Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(1): 261–275.
DOI: 10.3934/nhm.2026013
Received: 09 January 2026
Revised: 01 March 2026
Accepted: 03 March 2026
Published: 10 March 2026

*Research article*

# Analysis and numerical implementation of high-order method for distributed-order diffusion model

**Lingna Lu**[1] **and Changshun Hou**[2,*]

[1] School of Basic Sciences, Zhengzhou University of Technology, Zhengzhou 450044, China
[2] School of Mathematics and Statistics, Henan University of Technology, Zhengzhou 450001, China

* **Correspondence:** Email: chhaut2002@163.com.

**Abstract:** This work proposes a high-order discontinuous Galerkin (DG) formulation employing generalized alternating numerical fluxes for approximating solutions to the distributed-order diffusion equation. Such models often arise in ultraslow diffusion, where solutions decay only logarithmically as $t \to \infty$. Utilizing the Grünwald–Letnikov scheme and the DG method, we construct a fully discrete numerical algorithm. Using a rigorous induction argument, we prove unconditional stability and convergence of the proposed scheme. A comprehensive set of computational experiments is presented to validate the efficacy and performance of the method.

**Keywords:** diffusion model; fractional derivative; convergence; stability

## 1. Introduction

The distributed-order differential equation constitutes a significant extension of classical fractional calculus, providing a rigorous mathematical framework that generalizes both single-order and multi-term fractional models. By integrating over a continuous distribution of fractional orders, this formulation captures complex memory and heterogeneity effects that are beyond the reach of its simpler counterparts. A prominent application of this generalized framework is found in the modeling of ultraslow, a phenomenon wherein the mean square displacement of particles grows logarithmically with time, rather than according to a conventional power law [1–3]. This behavior is characteristic of strongly disordered media, such as certain porous geological formations or complex viscoelastic polymers.

Furthermore, interpreting the distribution of derivative orders as a spectral weighting of delay times allows these equations to describe systems governed by a continuum of relaxation or retardation mechanisms [4]. This makes them particularly adept at modeling materials with a broad, continuous spectrum of response times. Extending this physical insight, when the distribution spans the interval from zero to two, the model elegantly bridges distinct dynamic regimes: Orders near zero correspond to near-elastic behavior, orders near one dominate viscoelastic damping, and orders approaching two introduce significant inertial or wave-like effects [5, 6]. Consequently, the distributed-order operator provides a unified continuum description that seamlessly integrates viscoelastic and visco-inertial dynamics within a single constitutive law.

A variety of numerical strategies have been developed for distributed-order differential equations, reflecting the diverse structures of the underlying memory operators. Diethelm and Ford [7] developed an efficient numerical approach for both linear and nonlinear distributed-order fractional ordinary differential equations and established a rigorous convergence theory for the resulting scheme. Ye et al. [8] devised a compact finite-difference discretization for a distributed-order diffusion-wave model, while Morgado and Rebelo [9] developed a numerical scheme tailored for problems involving nonlinear source terms. Gao et al. [10] further advanced finite difference techniques, proposing schemes for both one- and two-dimensional settings, and provided detailed stability and convergence proofs. Alternative approaches include the work of Li and Wu [11], who approximated the distributed-order model using a multi-term fractional model and solved it, and Alikhanov [12] derived a finite-difference scheme for diffusion equations with multiple terms and a variable distributed-order structure. Additionally, Katsikadelis [13] presented a general numerical method applicable to both linear and nonlinear problems. More recently, high-accuracy discretizations for related (single- or multi-term) fractional diffusion-type equations have been actively studied, including high-precision interpolation-based approaches and compact finite difference schemes; see [14, 15] for representative examples.

Despite these contributions, the literature on numerical schemes for solving distributed-order differential equations remains relatively limited. To the best of our knowledge, there is a notable scarcity of published works addressing robust and versatile discretizations for this class of problems. Consequently, the development of effective, computationally tractable, and broadly applicable numerical schemes continues to be a significant and open challenge in the field. The DG method effectively synthesizes the strengths of finite element and finite volume frameworks [16–18]. By adopting a piecewise discontinuous polynomial space, the DG approach achieves high-order accuracy and exceptional flexibility in the numerical resolution of partial differential equations. This makes it particularly suitable for problems involving intricate geometries or solutions exhibiting sharp gradients and discontinuities. A defining advantage of the method is its local conservation property and inherent suitability for high-order polynomial approximations on non-conforming meshes, facilitating efficient h-p adaptivity. These algorithmic merits are underpinned by a solid theoretical foundation, encompassing comprehensive analyses of stability, convergence, and a priori error estimates.

This work is devoted to the formulation and analysis of a DG method with generalized alternating numerical fluxes for the initial-boundary value problem governed by a distributed-order diffusion equation. Let $\Omega = (a, b)$. We consider

$$\begin{cases} \mathcal{D}_t^w u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} - \rho u(x,t) + f(x,t), & x \in (a,b),\ t \in (0,T], \\ u(x,0) = 0, & x \in [a,b], \end{cases} \tag{1.1}$$

where $\rho > 0$ and the distributed-order fractional derivative $\mathcal{D}_t^w$ is defined by

$$\mathcal{D}_t^w u(x,t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(x,t)\, d\alpha,$$

with a non-negative weight function $w(\alpha) \geq 0$ satisfying $w \in C([0,1])$ and $\int_0^1 w(\alpha)d\alpha = c_0 > 0$. Here, ${}_0^C D_t^\alpha$ denotes the Caputo derivative of order $\alpha \in (0,1)$:

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad t > 0, $$

and $\Gamma(\cdot)$ is the Gamma function. In the subsequent analysis, we assume the solution $u(x,t)$ is either periodic on the spatial domain or possesses compact support.

The organization of the paper is arranged as follows: Section 2 fixes the notation, specifies the relevant function spaces, and collects auxiliary results used later. Section 3 formulates a fully discrete local discontinuous Galerkin (LDG) scheme for the distributed-order fractional diffusion equation. A rigorous analysis establishes the unconditional stability and convergence of the proposed method, achieving an error bound of order $O(h^{k+1}+\Delta t+\Delta\alpha^2)$. Section 4 presents a series of numerical experiments to validate the theoretical findings and illustrate the performance of the algorithm. Finally, concluding remarks and perspectives for future work are provided in Section 5.

## 2. Notations and auxiliary results

In this section, we begin by introducing the spatial discretization. Consider a partition of the spatial domain $\Omega = (a,b)$:

$$ a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b. $$

For $m = 1, \ldots, N$, define the cell $I_m = [x_{m-\frac{1}{2}}, x_{m+\frac{1}{2}}]$, its length $\Delta x_m = x_{m+\frac{1}{2}} - x_{m-\frac{1}{2}}$, and the maximal mesh size $h = \max_{1\leq m\leq N} \Delta x_m$. At an interface $x_{m+\frac{1}{2}}$, we denote by $u^+_{m+\frac{1}{2}}$ and $u^-_{m+\frac{1}{2}}$ the limits of a piecewise-defined function $u$ from the right cell $I_{m+1}$ and the left cell $I_m$, respectively.

The DG approximation space $V_h^k$ is defined as the collection of functions that restrict to polynomials of degree no greater than $k$ on every mesh element:

$$ V_h^k := \left\{ \phi \in L^2([a,b]) \,\middle|\, \phi|_{I_m} \in \mathbb{P}^k(I_m),\ m = 1, \ldots, N \right\}. $$

For the temporal discretization, we take a uniform partition on $[0,T]$ with time step $\Delta t := T/M$, where $M \in \mathbb{N}$, and the temporal nodes are denoted by $t_j = j\Delta t$ for $j = 0, 1, \ldots, M$.

Finally, to approximate the distributed-order integral, the interval $[0,1]$ is subdivided into $L$ equal subintervals of length $\Delta\alpha = 1/L$, with nodes $\alpha_n = n\Delta\alpha$, $n = 0, 1, \ldots, L$.

**Lemma 2.1** (Composite trapezoidal rule)**.** *Assume $s(\alpha) \in C^2[0,1]$. Then*

$$ \int_0^1 s(\alpha)\,d\alpha = \Delta\alpha \sum_{m=0}^{L} c_m\, s(\alpha_m) - \frac{\Delta\alpha^2}{12}\, s''(\eta), \qquad 0 < \eta < 1, $$

*where $\alpha_m = m\Delta\alpha$, $\Delta\alpha = 1/L$, and the weights are given by*

$$ c_j = \begin{cases} \frac{1}{2}, & j = 0 \text{ or } j = L, \\ 1, & \text{otherwise}. \end{cases} $$

Next, we recall the definition of the Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} \, ds. $$

**Lemma 2.2.** *[19] Let $0 < \alpha < 1$ and suppose that $f(0) = 0$. Then the Caputo derivative ${}_0^C D_t^\alpha f(t)$ agrees with the Riemann–Liouville derivative ${}_0D_t^\alpha f(t)$.*

**Remark 2.1.** *In problem (1.1), we have $u(x, 0) = 0$ for all $x \in [a, b]$. Therefore, for each fixed x, the time trace $t \mapsto u(x, t)$ satisfies the assumption in Lemma 2.2, and hence ${}_0^C D_t^\alpha u(x, t) = {}_0D_t^\alpha u(x, t)$ holds pointwise in x (for $t > 0$).*

**Lemma 2.3.** *Let $\alpha \in (0, 1)$ and define the function space*

$$ \mathcal{I}^{1+\alpha}(\mathbb{R}) = \Big\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} (1 + |\xi|)^{1+\alpha} |\widehat{f}(\xi)| \, d\xi < \infty \Big\}, $$

*where $\widehat{f}$ denotes the Fourier transform of $f$. If $f \in \mathcal{I}^{1+\alpha}(\mathbb{R})$, then*

$$ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^t \frac{f(s)}{(t-s)^\alpha} \, ds = \frac{1}{(\Delta t)^\alpha} \sum_{m=0}^{\infty} g_m^{(\alpha)} f(t - (m - r)\Delta t) + O(\Delta t), $$

*where r is an integer, and the Grünwald weights are given by*

$$ g_m^{(\alpha)} = (-1)^m \binom{\alpha}{m}, \qquad m = 0, 1, 2, \ldots. $$

**Lemma 2.4.** *[10] Assume that $w(\alpha) \geq 0$, $w \in C([0, 1])$, then there exists a constant $\delta > 0$, independent of $\Delta t$ and $\Delta\alpha$, such that*

$$ \Delta\alpha \sum_{j=0}^{L} c_j \, w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1} g_k^{(\alpha_j)} \ \geq \ \delta \ > \ 0, $$

*where $c_j$ are the trapezoidal weights given in Lemma 2.1 and $g_k^{(\alpha_j)}$ are the Grünwald weights defined in Lemma 2.3.*

Unless stated otherwise, $C > 0$ denotes a generic constant that may vary from occurrence to occurrence but is independent of $h$, $\Delta t$, and $\Delta\alpha$. Throughout the paper, $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product, and $\|\cdot\|$ is the associated norm.

## 3. The scheme

Invoking Lemmas 2.1–2.3, we arrive at the following discrete approximation of the distributed-order derivative at $t_n$ [10]:

$$ \begin{aligned} \mathcal{D}_t^w u(x, t_n) &= \int_0^1 w(\alpha) \, {}_0^C D_{t_n}^\alpha u(x, t_n) \, d\alpha \\ &= \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n} g_k^{(\alpha_j)} \, u(x, t_n - k\Delta t) \ + \ O(\Delta t + \Delta\alpha^2). \end{aligned} \tag{3.1} $$

Here, $c_j$ are the trapezoidal weights from Lemma 2.1 and $g_k^{(\alpha_j)}$ the Grünwald weights from Lemma 2.3. To set up the LDG method, we recast Eq (1.1) into an equivalent first-order system:

$$p = u_x, \qquad \mathcal{D}_t^w u = p_x - \rho u + f(x,t). \tag{3.2}$$

Let $u_h^n, p_h^n \in V_h^k$ be the DG approximations of $u(\cdot, t_n)$ and $p(\cdot, t_n)$, respectively, and define $f^n(x) := f(x, t_n)$. The fully discrete LDG scheme reads: Determine $u_h^n, p_h^n \in V_h^k$ such that, for every $v, \xi \in V_h^k$,

$$\begin{aligned}
(\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j} + \rho) \int_\Omega u_h^n v dx + \int_\Omega p_h^n v_x dx - \sum_{j=1}^{N} ((\widehat{p_h^n} v^-)_{j+\frac{1}{2}} - (\widehat{p_h^n} v^+)_{j-\frac{1}{2}}) \\
= \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=1}^{n} (-g_k^{\alpha_j}) \int_\Omega u_h^{n-k} v dx + \int_\Omega f^n v dx, \\
\int_\Omega p_h^n \xi dx + \int_\Omega u_h^n \xi_x dx - \sum_{j=1}^{N} ((\widehat{u_h^n} \xi^-)_{j+\frac{1}{2}} - (\widehat{u_h^n} \xi^+)_{j-\frac{1}{2}}) = 0.
\end{aligned} \tag{3.3}$$

The terms marked with a "hat" $(\widehat{u_h^n}, \widehat{p_h^n})$ in the interface summations arise from integration by parts and are referred to as numerical fluxes. The selection of these fluxes is crucial for both the stability analysis and the practical performance of the LDG method. In this work, we employ the generalized alternating numerical fluxes, which provide enhanced flexibility and a wider scope of applicability compared to conventional choices [20]. These fluxes are defined as

$$\widehat{u_h^n} = \vartheta(u_h^n)^- + (1-\vartheta)(u_h^n)^+, \qquad \widehat{p_h^n} = (1-\vartheta)(p_h^n)^- + \vartheta(p_h^n)^+, \tag{3.4}$$

where the parameter $\vartheta$ belongs to $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. The case $\vartheta = \frac{1}{2}$ is excluded here because it leads to a symmetric flux that complicates the analysis of uniqueness and approximation properties related to the underlying generalized Gauss–Radau projection.

**Remark 3.1.** *The parameter $\vartheta \in [0,1] \setminus \{\frac{1}{2}\}$ controls the generalized alternating numerical fluxes. In computations, one may simply take $\vartheta = 0$ or $\vartheta = 1$, which are standard alternating choices and are commonly used in LDG implementations; other values work equally well.*

For conciseness and without loss of generality, the case of $f = 0$ is considered in the subsequent numerical analysis.

### *3.1. Stability*

**Theorem 3.1.** *Assume periodic boundary conditions, or alternatively that the solution is compactly supported. Then the fully discrete LDG scheme* (3.3) *is unconditionally stable. Moreover, there exists a constant $C > 0$ depending solely on $T$ and $u$ such that, for all $n = 1, 2, \ldots, M$,*

$$\|u_h^n\| \le \|u_h^0\|. \tag{3.5}$$

*Proof.* By taking $v = u_h^n$ and $\xi = p_h^n$ in the discrete formulation (3.3) and employing the numerical fluxes prescribed in Eq (3.4), we derive the estimate

$$\begin{aligned}(\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\|u_h^n\|^2+\|p_h^n\|^2+\sum_{j=1}^{N}\Big(F_1(p_h^n,u_h^n)_{j+\frac{1}{2}}-F_1(p_h^n,u_h^n)_{j-\frac{1}{2}}+F_2(p_h^n,u_h^n)_{j-\frac{1}{2}}\Big)\\ \leq\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\,\|u_h^{n-k}\|\,\|u_h^n\|.\end{aligned}\tag{3.6}$$

In the above expression, the numerical flux terms are given by

$$F_1(p_h^n,u_h^n)=(p_h^n)^-(u_h^n)^- - \widehat{p_h^n}(u_h^n)^- - \widehat{u_h^n}(p_h^n)^-$$

and

$$F_2(p_h^n,u_h^n)=(p_h^n)^-(u_h^n)^- -(p_h^n)^+(u_h^n)^+ - \widehat{p_h^n}(u_h^n)^- + \widehat{p_h^n}(u_h^n)^+ - \widehat{u_h^n}(p_h^n)^- + \widehat{u_h^n}(p_h^n)^+.$$

After algebraic manipulation, it follows that $F_2(p_h^n,u_h^n)=0$.

Moreover, the inequality

$$-g_n^{\alpha_j}\leq\sum_{k=0}^{n-1}g_k^{\alpha_j}$$

holds for the coefficients of the time-stepping scheme.

Consequently, starting from inequality (3.6), we deduce the following estimate:

$$\begin{aligned}\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}\|u_h^n\|&\leq(\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\|u_h^n\|\\ &\leq\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n-1}(-g_k^{\alpha_j})\,\|u_h^{n-k}\|\\ &\quad+\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=0}^{n-1}g_k^{\alpha_j}\,\|u_h^0\|.\end{aligned}\tag{3.7}$$

We proceed to establish Theorem 3.1 via induction on $n$. Setting $n=1$ in inequality (3.7) yields

$$\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}\|u_h^1\|\leq\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}\|u_h^0\|,$$

which immediately implies $\|u_h^1\|\leq\|u_h^0\|$.

Assume now that the inductive hypothesis

$$\|u_h^l\|\leq\|u_h^0\|,\qquad l=1,2,\ldots,n-1$$

holds true.

From Eq (3.7), we obtain

$$\begin{aligned}
\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}\|u_h^n\| &\le (\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\|u_h^n\| \\
&\le \Big(\Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n-1}(-g_k^{\alpha_j}) \\
&\quad + \Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=0}^{n-1}g_k^{\alpha_j}\Big)\|u_h^0\| \\
&= \Delta\alpha\sum_{j=0}^{L}c_jw(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}\|u_h^0\|.
\end{aligned}$$

Thus, it follows that

$$\|u_h^n\| \le \|u_h^0\|.$$

*3.2. Convergence*

We begin by defining the generalized Gauss–Radau projection, which will serve as a key ingredient in the convergence analysis below.

Let $\varpi$ be a periodic function on $[a, b]$. Following [21], the generalized Gauss–Radau projection of $\varpi$, denoted by $Q_\vartheta\varpi$, is defined uniquely by the conditions below.

Let the associated projection error be defined as $\varpi^e = Q_\vartheta\varpi - \varpi$. For $\vartheta \neq \frac{1}{2}$, the projection satisfies the following conditions on each element $I_j$, $j = 1, 2, \ldots, N$:

$$\int_{I_j}\varpi^e\phi\,dx = 0, \quad \forall\phi \in P^{k-1}(I_j) \qquad \text{and} \qquad (\varpi^e)^{(\vartheta)}_{j+\frac{1}{2}} = 0. \tag{3.8}$$

As a consequence of the defining conditions, one obtains the following approximation estimate [21,22].

**Lemma 3.1.** *Assume $\vartheta \neq \frac{1}{2}$ and let $\varpi \in H^{s+1}[a, b]$. Then the projection error $\varpi^e$ satisfies the estimate*

$$\|\varpi^e\| + h^{\frac{1}{2}}\|\varpi^e\|_{L^2(\Gamma_h)} \le Ch^{\min(s+1,k+1)}\|\varpi\|_{s+1}, \tag{3.9}$$

*where $C > 0$ does not depend on the mesh size h or the function $\varpi$.*

**Theorem 3.2.** *Let $u(x, t_n)$ be the exact solution of Eq* (1.1) *and assume sufficient regularity. Let $u_h^n$ be the approximation produced by the fully discrete LDG scheme* (3.3)*. Then there exists a constant $C > 0$, independent of the mesh size h and the time step $\Delta t$, such that*

$$\|u(x, t_n) - u_h^n\| \le C(h^{k+1} + \Delta t + \Delta\alpha^2), \qquad n = 1, \ldots, M. \tag{3.10}$$

*Proof.* We introduce the following error decomposition:

$$\begin{aligned}
e_u^n &= u_h^n - u(x, t_n) = \xi_u^n - \eta_u^n, \quad \xi_u^n = Q_\vartheta e_u^n, \quad \eta_u^n = Q_\vartheta u - u, \\
e_p^n &= p_h^n - p(x, t_n) = \xi_p^n - \eta_p^n, \quad \xi_p^n = Q_{1-\vartheta}e_p^n, \quad \eta_p^n = Q_{1-\vartheta}p - p.
\end{aligned} \tag{3.11}$$

Using the numerical fluxes given in Eq (3.4), the following error equations can be derived:

$$
\begin{aligned}
&(\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\int_\Omega e_u^n v\,dx+\int_\Omega e_p^n v_x\,dx-\sum_{j=1}^{N}\Big(((e_p^n)^+v^-)_{j+\frac{1}{2}}-((e_p^n)^+v^+)_{j-\frac{1}{2}}\Big)\\
&\qquad=\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\int_\Omega e_u^{n-k}v\,dx-\int_\Omega r^n v\,dx,\\
&\int_\Omega e_p^n\xi\,dx+\int_\Omega e_u^n\xi_x\,dx-\sum_{j=1}^{N}\Big(((e_u^n)^-\xi^-)_{j+\frac{1}{2}}-((e_u^n)^-\xi^+)_{j-\frac{1}{2}}\Big)=0.
\end{aligned}
\tag{3.12}
$$

Substituting the error decompositions given in Eq (3.11) into the error equation (3.12), we obtain the following equivalent system:

$$
\begin{aligned}
&(\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\int_\Omega \xi_u^n v dx+\int_\Omega \xi_p^n v_x dx-\sum_{j=1}^{N}(((\xi_p^n)^+v^-)_{j+\frac{1}{2}}-((\xi_p^n)^+v^+)_{j-\frac{1}{2}})\\
&\quad+\int_\Omega \xi_p^n\xi dx+\int_\Omega \xi_u^n\xi_x dx-\sum_{j=1}^{N}(((\xi_u^n)^-\xi^-)_{j+\frac{1}{2}}-((\xi_u^n)^-\xi^+)_{j-\frac{1}{2}})\\
&=\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\int_\Omega \xi_u^{n-k}v dx\\
&\quad-\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\int_\Omega(\eta_u^{n-k})v dx-((\eta_p^n)^+v^+)_{j-\frac{1}{2}})\\
&\quad+(\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\int_\Omega(\eta_u^n)v dx+\int_\Omega(\eta_p^n)v_x dx\\
&\quad-\sum_{j=1}^{N}(((\eta_p^n)^+v^-)_{j+\frac{1}{2}}+\int_\Omega(\eta_p^n)\xi dx+\int_\Omega e_u^n\xi_x dx-\sum_{j=1}^{N}(((\eta_u^n)^-\xi^-)_{j+\frac{1}{2}}-((\eta_u^n)^-\xi^+)_{j-\frac{1}{2}}).
\end{aligned}
\tag{3.13}
$$

Selecting $v=\xi_u^n$ and $\xi=\xi_p^n$ as test functions in Eq (3.13) results in the following estimate:

$$
\begin{aligned}
&(\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\int_\Omega(\xi_u^n)^2dx+\int_\Omega(\xi_p^n)^2dx\\
&\quad=\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\int_\Omega \mathcal{P}^-e_u^{n-k}\xi_u^n dx\\
&\quad-\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}\sum_{k=1}^{n}(-g_k^{\alpha_j})\int_\Omega(\eta_u^{n-k})\xi_u^n dx\\
&\quad+(\Delta\alpha\sum_{j=0}^{L}c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}}g_0^{\alpha_j}+\rho)\int_\Omega(\eta_u^n)\xi_u^n dx+\int_\Omega(\eta_p^n)\xi_p^n dx+\int_\Omega r^n\xi_u^n dx.
\end{aligned}
$$

Consequently, we obtain the estimate

$$
\begin{aligned}
&(\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j} + \rho)\|\xi_u^n\|^2 + \|\xi_p^n\|^2 \\
&\leq \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=1}^{n-1} (-g_k^{\alpha_j})(\|\xi_u^{n-k}\| + \|\eta_u^{n-k}\|)\|\xi_u^n\| \\
&\quad + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1} (g_k^{\alpha_j})\|\xi_u^0\|\|\xi_u^n\| \\
&\quad + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1} (g_k^{\alpha_j})\|\eta_u^0\|\|\xi_u^n\| \\
&\quad + (\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j} + \rho)\|\eta_u^n\|\|\xi_u^n\| + \|\eta_p^n\|\|\xi_p^n\| + \|r^n\|\|\xi_u^n\|.
\end{aligned}
\tag{3.14}
$$

Applying the Cauchy–Schwarz inequality, Lemma 2.4 and standard norm estimates to the right-hand side of Eq (3.14) lead to the following inequality:

$$
\begin{aligned}
&(\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j} + \rho)\|\xi_u^n\| \\
&\leq \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=1}^{n-1} (-g_k^{\alpha_j})\|\xi_u^{n-k}\| \\
&\quad + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1} (g_k^{\alpha_j})(\|\xi_u^0\| + \frac{1}{\delta}(\|r^n\| + \|\eta_p^n\|)) \\
&\quad + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=1}^{n-1} (-g_k^{\alpha_j})\|\eta_u^{n-k}\| \\
&\quad + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1} (g_k^{\alpha_j})\|\eta_u^0\| \\
&\quad + (\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j) \frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j} + \rho)\|\eta_u^n\|.
\end{aligned}
\tag{3.15}
$$

The proof of the error estimate (3.10) proceeds by mathematical induction. We begin by establishing the base case $n = 1$. From inequality (3.15), we observe that

$$
\begin{aligned}
\|\xi_u^1\| \leq& \|\xi_u^0\| + \|\eta_u^0\| \\
&+ \|\eta_u^1\| + \frac{1}{\delta}(\|r^1\| + \|\eta_p^1\|).
\end{aligned}
\tag{3.16}
$$

Taking into account the identities $\mathcal{P}^- e_u^0 = 0$ and $\|r^1\| \leq C(\Delta t + \Delta\alpha^2)$, together with the approximation property stated in Lemma 3.1, we deduce that

$$
\|\xi_u^1\| \leq C(h^{k+1} + \Delta t + \Delta\alpha^2). \tag{3.17}
$$

Now we assume that the following error bound holds for all previous time steps:

$$\|\mathcal{P}^{-}e_u^m\| \le C(h^{k+1} + \Delta t + \Delta\alpha^2), \qquad m = 1, 2, \ldots, K. \tag{3.18}$$

Setting $n = K + 1$ in Eq (3.15) and observing that

$$\begin{aligned}\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=1}^{n-1}(-g_k^{\alpha_j}) + \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n-1}(g_k^{\alpha_j})\\ =\Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j)\frac{1}{(\Delta t)^{\alpha_j}} g_0^{\alpha_j},\end{aligned}$$

we immediately arrive at the following estimate:

$$\|\xi_u^{K+1}\| \le C(h^{k+1} + \Delta t + \Delta\alpha^2).$$

Finally, applying the triangle inequality in conjunction with the interpolation estimate (3.1) completes the proof of Theorem 3.2.

## 4. Numerical examples

In this section, we present the implementation details and a set of numerical tests that confirm the theoretical results and assess the efficiency of the fully discrete LDG method. Unless otherwise stated, the errors are evaluated at the final time $T$ in the discrete $L^2$- and $L^\infty$-norms.

### 4.1. Algorithm

**Algorithm. Fully discrete scheme for the distributed-order diffusion model**

**Require:** Polynomial degree $k$, mesh $\mathcal{T}_h$, final time $T$, time step $\Delta t$ with $t_n = n\Delta t$ ($n = 0, \ldots, M$), and number of order-quadrature nodes $L$. Weight function $w(\alpha)$, diffusion coefficient $\kappa$ (or $\kappa(x)$), source term $f(x, t)$, and initial data $u_0(x)$. Flux parameter $\vartheta \in [0, 1] \setminus \{\frac{1}{2}\}$.

Choose quadrature nodes and weights $\{(\alpha_\ell, \omega_\ell)\}_{\ell=1}^{L}$ on $[0, 1]$ (e.g., composite Simpson/Gauss); approximate $\int_0^1 w(\alpha) \cdot d\alpha \approx \sum_{\ell=1}^{L} \omega_\ell\, w(\alpha_\ell) \cdot$.

Construct LDG spaces $V_h^k$ (piecewise degree-$k$ polynomials) and define numerical fluxes (generalized alternating flux) with parameter $\vartheta$.

Initialize: Set $u_h^0 \leftarrow \Pi_h u_0$ (projection/interpolation) and initialize history variables for each $\alpha_\ell$ (see below).

**for** $n = 1$ to $M$ **do** ▹ time stepping

Evaluate $f^n(\cdot) \leftarrow f(\cdot, t_n)$.

**for** $\ell = 1$ to $L$ **do** ▹ per fractional order

Compute/update convolution (history) weights for order $\alpha_\ell$:

$$g_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \qquad j = 0, 1, 2, \ldots$$

and form the discrete fractional operator at $t_n$ (GL/L1-type, as used in the paper)

$$(\mathcal{D}_{\Delta t}^{\alpha_\ell} u_h)^n = \Delta\alpha \sum_{j=0}^{L} c_j w(\alpha_j)\, \frac{1}{(\Delta t)^{\alpha_j}} \sum_{k=0}^{n} g_k^{(\alpha_j)}\, u_h^{n-k}$$

(or the equivalent recursion used in the manuscript; store the history sum incrementally).
**end for**
Assemble the fully discrete LDG system at $t_n$:
Find $(u_h^n, p_h^n) \in V_h^k \times V_h^k$ such that for all test functions $(v_h, r_h)$,

$$\sum_{\ell=1}^{L} \omega_\ell w(\alpha_\ell)(\mathcal{D}_{\Delta t}^{\alpha_\ell} u_h)^n(v_h) + \kappa\, a_{\mathrm{LDG}}(u_h^n, q_h^n; v_h, r_h) = (f^n, v_h),$$

where $a_{\mathrm{LDG}}(\cdot)$ is the LDG bilinear form with the chosen numerical fluxes and boundary treatment.
Solve the resulting linear system to obtain $(u_h^n, p_h^n)$.
**end for**
**return** $\{u_h^n\}_{n=0}^M$ (and auxiliary variables if needed).

**Implementation cost and memory.** Let $N = \dim(V_h^k)$ denote the number of spatial degrees of freedom, $M = T/\Delta t$ the number of time steps, and $L$ the number of quadrature nodes used for the distributed order integral in $\alpha$.

At each time level $t_n$, the LDG discretization yields a linear algebraic system with $2N$ unknowns for $u_h^n, p_h^n$. The dominant additional cost compared with the classical diffusion case comes from the history (convolution) terms: For each quadrature node $\alpha_\ell$, the discrete fractional operator involves a sum over all previous time levels. A straightforward implementation therefore requires $O(n)$ operations per $\alpha_\ell$ at step $n$, hence $O(L\sum_{n=1}^M n) = O(LM^2)$ operations for the history part over the whole simulation.

Regarding memory, storing $\{u_h^n\}_{n=0}^M$ requires $O(MN)$ storage, while the history weights can be generated on the fly with negligible extra memory. In long-time simulations, one may reduce the cost by standard techniques such as short-memory, recursive updates for convolution sums, or sum-of-exponentials approximations.

### 4.2. Numerical experiments

**Example.** We solve Eq (1.1) on the domain $\Omega = [0, 1]$ up to the final time $T = 0.5$. The weight function is taken as $w(\alpha) = \Gamma(3 - \alpha)$, and the term $f(x, t)$ is manufactured so that the exact solution takes the form

$$u(x, t) = 4t^2 \sin(2\pi x).$$

To examine the spatial accuracy of the method, we fix the temporal step size $\Delta t = 1/500$ and the distributed-order discretization parameter $\Delta\alpha = 1/300$, which are sufficiently small to ensure that errors from time and order integration are negligible compared with the spatial error. The spatial mesh is refined successively with $h = 1/5, 1/10, 1/20, 1/40$. Tables 1 and 2 present the numerical errors and the corresponding convergence orders for the flux parameter $\vartheta = 0.6$ and $\vartheta = 0.3$, respectively, using piecewise $P^k$ polynomials with $k = 0, 1, 2$.

The results clearly demonstrate that the scheme achieves the optimal $(k + 1)$-th order convergence in both $L^2$- and $L^\infty$-norms. For $P^0$ elements ($k = 0$), we observe approximately first-order convergence; for $P^1$ elements ($k = 1$), second-order convergence is obtained; and for $P^2$ elements ($k = 2$), third-order convergence is confirmed. This behavior agrees perfectly with the theoretical error estimate (3.10), which predicts an $O(h^{k+1})$ spatial error. Moreover, the numerical results appear largely insensitive to the specific value of the flux parameter $\vartheta$, indicating the robustness of the LDG discretization with respect to this choice.

**Table 1.** Spatial accuracy test with piecewise $P^k$ elements, using $\rho = 0.5, \Delta\alpha = \frac{1}{300}, \Delta t = \frac{1}{500}, \vartheta = 0.6$, and $T = 0.5$.

| $\vartheta$ | $P^k$ | $N$ | $L^\infty$-error | order | $L^2$-error | order |
|---|---|---|---|---|---|---|
| $\vartheta = 0.6$ | $P^0$ | 5 | 0.821256543642554 | - | 0.882585555225640 | - |
| | | 10 | 0.407791855159989 | 1.01 | 0.463232391022541 | 0.93 |
| | | 20 | 0.211086301937670 | 0.95 | 0.238127813082612 | 0.96 |
| | | 40 | 0.108510372819908 | 0.96 | 0.124983369113747 | 0.93 |
| | $P^1$ | 5 | 0.465895434553645 | - | 0.486542451345121 | - |
| | | 10 | 0.118099773114141 | 1.98 | 0.128571037142656 | 1.92 |
| | | 20 | 0.030778742458649 | 1.94 | 0.033276271243925 | 1.95 |
| | | 40 | 0.007911012672206 | 1.96 | 0.008612439103767 | 1.95 |
| | $P^2$ | 5 | 0.042595522343425 | - | 0.059852453234533 | - |
| | | 10 | 0.005436318101684 | 2.97 | 0.007908140355095 | 2.92 |
| | | 20 | 0.000738479550611 | 2.88 | 0.001066836046182 | 2.89 |
| | | 40 | 0.000097573275947 | 2.92 | 0.000139984449546 | 2.93 |

**Table 2.** Spatial accuracy test with piecewise $P^k$ elements, using $\rho = 1.0, \Delta\alpha = \frac{1}{300}, \Delta t = \frac{1}{500}, \vartheta = 0.3$, and $T = 0.5$.

| $\vartheta$ | $P^k$ | $N$ | $L^\infty$-error | order | $L^2$-error | order |
|---|---|---|---|---|---|---|
| $\vartheta = 0.3$ | $P^0$ | 5 | 0.793268945565257 | - | 0.868845516551214 | - |
| | | 10 | 0.399393283712288 | 0.99 | 0.449742643761157 | 0.95 |
| | | 20 | 0.205310878629086 | 0.96 | 0.236051178246416 | 0.93 |
| | | 40 | 0.105541476540735 | 0.96 | 0.121343740076798 | 0.96 |
| | $P^1$ | 5 | 0.506879561559624 | - | 0.562626845612562 | - |
| | | 10 | 0.133945210240091 | 1.92 | 0.145616959618551 | 1.95 |
| | | 20 | 0.034908323875990 | 1.94 | 0.038479938332127 | 1.92 |
| | | 40 | 0.009224682025670 | 1.92 | 0.010098258313005 | 1.93 |
| | $P^2$ | 5 | 0.049265685556163 | - | 0.049265685556163 | - |
| | | 10 | 0.006554615840513 | 2.91 | 0.006832962589201 | 2.85 |
| | | 20 | 0.000830764358483 | 2.98 | 0.000896584353557 | 2.93 |
| | | 40 | 0.000113637525591 | 2.87 | 0.000119287206497 | 2.91 |

To investigate the temporal accuracy, we fix a fine spatial mesh with $N = 500$ elements (using piecewise $P^1$ polynomials) and $\Delta\alpha = 1/300$ while successively refining the time step as $\Delta t = 1/5, 1/10, 1/20, 1/40$. The numerical errors are reported in Table 3. The computed convergence rates are approximately first order in both $L^2$- and $L^\infty$-norms for two different values of the flux parameter $\vartheta$ (0.4 and 0.8). This result fully agrees with the theoretical error estimate (3.10), which predicts an $O(\Delta t)$ temporal error. The experiment confirms that the time discretization based on the first-order GL-type formula achieves the designed $O(\Delta t)$ accuracy and that the overall scheme remains stable for long-time simulations.

**Table 3.** Temporal errors and observed convergence rates for the $P^1$ DG approximation, with $\rho = 0.5$, $T = 0.5$, $N = 500$, and $\Delta\alpha = \frac{1}{300}$.

| | $M$ | $L^2$-error | order | $L^\infty$-error | order |
|---|---|---|---|---|---|
| | 5 | 0.122231232153252 | - | 0.081322554164165 | - |
| | 10 | 0.063270853623087 | 0.95 | 0.041804421177583 | 0.96 |
| $\vartheta = 0.4$ | 20 | 0.032300154376509 | 0.97 | 0.021489851714013 | 0.96 |
| | 40 | 0.016375523864371 | 0.98 | 0.010894919432353 | 0.98 |
| | 5 | 0.111254434435435 | - | 0.091255344554335 | - |
| | 10 | 0.057588906796408 | 0.95 | 0.046910440746288 | 0.96 |
| $\vartheta = 0.8$ | 20 | 0.030226018127902 | 0.93 | 0.024621336143745 | 0.93 |
| | 40 | 0.015645968177605 | 0.95 | 0.012569341376938 | 0.97 |

## 5. Conclusions

This work presents a high-order numerical method for solving a class of distributed-order time-fractional diffusion equations. The proposed scheme integrates a finite difference discretization in time with the LDG method in space. By careful selection of projections and numerical fluxes, we establish that the method is unconditionally stable and achieves a convergence rate of order $O(h^{k+1} + \Delta t + \Delta\alpha^2)$ in the $L^2$-norm.

We note that although the current analysis focuses on time-fractional derivatives, the present methodology can, in principle, be extended to problems involving spatial fractional derivatives. Adapting the scheme to such cases needs modifications to handle the fractional character of spatial fractional operators, which is an interesting direction for future research.

## Author contributions

Lingna Lu: Investigation, Methodology, Writing–original draft, Review and editing, Visualization, Project administration, Supervision. Changshun Hou: Writing – original draft, Investigation, Formal analysis, Programming, Revision and editing. All authors read and approved the final manuscript.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgements

The authors would like to thank the reviewers for their insightful comments and constructive suggestions, which have significantly improved the quality of this manuscript.

## Conflict of interest

The authors declare no conflicts of interest.

## References


1. C. P. Li, F. H. Zeng, *Numerical Methods for Fractional Calculus*, CRC Press, Boca Raton, FL, 2015.
2. M. Stynes, E. O'Riordan, J. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, **55** (2017), 1057–1079. https://doi.org/10.1137/16M1082329
3. A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.*, **340** (2008), 252–281. https://doi.org/10.1016/j.jmaa.2007.08.024
4. M. M. Meerschaert, E. Nane, P. Vellaisamy, Distributed-order fractional diffusions on bounded domains, *J. Math. Anal. Appl.*, **379** (2011), 216–228. https://doi.org/10.1016/j.jmaa.2010.12.056
5. C. F. Lorenzo, T. T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dyn.*, **29** (2002), 57–98. https://doi.org/10.1023/A:1016586905654
6. T. T. Hartley, C. F. Lorenzo, Fractional-order system identification based on continuous order-distributions, *Signal Process.*, **83** (2003), 2287–2300. https://doi.org/10.1016/S0165-1684(03)00182-8
7. K. Diethelm, N. J. Ford, Numerical analysis for distributed-order differential equations, *J. Comput. Appl. Math.*, **225** (2009), 96–104. https://doi.org/10.1016/j.cam.2008.07.018
8. H. Ye, F. Liu, V. Anh, Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains, *J. Comput. Phys.*, **298** (2015), 652–660. https://doi.org/10.1016/j.jcp.2015.06.025
9. M. L. Morgado, M. Rebelo, Numerical approximation of distributed order reaction-diffusion equations, *J. Comput. Appl. Math.*, **275** (2015), 216–227. https://doi.org/10.1016/j.cam.2014.07.029
10. G. H. Gao, H. W. Sun, Z. Z. Sun, Some high-order difference schemes for the distributed-order differential equations, *J. Comput. Phys.*, **298** (2015), 337–359. https://doi.org/10.1016/j.jcp.2015.05.047
11. X. Y. Li, B. Y. Wu, A numerical method for solving distributed order diffusion equations, *Appl. Math. Lett.*, **53** (2016), 92–99. https://doi.org/10.1016/j.aml.2015.10.009
12. A. A. Alikhanov, Numerical methods of solutions of boundary value problems for the multi-term variabledistributed order diffusion equation, *Appl. Math. Comput.*, **268** (2015), 12–22. https://doi.org/10.1016/j.amc.2015.06.045
13. J. T. Katsikadelis, Numerical solution of distributed order fractional differential equations, *J. Comput. Phys.*, **259** (2014), 11–22. https://doi.org/10.1016/j.jcp.2013.11.013
14. Y. Chen, X. Zhang, L. Wei, High precision interpolation method for mixed 2D time-space fractional convection-diffusion equation, *Int. J. Dyn. Control*, **13** (2025), 333. https://doi.org/10.1007/s40435-025-01842-z

15. X. Zhang, H. Wang, Z. Luo, L. Wei, A high-accuracy compact finite difference scheme for time-fractional diffusion equations, *Rev. Union Mat. Argent.*, **68** (2025), 589–609. https://doi.org/10.33044/revuma.4665

16. L. Wei, Y. F. Yang, Optimal order finite difference/local discontinuous Galerkin method for variable-order time-fractional diffusion equation, *J. Comput. Appl. Math.*, **383** (2021), 113129. https://doi.org/10.1016/j.cam.2020.113129

17. L. Wei, W. Li, Local discontinuous Galerkin approximations to variable-order time-fractional diffusion model based on the Caputo-Fabrizio fractional derivative, *Math. Comput. Simul.*, **188** (2021), 280–290. https://doi.org/10.1016/j.matcom.2021.04.001

18. Y. Liu, M. Zhang, H. Li, J. Li, High-order local discontinuous Galerkin method combined with WSGD-approximation for a fractional sub-diffusion equation, *Comput. Math. Appl.*, **73** (2017), 1298–1314. https://doi.org/10.1016/j.camwa.2016.08.015

19. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

20. Y. Xu, C. W. Shu, Local discontinuous Galerkin method for the Camassa-Holm equation, *SIAM J. Numer. Anal.*, **46** (2008), 1998–2021. https://doi.org/10.1137/070679764

21. Y. Cheng, X. Meng, Q. Zhang, Application of generalized Gauss-Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations, *Math. Comput.*, **86** (2017), 1233–1267. https://doi.org/10.1090/mcom/3141

22. L. Wei, H. Wang, Y. Chen, Local discontinuous Galerkin method for a hidden-memory variable order reaction–diffusion equation, *J. Appl. Math. Comput.*, **69** (2023), 2857–2872. https://doi.org/10.1007/s12190-023-01865-9