Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(1): 243–260.
DOI: 10.3934/nhm.2026012
Received: 18 January 2026
Revised: 06 February 2026
Accepted: 25 February 2026
Published: 03 March 2026

*Research Article*

# On a Hartree-type nonlinearity wave equation with distributed delay combined with a fractional condition

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**Abstract:** This work focused on the analysis of a nonlinear wave equation of Hartree-type that includes a distributed delay term, where the delay effects are governed with fractional conditions. Such a formulation allows the model to incorporate long-range memory effects and anomalous dissipation phenomena, which are characteristic of complex media. The model captures complex memory and nonlocal interaction effects that arise in various physical systems, such as quantum mechanics and nonlinear optics. In particular, the fractional delay mechanism provides a more accurate description of hereditary effects than classical integer-order delay models. We worked under a framework that allows for initial data with negative energy and imposed suitable assumptions on the kernel functions and nonlinear terms. Using energy methods and a concavity argument, we rigorously proved that the solution to the system cannot exist globally in time and must blow up in finite time. Compared with the classical Hartree wave equation without delay or fractional effects, our results show that the combined presence of distributed delay and fractional damping significantly enhances the instability mechanism.

**Keywords:** nonlinear equations; blow up; fractional damping; distributed delay; Hartree-type nonlinearity
**Mathematics Subject Classification:** 35B40, 35L70, 76Exx, 93D20

## 1. Introduction

In this work, we address the problem of a Hartree-type wave equation incorporating nonlocal nonlinearity and distributed delay. Such equations arise naturally in the modeling of wave propagation phenomena in complex media, where long-range interactions and memory effects play a crucial role.

$$
\begin{cases}
u_{tt} - \Delta u + a_1 u_t + \displaystyle\int_{t_1}^{t_2} a_2(\lambda)\partial_t^{\mu,\delta} u\,(x, t-\lambda)\,d\lambda = \mathbb{F}(u), & 0 < t, x \in \mho, \\
u(x,t) = 0, & 0 < t, x \in \partial\mho, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mho, \\
u_t(x,-t) = f_0(x,t), & t \in (0, t_2), x \in \mho.
\end{cases}
\tag{1.1}
$$

The function $\mathbb{F}$ is defined as

$$
\mathbb{F}(u) := \left(\frac{1}{|x|^{n-2}} * |u|^p\right)|u|^{p-2}u,
$$

where

$$
\frac{1}{|x|^{n-2}} * |u|^p = \int_{\mho} \frac{|u(y)|^p}{|x-y|^{n-2}}\,dy.
$$

The Hartree term $\left(\frac{1}{|x|} * |u|^2\right)u$ is generalized by the equation $\mathbb{F}(u)$. This nonlocal convolution structure models long-range interactions and appears prominently in quantum mechanics, nonlinear optics, and many-body physics.

The domain $\mho$ is bounded in $\mathbb{R}^n$, $n \geq 5$, and its boundary $\partial\mho$ is of class $C^2$ and sufficiently smooth. Moreover, the coefficients satisfy $a_1, b > 0$, $1 < p < \frac{n+4}{n-4}$, $a_2 \in L^\infty$, and $0 < t_1 \leq t_2$. The distributed delay term accounts for the influence of past states over a finite time interval, reflecting hereditary effects in the evolution process.

The generalized fractional Caputo derivative of order $0 < \mu < 1$ [1, 2] is denoted by $\partial_t^{\mu,\delta}$ and is defined by

$$
\partial_t^{\mu,\delta} u(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu} e^{-\delta(t-s)} u_s(s)\,ds, \quad \delta \geq 0,
$$

where

$$
\partial_t^{\mu,\delta} u(t) = I^{1-\mu,\delta} u_t(t),
\tag{1.2}
$$

and

$$
I^{\mu,\delta} u(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} e^{-\delta(t-s)} u(s)\,ds, \quad \delta \geq 0.
$$

Here $I^{\mu,\delta}$ represents the exponential fractional integral operator and $\Gamma$ denotes the Euler gamma function. The presence of the exponential kernel allows the fractional operator to capture fading memory effects, which are more realistic in many physical applications than classical integer-order derivatives.

Finally, Eq (1.1) is associated with the stationary problem

$$
-\Delta u + V(x)u = \left(\frac{1}{|x|^{\nu}} * |u|^p\right)|u|^{p-2}u,
\tag{1.3}
$$

where the condition $\frac{2n-\nu}{n} \leq p \leq \frac{2n-\nu}{n-2}$ holds. This stationary equation highlights the intrinsic connection between the dynamic model and the classical Hartree-type elliptic problem, providing additional insight into the underlying nonlocal structure. The stationary problem (1.3) reduces, in the special case $(n, p) = (1, 2)$, to a model describing the helium atom. This particular case is one of the earliest and most celebrated applications of the Hartree equation in quantum mechanics. For further physical background, we refer to [3]. Picard [4] also interpreted this equation as a model arising in the quantum theory of stationary polarons. Related developments can be found in the pioneering works of Choquard and Lieb [5], where the nonlocal interaction structure was first rigorously analyzed. For other parameter ranges, we refer the reader to [6] and the works of Petrovsky [7]. Hartree-type equations naturally appear in several branches of applied sciences and physics, including acoustics, optics, plasma physics, and quantum many-body systems.

In recent years, a growing body of literature has been devoted to the qualitative analysis of Hartree-type wave equations; see, for instance, [8]. In that work, the authors investigated the equation

$$u_{tt} - \Delta u = \left(\frac{1}{|x|^{n-2}} * |u|^p\right)|u|^{p-2}u, \tag{1.4}$$

posed on a bounded smooth convex domain with homogeneous Dirichlet boundary conditions. They established local existence of solutions by means of the semigroup theory and derived conditions for global existence of weak solutions using the potential well method. Moreover, by employing the convexity argument and potential well theory, the authors proved finite-time blow-up results for solutions corresponding to both negative and nonnegative initial energy levels. These results provide a fundamental reference point for the analysis of more complex Hartree-type models.

In [9], the authors considered the following Hartree-type Petrovsky equation:

$$z_{tt} + \Delta^2 z - \Delta z = \left(\frac{1}{|x|^{n-2}} * |z|^p\right)|z|^{p-2}z, \tag{1.5}$$

and proved the global existence of weak solutions by applying the potential well theory. Furthermore, they investigated the finite-time blow-up behavior of solutions under both nonnegative and negative initial energy conditions. Compared with these classical models, the presence of distributed delay and fractional damping in our setting introduces additional memory and dissipation mechanisms, leading to richer and more delicate blow-up dynamics.

Many natural and engineering processes depend crucially on delay effects, which may appear in the form of constant, time-varying, or distributed delays. Such delay phenomena reflect the fact that the present state of a system is influenced not only by its current configuration but also by its past history.

Concerning distributed delay, which is incorporated into the present work, we refer to the seminal contribution of Nicaise et al. [10], where the authors investigated the problem

$$u_{tt} - \Delta u + \alpha_1 u_t + \int_{\varrho_1}^{\varrho_2} \alpha_2(s) u(x, t-s)\, ds = 0.$$

Under suitable assumptions on the delay kernel, they established well-posedness and general decay results for the corresponding solutions. This work clearly demonstrates the stabilizing or destabilizing influence that distributed delays may exert on the long-time behavior of solutions. Following this pioneering study, many authors have incorporated distributed delay terms into various evolution equations and analyzed issues related to well-posedness, general decay, blow-up, exponential growth, and global existence; see, for example, [11–13].

Several investigations have employed different analytical techniques, such as the energy method combined with semigroup theory and the Faedo–Galerkin method, which are particularly effective for treating nonlinear and nonlocal problems. These approaches have proven to be powerful tools in capturing the delicate interplay between damping, delay, and source terms.

In [14], the authors studied a nonlinear viscoelastic Kirchhoff-type equation with a variable exponent and distributed delay. Under appropriate hypotheses, they proved the occurrence of finite-time blow-up solutions. When the source term is absent, general decay estimates were obtained by means of an integral inequality due to Komornik; see also [15–18]. These results highlight how memory effects induced by viscoelasticity and delay can significantly influence the stability of solutions.

Fractional derivatives in partial differential equations have attracted considerable attention in recent years. They provide an efficient mathematical framework for modeling anomalous diffusion, hereditary properties, and long-range temporal memory effects. Fractional boundary conditions and initial value problems have been successfully employed to describe various physical processes; see, for instance, Magin [6]. The theory of fractional calculus and several of its applications are comprehensively presented by Tarasov [19]. Further applications to the dynamics of particles, fields, and media can be found in the context of fractional dynamics, where systems with fractal structures, spatial nonlocality, and long-time memory are modeled. Such features cannot be adequately captured by classical integer-order models. Fractional calculus has been used to solve scientific and engineering problems since the nineteenth century; see Valério et al. [20].

Regarding nonlinear source terms, including logarithmic nonlinearities, several related results have been established for wave-type equations. In [21], the authors analyzed a nonlinear viscoelastic plate equation with a variable exponent and logarithmic source term. They proved global existence and general decay results using Komornik's method, and also showed that solutions with negative initial energy blow up in finite time. We also refer the reader to [22, 23] for further developments in this direction. These studies further motivate the investigation of blow-up phenomena in wave equations combining memory effects, nonlinear sources, and nonlocal interactions. Additionally, a nonlinear viscoelastic Kirchhoff-type equation with delay, Balakrishnan–Taylor damping, and logarithmic nonlinearity was studied in [24]. Under suitable assumptions and positive initial energy, the authors established exponential growth and finite-time blow-up of solutions. This work illustrates how the interaction between delay effects, strong damping mechanisms, and nonlinear sources can significantly influence the stability properties of wave-type systems. Similarly, [25] focused on a Kirchhoff-type equation for elastic membranes incorporating nonlinear viscoelastic behavior with distributed delay, logarithmic nonlinear effects, and Balakrishnan–Taylor damping. The occurrence of blow-up solutions under appropriate conditions was also confirmed; see [26].

Among studies closely related to the present work, we highlight [27], where the authors investigated the problem

$$y_{tt} - \Delta y + \partial_t^{\mu} y = |y|^{p-2} y.$$

They proved exponential growth of solutions. Concerning well-posedness and finite-time blow-up, the authors in [28] studied the equation

$$u_{tt} - \Delta u + \alpha_1 u_t + \alpha_2 u(x, t - \varrho) = |u|^{p-2} u \ln |u|^k,$$

where well-posedness was first established and blow-up in finite time was subsequently demonstrated.

Subsequently, Aounallah et al. [29] considered the problem

$$y_{tt} - \Delta y + a_1 \partial_t^{\mu,\beta} y(t - \varrho) + a_2 y_t = |y|^{p-2} y,$$

and employed the energy method to prove the existence and uniqueness of solutions. They also established global existence together with general decay results and obtained blow-up phenomena under suitable conditions. These results emphasize the significant role played by fractional damping terms combined with delay effects.

The incorporation of fractional conditions in Timoshenko-type systems was examined in [30], where the Faedo–Galerkin method was used to prove the existence and uniqueness of solutions, followed by general decay estimates. More recently, [31] extended this line of research to a viscoelastic wave equation with internal fractional feedback and time-delay:

$$w_{tt} - \Delta w + \int_0^t f(t-\varsigma)\Delta w(\varsigma)\,d\varsigma + \alpha_1 w_t + \alpha_2 \partial_t^{\mu,\beta} w(t-\varrho) = 0.$$

Using a combination of the energy method and the Faedo–Galerkin approach, the authors proved global existence of solutions under specific assumptions. They also demonstrated that suitably constructed Lyapunov functionals yield general decay results. These contributions further underline the effectiveness of fractional feedback mechanisms in controlling the long-time dynamics of wave equations.

The study in [24] also addressed a wave equation with fractional boundary conditions and acoustic coupling through delay and source terms, given by

$$\begin{aligned}
&u_{tt} - \Delta u + \mathfrak{R}_1 u_t + \mathfrak{R}_2 u(t-\varrho) = |u|^{p-2}u,\\
&\frac{\partial u}{\partial \nu} = -\mathcal{B}\partial_t^{\mu,\delta} u + \chi_t,\\
&u_t + P(x)\chi_t + Q(x)\chi = 0.
\end{aligned}$$

Under appropriate hypotheses, the authors proved global existence of solutions and established general decay results by constructing suitable Lyapunov functionals. In a related framework, the same authors studied a nonlinear viscoelastic Kirchhoff-type equation with logarithmic nonlinearity, delay, and Balakrishnan–Taylor damping, where exponential growth and blow-up results were obtained for positive initial energy.

Motivated by the above works, and in contrast to existing results that treat either fractional effects or delay mechanisms separately, the present paper considers the combined influence of a distributed delay and a fractional damping condition in a Hartree-type wave equation. This coupling leads to new analytical challenges and richer blow-up dynamics. The main objective of this paper is to investigate the finite-time blow-up behavior of solutions under suitable assumptions, even in the presence of negative initial energy.

The structure of the paper is organized as follows. In Section 2, we introduce the necessary functional setting, assumptions, and preliminary results, and define the associated energy functional. Section 3 is devoted to establishing the finite-time blow-up result for solutions of the considered problem. Finally, Section 4 presents concluding remarks together with perspectives for future work. In particular, we briefly discuss possible extensions of the present analysis to global existence and decay results under different assumptions on the initial energy and damping mechanisms. Throughout the paper, the notations $c$ and $C$ denote generic positive constants, whose values may change from line to line.

## 2. Preliminaries

This section is devoted to the preliminaries required for the analysis of our problem. We introduce the notations and assumptions used throughout the paper and establish several auxiliary lemmas that will play a fundamental role in the proofs of our main results.

**Lemma 2.1.** *[32]*

*Consider a number p with condition* $+\infty \geq p \geq 1$ *when* $N = 1, 2$ *or* $\frac{N+2}{N-2} \geq p \geq 1$ *when* $3 \leq N$*. Then, there exists a positive constant* $C_* = B_{p,\mho} > 0$ *such that*

$$\|u\|_{p+1} \leq C_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\mho).$$

**Lemma 2.2.** *(Hardy-Littlewood-Sobolev inequality) [33, 34]*

*Suppose that* $\varsigma, p > 1$ *and* $n > \alpha > 0$ *along with* $\frac{1}{\varsigma} + \frac{\alpha}{n} + \frac{1}{p} = 2$*,* $f \in L^{\varsigma}(\mathbb{R}^n)$ *and* $h \in L^s(\mathbb{R}^n)$*. There is a sharp constant* $C(\varsigma, n, \alpha, s)$*; independent of* $f, h$ *such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^{\alpha}} dxdy \leq C(\varsigma, n, \alpha, s)\|f\|_{\varsigma}\|h\|_p. \tag{2.1}$$

For all $u \in H^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}} dxdy$ is well defined by $\frac{2n-\alpha}{n} \leq p \leq \frac{2n-\alpha}{n-2}$.

From Lemma 2.2, for $u \in H_0^1(\mho)$, we define $u(x) = 0$ by $x \in \mathbb{R}^n/\mho$.

Therefore, for $u \in H^1(\mathbb{R}^n)$, i.e., for a general field, for $u \in L^{\varsigma}(\mathbb{R}^n)$ and $\mho \in L^s(\mathbb{R}^n)$, we find the Hardy-Littlewood-Sobolev inequality as follows:

$$\int_{\mho}\int_{\mho} \frac{u(x)\mho(y)}{|x-y|^{\alpha}} dxdy \leq C(\varsigma, n, \alpha, s, \mho)\|u\|_{\varsigma}\|\mho\|_p, \tag{2.2}$$

and if $\frac{2n-\alpha}{n} \leq p \leq \frac{2n-\alpha}{n-2}$, then $\int_{\mho}\int_{\mho} \frac{|u(x)|^p|\mho(y)|^p}{|x-y|^{\alpha}} dxdy$ is well defined for $u \in H_0^1(\mho)$.

Hence, for $u \in H_0^1(\mho)$, applying the Sobolev embedding theorem and the Hardy-Littlewood-Sobolev inequality (2.2) gives

$$\int_{\mho}\int_{\mho} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}} dxdy \leq C_1(n, p, \mho)\|u\|^{2p}_{\frac{2np}{n+2}} \leq C_2\|\nabla u\|^{2p}. \tag{2.3}$$

Here $C_2 = C_1(n, p, \mho)c_*^{2p}$, where $C_1$ and $c_*$ are the Hardy-Littlewood-Sobolev and the Sobolev embedding constants, respectively. Next, by applying the Sobolev embedding theorem,

$$\frac{n+2}{n} < p < \frac{n+2}{n-2}.$$

**Theorem 2.3.** *[35]*

*Consider a function* $\alpha$ *defined by*

$$\alpha(\zeta) = |\zeta|^{\frac{(2\mu-1)}{2}}, \quad 1 > \mu > 0,\ \zeta \in \mathbb{R}. \tag{2.4}$$

*Then, we can get*

$$O = I^{1-\mu,\delta}U, \tag{2.5}$$

*which expresses the connection between U and O of the system below:*

$$\partial_t\psi(x,\lambda,\zeta,t) + (\zeta^2+\delta)\psi(x,\lambda,\zeta,t) - U(x,\lambda,t)\alpha(\zeta) = 0, \quad t>0, \delta\geq 0, \ \zeta\in\mathbb{R}, \tag{2.6}$$

$$\psi(x,\lambda,\zeta,0) = 0, \tag{2.7}$$

$$O(x,\lambda,t) = \frac{\sin(\mu\pi)}{\pi}\int_{-\infty}^{+\infty}\psi(x,\lambda,\zeta,t)\alpha(\zeta)d\zeta, \ \zeta\in\mathbb{R}, \lambda\in[\varrho_1,\varrho_2], \ t>0. \tag{2.8}$$

**Lemma 2.4.** *[36] For all $\tau \in D_\delta := \{\tau\in\mathbb{C} : \Re e\tau+\delta>0\}\cup\{\tau\in\mathbb{C} : Im\tau\neq 0\}$,*

$$A_\tau = \int_{-\infty}^{+\infty}\frac{\alpha^2(\zeta)}{\tau+\delta+\zeta^2}d\zeta = \frac{\pi}{\sin(\mu\pi)}(\tau+\delta)^{\mu-1}. \tag{2.9}$$

In order to proceed our analysis, we have take the assumption on $a_2$ as:
**(H1)** $a_2 : [t_1,t_2]\to\mathbb{R}$ is a bounded function that holds the condition below:

$$a_1 > 2bA_0\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda. \tag{2.10}$$

As in [10], we set the transformation of variables as:

$$z(x,\omega,\lambda,t) = u_t(x,t-\lambda\omega),$$

where

$$(x,\omega,\lambda,t)\in\mathfrak{D} := \mho\times(0,1)\times(t_1,t_2)\times\mathbb{R}_+,$$

it holds that

$$\begin{cases} \lambda z_t(x,\omega,\lambda,t) + z_\omega(x,\omega,\lambda,t) = 0 \\ z(x,0,\lambda,t) = u_t(x,t). \end{cases} \tag{2.11}$$

Therefore, by virtue of system (1.2) together with Theorem 2.3, we deduce that

$$\begin{cases} u_{tt} - \Delta u + a_1u_t + b\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}a_2(\lambda)\psi(x,\zeta,\lambda,t)\alpha(\zeta)d\zeta d\lambda = \mathbb{F}(u), \\ \partial_t\psi(x,\zeta,\lambda,t) + (\zeta^2+\delta)\psi(x,\zeta,\lambda,t) - z(x,1,\lambda,t)\alpha(\zeta) = 0, \\ \lambda z_t(x,\omega,\lambda,t) + z_\omega(x,\omega,\lambda,t) = 0, \\ u(x,t) = 0, \qquad x\in\partial\mho, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ z(x,\omega,\lambda,0) = f_0(x,\lambda\omega), \ \ \lambda\in(0,t_2), \end{cases} \tag{2.12}$$

where

$$b = \frac{\sin(\mu\pi)}{\pi}, \quad (x,\omega,\lambda,t)\in\mathfrak{D}, \quad \text{and} \quad \zeta\in\mathbb{R}.$$

We now state the well-posedness result for system (2.12), which can be determined by means of the energy approach, combined with the approaches of [28, 29, 37], with the necessary changes.

**Theorem 2.5.** *Assume that inequality (2.10) holds, and then, for any $(u_0, u_1, \psi_0, f_0) \in \mathcal{H}$, $\exists$ a weak solution $(u, \psi, z)$ of the system (2.12) such that*

$$\begin{aligned}
&u, u_t \in C(]0, T[, H_0^1(\mho)) \cap C^1(]0, T[, L^2(\mho)),\\
&u_{tt} \in C(]0, T[, L^2(\mho)),\\
&\psi \in C([0, T]; L^2(\mho \times \mathbb{R} \times (t_1, t_2))),\\
&z \in C([0, T]; L^2(\mho \times (0, 1) \times (t_1, t_2))),
\end{aligned}$$

*where*

$$\mathcal{H} := H_0^1(\mho) \times L^2(\mho) \times L^2(\mho \times \mathbb{R} \times (t_1, t_2)) \times L^2(\mho \times (0, 1) \times (t_1, t_2)).$$

Next, we introduce $E$, the energy function of system (2.12), with a definition and proof.

**Lemma 2.6.** *Suppose $(u, \psi, z)$ is the solution of system (2.12), and then, we have the energy function given by*

$$\begin{aligned}
E(t) \quad = \quad & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\\
&+\frac{b}{2}\int_\mho\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx\\
&+bA_0\int_\mho\int_0^1\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx, \qquad (2.13)
\end{aligned}$$

*which satisfies*

$$E'(t) \le -C_0\|u_t\|_2^2 - \frac{b}{2}\int_\mho\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \le 0, \qquad (2.14)$$

*where*

$$C_0 = a_1 - 2bA_0\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda > 0.$$

*Proof.* To begin, we multiply Eq $(2.12)_1$ by $u_t$ and integrate over $\mho$. Using integration by parts, this yields

$$\begin{aligned}
&\int_\mho u_{tt}u_t - \int_\mho \Delta u u_t dx + a_1\|u_t\|_2^2 + b\int_\mho u_t\int_{t_1}^{t_2}a_2(\lambda)\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta d\lambda dx\\
&= \int_\mho \mathbb{F}(u)\, u_t\, dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\int_\mho\int_\mho\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right]\\
&+a_1\|u_t\|_2^2 + b\int_\mho u_t\int_{t_1}^{t_2}a_2(\lambda)\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta d\lambda dx = 0. \qquad (2.15)
\end{aligned}$$

Next, we multiply Eq $(2.12)_2$ by $b|a_2(\lambda)|\psi$ and integrate over $\mho \times (t_1, t_2) \times (-\infty, +\infty)$, and we find

$$\begin{aligned}
&\frac{b}{2}\frac{d}{dt}\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\
&+b\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\
&-b\int_{\mho}\int_{t_1}^{t_2}|a_2(\lambda)|z(x,1,\lambda,t)\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta d\lambda dx = 0.
\end{aligned} \tag{2.16}$$

After that, we multiply the Eq $(2.12)_3$ by $z|a_2(\lambda)|$ and integrate over $\mho \times (0,1) \times (t_1, t_2)$ by utilizing Eq $(2.11)_2$, and we find

$$\begin{aligned}
&\frac{d}{dt}bA_0\int_{\mho}\int_0^1\int_{t_1}^{t_2}\lambda|a_2(\lambda)|z^2(x,\omega,\lambda,t)d\lambda d\omega dx \\
= \; & bA_0\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda\|u_t\|_2^2 - bA_0\int_{t_1}^{t_2}|a_2(\lambda)|\|z(x,1,\lambda,t)\|_2^2 d\lambda.
\end{aligned} \tag{2.17}$$

Utilizing the Cauchy-Schwarz inequality, we deduce

$$\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta \le \Big(\int_{-\infty}^{+\infty}\frac{\alpha^2(\zeta)}{\zeta^2+\delta}d\zeta\Big)^{1/2}\Big(\int_{-\infty}^{+\infty}(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta\Big)^{1/2}.$$

Subsequently, the utilization of Young's inequality yields

$$\begin{aligned}
&b\int_{\mho}\int_{t_1}^{t_2}|a_2(\lambda)|z(x,1,\lambda,t)\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta d\lambda dx \\
\le \; & bA_0\int_{\mho}\int_{t_1}^{t_2}|a_2(\lambda)||z(x,1,\lambda,t)|^2 d\lambda dx \\
&+\frac{b}{4}\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx.
\end{aligned} \tag{2.18}$$

Moreover, by Young's inequality, we deduce

$$\begin{aligned}
b\int_{\mho}\int_{t_1}^{t_2}a_2(\lambda)u_t\int_{-\infty}^{+\infty}\alpha(\zeta)\psi(x,\zeta,\lambda,t)d\zeta d\lambda dx &\le bA_0(\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda)\|u_t\|_2^2 \\
&+\frac{b}{4}\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx.
\end{aligned} \tag{2.19}$$

From Eqs (2.15)–(2.19), we deduce Eq (2.13) with

$$\begin{aligned}
\frac{d}{dt}E(t) &= -\Big(a_1 - 2bA_0\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda\Big)\|u_t\|_2^2 \\
&\quad -\frac{b}{2}\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \le 0.
\end{aligned}$$

Taking condition (2.10) into account, it follows that

$$C_0 = a_1 - 2bA_0 \int_{t_1}^{t_2} |a_2(\lambda)| d\lambda > 0. \tag{2.20}$$

This leads to Eq (2.14).

Consequently

$$E(0) \geq E(t). \tag{2.21}$$

We now introduce the following lemmas, which will be particularly useful in the subsequent sections for proving our main results.

The proof of these lemmas is based mainly on the reference [28] with some basic changes.

**Lemma 2.7.** *There exists* $0 < c(\mho)$*, such that*

$$\Big( \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \Big)^{\frac{\varsigma}{2p}} \leq c\Big( \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy + \|\nabla u\|_2^2 \Big),$$

*for all* $2 \leq \varsigma \leq 2p$*, provided that* $\int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \geq 0$.

**Lemma 2.8.** *There exists* $c(\mho) > 0$*, such that*

$$\|u\|_{2p}^{2p} \leq c\Big( \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy + \|\nabla u\|_2^2 \Big),$$

*for all* $u \in L^{2p}(\mho)$ *and provided that* $\int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \geq 0$.

**Corollary 2.9.** *There exists* $c(\mho) > 0$*, such that*

$$\|u\|_2^2 \leq c\Big[ \Big( \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \Big)^{\frac{2}{2p}} + \|\nabla u\|_2^{\frac{4}{2p}} \Big],$$

*provided that* $\int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \geq 0$.

**Lemma 2.10.** *There exists* $c(\mho) > 0$*, such that*

$$\|u\|_{2p}^{\varsigma} \leq c\Big( \|u\|_{2p}^{2p} + \|\nabla u\|_2^2 \Big),$$

*for all* $u \in L^{2p}(\mho)$ *and* $2 \leq \varsigma \leq 2p$.

Before proving the blow-up results, we introduce the following functional:

$$\begin{aligned} \mathbb{H}(t) = -E(t) &= -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2p} \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \\ &\quad -\frac{b}{2} \int_{\mho} \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\lambda)| |\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \end{aligned}$$

$$-bA_0 \int_{\mho} \int_0^1 \int_{t_1}^{t_2} \lambda |a_2(\lambda)| |z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx. \tag{2.22}$$

Hence

$$\mathbb{H}'(t) \geq C_0 \|u_t\|_2^2 + \frac{b}{2} \int_{\mho} \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx. \tag{2.23}$$

Therefore, we have

$$\begin{aligned} \mathbb{H}'(t) &\geq C_0 \|u_t\|_2^2 \geq 0 \\ \mathbb{H}'(t) &\geq \frac{b}{2} \int_{\mho} \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \geq 0. \end{aligned} \tag{2.24}$$

From Eq (2.21), we have

$$0 < \mathbb{H}(0) \leq \mathbb{H}(t) \leq \frac{1}{2p} \int_{\mho} \int_{\mho} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dx dy. \tag{2.25}$$

## 3. Blow up

This section is devoted to proving the blow-up of solutions to system (2.12), specifically in the case of negative initial energy.

**Theorem 3.1.** *Suppose that Eq (2.10) and* $0 > E(0)$*, and then the solution of system (2.12) will blow up in finite time.*

*Proof.* To begin the proof, we define

$$\mathcal{R}(t) = \mathbb{H}^{1-\mu}(t) + \varepsilon \int_{\mho} u u_t dx + \frac{\varepsilon a_1}{2} \int_{\mho} u^2 dx, \tag{3.1}$$

where $0 < \varepsilon >$ denotes a small parameter, whose value will be specified later, and

$$\frac{p-1}{p^2} < \mu < \frac{p-1}{2p} < \frac{p-1}{p} < 1. \tag{3.2}$$

Multiplying Eq $(2.12)_1$ by $u$ and taking the derivative of Eq (3.1), we deduce

$$\begin{aligned} \mathcal{R}'(t) &= (1-\mu)\mathbb{H}^{-\mu}\mathbb{H}'(t) + \varepsilon \|u_t\|_2^2 + \varepsilon \int_{\mho} \mathbb{F}(u) u dx - \varepsilon \|\nabla u\|_2^2 \\ &\quad \underbrace{-\varepsilon b \int_{\mho} u \int_{t_1}^{t_2} a_2(\lambda) \int_{-\infty}^{+\infty} \alpha(\zeta)\psi(x,\zeta,\lambda,t) d\zeta d\lambda dx}_{J_{01}}. \end{aligned} \tag{3.3}$$

By applying the Cauchy-Schwarz and Young inequalities, we get for $\delta > 0$,

$$\begin{aligned} J_{01} &\leq \varepsilon \delta b A_0 \int_{t_1}^{t_2} |a_2(\lambda)| d\lambda \|u\|_2^2 \\ &\quad + \frac{b\varepsilon}{4\delta} \int_{\mho} \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx. \end{aligned} \tag{3.4}$$

By substituting Eq (3.4) in Eq (3.3) and recalling Eq (2.10), we find

$$\begin{aligned}\mathcal{R}'(t) \quad &\geq \quad (1-\mu)\mathbb{H}^{-\mu}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\ &\quad -\varepsilon\|\nabla u\|_2^2 - \varepsilon\delta bA_0\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda\|u\|_2^2 \\ &\quad -\varepsilon\frac{b}{4\delta}\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)|(\zeta^2+\delta)|\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx. \qquad (3.5)\end{aligned}$$

For the second case, $\delta$ is selected in the following appropriate way:

$$\frac{1}{2\delta} = \vartheta\mathbb{H}^{-\mu}(t).$$

By taking Eq (2.24) and substituting in Eq (3.5), we get

$$\begin{aligned}\mathcal{R}'(t) \quad &\geq \quad [(1-\mu)-\varepsilon\vartheta]\mathbb{H}^{-\mu}\mathbb{H}'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\ &\quad -\varepsilon\|\nabla u\|_2^2 - \varepsilon\left(\frac{bA_0H^{\mu}(t)}{2\vartheta}\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda\right)\|u\|_2^2. \qquad (3.6)\end{aligned}$$

Next, from the system (2.22) and for $1 > a > 0$, we deduce

$$\begin{aligned}\varepsilon\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \quad &= \quad \varepsilon a\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy + \varepsilon p(1-a)\|u_t\|_2^2 \\ &\quad +2\varepsilon p(1-a)\mathbb{H}(t) + \varepsilon p(1-a)\|\nabla u\|_2^2 \qquad (3.7) \\ &\quad +\varepsilon p(1-a)b\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\ &\quad +2\varepsilon bA_0 p(1-a)\int_{\mho}\int_0^1\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx.\end{aligned}$$

By substituting in Eq (3.6), we get the following estimate

$$\begin{aligned}\mathcal{R}'(t) \quad &\geq \quad [(1-\mu)-\varepsilon\vartheta]\mathbb{H}^{-\mu}\mathbb{H}'(t) - \varepsilon\left(\frac{bA_0H^{\mu}(t)}{2\vartheta}\int_{t_1}^{t_2}|a_2(\lambda)|d\lambda\right)\|u\|_2^2 \\ &\quad +\varepsilon a\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy + \varepsilon\Big(p(1-a)+1\Big)\|u_t\|_2^2 \\ &\quad +\varepsilon\Big(p(1-a)-1\Big)\|\nabla u\|_2^2 + 2\varepsilon p(1-a)\mathbb{H}(t) \\ &\quad +\varepsilon p(1-a)b\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\ &\quad +2\varepsilon bA_0 p(1-a)\int_{\mho}\int_0^1\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx. \qquad (3.8)\end{aligned}$$

According to Eq (2.25) and Corollary 2.9, together with Young's inequality, it follows that

$$\mathbb{H}^{\mu}(t)\|u\|_2^2 \quad \leq \quad \left(\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^{\mu}\|u\|_2^2$$

$$
\begin{aligned}
&\le c\Big[\Big(\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\mu+\frac{2}{2p}}+\Big(\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\mu}\|\nabla u\|_2^{\frac{4}{2p}}\Big]\\
&\le c\Big[\Big(\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\frac{(\mu p+1)}{p}}+\Big(\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\frac{\mu p}{(p-1)}}+\|\nabla u\|_2^2\Big].
\end{aligned}
$$

By Eq (3.2), this yields

$$2<2(\mu p+1)\le 2p \ \text{ and } \ 2<\frac{2\mu p^2}{p-1}\le 2p.$$

Thus, Lemma 2.7 yields

$$\mathbb{H}^{\mu}(t)\|u\|_2^2 \ \le \ c\Big(\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\|\nabla u\|_2^2\Big). \tag{3.9}$$

Combining Eqs (3.8) and (3.9), we get

$$
\begin{aligned}
\mathcal{R}'(t) \ \ge\ & \Big\{(1-\mu)-\varepsilon\vartheta\Big\}\mathbb{H}^{-\mu}\mathbb{H}'(t)+\varepsilon\Big\{p(1-a)+1\Big\}\|u_t\|_2^2\\
&+\varepsilon\Big(a-\frac{bA_0c}{2\vartheta}\Big)\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+2\varepsilon p(1-a)\mathbb{H}(t)\\
&+\varepsilon\Big\{p(1-a)-1-\frac{bA_0c}{2\vartheta}\Big\}\|\nabla u\|_2^2\\
&+\varepsilon p(1-a)b\int_\mho\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2d\zeta d\lambda dx\\
&+2\varepsilon bA_0p(1-a)\int_\mho\int_0^1\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2d\lambda d\omega dx.
\end{aligned} \tag{3.10}
$$

We now fix $0<a$ sufficiently small such that

$$\mathfrak{m}=(1-a)p-1>0.$$

Then we choose $\vartheta$ sufficiently large such that

$$a-\frac{bA_0c}{2\vartheta}>0 \quad \text{and} \quad \mathfrak{m}-\frac{bA_0c}{2\vartheta}>0.$$

Next, we fixed $\vartheta, a$, and we select $\varepsilon$ sufficiently small such that

$$(1-\mu)-\varepsilon\vartheta>0,$$

and

$$\mathcal{R}(0)>0.$$

Hence, estimate (3.8) becomes, for some $\mathfrak{m}_1>0$,

$$\mathcal{R}'(t) \ \ge\ \mathfrak{m}_1\Big\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\int_\mho\int_\mho \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy$$

$$
\begin{aligned}
&+\int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\
&+\int_{\mho}\int_{0}^{1}\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx\Big\}. \qquad (3.11)
\end{aligned}
$$

By utilizing the Young's and Holder's inequalities, we obtain

$$
\Big|\int_{\mho} uu_t dx\Big|^{\frac{1}{1-\mu}} \le c\Big[\|u\|_{2p}^{\frac{\theta}{1-\mu}} + \|u_t\|_2^{\frac{\alpha}{1-\mu}}\Big], \qquad (3.12)
$$

where $\frac{1}{\alpha}+\frac{1}{\theta}=1$.

We set $\alpha = 2(1-\mu)$ to get

$$
\frac{\theta}{1-\mu} = \frac{2}{2(1-\mu)-1} \le 2p,
$$

which is achieved according to relationship (3.2).

After that, estimate (3.12) for $s=\frac{\theta}{1-\mu}=\frac{2}{2(1-\mu)-1}$ gives

$$
\Big|\int_{\mho} uu_t dx\Big|^{\frac{1}{1-\mu}} \le c\Big[\|u\|_{2p}^{s} + \|u_t\|_2^{2}\Big].
$$

Then, Lemmas 2.10 and 2.8 yield

$$
\begin{aligned}
\Big|\int_{\mho} uu_t dx\Big|^{\frac{1}{1-\mu}} &\le c\Big[\|u\|_{2p}^{2p} + \|u_t\|_2^2 + \|\nabla u\|_2^2\Big] \\
&\le c\Big[\int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy + \|u_t\|_2^2 + \|\nabla u\|_2^2\Big]. \qquad (3.13)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{R}^{\frac{1}{1-\mu}}(t) &= \Big(\mathbb{H}^{1-\mu}(t) + \varepsilon\int_{\mho} uu_t dx + \varepsilon\frac{a_1}{2}\int_{\mho} u^2 dx\Big)^{\frac{1}{1-\mu}} \\
&\le c\Big(\mathbb{H}(t) + |\int_{\mho} uu_t dx|^{\frac{1}{1-\mu}} + \|u\|_2^{\frac{2}{1-\mu}}\Big) \\
&\le c\Big(\mathbb{H}(t) + |\int_{\mho} uu_t dx|^{\frac{1}{1-\mu}} + \|u\|_{2p}^{\frac{2}{1-\mu}}\Big) \\
&\le c\Big\{\mathbb{H}(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{2p}^{2p} + \int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big\} \\
&\le c\Big\{\mathbb{H}(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \int_{\mho}\int_{\mho}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\
&\quad + \int_{\mho}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\lambda)||\psi(x,\zeta,\lambda,t)|^2 d\zeta d\lambda dx \\
&\quad + \int_{\mho}\int_{0}^{1}\int_{t_1}^{t_2}\lambda|a_2(\lambda)||z(x,\omega,\lambda,t)|^2 d\lambda d\omega dx\Big\}. \qquad (3.14)
\end{aligned}
$$

From Eqs (3.11) and (3.14), we have

$$\mathcal{R}'(t) \geq \mathcal{B}\mathcal{R}^{\frac{1}{1-\mu}}(t), \tag{3.15}$$

where $\mathcal{B}(\mathfrak{m}_1, c) > 0$.

Finally, integrating Eq (3.15) over $(0, t)$ gives

$$\mathcal{R}^{\frac{\mu}{1-\mu}}(t) \geq \frac{1}{\mathcal{R}^{\frac{-\mu}{1-\mu}}(0) - \mathcal{B}\frac{\mu}{(1-\mu)}t}.$$

So, $\mathcal{R}(t)$ blows up in time

$$T \leq T^* = \frac{1-\mu}{\mathcal{B}\mu\mathcal{R}^{\mu/(1-\mu)}(0)}.$$

This completes the proof.

## 4. Conclusions

In this study, we investigated a Hartree-type wave equation incorporating a distributed delay term under fractional conditions. By establishing suitable assumptions and allowing for negative initial energy, we proved that the corresponding solution blows up in finite time. The obtained results highlight the strong destabilizing effect induced by the combined presence of nonlocal Hartree interactions, distributed delay, and fractional damping mechanisms. These findings contribute to the theoretical understanding of wave equations with memory effects and delayed interactions.

Compared with the classical Hartree wave equation without delay or fractional effects, our analysis shows that the introduction of fractional delay terms can significantly accelerate instability and lead to blow-up under weaker conditions. This demonstrates that memory and hereditary effects play a crucial role in the qualitative behavior of nonlinear wave propagation models.

These results extend and complement several earlier works devoted to Hartree-type and related wave equations. In particular, the present framework generalizes previous blow-up results by incorporating both fractional-order operators and distributed delay effects within a unified setting.

Possible extensions of this work include the investigation of global existence and decay results under different sign conditions on the initial energy or stronger damping assumptions. Moreover, by employing alternative techniques such as semigroup theory, it may be possible to establish the existence and uniqueness of global solutions in suitable energy spaces. Another interesting direction concerns the inclusion of additional damping mechanisms or different delay kernels, which may lead to new stability or decay phenomena. These topics will be addressed in future research.

Finally, numerical simulations illustrating the blow-up dynamics and the influence of fractional delay parameters could further enhance the understanding of the theoretical results obtained in this paper.

## Use of AI tools declaration

The authors declare that no artificial intelligence (AI) tools were used in the development, analysis, writing, or preparation of this manuscript.

## Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for the financial support (QU-APC-2026).

## Conflict of interest

The authors declare that there are no known financial or personal conflicts of interest that could have influenced the work reported in this paper.

## Author contributions

**Salah Boulaaras**: Conceptualization, data curation, formal analysis, software, validation, writing–original draft preparation, and writing–review and editing, and funding; **Abdelbaki Choucha:** Conceptualization, validation, writing–original draft preparation, and writing–review and editing.

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