + \bar{\mu}_\ell I_\ell}{1 + \bar{\mu}_\ell \check{I}_\ell} \right), \\ \ln \left( \frac{I_\ell(t - \omega_\ell)}{I_\ell} \right) &= \ln \left( \frac{I_\ell(t - \omega_\ell) \check{V}}{V \check{I}_\ell} \right) + \ln \left( \frac{\check{I}_\ell V}{I_\ell \check{V}} \right). \end{aligned} \quad (2.32)$$

Equation (2.31) becomes

$$\begin{aligned} \frac{d\check{\Xi}_1}{dt} &= -\bar{\eta}_1 \gamma_1 \left( \frac{(T_1 - \check{T}_1)^2}{T_1} \right) - \frac{\bar{\eta}_1 \beta_1 \check{T}_1 \check{V}}{1 + \mu_1 \check{V}} \left( \Theta \left( \frac{\check{T}_1}{T_1} \right) + \Theta \left( \frac{(1 + \mu_1 \check{V}) T_1(t - \sigma_1) V(t - \sigma_1) \check{I}_1}{(1 + \mu_1 V(t - \sigma_1)) \check{T}_1 \check{V} I_1} \right) \right. \\ &\quad \left. + \Theta \left( \frac{I_1(t - \omega_1) \check{V}}{V \check{I}_1} \right) + \Theta \left( \frac{1 + \mu_1 V}{1 + \mu_1 \check{V}} + \frac{\mu_1 (V - \check{V})^2}{(1 + \mu_1 V)(1 + \mu_1 \check{V}) \check{V}} \right) \right. \\ &\quad \left. - \frac{\bar{\eta}_1 \bar{\beta}_1 \check{T}_1 \check{I}_1}{1 + \bar{\mu}_1 \check{I}_1} \left( \Theta \left( \frac{\check{T}_1}{T_1} \right) + \Theta \left( \frac{(1 + \bar{\mu}_1 \check{I}_1) T_1(t - \sigma_1) I_1(t - \sigma_1)}{(1 + \bar{\mu}_1 I_1(t - \sigma_1)) \check{T}_1 \check{I}_1} \right) + \Theta \left( \frac{1 + \bar{\mu}_1 I_1}{1 + \bar{\mu}_1 \check{I}_1} \right) \right. \right. \\ &\quad \left. \left. + \frac{\bar{\mu}_1 (I_1 - \check{I}_1)^2}{(1 + \bar{\mu}_1 \check{I}_1)(1 + \bar{\mu}_1 I_1) \check{I}_1} \right) - \bar{\eta}_2 \gamma_2 \left( \frac{(T_2 - \check{T}_2)^2}{T_2} \right) \right. \\ &\quad \left. - \frac{\bar{\eta}_2 \beta_2 \check{T}_2 \check{V}}{1 + \mu_2 \check{V}} \left( \Theta \left( \frac{\check{T}_2}{T_2} \right) + \Theta \left( \frac{(1 + \mu_2 \check{V}) T_2(t - \sigma_2) V(t - \sigma_2) \check{I}_2}{(1 + \mu_2 V(t - \sigma_2)) \check{T}_2 \check{V} I_2} \right) + \Theta \left( \frac{\check{V} I_2(t - \omega_2)}{V \check{I}_2} \right) \right. \right. \\ &\quad \left. \left. + \Theta \left( \frac{1 + \mu_2 V}{1 + \mu_2 \check{V}} + \frac{\mu_2 (V - \check{V})^2}{(1 + \mu_2 V)(1 + \mu_2 \check{V}) \check{V}} \right) - \frac{\bar{\eta}_2 \bar{\beta}_2 \check{T}_2 \check{I}_2}{1 + \bar{\mu}_2 \check{I}_2} \left( \Theta \left( \frac{\check{T}_2}{T_2} \right) \right. \right. \\ &\quad \left. \left. + \Theta \left( \frac{(1 + \bar{\mu}_2 \check{I}_2) T_2(t - \sigma_2) I_2(t - \sigma_2)}{(1 + \bar{\mu}_2 I_2(t - \sigma_2)) \check{T}_2 \check{I}_2} \right) + \Theta \left( \frac{1 + \bar{\mu}_2 I_2}{1 + \bar{\mu}_2 \check{I}_2} \right) + \frac{\bar{\mu}_2 (I_2 - \check{I}_2)^2}{(1 + \bar{\mu}_2 \check{I}_2)(1 + \bar{\mu}_2 I_2) \check{I}_2} \right) \right). \end{aligned}$$

As a result, we have  $\frac{d\check{\Xi}_1}{dt} \leq 0$  for all  $T_1, I_1, T_2, I_2, V > 0$ . Moreover, equality holds, i.e.,  $\frac{d\check{\Xi}_1}{dt} = 0$ , if and only if

$$(T_1(t), I_1(t), T_2(t), I_2(t), V(t)) = (\check{T}_1, \check{I}_1, \check{T}_2, \check{I}_2, \check{V})$$

for all  $t$ . Therefore, the trajectories of system (2.6)–(2.10) converge to the largest invariant subset  $\check{\Gamma}'_1 = \{\check{\Pi}\}$ . By applying LaSalle's invariance principle (LIP), we conclude that  $\check{\Pi}$  is GAS when  $\mathcal{R}_0^* > 1$ .

### 3. Model with distributed delays

This section contains the formulation and analysis of the model with distributed delays.

#### 3.1. Model formulation

In this section, we extend system (2.1)–(2.5) by incorporating distributed time delays. The formulation in this system includes distributed delays to explain how viruses infect cells and make new viruses. This method is a natural extension of the discrete-delay framework, which lets the model take into account how long the eclipse phase lasts and how long it takes for the virus to be released. The distributed-delay structure incorporates a range of potential delay times, illustrating that infected cells do not advance through their life cycle at a consistent rate; rather, they display a variety of maturation times. This approach captures the natural variation in delay times while still keeping the main ideas of the discrete-delay model. In turn, the continuous-delay system offers a more flexible and biologically realistic view of the infection process. The resulting model is expressed as follows:

$$\dot{T}_1 = \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1}, \quad (3.1)$$

$$\dot{I}_1 = \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left[ \frac{\beta_1 T_1(t-\tau) V(t-\tau)}{1 + \mu_1 V(t-\tau)} + \frac{\bar{\beta}_1 T_1(t-\tau) I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} \right] d\tau - \delta_1 I_1, \quad (3.2)$$

$$\dot{T}_2 = \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2}, \quad (3.3)$$

$$\dot{I}_2 = \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left[ \frac{\beta_2 T_2(t-\tau) V(t-\tau)}{1 + \mu_2 V(t-\tau)} + \frac{\bar{\beta}_2 T_2(t-\tau) I_2(t-\tau)}{1 + \bar{\mu}_2 I_2(t-\tau)} \right] d\tau - \delta_2 I_2, \quad (3.4)$$

$$\dot{V} = b_1 \int_0^{h_3} g_1(\tau) \delta_1 e^{-\varepsilon_1 \tau} I_1(t-\tau) d\tau + b_2 \int_0^{h_4} g_2(\tau) \delta_2 e^{-\varepsilon_2 \tau} I_2(t-\tau) d\tau - \varphi V, \quad (3.5)$$

where, in each integral,  $\tau$  denotes the delay variable with probability density  $f_1(\tau)$ ,  $f_2(\tau)$ ,  $g_1(\tau)$ , or  $g_2(\tau)$ , supported on the interval  $[0, h_\ell]$ . Here,  $\ell = 1, \dots, 4$  indexes the corresponding delay, and  $h_\ell > 0$  represents the maximal (upper bound) delay duration.

We make the following assumptions:

- (I) The probability that an uninfected CD4<sup>+</sup> T cell or an uninfected macrophage exposed to HIV at time  $t - \tau$  remains alive for  $\tau$  units of time and becomes infected at time  $t$  is given by the expressions  $f_1(\tau)e^{-\varepsilon_1 \tau}$  and  $f_2(\tau)e^{-\varepsilon_2 \tau}$ , respectively.
- (II) The probability that a newly produced immature HIV particle present at time  $t - \tau$  survives for  $\tau$  time units and matures at time  $t$  is represented by  $g_1(\tau)e^{-\varepsilon_1 \tau}$  and  $g_2(\tau)e^{-\varepsilon_2 \tau}$ .

Let us assume that  $f_\ell(\tau)$  and  $g_\ell(\tau)$  satisfy  $f_\ell(\tau) > 0$ ,  $g_\ell(\tau) > 0$ ;  $\ell = 1, 2$  and

$$\int_0^{h_\ell} f_\ell(\tau) d\tau = 1, \quad \int_0^{h_\ell} f_\ell(\varpi) e^{-\vartheta \varpi} d\varpi < \infty,$$

$$\int_0^{h_{\ell+2}} g_\ell(\tau) d\tau = 1, \quad \int_0^{h_{\ell+2}} g_\ell(\varpi) e^{-\vartheta\varpi} d\varpi < \infty, \quad \ell = 1, 2,$$

where  $\vartheta > 0$ . Denote  $\varsigma_\ell = \int_0^{h_\ell} f_\ell(\tau) e^{-\epsilon_\ell\tau} d\tau$  and  $\varsigma_{\ell+2} = \int_0^{h_{\ell+2}} g_\ell(\tau) e^{-\epsilon_\ell\tau} d\tau$ ,  $\ell = 1, 2$ , and  $0 < \varsigma_\ell \leq 1$ , thus  $0 < \varsigma_\ell \leq 1$  [39, 40].

The initial conditions for system (3.1)–(3.5) are taken to be the same as those given in Eq (2.11). In this setting, we define  $\tau^* = \max\{h_1, h_2, h_3, h_4\}$ .

### 3.2. Properties of the solution

The nonnegativity and boundedness of the solution of system (3.1)–(3.5) are established by the following Lemma:

**Lemma 3.** *The solutions  $(T_1(t), I_1(t), T_2(t), I_2(t), \text{ and } V(t))$  of model (3.1)–(3.5) with initial conditions (2.11) are non-negative and are ultimately bounded.*

*Proof.* We have  $\dot{T}_1|_{T_1=0} = \lambda_1 > 0$  and  $\dot{T}_2|_{T_2=0} = \lambda_2 > 0$ ; hence,  $T_1(t) > 0$ ,  $T_2(t) > 0$  for all  $t \geq 0$  (see Proposition B.7 of [31]). From Eqs (3.2), (3.4), and (3.5), we have

$$\begin{aligned} I_1(t) &= e^{-\delta_1 t} \chi_2(0) + \int_0^t e^{-\delta_1(t-z)} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1\tau} \left[ \frac{\beta_1 T_1(z-\tau)V(z-\tau)}{1 + \mu_1 V(z-\tau)} + \frac{\bar{\beta}_1 T_1(z-\tau)I_1(z-\tau)}{1 + \bar{\mu}_1 I_1(z-\tau)} \right] d\tau dz, \\ I_2(t) &= e^{-\delta_2 t} \chi_4(0) + \int_0^t e^{-\delta_2(t-z)} \int_0^{h_2} f_2(\tau) e^{-\epsilon_2\tau} \left[ \frac{\beta_2 T_2(z-\tau)V(z-\tau)}{1 + \mu_2 V(z-\tau)} + \frac{\bar{\beta}_2 T_2(z-\tau)I_2(z-\tau)}{1 + \bar{\mu}_2 I_2(z-\tau)} \right] d\tau dz, \\ V(t) &= e^{-\varphi t} \chi_5(0) + \int_0^t e^{-\varphi(t-z)} \left[ b_1 \int_0^{h_3} g_1(\tau) \delta_1 e^{-\epsilon_1\tau} I_1(z-\tau) d\tau + b_2 \int_0^{h_4} g_2(\tau) \delta_2 e^{-\epsilon_2\tau} I_2(z-\tau) d\tau \right] dz, \end{aligned}$$

which yield that  $I_1(t) \geq 0$ ,  $I_2(t) \geq 0$ , and  $V(t) \geq 0$  for all  $t \in [0, \tau^*]$  [32]. Hence, by recursive argumentation, we obtain that  $I_1(t) \geq 0$ ,  $I_2(t) \geq 0$  and  $V(t) \geq 0$  for all  $t \geq 0$ .

Now from Eq (3.1), we have

$$\dot{T}_1(t) \leq \lambda_1 - \gamma_1 T_1(t),$$

which yields

$$0 < T_1(t) \leq \frac{\lambda_1}{\gamma_1} =: M_{T_1} \quad \text{for all } t \geq 0.$$

Since  $\frac{x}{1+\mu x} \leq \frac{1}{\mu}$  for  $x \geq 0$ , and using  $T_1(t) \leq M_{T_1}$ , we have

$$\frac{\beta_1 T_1(t-\tau)V(t-\tau)}{1 + \mu_1 V(t-\tau)} \leq \frac{\beta_1}{\mu_1} M_{T_1}, \quad \frac{\bar{\beta}_1 T_1(t-\tau)I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} \leq \frac{\bar{\beta}_1}{\bar{\mu}_1} M_{T_1}.$$

Next we consider the infected cells  $I_1(t)$  using Eq (3.2):

$$\begin{aligned} \dot{I}_1(t) &\leq \left( \frac{\beta_1}{\mu_1} + \frac{\bar{\beta}_1}{\bar{\mu}_1} \right) M_{T_1} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1\tau} d\tau - \delta_1 I_1(t) \\ &= \left( \frac{\beta_1}{\mu_1} + \frac{\bar{\beta}_1}{\bar{\mu}_1} \right) M_{T_1} \varsigma_1 - \delta_1 I_1(t). \end{aligned}$$

Hence,

$$\dot{I}_1(t) + \delta_1 I_1(t) \leq \left( \frac{\beta_1}{\mu_1} + \frac{\bar{\beta}_1}{\bar{\mu}_1} \right) M_{T_1}.$$

Thus,

$$0 \leq I_1(t) \leq \max \left\{ I_1(0), \left( \frac{\beta_1}{\mu_1} + \frac{\bar{\beta}_1}{\bar{\mu}_1} \right) \frac{M_{T_1}}{\delta_1} \right\} =: M_{I_1}.$$

Similarly, by applying the same argument to Eqs (3.3) and (3.4), we get

$$0 < T_2(t) \leq M_{T_2} := \frac{\lambda_2}{\gamma_2}, \quad 0 \leq I_2(t) \leq M_{I_2} := \max \left\{ I_2(0), \left( \frac{\beta_2}{\mu_2} + \frac{\bar{\beta}_2}{\bar{\mu}_2} \right) \frac{M_{T_2}}{\delta_2} \right\}.$$

Next, from Eq (3.5), using boundedness of  $I_1$  and  $I_2$ ,

$$\dot{V}(t) \leq b_1 \delta_1 M_{I_1} \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} d\tau + b_2 \delta_2 M_{I_2} \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} d\tau - \varphi V(t).$$

Then,

$$\dot{V}(t) + \varphi V(t) \leq b_1 \delta_1 M_{I_1} S_3 + b_2 \delta_2 M_{I_2} S_4.$$

Solving this inequality gives

$$0 \leq V(t) \leq \max \left\{ V(0), \frac{b_1 \delta_1 B_1 M_{I_1} + b_2 \delta_2 B_2 M_{I_2}}{\varphi} \right\} =: M_V.$$

Therefore, the variables  $T_1(t)$ ,  $I_1(t)$ ,  $T_2(t)$ ,  $I_2(t)$ , and  $V(t)$  are ultimately bounded.

### 3.3. Equilibria

Model (3.1)–(3.5) consistently possesses an infection-free equilibrium (IFE), denoted by  $\Pi_0 = (T_1^0, 0, T_2^0, 0, 0)$ , where  $T_1^0 = \frac{\lambda_1}{\gamma_1}$ ,  $T_2^0 = \frac{\lambda_2}{\gamma_2}$ , which represents the state where no infection persists in the system. In addition to  $\Pi_0$ , the model also admits a positive infection equilibrium (IPE), which corresponds to the persistent presence of the infection in vivo  $\tilde{\Pi} = (\tilde{T}_1, \tilde{I}_1, \tilde{T}_2, \tilde{I}_2, \tilde{V})$ , where

$$\begin{aligned} \tilde{T}_1 &= \frac{\lambda_1(1 + \mu_1 \tilde{V})(1 + \bar{\mu}_1 \tilde{I}_1)}{\gamma_1(1 + \mu_1 \tilde{V})(1 + \bar{\mu}_1 \tilde{I}_1) + \beta_1 \tilde{V}(1 + \bar{\mu}_1 \tilde{I}_1) + \bar{\beta}_1 \tilde{I}_1(1 + \mu_1 \tilde{V})}, \\ \tilde{I}_1 &= \frac{-\tilde{B}_1 + \sqrt{\tilde{B}_1^2 - 4\tilde{A}_1 \tilde{C}_1}}{2\tilde{A}_1}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \tilde{T}_2 &= \frac{\lambda_2(1 + \mu_2 \tilde{V})(1 + \bar{\mu}_2 \tilde{I}_2)}{\gamma_2(1 + \mu_2 \tilde{V})(1 + \bar{\mu}_2 \tilde{I}_2) + \beta_2 \tilde{V}(1 + \bar{\mu}_2 \tilde{I}_2) + \bar{\beta}_2 \tilde{I}_2(1 + \mu_2 \tilde{V})}, \\ \tilde{I}_2 &= \frac{-\tilde{B}_2 + \sqrt{\tilde{B}_2^2 - 4\tilde{A}_2 \tilde{C}_2}}{2\tilde{A}_2}, \end{aligned}$$

$$\begin{aligned} \tilde{A}_1 &= \delta_1 \left( (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1)(1 + \mu_1 \tilde{V}) + \tilde{V} \beta_1 \bar{\mu}_1 \right), \\ \tilde{B}_1 &= - \left( (\varsigma_1 \bar{\beta}_1 \lambda_1 - \delta_1 \gamma_1)(1 + \mu_1 \tilde{V}) - \tilde{V} \beta_1 (\delta_1 - \lambda_1 \bar{\mu}_1 \varsigma_1) \right), \\ \tilde{C}_1 &= -\varsigma_1 \tilde{V} \beta_1 \lambda_1, \\ \tilde{A}_2 &= \delta_2 \left( (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2)(1 + \mu_2 \tilde{V}) + \tilde{V} \beta_2 \bar{\mu}_2 \right), \\ \tilde{B}_2 &= - \left( (\varsigma_2 \bar{\beta}_2 \lambda_2 - \delta_2 \gamma_2)(1 + \mu_2 \tilde{V}) - \tilde{V} \beta_2 (\delta_2 - \lambda_2 \bar{\mu}_2 \varsigma_2) \right), \\ \tilde{C}_2 &= -\varsigma_2 \tilde{V} \beta_2 \lambda_2, \end{aligned} \tag{3.7}$$

and  $\tilde{V}$  satisfies the following equation:

$$\varphi \tilde{V} = \varsigma_3 b_1 \delta_1 \tilde{I}_1 + \varsigma_4 b_2 \delta_2 \tilde{I}_2. \quad (3.8)$$

### 3.3.1. The basic reproduction number

Using similar methodology as in Section 2.3.1, we determine the basic reproduction number as

$$\tilde{\mathcal{R}}_0 = \frac{1}{2} \left( \tilde{U} + \tilde{W} + \sqrt{(\tilde{U} - \tilde{W})^2 + 4\tilde{\mathcal{U}}_2\tilde{\mathcal{W}}_2} \right), \quad (3.9)$$

where

$$\begin{aligned} \tilde{U} &= \tilde{\mathcal{U}}_1 + \tilde{\mathcal{U}}_2, & \tilde{W} &= \tilde{\mathcal{W}}_1 + \tilde{\mathcal{W}}_2, \\ \tilde{\mathcal{U}}_1 &= \frac{\bar{\beta}_1 T_1^0 \varsigma_1}{\delta_1}, & \tilde{\mathcal{U}}_2 &= \frac{b_1 \beta_1 T_1^0 \varsigma_1 \varsigma_3}{\varphi}, \\ \tilde{\mathcal{W}}_1 &= \frac{\bar{\beta}_2 T_2^0 \varsigma_2}{\delta_2}, & \tilde{\mathcal{W}}_2 &= \frac{b_2 \beta_2 T_2^0 \varsigma_2 \varsigma_4}{\varphi}. \end{aligned} \quad (3.10)$$

In the next lemma, we identify a condition that guarantees the existence of a positive solution for  $\tilde{V}$ .

**Proposition 3.** *Assume that  $\tilde{\mathcal{R}}_0 > 1$ . If  $\tilde{U} < 1$  and  $\tilde{W} < 1$ , then the expression*

$$\tilde{M} = \frac{\tilde{\mathcal{U}}_2}{1 - \tilde{\mathcal{U}}_1} + \frac{\tilde{\mathcal{W}}_2}{1 - \tilde{\mathcal{W}}_1}$$

satisfies  $\tilde{M} > 1$ .

*Proof.* The proof follows the same reasoning as in Proposition 1.

**Proposition 4.** *If  $\tilde{\mathcal{R}}_0 < 1$ , then the infection-present equilibrium (IPE)  $\tilde{\Pi}$  exists.*

*Proof.* Any equilibrium must satisfy

$$0 = \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1}, \quad (3.11)$$

$$0 = \varsigma_1 \left[ \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right] - \delta_1 I_1, \quad (3.12)$$

$$0 = \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2}, \quad (3.13)$$

$$0 = \varsigma_2 \left[ \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right] - \delta_2 I_2, \quad (3.14)$$

$$0 = \varsigma_3 b_1 \delta_1 I_1 + \varsigma_4 b_2 \delta_2 I_2 - \varphi V. \quad (3.15)$$

For the equilibrium  $\tilde{\Pi}$ , we assume  $V \neq 0$ . From Eq (3.15), we obtain

$$\frac{b_1 \delta_1 \varsigma_3 I_1}{\varphi} + \frac{b_2 \delta_2 \varsigma_4 I_2}{\varphi} - V = 0.$$

Substituting the expressions from Eqs (3.11)–(3.14) into the equation above yields

$$s_1 s_3 \left( \frac{b_1 \lambda_1}{\varphi} - \frac{b_1 \gamma_1 T_1}{\varphi} \right) + s_2 s_4 \left( \frac{b_2 \lambda_2}{\varphi} - \frac{b_2 \gamma_2 T_2}{\varphi} \right) - V = 0.$$

Since  $T_1, I_1, T_2,$  and  $I_2$  are functions of  $V$ , we define the function  $\tilde{G}(V)$  as

$$\tilde{G}(V) = s_1 s_3 \left( \frac{b_1 \lambda_1}{\varphi} - \frac{b_1 \gamma_1 T_1}{\varphi} \right) + s_2 s_4 \left( \frac{b_2 \lambda_2}{\varphi} - \frac{b_2 \gamma_2 T_2}{\varphi} \right) - V,$$

where  $T_1, I_1, T_2,$  and  $I_2$  satisfy Eqs (3.11)–(3.14).

To demonstrate the existence of a positive root, consider the value

$$\tilde{V}^* = \frac{b_1 \lambda_1 s_1 s_3}{\varphi} + \frac{b_2 \lambda_2 s_2 s_4}{\varphi} > 0.$$

At  $V = \tilde{V}^*$ , we have  $T_1(\tilde{V}^*) > 0, I_1(\tilde{V}^*) > 0, T_2(\tilde{V}^*) > 0,$  and  $I_2(\tilde{V}^*) > 0$ . Evaluating  $\tilde{G}(\tilde{V}^*)$  gives

$$\tilde{G}(\tilde{V}^*) = -\frac{b_1 \gamma_1 T_1(\tilde{V}^*) s_1 s_3}{\varphi} - \frac{b_2 \gamma_2 T_2(\tilde{V}^*) s_2 s_4}{\varphi} < 0.$$

To determine the behavior of the function  $\tilde{G}(V)$  near  $V = 0$ , we first compute  $\tilde{G}(0)$  and its derivative as follows:

$$\begin{aligned} \tilde{G}(0) &= s_1 s_3 \left( \frac{b_1 \lambda_1}{\varphi} - \frac{b_1 \gamma_1}{\varphi} T_1(0) \right) + s_2 s_4 \left( \frac{b_2 \lambda_2}{\varphi} - \frac{b_2 \gamma_2}{\varphi} T_2(0) \right) \\ &= \frac{\tilde{\mathcal{U}}_2 \gamma_1 \bar{\beta}_1}{\beta_1 (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1)} \left( 1 - \frac{2}{2 - (1 - \tilde{\mathcal{U}}_1) + |1 - \tilde{\mathcal{U}}_1|} \right) + \frac{\tilde{\mathcal{W}}_2 \gamma_2 \bar{\beta}_2}{\beta_2 (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2)} \left( 1 - \frac{2}{2 - (1 - \tilde{\mathcal{W}}_1) + |1 - \tilde{\mathcal{W}}_1|} \right). \end{aligned}$$

Differentiating  $\tilde{G}(V)$  at  $V = 0$  gives

$$\tilde{G}'(0) = -s_1 s_3 \left( \frac{b_1 \gamma_1}{\varphi} T_1'(0) \right) - s_2 s_4 \left( \frac{b_2 \gamma_2}{\varphi} T_2'(0) \right) - 1.$$

The expressions for  $\tilde{G}(0)$  and  $\tilde{G}'(0)$  depend on the threshold quantities  $\tilde{\mathcal{U}}_1$  and  $\tilde{\mathcal{W}}_1$ , giving rise to the following cases.

**Case 1.** If  $\tilde{\mathcal{U}}_1 = 1$  and  $\tilde{\mathcal{W}}_1 \leq 1$ , then  $\tilde{G}(0) = 0$ , and

$$\lim_{\tilde{\mathcal{U}}_1 \rightarrow 1} \tilde{G}'(0) = +\infty.$$

**Case 2.** If  $\tilde{\mathcal{U}}_1 \leq 1$  and  $\tilde{\mathcal{W}}_1 = 1$ , then  $\tilde{G}(0) = 0$ , and

$$\lim_{\tilde{\mathcal{W}}_1 \rightarrow 1} \tilde{G}'(0) = +\infty.$$

**Case 3.** If  $\tilde{\mathcal{U}}_1 < 1$  and  $\tilde{\mathcal{W}}_1 < 1$ , then  $\tilde{G}(0) = 0$ , and by Proposition 3,

$$\tilde{G}'(0) = \tilde{M} - 1 > 0.$$

**Case 4.** If  $\tilde{\mathcal{U}}_1 \leq 1$  and  $\tilde{\mathcal{W}}_1 > 1$ , then

$$\tilde{G}(0) = \frac{\tilde{\mathcal{W}}_2 \gamma_2 \bar{\beta}_2}{\beta_2 (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2)} \left( \frac{\tilde{\mathcal{W}}_1 - 1}{\tilde{\mathcal{W}}_1} \right) > 0.$$

**Case 5.** If  $\tilde{\mathcal{U}}_1 > 1$  and  $\tilde{\mathcal{W}}_1 \leq 1$ , then

$$\tilde{G}(0) = \frac{\tilde{\mathcal{U}}_2 \gamma_1 \bar{\beta}_1}{\beta_1 (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1)} \left( \frac{\tilde{\mathcal{U}}_1 - 1}{\tilde{\mathcal{U}}_1} \right) > 0.$$

**Case 6.** If  $\tilde{\mathcal{U}}_1 > 1$  and  $\tilde{\mathcal{W}}_1 > 1$ , then

$$\tilde{G}(0) = \frac{\tilde{\mathcal{U}}_2 \gamma_1 \bar{\beta}_1}{\beta_1 (\gamma_1 \bar{\mu}_1 + \bar{\beta}_1)} \left( \frac{\tilde{\mathcal{U}}_1 - 1}{\tilde{\mathcal{U}}_1} \right) + \frac{\tilde{\mathcal{W}}_2 \gamma_2 \bar{\beta}_2}{\beta_2 (\gamma_2 \bar{\mu}_2 + \bar{\beta}_2)} \left( \frac{\tilde{\mathcal{W}}_1 - 1}{\tilde{\mathcal{W}}_1} \right) > 0.$$

In Cases 1–3, the function  $\tilde{G}(V)$  is increasing in a right neighborhood of  $V = 0$ , while in Cases 4–6, we have  $\tilde{G}(0) > 0$ . Since  $\tilde{G}(V) < 0$  for some sufficiently large  $V > 0$ , it follows that when  $\tilde{\mathcal{R}}_0 > 1$ , there exists a root

$$\tilde{V} \in (0, \tilde{V}^*) \quad \text{such that} \quad \tilde{G}(\tilde{V}) = 0.$$

By Eqs (3.6) and (3.7), the corresponding equilibrium satisfies

$$\tilde{T}_1 > 0, \quad \tilde{I}_1 > 0, \quad \tilde{T}_2 > 0, \quad \tilde{I}_2 > 0, \quad \tilde{V} > 0,$$

if  $\tilde{\mathcal{R}}_0 > 1$ .

From above, we can conclude that (i) if  $\tilde{\mathcal{R}}_0 \leq 1$ , then there will be only one equilibrium  $\Pi_0$ ; and (ii) if  $\tilde{\mathcal{R}}_0 > 1$ , then there will be two equilibria  $\Pi_0$  and  $\tilde{\Pi}$ .

### 3.4. Global stability

To analyze the global asymptotic stability of both equilibria  $\Pi_0$  and  $\tilde{\Pi}$  for system (3.1)–(3.5), let  $\tilde{\Gamma}'_i$  be the maximal invariant subset of

$$\tilde{\Gamma}'_i = \left\{ (T_1, I_1, T_2, I_2, V) : \frac{d\tilde{\Xi}_i}{dt} = 0 \right\}, \quad i = 1, 2,$$

and we need to present the following lemma:

**Lemma 4.** Assume that  $\tilde{\mathcal{R}}_0 \leq 1$ . Then:

- (i)  $\tilde{\mathcal{U}}_1 \leq 1$ ,  $\tilde{\mathcal{U}}_2 \leq 1$ ,  $\tilde{\mathcal{W}}_1 \leq 1$ ,  $\tilde{\mathcal{W}}_2 \leq 1$ ;
- (ii) If  $\tilde{M} = \tilde{U} + \tilde{W} - \tilde{U}\tilde{W} + \tilde{\mathcal{U}}_2\tilde{\mathcal{W}}_2$ , then  $0 < \tilde{M} \leq 1$ .

*Proof.* The proof follows the same structure as that of Lemma 2.

**Theorem 3.** For system (3.1)–(3.5), if  $\tilde{\mathcal{R}}_0 \leq 1$ , then the disease-free equilibrium  $\Pi_0$  is globally asymptotically stable.

*Proof.* Assume  $\tilde{\mathcal{R}}_0 \leq 1$ , and define the Lyapunov candidate function  $\tilde{\Xi}_0(T_1, I_1, T_2, I_2, V)$  as:

$$\begin{aligned} \tilde{\Xi}_0 = & \tilde{\eta}_1 \left[ T_1^0 \Theta \left( \frac{T_1}{T_1^0} \right) + \varsigma_1^{-1} I_1 \right] + \tilde{\eta}_1 \varsigma_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \int_{t-\tau}^t \left( \frac{\beta_1 T_1(\xi) V(\xi)}{1 + \mu_1 V(\xi)} + \frac{\bar{\beta}_1 T_1(\xi) I_1(\xi)}{1 + \bar{\mu}_1 I_1(\xi)} \right) d\xi d\tau \\ & + \tilde{\eta}_2 \left[ T_2^0 \Theta \left( \frac{T_2}{T_2^0} \right) + \varsigma_2^{-1} I_2 \right] + \tilde{\eta}_2 \varsigma_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \int_{t-\tau}^t \left( \frac{\beta_2 T_2(\xi) V(\xi)}{1 + \mu_2 V(\xi)} + \frac{\bar{\beta}_2 T_2(\xi) I_2(\xi)}{1 + \bar{\mu}_2 I_2(\xi)} \right) d\xi d\tau \\ & + \tilde{\eta}_3 \left( b_1 \delta_1 \int_0^{h_3} g_1(\tau) e^{-\epsilon_1 \tau} \int_{t-\tau}^t I_1(\xi) d\xi d\tau + b_2 \delta_2 \int_0^{h_4} g_2(\tau) e^{-\epsilon_2 \tau} \int_{t-\tau}^t I_2(\xi) d\xi d\tau \right) + \tilde{\eta}_3 V, \end{aligned}$$

where

$$\begin{aligned} \tilde{\eta}_1 &= \delta_1 \delta_2 b_1 \varsigma_1 \varsigma_3 (1 - \tilde{\mathcal{W}}_1), \quad \tilde{\eta}_2 = \delta_1 \delta_2 b_2 \varsigma_2 \varsigma_4 (1 - \tilde{\mathcal{U}}_1), \\ \tilde{\eta}_3 &= \delta_1 \delta_2 (1 - \tilde{\mathcal{U}}_1) (1 - \tilde{\mathcal{W}}_1). \end{aligned} \quad (3.16)$$

Clearly,  $\tilde{\Xi}_0(T_1, I_1, T_2, I_2, V) > 0$  for all  $T_1, I_1, T_2, I_2, V > 0$ , and  $\tilde{\Xi}_0(T_1^0, 0, T_2^0, 0, 0) = 0$ . We calculate  $\frac{d\tilde{\Xi}_0}{dt}$  as

$$\begin{aligned} \frac{d\tilde{\Xi}_0}{dt} = & \tilde{\eta}_1 \left( 1 - \frac{T_1^0}{T_1} \right) \dot{T}_1 + \tilde{\eta}_1 \varsigma_1^{-1} \dot{I}_1 \\ & + \tilde{\eta}_1 \varsigma_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\beta_1 T_1(t-\tau) V(t-\tau)}{1 + \mu_1 V(t-\tau)} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\bar{\beta}_1 T_1(t-\tau) I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} \right) d\tau \\ & + \tilde{\eta}_2 \left( 1 - \frac{T_2^0}{T_2} \right) \dot{T}_2 + \tilde{\eta}_2 \varsigma_2^{-1} \dot{I}_2 \\ & + \tilde{\eta}_2 \varsigma_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\beta_2 T_2(t-\tau) V(t-\tau)}{1 + \mu_2 V(t-\tau)} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\bar{\beta}_2 T_2(t-\tau) I_2(t-\tau)}{1 + \bar{\mu}_2 I_2(t-\tau)} \right) d\tau \\ & + \tilde{\eta}_3 \left( b_1 \delta_1 \int_0^{h_3} g_1(\tau) e^{-\epsilon_1 \tau} (I_1 - I_1(t-\tau)) d\tau + b_2 \delta_2 \int_0^{h_4} g_2(\tau) e^{-\epsilon_2 \tau} (I_2 - I_2(t-\tau)) d\tau \right) + \tilde{\eta}_3 \dot{V}. \end{aligned}$$

From system (3.1)–(3.5), we find

$$\begin{aligned} \frac{d\tilde{\Xi}_0}{dt} = & \tilde{\eta}_1 \left( 1 - \frac{T_1^0}{T_1} \right) \left( \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right) \\ & + \tilde{\eta}_1 \varsigma_1^{-1} \left( \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \left[ \frac{\beta_1 T_1(t-\tau) V(t-\tau)}{1 + \mu_1 V(t-\tau)} + \frac{\bar{\beta}_1 T_1(t-\tau) I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} \right] d\tau - \delta_1 I_1 \right) \\ & + \tilde{\eta}_1 \varsigma_1^{-1} \left( \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \left[ \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\beta_1 T_1(t-\tau) V(t-\tau)}{1 + \mu_1 V(t-\tau)} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\bar{\beta}_1 T_1(t-\tau) I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} \right] d\tau \right) \\ & + \tilde{\eta}_2 \left( 1 - \frac{T_2^0}{T_2} \right) \left( \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\ & + \tilde{\eta}_2 \varsigma_2^{-1} \left( \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \left[ \frac{\beta_2 T_2(t-\tau) V(t-\tau)}{1 + \mu_2 V(t-\tau)} + \frac{\bar{\beta}_2 T_2(t-\tau) I_2(t-\tau)}{1 + \bar{\mu}_2 I_2(t-\tau)} \right] d\tau - \delta_2 I_2 \right) \\ & + \tilde{\eta}_2 \varsigma_2^{-1} \left( \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \left[ \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\beta_2 T_2(t-\tau) V(t-\tau)}{1 + \mu_2 V(t-\tau)} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\bar{\beta}_2 T_2(t-\tau) I_2(t-\tau)}{1 + \bar{\mu}_2 I_2(t-\tau)} \right] d\tau \right) \end{aligned}$$

$$\begin{aligned}
& + \tilde{\eta}_3 \left( b_1 \delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} (I_1 - I_1(t-\tau)) d\tau + b_2 \delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} (I_2 - I_2(t-\tau)) d\tau \right) \\
& + \tilde{\eta}_3 \left( b_1 \int_0^{h_1} g_1(\tau) \delta_1 e^{-\varepsilon_1 \tau} I_1(t-\tau) d\tau + b_2 \int_0^{h_4} g_2(\tau) \delta_2 e^{-\varepsilon_2 \tau} I_2(t-\tau) d\tau - \varphi V \right). \tag{3.17}
\end{aligned}$$

After performing further calculations, Eq (3.17) becomes

$$\begin{aligned}
\frac{d\tilde{\Xi}_0}{dt} &= -\tilde{\eta}_1 \gamma_1 \frac{(T_1 - T_1^0)^2}{T_1} - \tilde{\eta}_2 \gamma_2 \frac{(T_2 - T_2^0)^2}{T_2} + \frac{\delta_1 \bar{\mu}_1}{1 + \bar{\mu}_1 I_1} (\tilde{\eta}_3 b_1 S_3 - \tilde{\eta}_1 S_1^{-1}) I_1^2 \\
&+ \frac{\delta_2 \bar{\mu}_2}{1 + \bar{\mu}_2 I_2} (\tilde{\eta}_3 b_2 S_4 - \tilde{\eta}_2 S_2^{-1}) I_2^2 - \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (1 - M) V \\
&+ \frac{1}{(1 + \mu_1 V)(1 + \mu_2 V)} (\tilde{\eta}_1 \beta_1 T_1^0 \mu_2 + \tilde{\eta}_2 \beta_2 T_2^0 \mu_1 - \tilde{\eta}_3 \varphi (\mu_1 + \mu_2 + \mu_1 \mu_2 V)) V^2 \\
&= -\tilde{\eta}_1 \gamma_1 \frac{(T_1 - T_1^0)^2}{T_1} - \tilde{\eta}_2 \gamma_2 \frac{(T_2 - T_2^0)^2}{T_2} + \frac{\delta_1^2 \delta_2 b_1 S_3 \bar{\mu}_1}{1 + \bar{\mu}_1 I_1} (1 - \tilde{\mathcal{W}}_1) \tilde{\mathcal{U}}_1 I_1^2 \\
&+ \frac{\delta_1 \delta_2^2 b_2 S_4 \bar{\mu}_2}{1 + \bar{\mu}_2 I_2} (1 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{W}}_1 I_2^2 - \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} (1 - M) V \\
&- \frac{\delta_1 \delta_2 \varphi}{(1 + \mu_1 V)(1 + \mu_2 V)} ((1 - \tilde{\mathcal{W}}_1)(1 - \tilde{\mathcal{U}}) \mu_2 + (1 - \tilde{\mathcal{U}}_1)(1 - \tilde{\mathcal{W}}) \mu_1) V^2 \\
&- \frac{\delta_1 \delta_2 \varphi \mu_1 \mu_2}{(1 + \mu_1 V)(1 + \mu_2 V)} (1 - \tilde{\mathcal{U}}_1)(1 - \tilde{\mathcal{W}}_1) V^3.
\end{aligned}$$

Consequently, we conclude that  $\tilde{\Gamma}'_0 = \{\tilde{\Pi}_0\}$ , and by invoking LIP, it follows that the disease-free equilibrium  $\tilde{\Pi}_0$  is globally asymptotically stable.

**Theorem 4.** For system (3.1)–(3.5), if  $\tilde{\mathcal{R}}_0 > 1$ , then the endemic equilibrium  $\tilde{\Pi}$  is globally asymptotically stable.

*Proof.* To establish global stability of  $\tilde{\Pi}$ , we construct a Lyapunov functional  $\tilde{\Xi}_1(T_1, I_1, T_2, I_2, V)$  defined as:

$$\begin{aligned}
\tilde{\Xi}_1 &= \tilde{\eta}_1 \left[ \tilde{T}_1 \Theta \left( \frac{T_1}{\tilde{T}_1} \right) + S_1^{-1} \tilde{I}_1 \Theta \left( \frac{I_1}{\tilde{I}_1} \right) \right] \\
&+ \frac{\tilde{\eta}_1 \beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \int_{t-\tau}^t \Theta \left( \frac{T_1(\xi) V(\xi) (1 + \mu_1 \tilde{V})}{(1 + \mu_1 V(\xi)) \tilde{T}_1 \tilde{V}} \right) d\xi d\tau \\
&+ \frac{\tilde{\eta}_1 \beta_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \int_{t-\tau}^t \Theta \left( \frac{T_1(\xi) I_1(\xi) (1 + \bar{\mu}_1 \tilde{I}_1)}{(1 + \bar{\mu}_1 I_1(\xi)) \tilde{T}_1 \tilde{I}_1} \right) d\xi d\tau \\
&+ \tilde{\eta}_2 \left[ \tilde{T}_2 \Theta \left( \frac{T_2}{\tilde{T}_2} \right) + S_2^{-1} \tilde{I}_2 \Theta \left( \frac{I_2}{\tilde{I}_2} \right) \right] \\
&+ \frac{\tilde{\eta}_2 \beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \int_{t-\tau}^t \Theta \left( \frac{T_2(\xi) V(\xi) (1 + \mu_2 \tilde{V})}{(1 + \mu_2 V(\xi)) \tilde{T}_2 \tilde{V}} \right) d\xi d\tau
\end{aligned}$$

$$\begin{aligned}
& + \frac{\dot{\eta}_2 \bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \int_{t-\tau}^t \Theta \left( \frac{T_2(\xi) I_2(\xi) (1 + \bar{\mu}_2 \tilde{I}_2)}{(1 + \bar{\mu}_2 I_2(\xi)) \tilde{T}_2 \tilde{I}_2} \right) d\xi d\tau \\
& + \tilde{V} \Theta \left( \frac{V}{\tilde{V}} \right) + b_1 \delta_1 \tilde{I}_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} \int_{t-\tau}^t \Theta \left( \frac{I_1(\xi)}{\tilde{I}_1} \right) d\xi d\tau \\
& + b_2 \delta_2 \tilde{I}_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} \int_{t-\tau}^t \Theta \left( \frac{I_2(\xi)}{\tilde{I}_2} \right) d\xi d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\dot{\eta}_1 & = b_1 \delta_1 \tilde{I}_1 \mathcal{S}_3 = b_1 \mathcal{S}_1 \mathcal{S}_3 \left( 1 + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1 (1 + \mu_1 \tilde{V})}{\beta_1 \tilde{T}_1 \tilde{V} (1 + \bar{\mu}_1 \tilde{I}_1)} \right), \\
\dot{\eta}_2 & = b_2 \delta_2 \tilde{I}_2 \mathcal{S}_4 = b_2 \mathcal{S}_2 \mathcal{S}_4 \left( 1 + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2 (1 + \mu_2 \tilde{V})}{\beta_2 \tilde{T}_2 \tilde{V} (1 + \bar{\mu}_2 \tilde{I}_2)} \right).
\end{aligned} \tag{3.18}$$

It is clear that  $\tilde{\Xi}_1(T_1, I_1, T_2, I_2, V) > 0$  for all  $T_1, I_1, T_2, I_2, V > 0$ , and  $\tilde{\Xi}_1(\tilde{T}_1, \tilde{I}_1, \tilde{T}_2, \tilde{I}_2, \tilde{V}) = 0$ . Calculating  $\frac{d\tilde{\Xi}_1}{dt}$  along the trajectories of Eqs (3.1)–(3.5), we get

$$\begin{aligned}
\frac{d\tilde{\Xi}_1}{dt} & = \dot{\eta}_1 \left( 1 - \frac{\tilde{T}_1}{T_1} \right) \left( \lambda_1 - \gamma_1 T_1 - \frac{\beta_1 T_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} \right) \\
& + \dot{\eta}_1 \mathcal{S}_1^{-1} \left( 1 - \frac{\tilde{I}_1}{I_1} \right) \left( \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left[ \frac{\beta_1 T_1 (t-\tau) V (t-\tau)}{1 + \mu_1 V (t-\tau)} + \frac{\bar{\beta}_1 T_1 (t-\tau) I_1 (t-\tau)}{1 + \bar{\mu}_1 I_1 (t-\tau)} \right] d\tau - \delta_1 I_1 \right) \\
& + \frac{\dot{\eta}_1 \beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} \left[ \frac{(1 + \mu_1 \tilde{V}) T_1 V}{(1 + \mu_1 V) \tilde{T}_1 \tilde{V}} - \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \frac{(1 + \mu_1 \tilde{V}) T_1 (t-\tau) V (t-\tau)}{(1 + \mu_1 V (t-\tau)) \tilde{T}_1 \tilde{V}} d\tau \right. \\
& \left. + \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \ln \left( \frac{(1 + \mu_1 V) T_1 (t-\tau) V (t-\tau)}{(1 + \mu_1 V (t-\tau)) T_1 V} \right) d\tau \right] \\
& + \frac{\dot{\eta}_1 \bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \left[ \frac{(1 + \bar{\mu}_1 \tilde{I}_1) T_1 I_1}{(1 + \bar{\mu}_1 I_1) \tilde{T}_1 \tilde{I}_1} - \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \frac{(1 + \bar{\mu}_1 \tilde{I}_1) T_1 (t-\tau) I_1 (t-\tau)}{(1 + \bar{\mu}_1 I_1 (t-\tau)) \tilde{T}_1 \tilde{I}_1} d\tau \right. \\
& \left. + \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \ln \left( \frac{(1 + \bar{\mu}_1 I_1) T_1 (t-\tau) I_1 (t-\tau)}{(1 + \bar{\mu}_1 I_1 (t-\tau)) T_1 I_1} \right) d\tau \right] \\
& + \dot{\eta}_2 \left( 1 - \frac{\tilde{T}_2}{T_2} \right) \left( \lambda_2 - \gamma_2 T_2 - \frac{\beta_2 T_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\
& + \dot{\eta}_2 \mathcal{S}_2^{-1} \left( 1 - \frac{\tilde{I}_2}{I_2} \right) \left( \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left[ \frac{\beta_2 T_2 (t-\tau) V (t-\tau)}{1 + \mu_2 V (t-\tau)} + \frac{\bar{\beta}_2 T_2 (t-\tau) I_2 (t-\tau)}{1 + \bar{\mu}_2 I_2 (t-\tau)} \right] d\tau - \delta_2 I_2 \right) \\
& + \frac{\dot{\eta}_2 \beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} \left[ \frac{(1 + \mu_2 \tilde{V}) T_2 V}{(1 + \mu_2 V) \tilde{T}_2 \tilde{V}} - \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \frac{(1 + \mu_2 \tilde{V}) T_2 (t-\tau) V (t-\tau)}{(1 + \mu_2 V (t-\tau)) \tilde{T}_2 \tilde{V}} d\tau \right. \\
& \left. + \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \ln \left( \frac{(1 + \mu_2 V) T_2 (t-\tau) V (t-\tau)}{(1 + \mu_2 V (t-\tau)) T_2 V} \right) d\tau \right] \\
& + \frac{\dot{\eta}_2 \bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \left[ \frac{(1 + \bar{\mu}_2 \tilde{I}_2) T_2 I_2}{(1 + \bar{\mu}_2 I_2) \tilde{T}_2 \tilde{I}_2} - \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \frac{(1 + \bar{\mu}_2 \tilde{I}_2) T_2 (t-\tau) I_2 (t-\tau)}{(1 + \bar{\mu}_2 I_2 (t-\tau)) \tilde{T}_2 \tilde{I}_2} d\tau \right.
\end{aligned}$$

$$\begin{aligned}
& + S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \ln \left( \frac{(1 + \bar{\mu}_2 I_2) T_2(t - \tau) I_2(t - \tau)}{(1 + \bar{\mu}_2 I_2(t - \tau)) T_2 I_2} \right) d\tau \Big] \\
& + \left( 1 - \frac{\tilde{V}}{V} \right) \left( b_1 \delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} I_1(t - \tau) d\tau + b_2 \delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} I_2(t - \omega_2) d\tau - \varphi V \right) \\
& + b_1 \delta_1 \tilde{I}_1 \left[ S_3 \frac{I_1}{\tilde{I}_1} - \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} \frac{I_1(t - \tau)}{\tilde{I}_1} d\tau + \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} \ln \left( \frac{I_1(t - \tau)}{I_1} \right) d\tau \right] \\
& + b_2 \delta_2 \tilde{I}_2 \left[ S_4 \frac{I_2}{\tilde{I}_2} - \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} \frac{I_2(t - \tau)}{\tilde{I}_2} d\tau + \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} \ln \left( \frac{I_2(t - \tau)}{I_2} \right) d\tau \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d\tilde{\Xi}_1}{dt} = & \hat{\eta}_1 \left( 1 - \frac{\tilde{T}_1}{T_1} \right) (\lambda_1 - \gamma_1 T_1) - \left( \frac{\beta_1 T_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - \frac{\beta_1 \tilde{T}_1 V}{1 + \mu_1 V} - \frac{\bar{\beta}_1 \tilde{T}_1 I_1}{1 + \bar{\mu}_1 I_1} \right) \\
& + \hat{\eta}_1 S_1^{-1} \left( \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left[ \frac{\beta_1 T_1(t - \tau) V(t - \tau)}{1 + \mu_1 V(t - \tau)} + \frac{\bar{\beta}_1 T_1(t - \tau) I_1(t - \tau)}{1 + \bar{\mu}_1 I_1(t - \tau)} \right] d\tau - \delta_1 I_1 \right. \\
& \left. - \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left[ \frac{\beta_1 T_1(t - \tau) V(t - \tau)}{1 + \mu_1 V(t - \tau)} + \frac{\bar{\beta}_1 T_1(t - \tau) I_1(t - \tau)}{1 + \bar{\mu}_1 I_1(t - \tau)} \right] d\tau \frac{\tilde{I}_1}{I_1} - \delta_1 \tilde{I}_1 \right) \\
& + \hat{\eta}_1 \left[ \frac{\beta_1 T_1 V}{1 + \mu_1 V} - S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \frac{\beta_1 T_1(t - \tau) V(t - \tau)}{1 + \mu_1 V(t - \tau)} d\tau \right. \\
& \left. + \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \ln \left( \frac{T_1(t - \tau) V(t - \tau) (1 + \mu_1 V)}{(1 + \mu_1 V(t - \tau)) T_1 V} \right) d\tau \right] \\
& + \hat{\eta}_1 \left[ \frac{\bar{\beta}_1 T_1 I_1}{1 + \bar{\mu}_1 I_1} - S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \frac{\bar{\beta}_1 T_1(t - \tau) I_1(t - \tau)}{1 + \bar{\mu}_1 I_1(t - \tau)} d\tau \right. \\
& \left. + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \ln \left( \frac{T_1(t - \tau) I_1(t - \tau) (1 + \bar{\mu}_1 I_1)}{(1 + \bar{\mu}_1 I_1(t - \tau)) T_1 I_1} \right) d\tau \right] \\
& + \hat{\eta}_2 \left( 1 - \frac{\tilde{T}_2}{T_2} \right) (\lambda_2 - \gamma_2 T_2) - \left( \frac{\beta_2 T_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - \frac{\beta_2 \tilde{T}_2 V}{1 + \mu_2 V} - \frac{\bar{\beta}_2 \tilde{T}_2 I_2}{1 + \bar{\mu}_2 I_2} \right) \\
& + \hat{\eta}_2 S_2^{-1} \left( \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left[ \frac{\beta_2 T_2(t - \tau) V(t - \tau)}{1 + \mu_2 V(t - \tau)} + \frac{\bar{\beta}_2 T_2(t - \tau) I_2(t - \tau)}{1 + \bar{\mu}_2 I_2(t - \tau)} \right] d\tau - \delta_2 I_2 \right. \\
& \left. - \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left[ \frac{\beta_2 T_2(t - \tau) V(t - \tau)}{1 + \mu_2 V(t - \tau)} + \frac{\bar{\beta}_2 T_2(t - \tau) I_2(t - \tau)}{1 + \bar{\mu}_2 I_2(t - \tau)} \right] d\tau \frac{\tilde{I}_2}{I_2} + \delta_2 \tilde{I}_2 \right) \\
& + \hat{\eta}_2 \left[ \frac{\beta_2 T_2 V}{1 + \mu_2 V} - S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \frac{\beta_2 T_2(t - \tau) V(t - \tau)}{1 + \mu_2 V(t - \tau)} d\tau \right. \\
& \left. + \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \ln \left( \frac{T_2(t - \tau) V(t - \tau) (1 + \mu_2 V)}{(1 + \mu_2 V(t - \tau)) T_2 V} \right) d\tau \right] \\
& + \hat{\eta}_2 \left[ \frac{\bar{\beta}_2 T_2 I_2}{1 + \bar{\mu}_2 I_2} - S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \frac{\bar{\beta}_2 T_2(t - \tau) I_2(t - \tau)}{1 + \bar{\mu}_2 I_2(t - \tau)} d\tau \right. \\
& \left. + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \ln \left( \frac{T_2(t - \tau) I_2(t - \tau) (1 + \bar{\mu}_2 I_2)}{(1 + \bar{\mu}_2 I_2(t - \tau)) T_2 I_2} \right) d\tau \right]
\end{aligned}$$

$$\begin{aligned}
& + b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} I_1(t-\tau) d\tau + b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} I_2(t-\omega_2) d\tau - \varphi V \\
& - b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} I_1(t-\tau) d\tau \frac{\tilde{V}}{V} - b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} I_2(t-\omega_2) d\tau \frac{\tilde{V}}{V} + \varphi \tilde{V} \\
& + b_1\delta_1 I_1 \mathcal{S}_3 - b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} \frac{I_1(t-\tau)}{\tilde{I}_1} d\tau + b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} \ln\left(\frac{I_1(t-\tau)}{I_1}\right) d\tau \\
& + b_2\delta_2 I_2 \mathcal{S}_4 - b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} \frac{I_2(t-\tau)}{\tilde{I}_2} d\tau + b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} \ln\left(\frac{I_2(t-\tau)}{I_2}\right) d\tau. \quad (3.19)
\end{aligned}$$

Simplify Eq (3.19). Then we have

$$\begin{aligned}
\frac{d\tilde{\Xi}_1}{dt} = & \dot{\eta}_1 \left(1 - \frac{\tilde{T}_1}{T_1}\right) (\lambda_1 - \gamma_1 T_1) + \dot{\eta}_1 \left(\frac{\beta_1 \tilde{T}_1 V}{1 + \mu_1 V} + \frac{\bar{\beta}_1 \tilde{T}_1 I_1}{1 + \bar{\mu}_1 I_1}\right) - \dot{\eta}_1 \delta_1 I_1 \mathcal{S}_1^{-1} + \dot{\eta}_1 \mathcal{S}_1^{-1} \delta_1 \tilde{I}_1 \\
& - \dot{\eta}_1 \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1\tau} \frac{\beta_1 T_1(t-\tau) V(t-\tau)}{1 + \mu_1 V(t-\tau)} d\tau \frac{\tilde{I}_1}{I_1} \\
& + \dot{\eta}_1 \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1\tau} \frac{\bar{\beta}_1 T_1(t-\tau) I_1(t-\tau)}{1 + \bar{\mu}_1 I_1(t-\tau)} d\tau \frac{\tilde{I}_1}{I_1} \\
& + \dot{\eta}_1 \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1\tau} \ln\left(\frac{T_1(t-\tau) V(t-\tau) (1 + \mu_1 V)}{T_1 V (1 + \mu_1 V(t-\tau))}\right) d\tau \\
& + \dot{\eta}_1 \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \mathcal{S}_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1\tau} \ln\left(\frac{T_1(t-\tau) I_1(t-\tau) (1 + \bar{\mu}_1 I_1)}{T_1 I_1 (1 + \bar{\mu}_1 I_1(t-\tau))}\right) d\tau \\
& + \dot{\eta}_2 \left(1 - \frac{\tilde{T}_2}{T_2}\right) (\lambda_2 - \gamma_2 T_2) + \dot{\eta}_2 \left(\frac{\beta_2 \tilde{T}_2 V}{1 + \mu_2 V} + \frac{\bar{\beta}_2 \tilde{T}_2 I_2}{1 + \bar{\mu}_2 I_2}\right) - \dot{\eta}_2 \delta_2 I_2 \mathcal{S}_2^{-1} + \dot{\eta}_2 \delta_2 \tilde{I}_2 \mathcal{S}_2^{-1} \\
& + \dot{\eta}_2 \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2\tau} \frac{\beta_2 T_2(t-\tau) V(t-\tau)}{1 + \mu_2 V(t-\tau)} d\tau \frac{\tilde{I}_2}{I_2} \\
& + \dot{\eta}_2 \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2\tau} \frac{\bar{\beta}_2 T_2(t-\tau) I_2(t-\tau)}{1 + \bar{\mu}_2 I_2(t-\tau)} d\tau \frac{\tilde{I}_2}{I_2} \\
& + \dot{\eta}_2 \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2\tau} \ln\left(\frac{T_2(t-\tau) V(t-\tau) (1 + \mu_2 V)}{T_2 V (1 + \mu_2 V(t-\tau))}\right) d\tau \\
& + \dot{\eta}_2 \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \mathcal{S}_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2\tau} \ln\left(\frac{T_2(t-\tau) I_2(t-\tau) (1 + \bar{\mu}_2 I_2)}{T_2 I_2 (1 + \bar{\mu}_2 I_2(t-\tau))}\right) d\tau \\
& - \varphi V - b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} I_1(t-\tau) d\tau \frac{\tilde{V}}{V} - b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} I_2(t-\omega_2) d\tau \frac{\tilde{V}}{V} + \varphi \tilde{V} \\
& + b_1\delta_1 I_1 \mathcal{S}_3 + b_1\delta_1 \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1\tau} \ln\left(\frac{I_1(t-\tau)}{I_1}\right) d\tau \\
& + b_2\delta_2 I_2 \mathcal{S}_4 + b_2\delta_2 \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2\tau} \ln\left(\frac{I_2(t-\tau)}{I_2}\right) d\tau. \quad (3.20)
\end{aligned}$$

By organizing the components of Eq (3.20) and substituting the equilibrium values associated with

$\tilde{\Pi}$ , we proceed with the analysis.

$$\begin{aligned}\lambda_1 &= \gamma_1 \tilde{T}_1 + \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1}, \\ \lambda_2 &= \gamma_2 \tilde{T}_2 + \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2}, \\ \delta_1 \tilde{I}_1 &= \varsigma_1 \left( \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \right), \\ \delta_2 \tilde{I}_2 &= \varsigma_2 \left( \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \right), \\ \varphi \tilde{V} &= b_1 \delta_1 \tilde{I}_1 \varsigma_3 + b_2 \delta_2 \tilde{I}_2 \varsigma_4 = \frac{\dot{\eta}_1 \beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\dot{\eta}_2 \beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} \\ &= b_1 \varsigma_1 \varsigma_3 \left( \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \right) + b_2 \varsigma_2 \varsigma_4 \left( \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \right),\end{aligned}$$

we get

$$\begin{aligned}\frac{d\tilde{\Xi}_1}{dt} &= -\dot{\eta}_1 \gamma_1 \frac{(T_1 - \tilde{T}_1)^2}{T_1} + \dot{\eta}_1 \left( \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \right) \left( 1 - \frac{\tilde{T}_1}{T_1} \right) \\ &+ \dot{\eta}_1 \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} \varsigma_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \left( \frac{(1 + \mu_1 \tilde{V}) V}{(1 + \mu_1 V) \tilde{V}} + 1 - \frac{V}{\tilde{V}} - \frac{T_1(t - \tau) V(t - \tau) (1 + \mu_1 \tilde{V}) \tilde{I}_1}{(1 + \mu_1 V(t - \tau)) \tilde{T}_1 \tilde{V} I_1} \right. \\ &+ \left. \ln \left( \frac{T_1(t - \tau) V(t - \tau) (1 + \mu_1 V)}{(1 + \mu_1 V(t - \tau)) T_1 V} \right) \right) d\tau + \dot{\eta}_1 \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \varsigma_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\epsilon_1 \tau} \left( \frac{(1 + \bar{\mu}_1 \tilde{I}_1) I_1}{(1 + \bar{\mu}_1 I_1) \tilde{I}_1} \right. \\ &+ \left. 1 - \frac{I_1}{\tilde{I}_1} - \frac{T_1(t - \tau) I_1(t - \tau) (1 + \mu_1 \tilde{I}_1)}{(1 + \mu_1 I_1(t - \tau)) \tilde{T}_1 I_1} + \ln \left( \frac{T_1(t - \tau) I_1(t - \tau) (1 + \bar{\mu}_1 I_1)}{(1 + \bar{\mu}_1 I_1(t - \tau)) T_1 I_1} \right) \right) d\tau \\ &+ \dot{\eta}_1 \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} \varsigma_3^{-1} \int_0^{h_3} g_1(\tau) e^{-\epsilon_1 \tau} \left( 1 - \frac{I_1(t - \tau) \tilde{V}}{V \tilde{I}_1} + \ln \left( \frac{I_1(t - \tau)}{I_1} \right) \right) \\ &- \dot{\eta}_2 \gamma_2 \frac{(T_2 - \tilde{T}_2)^2}{T_2} + \dot{\eta}_2 \left( \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \right) \left( 1 - \frac{\tilde{T}_2}{T_2} \right) \\ &+ \dot{\eta}_2 \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} \varsigma_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \left( \frac{(1 + \mu_2 \tilde{V}) V}{(1 + \mu_2 V) \tilde{V}} + 1 - \frac{V}{\tilde{V}} - \frac{T_2(t - \tau) V(t - \tau) (1 + \mu_2 \tilde{V}) \tilde{I}_2}{(1 + \mu_2 V(t - \tau)) \tilde{T}_2 \tilde{V} I_2} \right. \\ &+ \left. \ln \left( \frac{T_2(t - \tau) V(t - \tau) (1 + \mu_2 V)}{(1 + \mu_2 V(t - \tau)) T_2 V} \right) \right) d\tau + \dot{\eta}_2 \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \varsigma_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\epsilon_2 \tau} \left( \frac{(1 + \bar{\mu}_2 \tilde{I}_2) I_2}{(1 + \bar{\mu}_2 I_2) \tilde{I}_2} \right. \\ &+ \left. 1 - \frac{I_2}{\tilde{I}_2} - \frac{T_2(t - \tau) I_2(t - \tau) (1 + \mu_2 \tilde{I}_2)}{(1 + \mu_2 I_2(t - \tau)) \tilde{T}_2 I_2} + \ln \left( \frac{T_2(t - \tau) I_2(t - \tau) (1 + \bar{\mu}_2 I_2)}{(1 + \bar{\mu}_2 I_2(t - \tau)) T_2 I_2} \right) \right) d\tau \\ &+ \dot{\eta}_2 \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} \varsigma_4^{-1} \int_0^{h_4} g_2(\tau) e^{-\epsilon_2 \tau} \left( 1 - \frac{I_2(t - \tau) \tilde{V}}{V \tilde{I}_2} + \ln \left( \frac{I_2(t - \tau)}{I_2} \right) \right) d\tau.\end{aligned}\tag{3.21}$$

Using the following equalities, for all  $\ell = 1, 2$ , we get

$$\begin{aligned} \ln\left(\frac{T_\ell(t-\tau)V(t-\tau)(1+\mu_\ell V)}{(1+\mu_\ell V(t-\tau))T_\ell V}\right) &= \ln\left(\frac{\tilde{T}_\ell}{T_\ell}\right) + \ln\left(\frac{T_\ell(t-\tau)V(t-\tau)(1+\mu_\ell \tilde{V})\tilde{I}_\ell}{(1+\mu_\ell V(t-\tau))\tilde{T}_\ell \tilde{V} I_\ell}\right) \\ &\quad + \ln\left(\frac{\tilde{V} I_\ell}{V \tilde{I}_\ell}\right) + \ln\left(\frac{1+\mu_\ell V}{1+\mu_\ell \tilde{V}}\right), \\ \ln\left(\frac{T_\ell(t-\tau)I_\ell(t-\tau)(1+\bar{\mu}_\ell I_\ell)}{(1+\bar{\mu}_\ell I_\ell(t-\tau))T_\ell I_\ell}\right) &= \ln\left(\frac{\tilde{T}_\ell}{T_\ell}\right) + \ln\left(\frac{T_\ell(t-\tau)I_\ell(t-\tau)(1+\bar{\mu}_\ell \tilde{I}_\ell)}{(1+\bar{\mu}_\ell I_\ell(t-\tau))\tilde{T}_\ell \tilde{I}_\ell}\right) \\ &\quad + \ln\left(\frac{1+\bar{\mu}_\ell I_\ell}{1+\bar{\mu}_\ell \tilde{I}_\ell}\right), \\ \ln\left(\frac{I_\ell(t-\tau)}{I_\ell}\right) &= \ln\left(\frac{I_\ell(t-\tau)\tilde{V}}{V \tilde{I}_\ell}\right) + \ln\left(\frac{V \tilde{I}_\ell}{\tilde{V} I_\ell}\right). \end{aligned} \quad (3.22)$$

Equation (3.21) becomes

$$\begin{aligned} \frac{d\tilde{\Xi}_1}{dt} &= -\dot{\eta}_1 \gamma_1 \frac{(T_1 - \tilde{T}_1)^2}{T_1} - \dot{\eta}_1 \left( \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} + \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} \right) \Theta\left(\frac{\tilde{T}_1}{T_1}\right) \\ &\quad - \dot{\eta}_1 \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left( \Theta\left(\frac{1 + \mu_1 V}{1 + \mu_1 \tilde{V}}\right) + \Theta\left(\frac{T_1(t-\tau)V(t-\tau)(1 + \mu_1 \tilde{V})\tilde{I}_1}{(1 + \mu_1 V(t-\tau))\tilde{T}_1 \tilde{V} I_1}\right) \right. \\ &\quad \left. + \frac{\mu_1 (V - \tilde{V})^2}{(1 + \mu_1 V)(1 + \mu_1 \tilde{V})\tilde{V}} \right) d\tau - \dot{\eta}_1 \frac{\bar{\beta}_1 \tilde{T}_1 \tilde{I}_1}{1 + \bar{\mu}_1 \tilde{I}_1} S_1^{-1} \int_0^{h_1} f_1(\tau) e^{-\varepsilon_1 \tau} \left( \Theta\left(\frac{1 + \bar{\mu}_1 I_1}{1 + \bar{\mu}_1 \tilde{I}_1}\right) \right. \\ &\quad \left. + \Theta\left(\frac{T_1(t-\tau)I_1(t-\tau)(1 + \mu_1 \tilde{I}_1)}{(1 + \mu_1 I_1(t-\tau))\tilde{T}_1 I_1}\right) + \frac{\bar{\mu}_1 (I_1 - \tilde{I}_1)^2}{(1 + \bar{\mu}_1 I_1)(1 + \bar{\mu}_1 \tilde{I}_1)\tilde{I}_1} \right) d\tau \\ &\quad - \dot{\eta}_1 \frac{\beta_1 \tilde{T}_1 \tilde{V}}{1 + \mu_1 \tilde{V}} S_3^{-1} \int_0^{h_3} g_1(\tau) e^{-\varepsilon_1 \tau} \Theta\left(\frac{I_1(t-\tau)\tilde{V}}{V \tilde{I}_1}\right) d\tau \\ &\quad - \dot{\eta}_2 \gamma_2 \frac{(T_2 - \tilde{T}_2)^2}{T_2} - \dot{\eta}_2 \left( \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} + \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} \right) \Theta\left(\frac{\tilde{T}_2}{T_2}\right) \\ &\quad - \dot{\eta}_2 \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left( \Theta\left(\frac{1 + \mu_2 V}{1 + \mu_2 \tilde{V}}\right) + \Theta\left(\frac{T_2(t-\tau)V(t-\tau)(1 + \mu_2 \tilde{V})\tilde{I}_2}{(1 + \mu_2 V(t-\tau))\tilde{T}_2 \tilde{V} I_2}\right) \right. \\ &\quad \left. + \frac{\mu_2 (V - \tilde{V})^2}{(1 + \mu_2 V)(1 + \mu_2 \tilde{V})\tilde{V}} \right) d\tau - \dot{\eta}_2 \frac{\bar{\beta}_2 \tilde{T}_2 \tilde{I}_2}{1 + \bar{\mu}_2 \tilde{I}_2} S_2^{-1} \int_0^{h_2} f_2(\tau) e^{-\varepsilon_2 \tau} \left( \Theta\left(\frac{1 + \bar{\mu}_2 I_2}{1 + \bar{\mu}_2 \tilde{I}_2}\right) \right. \\ &\quad \left. + \Theta\left(\frac{T_2(t-\tau)I_2(t-\tau)(1 + \mu_2 \tilde{I}_2)}{(1 + \mu_2 I_2(t-\tau))\tilde{T}_2 I_2}\right) + \frac{\bar{\mu}_2 (I_2 - \tilde{I}_2)^2}{(1 + \bar{\mu}_2 I_2)(1 + \bar{\mu}_2 \tilde{I}_2)\tilde{I}_2} \right) d\tau \\ &\quad - \dot{\eta}_2 \frac{\beta_2 \tilde{T}_2 \tilde{V}}{1 + \mu_2 \tilde{V}} S_4^{-1} \int_0^{h_4} g_2(\tau) e^{-\varepsilon_2 \tau} \Theta\left(\frac{I_2(t-\tau)\tilde{V}}{V \tilde{I}_2}\right) d\tau. \end{aligned}$$

As a result, we have  $\frac{d\tilde{E}_1}{dt} \leq 0$  for all  $T_1, I_1, T_2, I_2, V > 0$ . Moreover, equality holds, i.e.,  $\frac{d\tilde{E}_1}{dt} = 0$ , if and only if  $(T_1(t), I_1(t), T_2(t), I_2(t), V(t)) = (\tilde{T}_1, \tilde{I}_1, \tilde{T}_2, \tilde{I}_2, \tilde{V})$ . This implies that the trajectories of system (3.1)–(3.5) converge to the invariant set  $\tilde{\Gamma}_1 = \{\tilde{\Pi}\}$ . Applying LaSalle's invariance principle, we conclude that the endemic equilibrium  $\tilde{\Pi}$  is globally asymptotically stable whenever  $\tilde{\mathcal{R}}_0 > 1$ .

#### 4. Numerical simulations

We use dde23 solver in Matlab (R2016a) to simulate the model with discrete delay. To simulate the system with distributed delay, we use quadrature approximation to replace the continuous delay integral with a finite weighted sum of discrete delays. The delay kernel is discretized into selected points with associated quadrature weights, and the distributed delay term is approximated as a linear combination of past states at those points. This transforms the original distributed delay model into a system of delay differential equations with discrete delays, which is solved efficiently using dde23 solver.

We firstly simulate the model with discrete delay, then simulate the model with distributed delay in the following subsections.

##### 4.1. Simulation of the model with discrete delay

In this section, we have provided a detail numerical investigation of system (2.6)–(2.10), focusing on sensitivity analysis and illustrating the implications of Theorems 1 and 2. Furthermore, we have explored the influence of time delays on the system's dynamic behavior. The simulations are carried out using the parameter values sourced from the established studies in HIV-1 modeling, as given in Table 2.

**Table 2.** Parameters and their corresponding values of system (3.1)–(3.5).

Parameter	Value	References	Parameter	Value	References	Parameter	Value	References
$\lambda_1$	10	[41, 42]	$\lambda_2$	0.03196	[43]	$\epsilon_1$	0.2	[44, 45]
$\gamma_1$	0.01	[41]	$\gamma_2$	0.01	[46]	$\epsilon_2$	0.2	[47]
$\delta_1$	0.5	[44, 47]	$\delta_2$	0.1	[44]	$\epsilon_1$	0.2	[48]
$b_1$	6	[49]	$b_2$	6	[49]	$\epsilon_2$	0.2	[44]
$\varphi$	2	[41]	$\beta_1$	$2 \times 10^{-5}$	[41, 42]	$\beta_2$	$4 \times 10^{-6}$	[41, 50]

##### 4.2. Sensitivity analysis

Determining which parameters most strongly influence infection control is a central objective in infectious disease modeling. A widely used method for this purpose is sensitivity analysis—particularly forward sensitivity analysis—which evaluates how variations in model parameters affect disease dynamics and transmission potential [51].

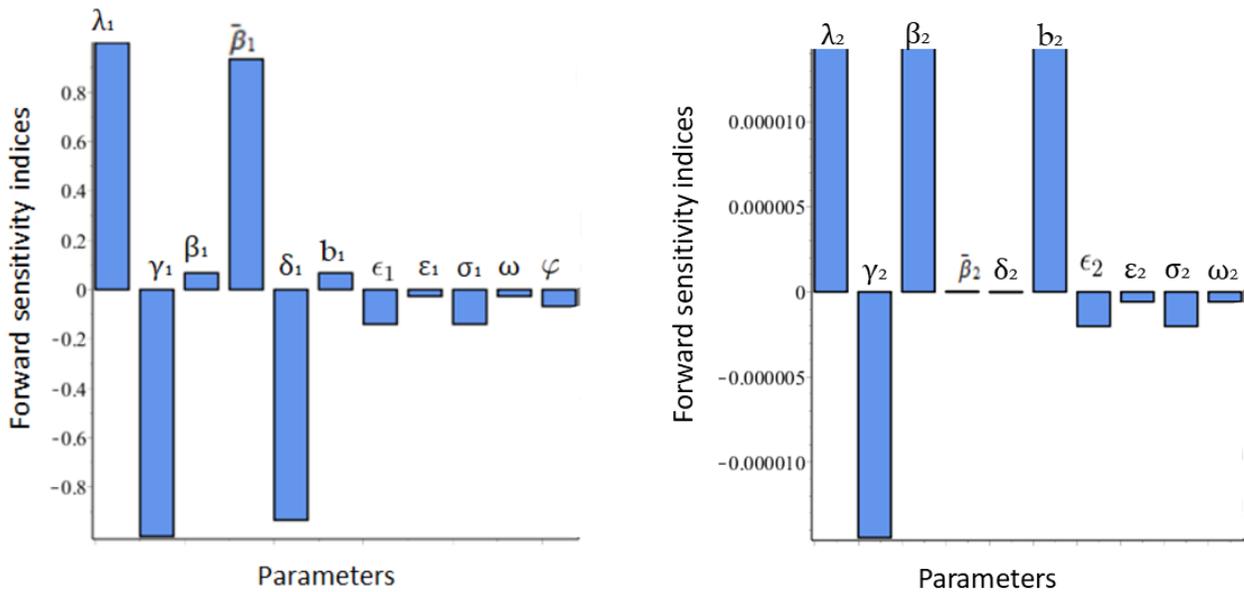
The normalized forward sensitivity index of the basic reproduction number  $\mathcal{R}_0^*$  with respect to a parameter  $\varkappa$  is defined by:

$$S_{\varkappa}^{\mathcal{R}_0^*} = \frac{\varkappa}{\mathcal{R}_0^*} \cdot \frac{\partial \mathcal{R}_0^*}{\partial \varkappa}. \quad (4.1)$$

This index is computed for each parameter contributing to  $\mathcal{R}_0^*$ . For instance, the sensitivity of  $\mathcal{R}_0^*$  with respect to  $\beta_1$  is given by:

$$S_{\beta_1}^{\mathcal{R}_0^*} = \frac{\beta_1}{\mathcal{R}_0^*} \cdot \frac{\partial \mathcal{R}_0^*}{\partial \beta_1} = \frac{1}{2\mathcal{R}_0^*} \left( \mathcal{U}_2^* + \frac{(\mathbb{U}^* - \mathbb{W}^*)\mathcal{U}_2^* + 2\mathcal{U}_2^*\mathcal{W}_2^*}{\sqrt{(\mathbb{U}^* - \mathbb{W}^*)^2 + 4\mathcal{U}_2^*\mathcal{W}_2^*}} \right).$$

Figure 1 and Table 3 present the computed sensitivity indices of  $\mathcal{R}_0^*$ , based on the parameter values listed in Table 2, along with the specific values:  $\sigma_1 = \sigma_2 = 0.7$ ,  $\omega_1 = \omega_2 = 0.2$ ,  $\beta_1 = 0.00002$ ,  $\beta_2 = 0.00002$ ,  $\bar{\beta}_1 = 0.0004$ , and  $\bar{\beta}_2 = 0.00001$ .



**Figure 1.** Forward sensitivity analysis of the parameters using  $\mathcal{R}_0^*$ .

**Table 3.** Sensitivity index of  $\mathcal{R}_0^*$ .

Parameter	Sensitivity index	Parameter	Sensitivity index	Parameter	Sensitivity index
$\lambda_1$	0.999	$\lambda_2$	$1.447 \times 10^{-5}$	$\epsilon_2$	$-2.025 \times 10^{-6}$
$\gamma_1$	-0.999	$\gamma_2$	$-1.447 \times 10^{-5}$	$\epsilon_1$	$-2.689 \times 10^{-3}$
$\beta_1$	$6.723 \times 10^{-2}$	$\beta_2$	$1.446 \times 10^{-5}$	$\epsilon_2$	$-5.784 \times 10^{-7}$
$\bar{\beta}_1$	$9.328 \times 10^{-1}$	$\bar{\beta}_2$	$5.394 \times 10^{-9}$	$\sigma_1$	$-1.399 \times 10^{-1}$
$\delta_1$	$-9.328 \times 10^{-1}$	$\delta_2$	$-5.394 \times 10^{-9}$	$\sigma_2$	$-2.025 \times 10^{-6}$
$b_1$	$6.723 \times 10^{-2}$	$b_2$	$1.446 \times 10^{-5}$	$\omega_1$	$-2.689 \times 10^{-3}$
$\varphi$	$-6.724 \times 10^{-2}$	$\epsilon_1$	$-1.399 \times 10^{-1}$	$\omega_2$	$-5.784 \times 10^{-7}$

The results indicate that the sensitivity indices for  $\lambda_1, \lambda_2, \beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2, b_1$ , and  $b_2$  are all positive. Among these,  $\lambda_1$  has the highest impact, while  $\bar{\beta}_2$  contributes the least. This suggests that increasing any of these parameters—while keeping others constant—leads to a rise in  $\mathcal{R}_0^*$ , thereby enhancing the potential for disease persistence.

Conversely, parameters such as  $\gamma_1, \gamma_2, \delta_1, \delta_2, \varphi, \epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \omega_1$ , and  $\omega_2$  exhibit negative

sensitivity indices. This implies that increasing these values contributes to a reduction in  $\mathcal{R}_0^*$ , thereby promoting infection control.

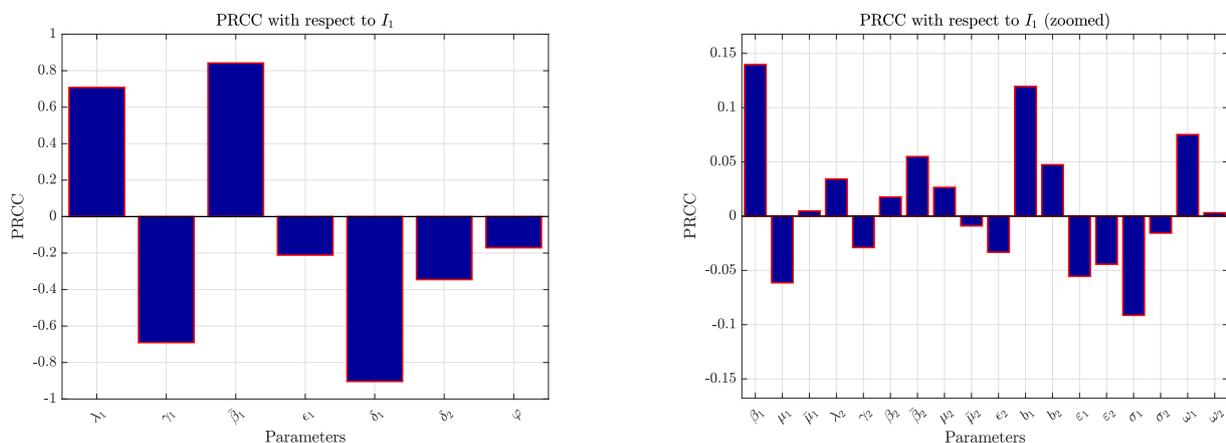
The sensitivity analysis performed in this part is local. The sensitivity indices were obtained by varying one parameter at a time around its baseline value given in Table 2, while all other parameters were held constant. This procedure measures how small changes in individual parameters affect the basic reproduction number  $\mathcal{R}_0^*$  near the chosen reference values.

#### 4.3. PRCC: Global sensitivity analysis

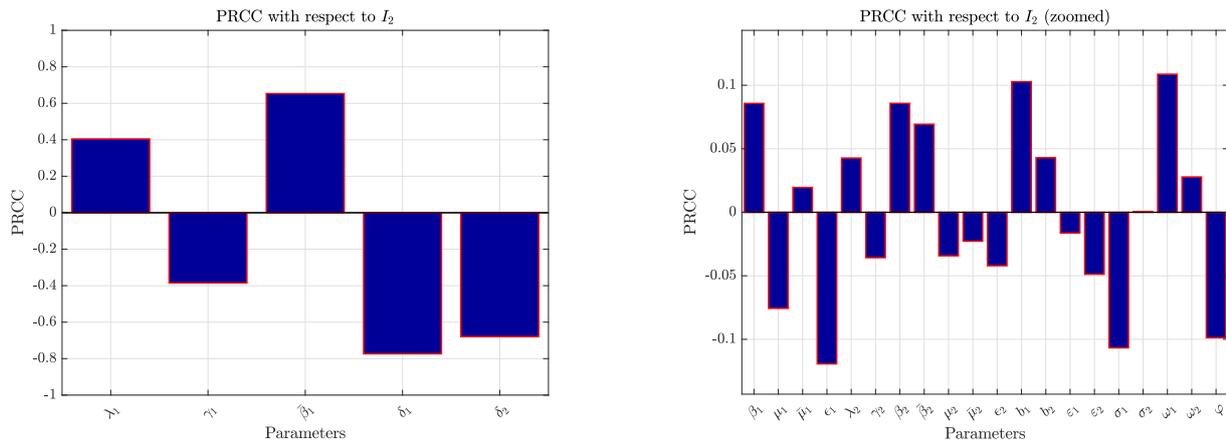
To study how uncertainties in the model parameters affect the infection dynamics, we perform a global sensitivity analysis based on the partial rank correlation coefficient (PRCC) method combined with Latin hypercube sampling (LHS). This technique is well suited for nonlinear delay models and allows us to assess the influence of each parameter over a wide range of possible values.

In the LHS procedure, the range of each parameter is divided into intervals of equal probability, and representative samples are drawn to ensure a good coverage of the parameter space. For every sampled parameter set, the delay model is solved numerically and relevant outputs—such as the peak or final value of the infected  $\text{CD4}^+$  T-cell population  $I_1(t)$ —are recorded. PRCC values are then calculated using rank-transformed data, which makes the analysis robust to nonlinear effects and isolates the impact of each parameter while accounting for the presence of the others.

The resulting PRCC values are displayed as bar charts in Figures 2 and 3. Parameters with positive PRCC values (for example,  $\lambda_i$  and  $\beta_i$ ,  $i = 1, 2$ ) tend to increase infection levels when their values grow, whereas parameters with negative PRCC values (such as  $\gamma_1$  and  $\gamma_2$ ) are associated with a reduction in infection. The absolute size of the PRCC indicates the strength of the parameter's influence, with larger magnitudes corresponding to stronger effects and values close to zero indicating a minor role.



**Figure 2.** PRCC analysis of the model parameters with respect to the infected  $\text{CD4}^+$ T cell population  $I_1$  for model (2.6)–(2.10).



**Figure 3.** PRCC analysis of the model parameters with respect to the infected macrophage population  $I_2$  for model (2.6)–(2.10).

#### 4.4. Numerical stability of the equilibria

While HIV disease progression occurs over years, the intracellular viral replication and maturation processes represented by the delay terms take place on a much shorter timescale, typically ranging from several hours to about one–two days. Accordingly, the delay parameters are chosen within biologically plausible ranges consistent with these intracellular dynamics [41, 44, 45, 47, 48].

In this subsection, we use MATLAB solver dde23 to numerically investigate the dynamic behavior of model (2.6)–(2.10). We take into account the following factors when performing the numerical simulation:

(i) We choose the parameters as  $\sigma_1 = \sigma_2 = 0.7$ ,  $\omega_1 = \omega_2 = 0.2$ ,  $\mu_1 = \mu_2 = \bar{\mu}_1 = \bar{\mu}_2 = 0.0001$ , and pick three different initial conditions for system (2.6)–(2.10):

**I.1 :**  $(T_1(\xi), I_1(\xi), T_2(\xi), I_2(\xi), V(\xi)) = (900, 8, 2.5, 0.008, 17)$  ;

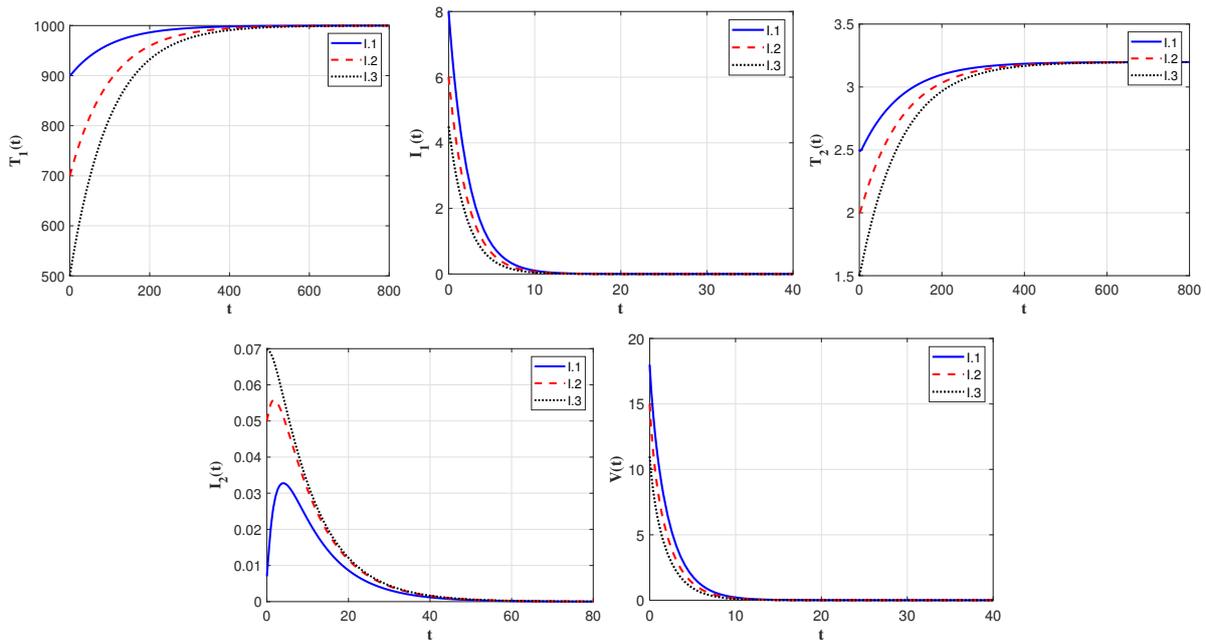
**I.2 :**  $(T_1(\xi), I_1(\xi), T_2(\xi), I_2(\xi), V(\xi)) = (700, 6, 2, 0.05, 15)$  ;

**I.3 :**  $(T_1(\xi), I_1(\xi), T_2(\xi), I_2(\xi), V(\xi)) = (500, 4.3, 1.5, 0.07, 11)$ , where

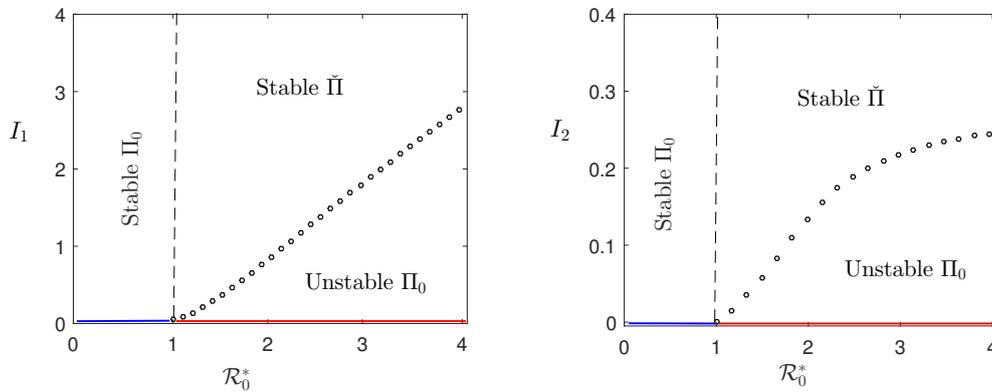
$\xi \in [-\max\{\sigma_1, \sigma_2, \omega_1, \omega_2\}, 0]$ . By choosing different parameter values of the infection rates  $\beta_1, \beta_2, \bar{\beta}_1$ , and  $\bar{\beta}_2$ , we have the the following outcomes:

**Scenario 1 (Stability of  $\Pi_0$ ):** We select the parameters  $\beta_1 = 0.00002$ ,  $\beta_2 = 0.0004$ ,  $\bar{\beta}_1 = 0.00002$ , and  $\bar{\beta}_2 = 0.00001$ , which yield  $\mathcal{R}_0^* = 0.0868 < 1$ . In this environment, the solution that solves system (2.6)–(2.10) tends towards the infection-free equilibrium state (IFE)  $\Pi_0 = (1000, 0, 3.196, 0, 0)$  for all  $t \rightarrow \infty$ . As noticed from Figure 4, while the amounts of all infectious components tend towards zero, the amount of healthy and uninfected CD4+ T cells and uninfected macrophages tends towards safe levels  $T_1 = 1000$  and  $T_2 = 3.196$ , respectively. This analysis illustrates that IFE  $\Pi_0$  remains globally stable and agrees with the theoretical analyses defined by Theorem 1. In this case, there would be an eradication.

Figure 5 depicts a forward bifurcation occurring at  $\mathcal{R}_0^* = 1$ , distinguishing the transition between disease-free and endemic equilibria. The figure indicates that the system remains disease-free when the infection rate is below a critical threshold, and becomes endemic only when this threshold is surpassed. If we can regulate the rates (such as  $\lambda_1, \lambda_2$  etc.) using drugs, the disease can be managed.



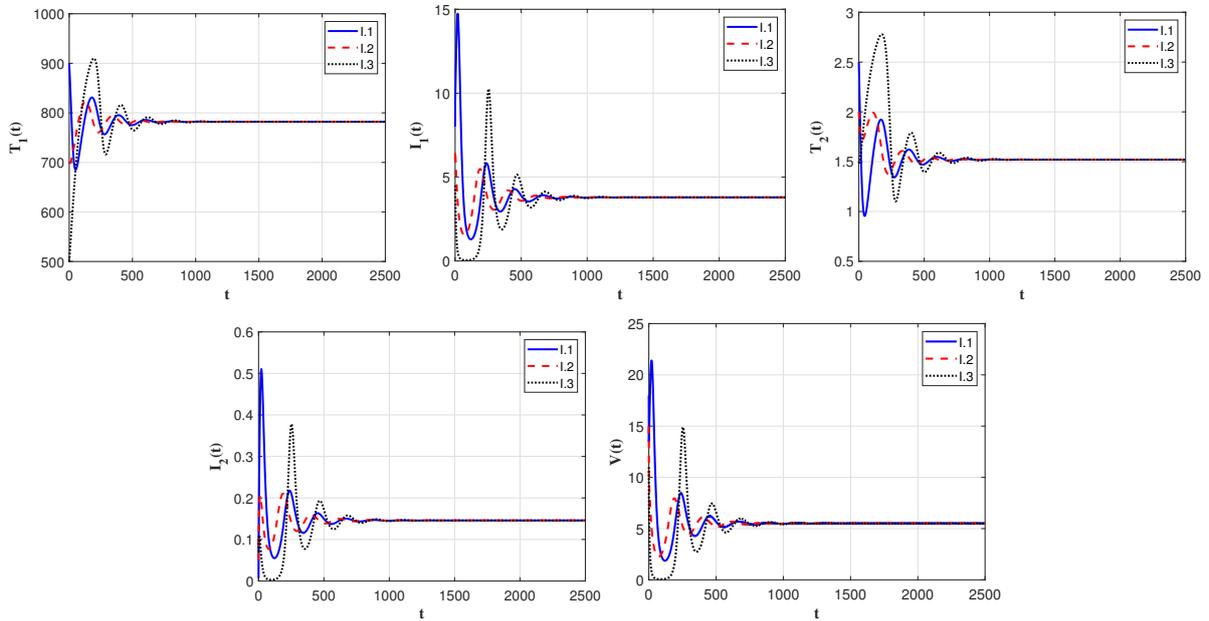
**Figure 4.** The numerical solutions of model (2.6)–(2.10) for  $\beta_1 = 0.00002$ ,  $\beta_2 = 0.0004$ ,  $\bar{\beta}_1 = 0.00002$ ,  $\bar{\beta}_2 = 0.00001$  with three different initial conditions. The infection-free equilibrium  $\Pi_0 = (1000, 0, 3.196, 0, 0)$  is globally asymptotically stable where  $\mathcal{R}_0^* = 0.0868 < 1$ .



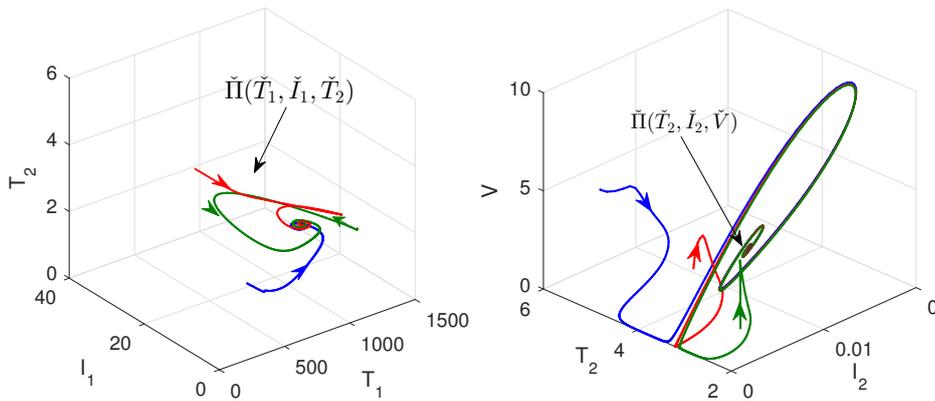
**Figure 5.** Forward bifurcation: Steady state values of infected populations are plotted with  $\mathcal{R}_0^*$ . We have varied  $\beta_1$  to plot this figure. The rest of the parameters are the same as in Figure 4.

**Scenario 2 (Stability of  $\check{\Pi}$ ):** To achieve  $\mathcal{R}_0^* = 1.2829 > 1$ , we use the parameter values  $\beta_1 = 0.0003$ ,  $\beta_2 = 0.002$ ,  $\bar{\beta}_1 = 0.0003$ , and  $\bar{\beta}_2 = 0.0002$ . The system (2.6)–(2.10) then converges to the infected persistent equilibrium (IPE)  $\check{\Pi} = (782.1368, 3.788, 1.52, 0.1457, 5.5013)$ . The stability and existence of  $\check{\Pi}$  are supported by Proposition 2 and Theorem 2, as shown in Figure 6. This equilibrium characterizes the state of an individual living with HIV-1, indicating ongoing infection. Therefore, the infection rate plays a crucial role in determining disease progression and control.

In Figure 7, phase trajectories are plotted for three different initial conditions. All trajectories converge to the same steady point. This shows the nonlinear stability of the endemic equilibrium.



**Figure 6.** The numerical solutions of model (2.6)–(2.10) for  $\beta_1 = 0.0003$ ,  $\beta_2 = 0.002$ ,  $\bar{\beta}_1 = 0.0003$ , and  $\bar{\beta}_2 = 0.0002$  with three different initial conditions. The infection-present equilibrium  $\check{\Pi} = (782.1368, 3.788, 1.52, 0.1457, 5.5013)$  is globally asymptotically stable where  $\mathcal{R}_0^* = 1.2829 > 1$ .



**Figure 7.** Global stability: Phase portraits of system (2.6)–(2.10) are plotted with different initial conditions. Parameter values are the same as those used in Figure 6.

**Impact of saturated infection on model (2.6)–(2.10):** We set  $\sigma_1 = \sigma_2 = 0.7$  and  $\omega_1 = \omega_2 = 0.2$ , and apply initial condition **I.1** along with the parameter values from **Scenario 2**. Figure 8 illustrates the influence of saturation on infection dynamics. The following cases are considered:

**Case A:**  $\mu_1 = \mu_2 = 0.0000002$ ,  $\bar{\mu}_1 = \bar{\mu}_2 = 0.0000001$ .

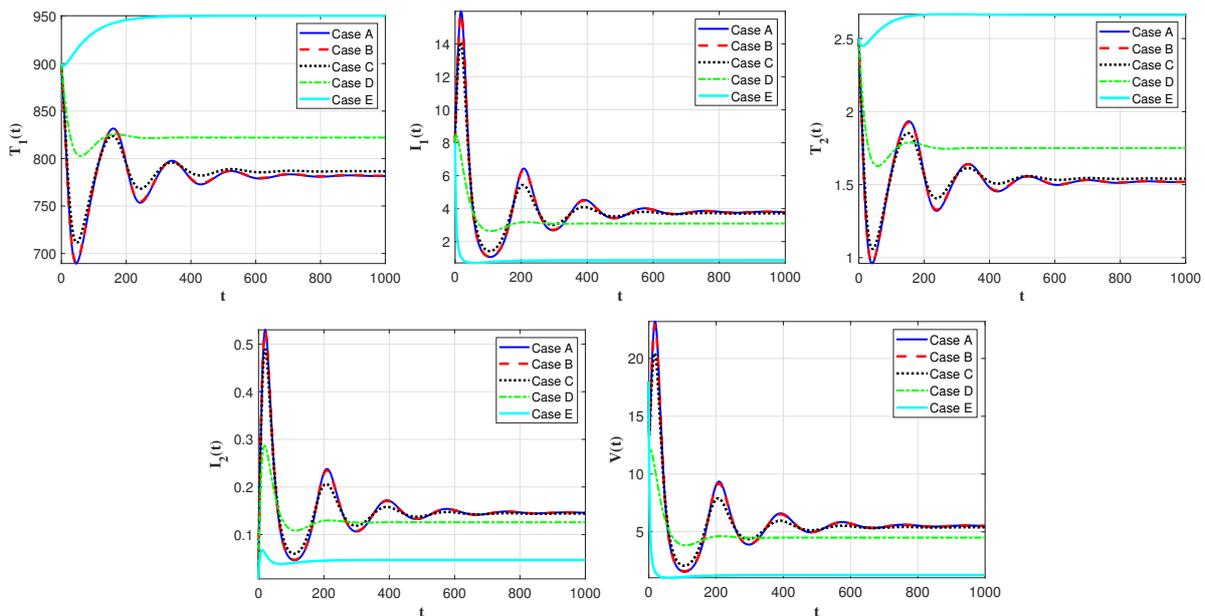
**Case B:**  $\mu_1 = \mu_2 = \bar{\mu}_1 = \bar{\mu}_2 = 0.0001$ .

**Case C:**  $\mu_1 = \bar{\mu}_1 = 0.001$ ,  $\mu_2 = 0.002$ ,  $\bar{\mu}_2 = 0.005$ .

**Case D:**  $\mu_1 = 0.01$ ,  $\mu_2 = 0.02$ ,  $\bar{\mu}_1 = 0.02$ ,  $\bar{\mu}_2 = 0.04$ .

**Case E:**  $\mu_1 = \mu_2 = \bar{\mu}_1 = \bar{\mu}_2 = 0.2$ .

As the values of  $\mu_1$ ,  $\mu_2$ ,  $\bar{\mu}_1$ , and  $\bar{\mu}_2$  increase, the infection rates decline. This leads to higher concentrations of uninfected cells and reduced levels of infected cells and free virus particles. Notably,  $\mathcal{R}_0^*$  remains unaffected by these parameters, indicating that saturation does not alter the stability of equilibria. However, in the presence of infection, saturation significantly influences the population of infected cells and the equilibrium values of  $\check{\Pi}$ .



**Figure 8.** The numerical solutions of model (2.6)–(2.10) with five different sets of saturation parameters, with  $\beta_1 = 0.0003$ ,  $\beta_2 = 0.002$ ,  $\bar{\beta}_1 = 0.0003$ ,  $\bar{\beta}_2 = 0.0002$ ,  $\sigma_1 = \sigma_2 = 0.7$ ,  $\omega_1 = \omega_2 = 0.2$ .

**Effect of different delays on the system dynamics:** Equation (2.21), which determines the threshold parameter  $\mathcal{R}_0^*$ , is influenced by the time delay coefficients  $\sigma_\ell$  and  $\varpi_\ell$  for  $\ell = 1, 2$ . These delay parameters play a critical role in shaping the stability behavior of the system's equilibria. To examine how variations in these coefficients affect the dynamics, we consider the parameter set  $\beta_1 = 0.0005$ ,  $\beta_2 = 0.004$ ,  $\bar{\beta}_1 = 0.0003$ , and  $\bar{\beta}_2 = 0.0003$ . Additionally, we adopt the values of  $\mu_1$ ,  $\mu_2$ ,  $\bar{\mu}_1$ , and  $\bar{\mu}_2$  from **Case B**. The parameters  $\sigma_1$ ,  $\sigma_2$ ,  $\omega_1$ , and  $\omega_2$  are then varied to explore different scenarios, as outlined below:

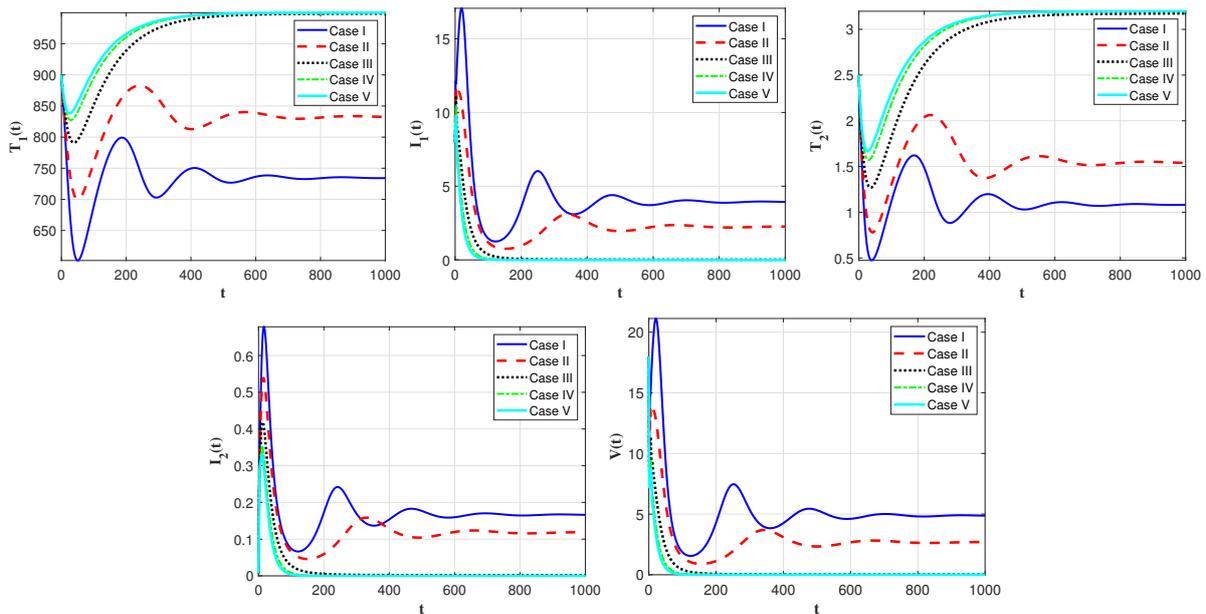
- Case I :**  $\sigma_1 = 1.5, \sigma_2 = 1.2, \omega_1 = 1, \omega_2 = 1.$
- Case II :**  $\sigma_1 = 2, \sigma_2 = 1.7, \omega_1 = 1.2, \omega_2 = 1.2.$
- Case III :**  $\sigma_1 = 2.5, \sigma_2 = 2.2, \omega_1 = 1.876, \omega_2 = 1.876.$
- Case IV :**  $\sigma_1 = 3, \sigma_2 = 2.8, \omega_1 = 2.5, \omega_2 = 2.5.$
- Case V :**  $\sigma_1 = 3.5, \sigma_2 = 3.2, \omega_1 = 2.6, \omega_2 = 2.6.$

In Case  $\sigma_1 = \sigma_2 = \omega_1 = \omega_2 = 0, \mathcal{R}_0^*$  becomes the form given by

$$\tilde{\mathcal{R}}_0 = \frac{1}{2} \left( \frac{\delta_2 \bar{\beta}_1 T_1^0 + \delta_1 \bar{\beta}_2 T_2^0}{\delta_1 \delta_2} + \frac{b_1 \beta_1 T_1^0 + b_2 \beta_2 T_2^0}{\varphi} \right) + \frac{1}{2} \sqrt{\left( \frac{\delta_2 \bar{\beta}_1 T_1^0 - \delta_1 \bar{\beta}_2 T_2^0}{\delta_1 \delta_2} + \frac{b_1 \beta_1 T_1^0 - b_2 \beta_2 T_2^0}{\varphi} \right)^2 + 4 \frac{b_1 b_2 \beta_1 \beta_2 T_1^0 T_2^0}{\varphi^2}},$$

and the system (2.6)–(2.10) is reduced to system (2.1)–(2.5).

Using the specified parameter values and initial conditions I.1, the influence of the time delay parameters on the system’s behavior is illustrated in Figure 9. In addition, the basic reproduction number  $\mathcal{R}_0^*$  is computed based on these inputs. The corresponding results are summarized in Table 4.



**Figure 9.** The numerical solutions of model (2.6)–(2.10) with five different sets of delay parameters and with  $\beta_1 = 0.0003, \beta_2 = 0.002, \bar{\beta}_1 = 0.0003, \bar{\beta}_2 = 0.0002$  and  $\mu_1 = \mu_2 = \bar{\mu}_1 = \bar{\mu}_2 = 0.0001.$

By applying the parameter values listed in Tables 2 and 4, we observe the following outcomes:

- (i) When  $\sigma_1 \geq 2.5, \sigma_2 \geq 2.2,$  and  $\omega_1 = \omega_2 \geq 1.876,$  the basic reproduction number satisfies  $\mathcal{R}_0^* \leq 1,$  indicating that the infection-free equilibrium (IFE)  $\Pi_0$  is globally asymptotically stable (GAS).
- (ii) Conversely, if  $\sigma_1 < 2.5, \sigma_2 < 2.2$  and  $\omega_1 = \omega_2 < 1.876,$  then  $\mathcal{R}_0^* > 1,$  and the equilibrium  $\Pi_0$  loses stability.

**Table 4.** Cases of time delay and  $\mathcal{R}_0^*$  values.

Case	$\sigma_1$	$\sigma_2$	$\omega_1$	$\omega_2$	$\mathcal{R}_0^*$
<b>I</b>	1.5	1.2	1	1	1.3711
<b>II</b>	2	1.7	1.2	1.2	1.2075
<b>III</b>	2.5	2.2	1.876	1.876	1
<b>IV</b>	3	2.8	2.5	2.5	0.8367
<b>V</b>	3.5	3.2	2.6	2.6	0.7481

These results clearly demonstrate that increasing the time delay parameters leads to a reduction in  $\mathcal{R}_0^*$ , which aligns with the findings discussed in Section 4.2.

In a nutshell, smaller time delays increase the number of infected cells; larger time delays suppress infection. This is because smaller delays increase the interaction rate between virus-infected cells and susceptible target cells, quickening the pace of infection. These observations suggest the strong impact of temporal time delays on the course of infection and that further investigation of time delay effects is necessary for the understanding of the disease.

#### 4.5. Simulation of the model with distributed delay

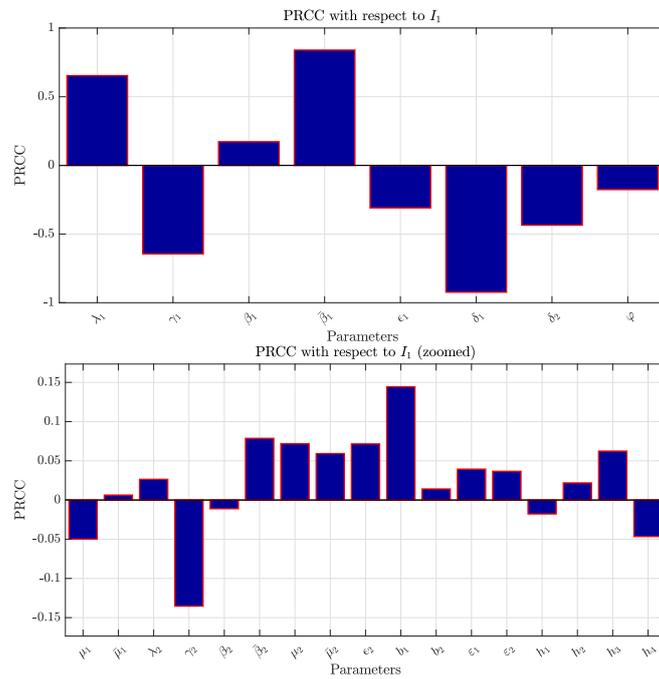
Here, we provide the simulations of the model with distributed delay. First, we plotted the global sensitivity indices of the parameters. Then, we vary the delay parameter to show its impact on the dynamics of the HIV-1 transmission.

Figures 10 and 11 illustrate the global sensitivity analysis carried out using PRCC in combination with LHS for model (3.1)–(3.5). In those Figures, the PRCC indices are evaluated with respect to the infected  $CD4^+$ T cell population  $I_1$  and the infected macrophage population  $I_2$ , as well as the corresponding results.

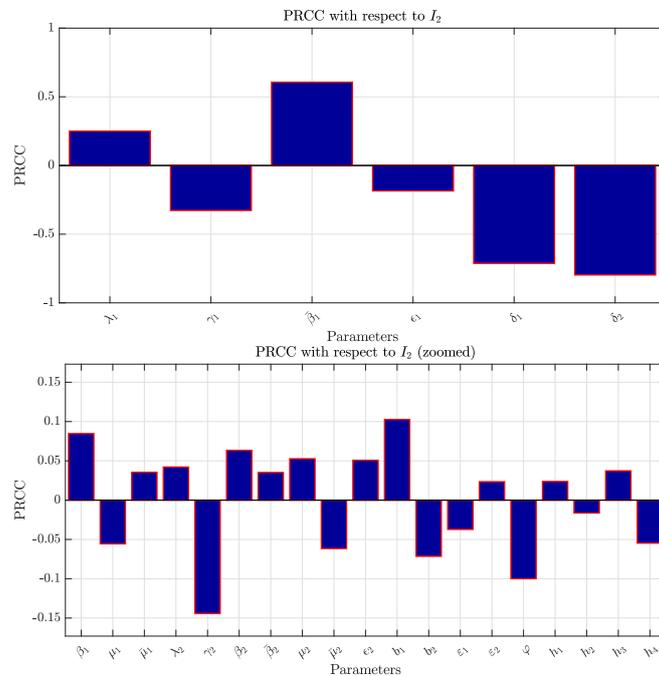
The obtained outcomes showed that the parameters related to infection and transmission processes exhibit positive PRCC values, indicating that increases in these parameters lead to higher levels of infected cells. In contrast, parameters associated with natural death or clearance, such as  $\gamma_1$  and  $\gamma_2$ , show negative PRCC values, reflecting their inhibitory effect on the infection dynamics. Overall, this analysis identifies the parameters with the strongest influence on the system and highlights the role of delay-related effects in shaping the behavior of the model.

The numerical simulations of the model system (3.1)–(3.5) are illustrated in Figure 12 for two distinct values of the delay parameter. The results show that increasing the delay leads to higher levels of susceptible cell populations, whereas shorter delays promote a stronger infection response and result in increased infected populations.

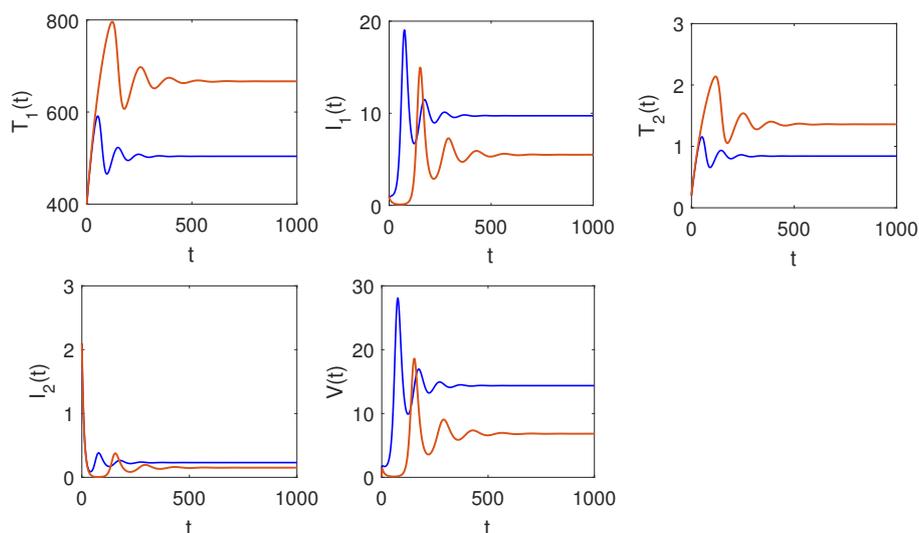
Figure 13 demonstrates the global stability behavior of the system. Phase portraits are generated using various initial conditions, and the trajectories are observed to converge toward a unique coexistence equilibrium. This confirms that the equilibrium point is globally asymptotically stable.



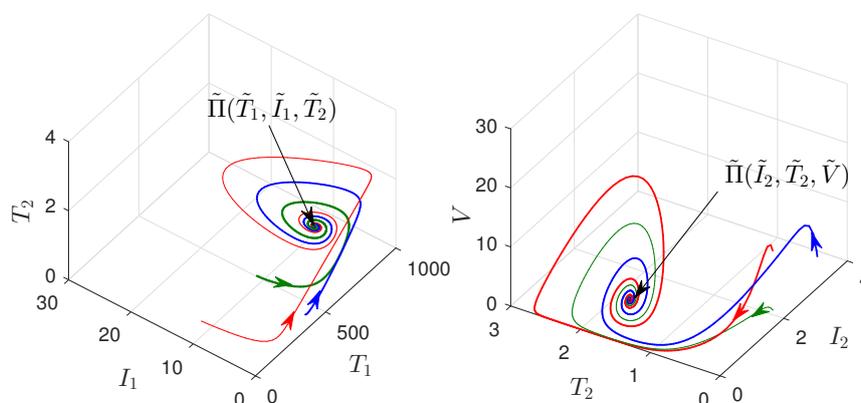
**Figure 10.** PRCC analysis of the model parameters with respect to the infected CD4<sup>+</sup>T cell population  $I_1$  for model (3.1)–(3.5).



**Figure 11.** PRCC analysis of the model parameters with respect to the infected macrophage population  $I_2$  for model (3.1)–(3.5).



**Figure 12.** Numerical solutions of system (3.1)–(3.5) for different values of delay parameter  $\tau$ . Parameter values are:  $\beta_1 = 0.0005$ ,  $\beta_2 = 0.002$ ,  $\bar{\beta}_1 = 0.0003$ ,  $\bar{\beta}_2 = 0.0002$ ,  $\lambda_1 = 10$ ,  $\lambda_2 = 0.03196$ ,  $\gamma_1 = 0.01$ ,  $\gamma_2 = 0.01$ ,  $\delta_1 = 0.5$ ,  $\delta_2 = 0.1$ ,  $b_1 = 6$ ,  $b_2 = 6$ ,  $\phi = 2$ ,  $\epsilon_1 = 0.2$ ,  $\epsilon_2 = 0.2$ ,  $\varepsilon_1 = 0.2$ ,  $\varepsilon_2 = 0.2$ ,  $\mu_1 = 0.002$ ,  $\mu_2 = 0.002$ ,  $\bar{\mu}_1 = 0.002$ ,  $\bar{\mu}_2 = 0.002$ . Blue lines represent  $\tau = 0.2$ , and red lines represent  $\tau = 5$ .



**Figure 13.** Global stability: Phase portraits of system (3.1)–(3.5) for different initial conditions. Parameter values are taken from Figure 6.

## 5. Discussions

In this study, we aim to show the global stability of the equilibria in HIV-1 models with both discrete and distributed delays. The *discrete-delay model* uses a fixed delay for all cells. This makes the model simple and efficient, but it is less realistic biologically. On the other hand, the *distributed-delay model* allows variability by using probability density functions. This makes the model closer to real biological behavior, but it also adds more mathematical and computational complexity [30, 52].

We examined the two mathematical models of HIV-1 infection with saturated incidence rates for two major target cell types: CD4<sup>+</sup> T-cells and macrophages. The models considered include two types of transmission mechanisms: virus-to-cell transmission and cell-to-cell transmission. To provide more realistic models of the biological systems under consideration, four discrete time delays are inserted into Eqs (2.1)–(2.5) to give the corresponding equation systems (2.6)–(2.10). Moreover, another model is developed using Eqs (2.1)–(2.5) into which another four time delays are inserted so as to form the equation systems (3.1)–(3.5).

We proved that the solutions of the proposed models are bounded and nonnegative. A sensitivity analysis was done to examine the effect of the parameters on the basic reproductive number  $\mathcal{R}_0^*$ . From the results, a direct relation existed between the parameters  $\lambda_1$ ,  $\lambda_2$ ,  $\beta_1$ ,  $\beta_2$ ,  $\bar{\beta}_1$ ,  $\bar{\beta}_2$ ,  $b_1$ , and  $b_2$  and the persistence of the disease within the population.

In this research, we prove that for the delayed systems, there exist two equilibrium solutions: infection-free equilibrium (IFE) and infection-present equilibrium (IPE). The existence of IPE requires that the basic reproduction number  $\mathcal{R}_0^*$  be greater than unity. In this case,  $\mathcal{R}_0^*$  becomes an important threshold quantity for system (1.1) to be asymptotically stable. More precisely, if  $\mathcal{R}_0^* \leq 1$ , then IFE is globally asymptotically stable (GAS), and it implies that infection will be eradicated. Otherwise, for  $\mathcal{R}_0^* > 1$ , IPE is GAS; that is, infection persists. In addition to these mathematical analyses above, simulations are employed to verify these findings and examine further effects of delay parameters.

From Figure 4, it is clear that if  $\mathcal{R}_0^* \leq 1$ , then the trajectories of this system tend to IFE. Additionally, it tends to IPE for different initial conditions in Figure 6 if  $\mathcal{R}_0^* > 1$ . This depicts the global stability of IFE, which denotes the uninfected condition of a cell. This scenario highlights the stability of IPE, which denotes an uninfected cell. Since HIV-1 infection persists within a cell, its effect upon the cell manifests. This scenario supports the ongoing effect of HIV-1 within a cell. The effect of saturations of infection rates within a cell can be observed in Figure 8.

When these saturation parameters,  $\mu_1$ ,  $\mu_2$ ,  $\bar{\mu}_1$ , and  $\bar{\mu}_2$  are increased, it shows a reduction in infection rates, an increase in uninfected cell levels, a reduction in levels of infected cells, and a reduction in levels of free viral particles. This indicates that higher saturation effects act as limiting factors on viral replication and spread, thereby contributing to system stability and lowering the basic reproductive number. This depicts an effect of saturation on an uninfected cell. Moreover, it shows that these saturation parameters do not affect  $\mathcal{R}_0^*$ . This suggests that IFE and IPE within an uninfected cell are independent. As indicated in Figure 9 and Table 4 above, it depicts that an increase in time delays contributes to a reduction in  $\mathcal{R}_0^*$ .

Thus, analytical and numerical studies lead us to conclude, in a broad sense, that susceptible CD4<sup>+</sup>T cells, macrophages, infected cells, and viral particles interact in an extremely complex manner. It should be mentioned that HIV-1 replication is supported through target cell infection, intracellular viral replication, and subsequently produced viral particles, hence ensuring that an overall infection cycle exists. The further development and maintenance of this particular viral threat in an infected individual is determined through an extremely complicated equilibrium of cell division, infectious dynamics, and eventual immune cell loss. Mathematical models describing this process through discrete and distributed delays have ensured that an appropriate mathematical tool exists for analyzing these complicated interactions. At the same time, they have managed to offer invaluable perspectives on potential therapeutic strategies in the form of antiretroviral drugs, which have an ability to inhibit significant stages of the viral lifecycle, potentially ensuring viral loads and immune system

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functionalities remain low.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interests

Fahad Al Basir is a Guest Editor of this Special Issue and was not involved in the editorial review or decision-making process for this manuscript. All authors declare that there are no competing interests.

### Author contributions

Aeshah A. Raezah: Conceptualization, Methodology, Validation, Software, Writing original draft preparation; Ghadeer Al Bogami: Methodology, Validation, Writing original draft preparation; A. S. Shflot: Methodology, Validation, Software, Visualization; Fahad Al Basir: Validation, Software, Writing original draft preparation, Supervision. All authors have read and agreed to publish the manuscript.

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