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Research article

The complex center and complex isochronous center problems for a complex quartic polynomial system

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Abstract: In this paper, we first study the complex center and complex isochronous center problems for a complex quartic polynomial differential system. More precisely, by calculating and decomposing the variety of the ideal generated by the singular point quantities (resp. complex period quantities), we obtain the necessary conditions for the resonant elementary equilibrium (resp. complex center) of the system to be a complex center (resp. a complex isochronous center). Using time-reversibility and the Darboux integrable theory, we rigorously prove that these conditions are also sufficient. Furthermore, when the coefficients and the variables of the system are complex conjugates, we not only derive the necessary and sufficient conditions for the center-type equilibrium to be both a center and an isochronous center, but we also give the parametric conditions under which the center-type equilibrium of the system becomes a weak focus of order 10. In this case, although the independence of the focal quantities is not satisfied, we still prove that there exist parametric conditions under which 10 limit cycles can bifurcate from this weak focus.

Keywords: quartic polynomial differential system; center; isochronous center; limit cycle; focal quantities

1. Introduction and the main results

Consider planar complex differential autonomous systems with a $1 : -1$ resonant elementary equilibrium at the origin $O : (0, 0)$ as follows:

$$\frac{\mathrm{d}z}{\mathrm{d}T} = z + \sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta}, \quad \frac{\mathrm{d}w}{\mathrm{d}T} = -w - \sum_{k=2}^{\infty} \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}, \tag{1.1}$$

where $z, w, T, a_{\alpha\beta}, b_{\alpha\beta} \in \mathbb{C}$. The first problem that needs to be solved for such systems is to determine the parametric conditions under which the resonant elementary equilibrium becomes a *resonant center*, also called a *complex center*, which will be explained later. Following Dulac [1], the origin O of system (1.1) is a complex center if and only if there exists an analytic first integral of the following form

$$\mathcal{H}(z,w) = zw + \sum_{j+k\geq 3} \psi_{j,k} z^j w^k. \tag{1.2}$$

It follows that the complex center problem for System (1.1) is reduced to find parametric conditions under which the system has a analytic first integral of the form (1.2). This problem goes back to Dulac [1], who gave a complete classification when System (1.1) is a complex quadratic polynomial system. For System (1.1) with $n = 3$, only the homogeneous case was completely solved by Sadovskiĭ [2], and also by Liu and Li [3] independently, but the general case remains open. However, there are fewer specific polynomial families for System (1.1) with cubic polynomials being studied, as seen in [4–8] and the references therein. For System (1.1) with quartic polynomials, it remains open for the homogeneous case, with no mention of the general case, as seen in [9, 10] and the references therein.

The second problem that needs to be solved for System (1.1) is to determine the parameters conditions under which the system can be reduced to the linear one $\dot{\xi} = \xi, \dot{\eta} = -\eta$ by means of the formal change of variables $\xi = z + o(|(z,w)|), \eta = w + o(|(z,w)|)$; in this case the resonant elementary equilibrium is called as a *complex isochronous center*, which will be explained later. For the complex isochronous center problem of System (1.1), the quadratic and cubic homogeneous cases were completely solved by Christopher and Rousseau [11]. The time–reversible and Lotka–Volterra complex cubic systems were studied by Chen and Romanovski [12], and Giné and Romanovski [13], respectively. However, the general complex cubic systems remain open. Moreover, limited work has been done on the complex isochronous center in complex quartic systems, as seen in time–reversible and Lotka–Volterra complex quartic homogeneous systems ([14] and [15], respectively, and the references therein).

In short, the complete classification of the complex center and complex isochronous center for complex quartic polynomial systems remains open. Therefore, the first main goal of this paper is to discuss these two problems in the following complex quartic polynomial systems

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}T} &= (1 - \mathbf{i}\delta)z + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{31}z^3w + a_{30}z^2w^2 + nzw^3, \\ \frac{\mathrm{d}w}{\mathrm{d}T} &= -(1 + \mathbf{i}\delta)w - \left(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{31}w^3z + b_{30}w^2z^2 + nwz^3\right), \end{aligned} \tag{1.3}$$

where $a_{ij}, b_{ij} \in \mathbb{C}$, $b_{ij} = \overline{a_{ij}}$, and $\delta, n \in \mathbb{R}$. By calculating and decomposing the ideal generated by the singular point quantities, denoted by μ_k, (equivalently, focal quantities, denoted by v_{2k+1}), we derive the necessary conditions for the complex center of System (1.3) at the origin. Then, using time–reversibility and the Darboux integrable theory, we prove that these conditions are also sufficient. The following theorem, whose proof is given in Section 3, provides a complete classification for the complex center of System (1.3) at the origin.

Theorem 1.1. *Assume that* $\delta = 0$. *For System* (1.3)*, the origin is a complex center if and only if one of* *the following seven conditions is satisfied*

$C_1: a_{21}=b_{21}, \quad a_{30}=b_{30}=a_{31}=b_{31}=0,$
$C_2: a_{21}=b_{21}, \quad a_{30}=b_{30}=0, \quad a_{12}=b_{12}, \quad a_{31}=b_{31}, \quad a_{31}b_{31}n\neq 0,$
$C_3: a_{21}=b_{21}, \quad a_{30}=b_{30}=a_{31}^2+a_{31}b_{31}+b_{31}^2=0, \quad a_{12}b_{31}=a_{31}b_{12}, \quad a_{31}b_{31}n\neq 0,$
$C_4: a_{21}=b_{21}, \quad a_{30}=b_{30}=n=0, \quad a_{12}a_{31}^2=b_{12}b_{31}^2, \quad a_{31}b_{31}\neq 0,$
$C_5: a_{21}=b_{21}, \quad a_{12}=b_{12}, \quad a_{31}=b_{31}, \quad a_{30}=b_{30}, \quad a_{30}b_{30}\neq 0,$
$C_6: a_{21}=b_{21}, \quad a_{12}a_{30}=b_{12}b_{30}, \quad a_{30}a_{31}=b_{30}b_{31}, \quad a_{30}^2+a_{30}b_{30}+b_{30}^2=0, \quad a_{30}b_{30}\neq 0,$
$C_7: a_{21}=b_{21}, \quad a_{12}a_{30}=b_{12}b_{30}, \quad a_{30}a_{31}=b_{30}b_{31}, \quad 2a_{12}a_{31}=b_{30}(a_{12}+b_{30}),$
$2b_{12}b_{31}=a_{30}(b_{12}+a_{30}), \quad n=0, \quad a_{30}b_{30}a_{12}b_{12}\neq 0.$

Based on Theorem 1.1, the following theorem, whose proof is given in Section 4, gives the classification for the complex isochronous center of System (1.3) at the origin.

Theorem 1.2. *Assume that $\delta=0$. The origin of System* (1.3) *is a complex isochronous center if and only if one of the following five conditions holds*

$\mathcal{I}_1: a_{21}=b_{21}=a_{12}=b_{12}=a_{30}=b_{30}=a_{31}=b_{31}=n=0,$
$\mathcal{I}_2: a_{21}=b_{21}=a_{12}=b_{12}=a_{30}=b_{30}=a_{31}-b_{31}=b_{31}+n=0, n\neq 0,$
$\mathcal{I}_3: a_{21}=b_{21}=a_{12}=b_{12}=a_{30}=b_{30}=a_{31}-\left(a\pm\sqrt{3}a\mathbf{i}\right)=b_{31}-\left(a\mp\sqrt{3}a\mathbf{i}\right)=n-2a=0, a\neq 0,$
$\mathcal{I}_4: a_{21}=b_{21}=n=0, \quad a_{12}=b_{12}=a_{31}=b_{31}=-a_{30}=-b_{30}\neq 0,$
$\mathcal{I}_5: a_{21}=b_{21}=n=0, \quad a_{12}=a_{31}=-b_{30}=-b+\sqrt{3}b\mathbf{i}, \quad b_{12}=b_{31}=-a_{30}=-b-\sqrt{3}b\mathbf{i}, \quad b\neq 0,$

where $a,b\in\mathbb{R}$.

Note that when the coefficients and the variables of System (1.1) are complex conjugates, i.e.,

$$w=\bar{z}, \quad b_{\alpha\beta}=\overline{a_{\alpha\beta}}, \quad \alpha\geq 0, \quad \beta\geq 0, \quad \alpha+\beta\geq 2, \tag{1.4}$$

and then by the change of variables

$$x=\frac{z+w}{2}, \quad y=\frac{(w-z)\mathbf{i}}{2}, \quad t=-T\mathbf{i}, \quad \mathbf{i}=\sqrt{-1}, \tag{1.5}$$

the system is reduced to the following

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-y+X(x,y), \quad \frac{\mathrm{d}y}{\mathrm{d}t}=x+Y(x,y), \tag{1.6}$$

where $x,y\in\mathbb{R}$, and X and Y are polynomials of a degree at least 2. In this case, we say that System (1.6) is the concomitant real system of (1.1), and System (1.1) is the complexification of the real System (1.6). Therefore, finding the parametric conditions of center (resp. isochronicity) of System (1.6) is equivalent to finding the parametric conditions of resonant center (resp. linearizability) of the complex System (1.1). Note that this is the reason we refer to the resonant center (resp. linearizability) of the complex System (1.1) as the complex center (resp. the complex isochronous center).

Note that the origin O of System (1.6) is a center-type equilibrium. Therefore, the next step is to distinguish between a center and a focus, which is an important task in the qualitative theory of differential equations. According to the Poincaré–Lyapunov theorem [16, 17], the necessary and sufficient condition for the origin O of System (1.6) to be a center is that one can find the first integral of the following form

$$\widetilde{\mathcal{H}}(x,y) = x^2 + y^2 + \sum_{j+k\geq 3} \widetilde{\psi}_{j,k} x^j y^k,$$

where the series converge in a neighborhood of the origin. Remark that the first integral $\widetilde{\mathcal{H}}(x,y)$ can be directly obtained from (1.2) using the change of variables (1.5). The center problem of (1.6) is completely solved in quadratic polynomial systems [18], and in cubic homogeneous polynomial systems [19]. However, the general real cubic polynomial remains open, and fewer specific polynomial families for System (1.6) with cubic polynomials were studied, as seen in [4, 5] and the references therein. For real quartic polynomial systems, limited work has been done on the center problem for such systems. For instance, the center problem for real quartic homogeneous polynomial systems was completely solved by Ferčec, Giné, Liu, and Romanovski in [10]. However, the center problem of the general real quartic polynomial systems remains open.

A further question in the study of centers is isochronicity, which deals with the synchronicity of all oscillations. By the definition in [20], a center is *isochronous* if all surrounding closed orbits near the center have the same period. An earlier result on isochronous centers was given by Loud [21] in 1964, who completely classified isochronous centers for the quadratic polynomial differential systems. Five years later, Pleshkan [22] gave necessary and sufficient conditions for the cubic polynomial differential systems with homogeneous nonlinearities having an isochronous center at the origin. Although the isochronous center problem of the general cubic polynomial systems remains open, some of specific cubic polynomial systems were studied, as seen in [7, 12] and the references therein. In the case when both X and Y are quartic polynomials, the problem was only solved in the homogeneous and time-reversible cases (see [23] and [14] respectively), as well as in some specific cases (see [24, 25]). However, the general case is still far from being solved.

In summary, the complete classification of the center and isochronous center for real quartic polynomial systems remains open. Thus, the main goal of this paper is to study these two problems for the concomitant real system of (1.3) at the origin. More precisely, by setting $w = \bar{z}$ and $b_{ij} = \overline{a_{ij}} = \overline{A_{ij} + \mathbf{i}B_{ij}}$ $(ij = 30, 21, 12, 31)$ with $A_{ij}, B_{ij} \in \mathbb{R}$, and together with the change of variables (1.5), System (1.3) can be transformed into its concomitant real system as follows:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} =& \delta x - y - (B_{12} + B_{21} + B_{30})x^3 + (A_{12} - A_{21} - 3A_{30})x^2 y \\
& - (B_{12} + B_{21} - 3B_{30})xy^2 + (A_{12} - A_{21} + A_{30})y^3 - (B_{30} + B_{31})x^4 \\
& - 2(A_{31} - n)x^3 y - 2B_{30}x^2y^2 - 2(A_{31} - n)xy^3 - (B_{30} - B_{31})y^4, \\
\frac{\mathrm{d}y}{\mathrm{d}t} =& x + \delta y + (A_{12} + A_{21} + A_{30})x^3 + (B_{12} - B_{21} - 3B_{30})x^2 y \\
& + (A_{12} + A_{21} - 3A_{30})xy^2 + (B_{12} - B_{21} + B_{30})y^3 + (A_{30} + A_{31} + n)x^4 \\
& - 2B_{31}x^3 y + 2A_{30}x^2y^2 - 2B_{31}xy^3 + (A_{30} - A_{31} - n)y^4,
\end{aligned} \tag{1.7}$$

where $\delta, n \in \mathbb{R}$. We can directly derive the center and isochronous center conditions, which are given in Corollaries 1 and 2, for System (1.7) at the origin by substituting $b_{ij} = \overline{a_{ij}} = \overline{A_{ij} + \mathbf{i}B_{ij}}$ $(ij = 30, 21, 12, 31)$ into the conditions of Theorems 1.1 and 1.2.

Finally, we study the limit cycles bifurcation for System (1.7) at the origin. As indicated in [26, Theorem 1.3.4, p.16], the first non-vanishing focal quantity v_{2k+1} of System $(1.7)|_{\delta=0}$ and the first non-vanishing singular point quantity μ_k of System (1.3) at the origin satisfy the following relation

$$v_{2k+1} = \mathbf{i}\pi\mu_k,\ k = 1, 2, \cdots . \tag{1.8}$$

It follows that we can determine the highest order for the center-type equilibrium of System (1.7) at the origin to be a weak focus; then we discuss the maximum number of limit cycles that bifurcate from the weak focus. The following theorem, whose proof is given in Section 5, presents the parametric conditions under which 10 limit cycles can bifurcate from the weak focus.

Theorem 1.3. *For System* (1.7)*, there are ten small-amplitude limit cycles bifurcating from a weak focus at the origin when the system parameters are perturbed as follows,*

$$\begin{aligned}
\delta &= -3.9270631957 \times 10^{10}\epsilon^{20},\\
B_{30} &= -1.9696143106 + 141.2368933207\epsilon^2 + 38917.8125355654\epsilon^4,\\
A_{30} &= -0.5462851325 - 509.2252922536\epsilon^2,\\
A_{21} &= -3.6208506663 + 2.1431386629 \times 10^{-99}\epsilon^2 + 4099.7455358899\epsilon^4\\
&\quad -888619.6356280381\epsilon^6 + 7.3232245633 \times 10^7\epsilon^8,\\
B_{21} &= -1.1502243431 \times 10^{11}\epsilon^{18},\\
A_{12} &= -0.1092570265 - 101.8450584507\epsilon^2 + 5.8554736737 \times 10^7\epsilon^{10}\\
&\quad +5.4582398737 \times 10^{10}\epsilon^{12},\\
B_{12} &= 0.3939228621 - 28.2473786641\epsilon^2 - 7783.5625071131\epsilon^4\\
&\quad -2.1111730955 \times 10^8\epsilon^{10} + 1.5138777560 \times 10^{10}\epsilon^{12}\\
&\quad +4.1714887184 \times 10^{12}\epsilon^{14} + 2.5264892267 \times 10^{11}\epsilon^{16},\\
A_{31} &= -1.6388553976 - 1527.6758767607\epsilon^2 - 7.3193420921 \times 10^8\epsilon^{10}\\
&\quad -6.7368775156 \times 10^{11}\epsilon^{12} + 8.0093378406 \times 10^{12}\epsilon^{14},\\
B_{31} &= 5.9088429317 - 423.7106799620\epsilon^2 - 116753.4376066961\epsilon^4\\
&\quad +2.6389663694 \times 10^9\epsilon^{10} + 2.2021375112 \times 10^{11}\epsilon^{12}\\
&\quad -5.0088760458 \times 10^{13}\epsilon^{14},\\
n &= -2.4792533315 + 1.0715693314 \times 10^{-98}\epsilon^2 + 20498.7276794495\epsilon^4\\
&\quad -4.4430981781 \times 10^6\epsilon^6,
\end{aligned}$$

where $0 < \epsilon \ll 1$. *Remark that the parameters are retained with a precision of 10 decimal places here and in Section 5 for brevity, and that higher precision values are available from the authors.*

The rest of this paper is arranged as follows. In Section 2, we introduce the algorithm to calculate the singular point quantity, and the definition of the complex center and the complex isochronous center; moreover, we present the relation of the complex center and the complex isochronous center for a complex polynomial system, and the center and the isochronous center for its concomitant real system.

In Section 3, we first summarize the expressions of the first twelve singular point quantities, and then discuss the complex center problem for System (1.3) at the origin. Based on the center conditions given in Section 3, we further investigate the complex isochronous center problem of System (1.3) at the origin in Section 4. Finally, using the Relation (1.8), we derive the parametric conditions for the origin of System (1.7) to be a weak focus of order 10, and then prove that there exist parametric conditions under which 10 limit cycles bifurcate from this weak focus. Furthermore, we present a numerical example of System (1.7), which has exactly ten limit cycles bifurcating from the weak focus at the origin. It is worth mentioning that since the independence of focal quantities is not satisfied, the method used here to prove the existence of 10 small-amplitude limit cycles is different from the one based on the independence of focal quantities.

2. Focal quantity, singular point quantity, complex center, complex isochronous center

First, we introduce the relation between the focal quantity and the singular point quantity. From [27, p.70], we know that by using normal transformations, the real normal form of System (1.6) is of the following form:

$$\dot{u} = -v - vR(r^2) - uG(r^2), \qquad \dot{v} = u + uR(r^2) - vG(r^2),$$

where $r^2 = u^2 + v^2$, $G(r^2) = \sum\limits_{k=1}^{\infty} v_{2k+1} r^{2k}$, and $R(r^2) = \sum\limits_{k=1}^{\infty} p_{2k} r^{2k}$. If there exists an integer N such that $v_{2k-1} = 0$, $v_{2N+1} \neq 0$ for $k < N$, then the origin of System (1.6) is referred to as a *weak focus*, v_{2N+1} is *the Nth focal quantity*, and the number of N is called the *order* of this weak focus. In particular, the origin of System (1.6) is a center if v_{2k+1} vanish for any positive integer k.

Now, we turn to introduce the concept of the singular point quantity. For System (1.1), Amelbkin, Lukashevich, and Sadovskii [28] proved the following result, which was also given in [26, Theorem 1.8.1, p.40].

Lemma 2.1 ([28]). *For System* (1.1)*, one can uniquely derive the following formal series*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j,$$

where $c_{k+1,k} = d_{k+1,k} = 0,\ k = 1, 2, \cdots$, *such that*

$$\frac{\mathrm{d}\xi}{\mathrm{d}T} = \xi\left(1 + \sum_{j=1}^{\infty} p_j (\xi\eta)^j\right), \quad \frac{\mathrm{d}\eta}{\mathrm{d}T} = -\eta\left(1 + \sum_{j=1}^{\infty} q_j (\xi\eta)^j\right).$$

Definition 2.2 ([3, 29]). Let $\mu_0 = 0$, $\mu_k = p_k - q_k$, $\tau_k = p_k + q_k$, $k = 1, 2, \cdots$. μ_k and τ_k are called the k-th singular point quantity and the k-th complex period constant of the origin of System (1.1), respectively. Moreover, the origin of System (1.1) is referred to as a *weak singular point of order* $\mathcal{N}$ if there exists an integer $\mathcal{N}$ such that $\mu_1 = \mu_2 = \cdots = \mu_{\mathcal{N}-1} = 0$, $\mu_{\mathcal{N}} \neq 0$.

The algorithms to calculate the singular point quantity μ_k and the complex period quantity τ_k at the origin of System (1.1) are provided in [29, Theorem B] and [29, Theorem 3.1], respectively. Moreover, as indicated in [26, Theorem 1.3.4, p.16], the first non-zero focal quantity v_{2k+1} of System (1.6) and the first non-zero singular point quantity μ_k of System (1.1) at the origin satisfy the Relation (1.8).

Lemma 2.3 ([3,29]). *The origin of System* (1.1) (*or System* (1.6)) *is a complex center* (*or a center*) *if and only if* $\mu_k = 0$ *for any positive integer k, and it is a complex isochronous center* (*or an isochronous center*) *if and only if* $\mu_k = \tau_k = 0$ *for any positive integer k.*

The following result gives the relation between the center and the isochronous center for System (1.6) and the complex center and the complex isochronous center for System (1.1) with Eq (1.4).

Lemma 2.4 ([3, 29]). *If the variables and the coefficients of System* (1.1) *satisfy Eq* (1.4)*, then System* (1.6) *has a center* (*resp. an isochronous center*) *at the origin if and only if its associated complex System* (1.1) *has a complex center* (*resp. a complex isochronous center*) *at the origin.*

3. Proof of Theorem 1.1

In this section, we provide a detailed proof of Theorem 1.1. First, we deduce the expressions of the first twelve singular point quantities of System $(1.3)|_{\delta=0}$ at the origin, which can be derived by the recursive formulas of [29, Theorem B] via the computer algebraic systems. Then, we derive the necessary and sufficient conditions for the first twelve singular point quantities vanishing. Finally, we provide a complete classification for the origin of Systems (1.3) (resp. Eq (1.7)) to be a complex center (resp. a center).

Lemma 3.1. *Assume that* $\delta = 0$*. The first twelve singular point quantities of System* (1.3) *at the origin are given as follows:*

$$\begin{aligned}\mu_1 =& a_{21} - b_{21},\\ \mu_2 =& -a_{12}a_{30} + b_{12}b_{30}.\end{aligned}$$

Case 1 $a_{30} = b_{30} = 0$,

$$\mu_3 = 0, \qquad \mu_4 = \frac{4}{3}(a_{31}b_{12} - a_{12}b_{31})n.$$

Case 1.1 $a_{31} = b_{31} = 0$,

$$\mu_5 = \mu_6 = \cdots = \mu_{12} = 0.$$

Case 1.2 $a_{31}b_{31}n \neq 0$, $a_{12} = pa_{31}$, $b_{12} = pb_{31}$,

$$\begin{aligned}\mu_5 =& -\frac{1}{12}(a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)p(25np - 12b_{21}),\\ \mu_6 =& \frac{1}{75}(a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)p(25a_{31}b_{31}p^2 + 48b_{21}^2).\end{aligned}$$

Let $p = 0$*; then, we have the following*

$$\begin{aligned}\mu_7 =& (a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)nb_{21},\\ \mu_8 =& -5(a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)nb_{21},\\ \mu_9 =& \frac{1}{4}(a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)n(5a_{31}b_{31} - 6n^2),\\ \mu_{10} =& \mu_{11} = 0,\\ \mu_{12} =& \frac{67}{75}(a_{31} - b_{31})(a_{31}^2 + a_{31}b_{31} + b_{31}^2)n^5.\end{aligned}$$

Case 1.3 $a_{31}b_{31} \neq 0,\ n = 0,$

$$\mu_5 = (a_{12}a_{31}^2 - b_{12}b_{31}^2)b_{21}.$$

Case 1.3.1 $a_{12} = hb_{31}^2,\ b_{12} = ha_{31}^2$

$$\mu_6 = \cdots = \mu_{12} = 0.$$

Case 1.3.2 $a_{12}a_{31}^2 - b_{12}b_{31}^2 \neq 0,\ b_{21} = 0,$

$$\mu_6 = \frac{1}{3}a_{12}b_{12}(a_{12}a_{31}^2 - b_{12}b_{31}^2) \neq 0.$$

Case 2 $a_{30}b_{30} \neq 0,\ a_{12} = qb_{30},\ b_{12} = qa_{30},$

$$\mu_3 = -a_{30}a_{31} + b_{30}b_{31}.$$

Let $a_{31} = sb_{30},\ b_{31} = sa_{30}$*; then, we get the following*

$$\mu_4 = (a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)(1 + q - 2qs).$$

Let $1 + q - 2qs = 0$*; then, we have the following*

$$\mu_5 = \frac{1}{8q}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n(1 + 5q)(1 - 5q).$$

Case 2.1 $(a_{30} - b_{30})\left(a_{30}^2 + a_{30}b_{30} + b_{30}^2\right)n \neq 0,\ q = \frac{1}{5},$

$$\begin{aligned}
\mu_6 =& \frac{1}{25}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n(-125 + 8n - 40b_{21}),\\
\mu_7 =& \frac{1}{36000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\mu_{70}(n, a_{30}, b_{30}),\\
\mu_8 =& -\frac{1}{3360000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\mu_{80}(n, a_{30}, b_{30}),\\
\mu_9 =& \frac{1}{134400000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\mu_{90}(n, a_{30}, b_{30}),\\
\mu_{10} =& -\frac{1}{435456000000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\mu_{100}(n, a_{30}, b_{30}),
\end{aligned} \tag{3.1}$$

where the expressions of $\mu_{j0}(k = 7, 8, 9, 10)$ *are given in Appendix.*

Case 2.2 $(a_{30} - b_{30})\left(a_{30}^2 + a_{30}b_{30} + b_{30}^2\right)n \neq 0,\ q = -\frac{1}{5},$

$$\begin{aligned}
\mu_6 =& -\frac{2}{25}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n(250 + n + 30b_{21}),\\
\mu_7 =& -\frac{1}{9000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\widetilde{\mu}_{70}(n, a_{30}, b_{30}),\\
\mu_8 =& -\frac{1}{630000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\widetilde{\mu}_{80}(n, a_{30}, b_{30}),\\
\mu_9 =& -\frac{1}{18900000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\widetilde{\mu}_{90}(n, a_{30}, b_{30}),\\
\mu_{10} =& -\frac{1}{1701000000}(a_{30} - b_{30})(a_{30}^2 + a_{30}b_{30} + b_{30}^2)n\widetilde{\mu}_{100}(n, a_{30}, b_{30}),
\end{aligned}$$

where the expressions of $\widetilde{\mu}_{j0}(k = 7, 8, 9, 10)$ *are given in Appendix. Remark that every* μ_k *is deduced from* $\mu_1 = \cdots = \mu_{k-1} = 0,\ k = 2, 3, \cdots, 12.$

Next we discuss the center problem of System (1.3). From Lemma 3.1, we have the following result.

Lemma 3.2. *For System* (1.3)$|_{\delta=0}$, *the first twelve singular point quantities at the origin vanish if and only if one of conditions* $C_i(i = 1, 2, \cdots, 7)$ *is satisfied, where* C_i *is given in Theorem* 1.1.

Proof. The sufficiency of the lemma can be directly obtained from Lemma 3.1, therefore, it suffices to prove the necessity of the lemma. For the case of **Case 1.1** in Lemma 3.1, we derive condition C_1 when $\mu_1 = \mu_2 = \cdots = \mu_{12} = 0$ is satisfied. For the case of **Case 1.2** in Lemma 3.1, we can obtain either condition C_2, or condition C_3 when $\mu_1 = \mu_2 = \cdots = \mu_{12} = 0$ is satisfied. Otherwise, one can conclude that the origin of System (1.3) is a weak singular point of an order at most 12. For the case of **Case 1.3** in Lemma 3.1, we can obtain condition C_4 when $\mu_1 = \mu_2 = \cdots = \mu_{12} = 0$ is satisfied. Otherwise, the origin of System (1.3) is a weak singular point of order at most 6.

For the case of **Case 2** in Lemma 3.1, we can obtain either condition C_5, condition C_6, or condition C_7 when $\mu_1 = \mu_2 = \cdots = \mu_{12} = 0$ is satisfied. Otherwise, we claim that the origin of System (1.3) is a weak singular point of an order at most 10 when **Cases 2.1** and **2.2** are satisfied. For our purpose, we only need to prove that polynomials μ_{70}, μ_{80}, μ_{90}, and μ_{100} (resp. $\widetilde{\mu}_{70}$, $\widetilde{\mu}_{80}$, $\widetilde{\mu}_{90}$, and $\widetilde{\mu}_{100}$) do have common zeros for the case of **Case 2.1** (resp. **Case 2.2**). Remark that $b_{30} = \overline{a_{30}}$, $a_{30} = A_{30} + B_{30}\mathbf{i}$ and $b_{30} = A_{30} - B_{30}\mathbf{i}$. Applying the command of "GroebnerBasis[$\{\mu_{70}, \mu_{80}, \mu_{90}, \mu_{100}\}, \{A_{30}, B_{30}, n\}$]" in Mathematica yields the following

$$\text{GroebnerBasis}[\{\mu_{70}, \mu_{80}, \mu_{90}, \mu_{100}\}, \{A_{30}, B_{30}, n\}] = \{1\}.$$

It follows from [30, Theorem 1.3.10], which described that a given polynomial system $p_1(x_1, x_2, \cdots, x_n) = p_2(x_1, x_2, \cdots, x_n) = \cdots = p_s(x_1, x_2, \cdots, x_n) = 0$ does not have a solution in $\mathbb{C}$ if and only if the reduced Grobner basis for $< p_1, p_2, \cdots, p_s >$ with respect to any term order $[x_1, x_2, \cdots, x_n]$ equals 1, that polynomials $\mu_{j0}(j = 7, 8, 9, 10)$ do not have common zeros. Similarly, we can also demonstrate that polynomials $\widetilde{\mu}_{j0}(j = 7, 8, 9, 10)$ do not have common zeros. Therefore, the claim is completely proven, and the necessity of the lemma is completed.

In summary, we complete the proof of Lemma 3.2.

Proof of Theorem 1.1. For the necessity of Theorem 1.1, it is directly derived from Lemma 3.2. Next we give a proof for the sufficiency of Theorem 1.1. If condition C_1 holds, then using the Darboux integrable theory [31, 32], we can construct an integrating factor of the form $\dfrac{1}{zw}$ because in this case System (1.3) has two invariant curves, $z = 0$ and $w = 0$. It follows that the origin of System (1.3) is a complex center. Similarly, we can prove that under condition C_7, the origin of System (1.3) is a complex center. Combined with the **Case 2** of Lemma 3.1, it follows that Condition C_7 is equivalent to the following conditions:

$$a_{21} = b_{21}, \quad a_{12} = qb_{30}, \quad b_{12} = qa_{30}, \quad a_{31} = \frac{1+q}{2q}b_{30}, \quad b_{31} = \frac{1+q}{2q}a_{30}, \quad a_{30}b_{30}q \neq 0, \quad n = 0. \tag{3.2}$$

Under Condition 3.2, System (1.3) has an integrating factor of the following form:

$$\begin{cases} \frac{1}{z^2w^2}, & \text{for} \quad q + 1 = 0, \\ z^7w^7\left(\frac{q}{b_{30}} + \frac{a_{30}q}{b_{30}}z^2 + \frac{2b_{21}q}{b_{30}(1+q)}zw + qw^2 + z^2w + \frac{a_{30}}{b_{30}}zw^2\right)^{4q-5}, & \text{for} \quad q(q+1) \neq 0. \end{cases}$$

Before proving the sufficiency of condition C_2, we first present the necessary and sufficient conditions for the time reversibility of the following complex quartic polynomial differential systems,

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= z + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{40}z^4 + a_{31}z^3w + a_{22}z^2w^2 + a_{13}zw^3,\\ \frac{\mathrm{d}w}{\mathrm{d}t} &= -w - \left(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{40}w^4 + b_{31}w^3z + b_{22}w^2z^2 + b_{13}wz^3\right).\end{aligned} \tag{3.3}$$

Using the algorithm provided in page 7 of [33], we obtain that System (3.3) is time reversible if and only if the Sibirsky ideal I_S of System (3.3) vanishes, where

$$\begin{aligned}I_S :=\langle & a_{21} - b_{21}, a_{12}a_{30} - b_{12}b_{30}, a_{13}^2b_{12}^3 - a_{12}^3b_{13}^2, a_{13}^2a_{30}b_{12}^2 - a_{12}^2b_{13}^2b_{30}, a_{13}^2a_{30}^2b_{12} - a_{12}b_{13}^2b_{30}^2,\\ & a_{13}^2a_{30}^3 - b_{13}^2b_{30}^3, a_{12}a_{22}b_{13} - a_{13}b_{12}b_{22}, a_{22}b_{13}b_{30} - a_{13}a_{30}b_{22}, a_{13}a_{22}b_{12}^2 - a_{12}^2b_{13}b_{22},\\ & a_{13}a_{22}a_{30}b_{12} - a_{12}b_{13}b_{22}b_{30}, a_{13}a_{22}a_{30}^2 - b_{13}b_{22}b_{30}^2, a_{22}^2b_{12} - a_{12}b_{22}^2, a_{22}^2a_{30} - b_{22}^2b_{30},\\ & a_{22}^3b_{13} - a_{13}b_{22}^3, a_{12}^2a_{31}b_{13} - a_{13}b_{12}^2b_{31}, a_{12}a_{31}b_{13}b_{30} - a_{13}a_{30}b_{12}b_{31}, a_{31}b_{13}b_{30}^2 - a_{13}a_{30}^2b_{31},\\ & a_{13}a_{31}b_{12} - a_{12}b_{13}b_{31}, a_{13}a_{30}a_{31} - b_{13}b_{30}b_{31}, a_{12}a_{31}b_{22} - a_{22}b_{12}b_{31}, a_{31}b_{22}b_{30} - a_{22}a_{30}b_{31},\\ & a_{13}a_{31}b_{22}^2 - a_{22}^2b_{13}b_{31}, a_{22}a_{31} - b_{22}b_{31}, a_{12}a_{31}^2 - b_{12}b_{31}^2, a_{31}^2b_{30} - a_{30}b_{31}^2, a_{13}a_{31}^2b_{22} - a_{22}b_{13}b_{31}^2,\\ & a_{13}a_{31}^3 - b_{13}b_{31}^3\rangle.\end{aligned}$$

Since System (1.3) is a subfamily of System (3.3), we can deduce that the Sibirsky ideal $I_S^{(1.3)}$ of System (1.3) is of the following form

$$I_S^{(1.3)} := \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7\rangle,$$

where

$$\begin{aligned}&f_1 = a_{21} - b_{21}, && f_2 = n^2(b_{12}^3 - a_{12}^3), \quad f_3 = n(a_{12}^2a_{31} - b_{12}^2b_{31}),\\ &f_4 = n(a_{31}b_{12} - a_{12}b_{31}), && f_5 = n(a_{31}^3 - b_{31}^3), \quad f_6 = a_{12}a_{31}^2 - b_{12}b_{31}^2, \quad f_7 = a_{30}^3 - b_{30}^3.\end{aligned}$$

Now, we are in a position to prove the sufficiency of condition C_2. When condition C_2 is satisfied, one can check that $I_S^{(1.3)}$ vanishes. Similarly, we can verify that $I_S^{(1.3)}$ also vanishes when conditions C_i, $i = 3, 4, 5, 6$ are satisfied. Thus, System (1.3) is time reversible when conditions C_i, $i = 2, 3, 4, 5, 6$ hold.

In short, the sufficiency of Theorem 1.1 is proven. Therefore, we complete the proof of the theorem.

At the end of this section, we provide the center conditions for System (1.7) at the origin.

Corollary 1. *For $\delta = 0$, the origin of System* (1.7) *is a center if and only if one of the following seven conditions holds*

$$\begin{aligned}&C_1: \ B_{21} = A_{30} = B_{30} = A_{31} = B_{31} = 0,\\ &C_2: \ A_{30} = B_{30} = B_{21} = B_{12} = B_{31} = 0, \ A_{31}n \neq 0,\\ &C_3: \ A_{30} = B_{30} = B_{21} = 3A_{31}^2 - B_{31}^2 = A_{12}B_{31} - A_{31}B_{12} = 0, \ n(A_{31}^2 + B_{31}^2) \neq 0,\\ &C_4: \ A_{30} = B_{30} = B_{21} = n = 0, \ A_{31}^2B_{12} + 2A_{12}A_{31}B_{31} - B_{12}B_{31}^2 = 0, \ A_{31}^2 + B_{31}^2 \neq 0,\\ &C_5: \ B_{21} = B_{12} = B_{30} = B_{31} = 0, \ A_{30} \neq 0,\\ &C_6: \ B_{21} = 0, \ A_{30}B_{12} + A_{12}B_{30} = A_{31}B_{30} + A_{30}B_{31} = 3A_{30}^2 - B_{30}^2 = 0, \ A_{30}^2 + B_{30}^2 \neq 0,\\ &C_7: \ B_{21} = 0, \ A_{30}B_{12} + A_{12}B_{30} = A_{31}B_{30} + A_{30}B_{31} = n = 0,\\ &\qquad 2A_{12}A_{31} - A_{30}(A_{12} + A_{30}) + B_{30}^2 - B_{12}(B_{30} + 2B_{31}) = 0,\\ &\qquad A_{30}(B_{12} - 2B_{30}) - 2A_{31}B_{12} - A_{12}(B_{30} + 2B_{31}) = 0, \ (A_{30}^2 + B_{30}^2)(A_{12}^2 + B_{12}^2) \neq 0.\end{aligned}$$

4. Proof of Theorem 1.2

Having determined the origin of System (1.3) to be a complex center, in this section, we focus on finding the parametric conditions under which this center can be a complex isochronous center when the condition C_i $(i = 1, 2, \cdots, 7)$ is satisfied. The key to finding the parametric conditions for the origin of System (1.3) is to deduce the expressions of the complex period quantity (equivalently, period quantity), and to decompose the variety of the ideal composed of these complex period quantities.

Proof of Theorem 1.2. First, we prove the necessity of the theorem. When condition C_1 in Theorem 1.1 holds, by using the recursive formulas of [29, Theorem 3.1], we can derive the first three complex period quantities for System (1.3) as follows

$$\tau_1 = 2b_{21}, \quad \tau_2 = -2a_{12}b_{12}, \quad \tau_3 = -2n^2.$$

Taking $b_{12} = \overline{a_{12}}$ into account, we have $b_{21} = a_{12} = b_{12} = n = 0$ by setting $\tau_1 = \tau_2 = \tau_3 = 0$. Furthermore, in this case, we have $\tau_i = 0$ for $i = 4, 5, \cdots, 9$. It follows that if condition $\mathcal{I}_1$ in Theorem 1.2 holds, then the origin of System (1.3) is a weak center of an order at least 10.

When condition C_2 holds, combining with the **Case 1.2** of Lemma (3.1) shows that condition C_2 is equivalent to the following condition:

$$a_{21} = b_{21}, \quad a_{30} = b_{30} = 0, \quad a_{12} = pa_{31}, \quad b_{12} = pb_{31}, \quad a_{31} = b_{31}, \quad a_{31}b_{31}n \neq 0.$$

Therefore, the first three complex period quantities for System (1.3) are as follows

$$\tau_1 = 2b_{21}, \quad \tau_2 = -2b_{31}^2p^2, \quad \tau_3 = 2(b_{31} - n)(b_{31} + n).$$

It follows from $\tau_1 = \tau_2 = \tau_3 = 0$ that either (i) $b_{21} = p = b_{31} - n = 0$ or (ii) $b_{21} = p = b_{31} + n = 0$. For the case of (i), we obtain that $\tau_4 = \cdots = \tau_8 = 0$, $\tau_9 = -8n^6 \neq 0$ because $b_{31}n \neq 0$ in condition C_2. It means that the origin of System (1.3) cannot be a complex isochronous center. For the case of (ii), we obtain that $\tau_i = 0$ for $i = 4, 5, \cdots, 9$. Hence, we also conclude that the origin of System (1.3) is a weak center of an order at least 10. Similarly, we can also deduce that condition $\mathcal{I}_3$ in Theorem 1.2 is a candidate complex isochronous center condition under condition C_3.

When condition C_4 is satisfied, we claim that the origin of System (1.3) cannot be a complex isochronous center. Under condition C_4, let $a_{12} = hb_{31}^2$, $b_{12} = ha_{31}^2$, and the first three complex period quantities for system (1.3) are as follows

$$\tau_1 = 2b_{21}, \quad \tau_2 = -2a_{31}^2b_{31}^2h^2, \quad \tau_3 = 2a_{31}b_{31}.$$

Taking $a_{31}b_{31} \neq 0$ in condition C_4 into account, we obtain that $\tau_3 \neq 0$. Therefore, the claim is proven. Similarly, we can conclude that the origin of System (1.3) can not be a complex isochronous center when condition C_7 holds.

When condition C_5 holds, combining with the **Case 2** of Lemma (3.1) yields that condition C_5 is equivalent to the following condition

$$a_{21} = b_{21}, \quad a_{12} = qb_{30}, \quad b_{12} = qa_{30}, \quad a_{31} = sb_{30}, \quad b_{31} = sa_{30}, \quad a_{30} = b_{30}, \quad a_{30}b_{30} \neq 0.$$

Furthermore, the first two complex period quantities for system (1.3) are as follows

$$\tau_1 = 2b_{21}, \quad \tau_2 = -2b_{30}^2 q(1+q).$$

Together with $b_{30} \neq 0$, we have either (i) $b_{21} = q = 0$ or (ii) $b_{21} = q + 1 = 0$ when $\tau_1 = \tau_2 = 0$.

For the case of (i), we calculate the following three complex period quantities, yielding the following

$$\tau_3 = 2(-2b_{30}^2 - n^2 - b_{30}^2 s + b_{30}^2 s^2), \quad \tau_4 = \frac{2}{3} b_{30}^2 (3b_{30} + 8n - 12b_{30}s),$$
$$\tau_5 = -\frac{1}{12} b_{30}^2 (-196b_{30}^2 + 7b_{30}n - 42n^2 - 66b_{30}^2 s - 48b_{30}ns + 90b_{30}^2 s^2).$$

Taking $\tau_4 = 0$, we obtain that $n = \frac{1}{8}(12b_{30}s - 3b_{30})$. Using this condition to simplify τ_3 and τ_5, we have

$$\tau_3 = -\frac{1}{32} b_{30}^2 \tau_{30}, \quad \tau_5 = -\frac{1}{384} b_{30}^4 \tau_{50},$$

where

$$\tau_{30} := 137 - 8s + 80s^2, \qquad \tau_{50} := -6545 + 312s - 2448s^2.$$

Let $\tau_{30} = 0$; then, $s = \frac{1}{20}(1 \pm 6\sqrt{19}\mathbf{i})$. Substituting these values into τ_{50} yields $\tau_{50} \neq 0$, which indicates that τ_{30} and τ_{50} do not have common zeros. It follows that τ_3 and τ_5 do not have common zeros, which implies that the origin of System (1.3) cannot be a complex isochronous center in this case.

For the case of (ii), we calculate the following two complex period quantities, yielding the following

$$\tau_3 = -2(2b_{30}^2 + n^2 + b_{30}^2 s - b_{30}^2 s^2), \quad \tau_4 = -4b_{30}^2 (3b_{30} + 4n + 3b_{30}s).$$

Since $b_{30} \neq 0$ in condition C_5, we have $n = -\frac{3}{4} b_{30}(1+s)$ by solving $\tau_4 = 0$ for n. Using this condition to simplify τ_3, we get

$$\tau_3 = \frac{1}{8} b_{30}^2 (1+s)(41 - 7s).$$

It follows from $\tau_3 = 0$ and together with $b_{30} \neq 0$ that either $1 + s = 0$, or $41 - 7s = 0$. Furthermore, we have $\tau_i = 0$ for $i = 5, 6, \cdots, 9$ when $1 + s = 0$ holds, and $\tau_5 = -\frac{1992}{49} b_{30}^4 \neq 0$ when $41 - 7s = 0$ holds. Therefore, condition $\mathcal{I}_4$ in Theorem 1.2 is also a candidate necessary condition for the origin of System (1.3) to be a complex isochronous center. Similarly, we also conclude that condition $\mathcal{I}_5$ in Theorem 1.2 is a candidate complex isochronous center condition under condition C_6.

In summary, conditions $\mathcal{I}_i$, $i = 1, 2, \cdots, 5$ in Theorem 1.2 are the candidate complex isochronous center conditions for System (1.3).

It what follows, we turn to prove the sufficiency of Theorem 1.2.

When condition $\mathcal{I}_1$ holds, System (1.3) is reduced to the linear one $\dot{z} = z$, $\dot{w} = -w$. It follows that the origin of System (1.3) is a complex isochronous center.

When conditions $\mathcal{I}_2$ and $\mathcal{I}_3$ are satisfied, System (1.3) becomes:

$$\frac{\mathrm{d}z}{\mathrm{d}T} = z(1 - nz^2 w + nw^3), \qquad \frac{\mathrm{d}w}{\mathrm{d}T} = -w(1 - nw^2 z + nz^3), \tag{4.1}$$

and

$$\frac{dz}{dT} = z\left(1 + \left(1 \pm \mathbf{i}\sqrt{3}\right) a z^2 w + 2a w^3\right), \qquad \frac{dw}{dT} = -w\left(1 + \left(1 \mp \mathbf{i}\sqrt{3}\right) a w^2 z + 2a z^3\right), \tag{4.2}$$

respectively. We can check that Systems (4.1) and (4.2) are both subfamilies of the following system,

$$\frac{dz}{dT} = z\left(1 + \frac{b_3^2}{a_1} z^2 w - a_1 w^3\right), \qquad \frac{dw}{dT} = -w\left(1 + \frac{b_3^2}{a_1^2} z^3 - b_3 w^2 z\right),$$

which is linearizable as indicated in [14, p.1531]. Thus, under conditions $\mathcal{I}_2$ and $\mathcal{I}_3$, the origin of System (1.3) is a complex isochronous center.

When condition $\mathcal{I}_4$ holds, System (1.3) becomes:

$$\begin{aligned} \frac{dz}{dT} &= \quad z + b_{30} z^3 - b_{30} z w^2 - b_{30} z^3 w + b_{30} z^2 w^2, \\ \frac{dw}{dT} &= -w - \left(b_{30} w^3 - b_{30} w z^2 - b_{30} w^3 z + b_{30} w^2 z^2\right). \end{aligned} \tag{4.3}$$

Using the change of variables (1.5) and $x = r\cos\theta, y = r\sin\theta$, System (4.3) is written as:

$$\frac{dr}{dt} = 2b_{30} r^3 (\sin\theta - \sin 2\theta), \qquad \frac{d\theta}{dt} = 1.$$

It follows that the origin is a complex isochronous center for System (4.3), because the origin of System (4.3) is a center under condition $\mathcal{I}_4$, which is a subfamily of center condition C_5.

Similar to the proof of the sufficiency of condition $\mathcal{I}_4$, we can obtain that the origin is a complex isochronous center for System (1.3) under condition $\mathcal{I}_5$.

Thus, the proof of the sufficiency of the theorem is completed, and thus we complete the proof of Theorem 1.2.

Finally, we provide the isochronous center conditions for System (1.7).

Corollary 2. *For $\delta = 0$, the origin of System* (1.7) *is an isochronous center if and only if one of the following five conditions holds,*

$\mathcal{I}_1 :\ A_{30} = A_{21} = A_{12} = A_{31} = B_{30} = B_{21} = B_{12} = B_{31} = n = 0,$

$\mathcal{I}_2 :\ A_{30} = A_{21} = A_{12} = B_{30} = B_{21} = B_{12} = B_{31} = A_{31} + n = 0, n \neq 0,$

$\mathcal{I}_3 :\ A_{30} = A_{12} = A_{21} = B_{30} = B_{21} = B_{12} = 3A_{31}^2 - B_{31}^2 = 2A_{31} - n = 0, n \neq 0,$

$\mathcal{I}_4 :\ A_{21} = B_{30} = B_{21} = B_{12} = B_{31} = n = A_{12} + A_{30} = A_{31} + A_{30} = 0, A_{30} \neq 0,$

$\mathcal{I}_5 :\ A_{21} = B_{21} = n = A_{12} + A_{30} = A_{31} + A_{30} = B_{12} + B_{30} = B_{31} + B_{30} = 3A_{30}^2 - B_{30}^2 = 0, A_{30} \neq 0.$

5. Proof of Theorem 1.3

In the proof of Lemma 3.2, we know that in the case of **Case 2**, the maximal order for the origin of System (1.3) to be a weak singular point is at most 10. It follows that the maximal order for the origin of System (1.7) to be a weak focus is at most 10, which means that the maximal number of limit cycles that bifurcate from the weak focus is at most 10. Therefore, in this section, under the case of

Case 2, we first derive the parametric conditions under which the origin of System (1.3) becomes a weak singular point of order 10. Correspondingly, the origin of System (1.7) is a weak focus of order 10. Then, although the independence of the focus quantities are not satisfied, we will still prove that exactly 10 limit cycles bifurcate from the weak focus of order 10.

Lemma 5.1. *The origin of System* (1.3) *is a tenth-order weak singular point if and only if* $\delta = 0$ *and one of the following two families of conditions is satisfied,*

$$\begin{aligned}
\mathcal{F}_1^w: &\ a_{21} = b_{21} = \frac{8n-125}{40}, \quad a_{12} = \frac{1}{5}b_{30}, \quad b_{12} = \frac{1}{5}a_{30}, \quad a_{31} = 3b_{30}, \quad b_{31} = 3a_{30}, \quad n \neq 0, \\
&\ \mu_{70}(n, a_{30}, b_{30}) = \mu_{80}(n, a_{30}, b_{30}) = \mu_{90}(n, a_{30}, b_{30}) = 0, \quad a_{30} - b_{30} \neq 0, \quad a_{30}^2 + a_{30}b_{30} + b_{30}^2 \neq 0, \\
\mathcal{F}_2^w: &\ a_{21} = b_{21} = -\frac{n+250}{30}, \quad a_{12} = -\frac{1}{5}b_{30}, \quad b_{12} = -\frac{1}{5}a_{30}, \quad a_{31} = -2b_{30}, \quad b_{31} = -2a_{30}, \quad n \neq 0, \\
&\ \widetilde{\mu}_{70}(n, a_{30}, b_{30}) = \widetilde{\mu}_{80}(n, a_{30}, b_{30}) = \widetilde{\mu}_{90}(n, a_{30}, b_{30}) = 0, \quad a_{30} - b_{30} \neq 0, \quad a_{30}^2 + a_{30}b_{30} + b_{30}^2 \neq 0,
\end{aligned}$$

where the expressions of μ_{j0} *and* $\widetilde{\mu}_{j0}$*,* $j = 7, 8, 9, 10$*, are given in the Appendix. Correspondingly, when the variables of System* (1.3) *are complex conjugates, i.e.,* $w = \bar{z}$*, the origin of System* (1.7) *is a weak focus of order* 10 *if and only if* $\delta = 0$ *and one of the conditions* $\mathcal{F}_1^w$ *and* $\mathcal{F}_2^w$ *holds.*

Proof. From the proof of Lemma 3.2, we can conclude that conditions $\mathcal{F}_1^w$ and $\mathcal{F}_2^w$ are the necessary and sufficient conditions for the origin of System (1.3), which is a weak singular point of an order at most 10. In what follows, we claim that this maximal number can be realized. More precisely, one needs to prove that there exist parametric conditions such that $\mu_1 = \mu_2 = \cdots = \mu_8 = \mu_9 = 0$, and $\mu_{10} \neq 0$. As indicated in the proof of Lemma 3.2, the above claim holds for complex domains. In the following, we will show that the above claim also holds for real domains. To achieve this, we first prove that $\mu_{70} = \mu_{80} = \mu_{90} = 0$ and $\widetilde{\mu}_{70} = \widetilde{\mu}_{80} = \widetilde{\mu}_{90} = 0$ have common zeros in real domains. We only present the detailed proof for $\mu_{70} = \mu_{80} = \mu_{90} = 0$ vanishing in real domains, and in a similar way, we can conclude that $\widetilde{\mu}_{70} = \widetilde{\mu}_{80} = \widetilde{\mu}_{90} = 0$ vanishing in real domains. Since $b_{ij} = \overline{a_{ij}}$, we let $a_{30} = A_{30} + B_{30}\mathbf{i}$ and $b_{30} = A_{30} - B_{30}\mathbf{i}$. Then, applying the command

$$\text{NSolve}[\{\mu_{70} == 0, \mu_{80} == 0, \mu_{90} == 0\}, \{A_{30}, B_{30}, n\}, 50]$$

in Mathematica, we get six families of real solutions for A_{30}, B_{30}, and n, one of which for these real solutions with 50 digits precision is given as follows,

$$\begin{aligned}
n^* &= -2.4792533314854845208097125591059376607101481562540, \\
{A_{30}}^* &= -0.54628513252937335542148550168262333706352139652\,66, \\
{B_{30}}^* &= -1.9696143105786187333150807889138760477461373761488.
\end{aligned} \tag{5.1}$$

For the other coefficients of System (1.3), based on the case of **Case 2.1**, we can solve $\mu_1 = \mu_2 = \cdots = \mu_6 = 0$ for a_{21}, a_{12}, b_{12}, a_{31}, b_{31}, s, q, and b_{21}, and their relations of which are expressed in conditions $\mathcal{F}_1^w$. Therefore, we can conclude that there exists parametric values of the parameters of System (1.3) that satisfies $\mu_i = 0 (i = 1, 2, \cdots, 9)$ and $\mu_{10} \neq 0$; then, the origin is a tenth-order weak singular point if condition $\mathcal{F}_1^w$ holds. Similarly, the conclusion also holds for condition $\mathcal{F}_2^w$.

Thus, the proof of Lemma 5.1 is completed.

After proving the reachability of the tenth-order weak focus, we proceed to demonstrate that there exist parametric conditions under which ten limit cycles can bifurcate from this weak focus.

Proof of Theorem 1.3. To demonstrate that ten limit cycles bifurcate from the origin of System (1.7), we only need to find the perturbed parameter values of System (1.7) from condition $\mathcal{F}_1^w$ because the case of condition $\mathcal{F}_2^w$ is similar. As noted in [34], to achieve this goal, we further need to determine the maximum number of independent sign changes in the displacement function near this weak focus, which can be obtained by verifying the independence of the focus quantities. More precisely, we need to verify that under conditions $\mathcal{F}_1^w$ and $\mathcal{F}_2^w$, the Jacobian determinant $\mathcal{J}_0$ of $v_1, v_3, v_5, \cdots, v_{19}$ with respect to parameters δ, A_{30}, B_{30}, A_{12}, B_{12}, A_{21}, B_{21}, A_{31}, B_{31}, and n is different from zero, where $v_1 = e^{2\pi\delta}$, $v_{2k+1} = \mathbf{i}\pi\mu_k$, and μ_k is given in Lemma 3.1. However, straightforward calculations show that $\mathcal{J}_0 = 0$ when the conditions $\mathcal{F}_1^w$ and $\mathcal{F}_2^w$ are satisfied. Therefore, the focus quantities v_1, v_3, $\cdots$, v_{19} are dependent since $\mathcal{J}_0$ vanishes under the conditions $\mathcal{F}_1^w$ and $\mathcal{F}_2^w$. Namely, the independence condition is not satisfied. Motivated by the idea of [35], we apply a more systematic method to determine the maximum number of independent sign changes in the displacement function near this weak focus. First, we assume that the first ten focal quantities of System (1.7) at the origin are perturbed as follows

$$\begin{aligned}
v_1(2\pi,\epsilon,\delta) &= e^{2\pi\delta} = 1 + c_1\pi\epsilon^{20} + o(\epsilon^{20}), & v_3(2\pi,\epsilon,\delta) &= c_2\pi\epsilon^{18} + o(\epsilon^{18}),\\
v_5(2\pi,\epsilon,\delta) &= c_3\pi\epsilon^{16} + o(\epsilon^{16}), & v_7(2\pi,\epsilon,\delta) &= c_4\pi\epsilon^{14} + o(\epsilon^{14}),\\
v_9(2\pi,\epsilon,\delta) &= c_5\pi\epsilon^{12} + o(\epsilon^{12}), & v_{11}(2\pi,\epsilon,\delta) &= c_6\pi\epsilon^{10} + o(\epsilon^{10}),\\
v_{13}(2\pi,\epsilon,\delta) &= c_7\pi\epsilon^{8} + o(\epsilon^{8}), & v_{15}(2\pi,\epsilon,\delta) &= c_8\pi\epsilon^{6} + o(\epsilon^{6}),\\
v_{17}(2\pi,\epsilon,\delta) &= c_9\pi\epsilon^{4} + o(\epsilon^{4}), & v_{19}(2\pi,\epsilon,\delta) &= c_{10}\pi\epsilon^{2} + o(\epsilon^{2}),\\
v_{21}(2\pi,\epsilon,\delta) &= v_{21}|_{\epsilon=0} + o(1) = j_0 + o(1), & &
\end{aligned} \tag{5.2}$$

where $0 < \epsilon \ll 1$ and $c_i (i = 1, 2, \cdots, 10)$ are real numbers such that $0 < |v_{2i-1}| \ll |v_{2i+1}|$ and $v_{2i-1}v_{2i+1} < 0$. It follows that we have the following displacement function:

$$\begin{aligned}
d(\epsilon h) &= r(2\pi, \epsilon h) - \epsilon h\\
&= (v_1(2\pi,\epsilon,\delta) - 1)\,\epsilon h + v_2(2\pi,\epsilon,\delta)(\epsilon h)^2 + v_3(2\pi,\epsilon,\delta)(\epsilon h)^3 + \cdots + v_{21}(2\pi,\epsilon,\delta)(\epsilon h)^{21} + \cdots\\
&= \pi\epsilon^{21}h[g(h) + \epsilon h G(h,\epsilon)],
\end{aligned} \tag{5.3}$$

where $G(h,\epsilon)$ is analytic at $(0,0)$. Taking the fact that $v_{2k} \in \langle v_1, v_3, \cdots, v_{2k-1}\rangle$ into account, we have the following:

$$g(h) = c_1 + c_2h^2 + c_3h^4 + c_4h^6 + c_5h^8 + c_6h^{10} + c_7h^{12} + c_8h^{14} + c_9h^{16} + c_{10}h^{18} + \frac{1}{\pi}j_0h^{20}. \tag{5.4}$$

For the existence of ten limit cycles, the equation $g(h) = 0$ in Eq (5.4) should have ten positive zeros of h. Without a loss of generality, we assume that

$$g(h) = \frac{1}{\pi}j_0(h^2-1)(h^2-2)(h^2-3)(h^2-4)(h^2-5)(h^2-6)(h^2-7)(h^2-8)(h^2-9)(h^2-10).$$

Then, taking Eq (5.4) into account, we obtain that

$$c_1 = \frac{3628800 j_0}{\pi}, \quad c_2 = \frac{-10628640 j_0}{\pi}, \quad c_3 = \frac{12753576 j_0}{\pi}, \quad c_4 = \frac{-8409500 j_0}{\pi},$$
$$c_5 = \frac{3416930 j_0}{\pi}, \quad c_6 = \frac{-902055 j_0}{\pi}, \quad c_7 = \frac{157773 j_0}{\pi}, \quad c_8 = \frac{-18150 j_0}{\pi},$$
$$c_9 = \frac{1320 j_0}{\pi}, \quad c_{10} = \frac{-55 j_0}{\pi}.$$

From Eqs (3.1), (5.1), and (1.8), we get $j_0 = -67996.2129945366$, and the focal quantities v_{2k+1} in Eq (5.2) become

$$\begin{aligned} v_1 - 1 &= K_1 \epsilon_0^{10} + o(\epsilon_0^{10}), \quad v_3 = K_2 \epsilon_0^9 + o(\epsilon_0^9), \quad v_5 = K_3 \epsilon_0^8 + o(\epsilon_0^8), \quad v_7 = K_4 \epsilon_0^7 + o(\epsilon_0^7), \\ v_9 &= K_5 \epsilon_0^6 + o(\epsilon_0^6), \quad v_{11} = K_6 \epsilon_0^5 + o(\epsilon_0^5), \quad v_{13} = K_7 \epsilon_0^4 + o(\epsilon_0^4), \quad v_{15} = K_8 \epsilon_0^3 + o(\epsilon_0^3), \\ v_{17} &= K_9 \epsilon_0^2 + o(\epsilon_0^2), \quad v_{19} = K_{10} \epsilon_0 + o(\epsilon_0), \quad v_{21} = K_{11} + o(1), \quad \epsilon_0 = \epsilon^2, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} &K_1 = -2.4674465771 \times 10^{11}, \ K_2 = 7.2270726928 \times 10^{11}, \quad K_3 = -8.6719487014 \times 10^{11}, \\ &K_4 = 5.7181415317 \times 10^{11}, \quad K_5 = -2.3233830007 \times 10^{11}, \ K_6 = 6.1336323913 \times 10^{10}, \\ &K_7 = -1.0727966513 \times 10^{10}, \ K_8 = 1.2341312659 \times 10^{9}, \quad K_9 = -8.9755001153 \times 10^{7}, \\ &K_{10} = 3.7397917147 \times 10^{6}, \quad K_{11} = -67996.2129945366. \end{aligned}$$

Next, we present how to determine the perturbations on the parameter values of System (1.7) step by step. From **Case 2.1** of Lemma 3.1, we can linearly solve $v_1 - 1 = e^{2\pi\delta} - 1 = 0$ and $\mu_k = 0 (k = 2, 3, 4, 5, 6)$ for $\delta, b_{21}, a_{12}, b_{12}, a_{31}, b_{31}, s, q$, and a_{21}. Correspondingly, we can find proper perturbations on δ for v_1, B_{21} for v_3, A_{12} and B_{12} for v_5, A_{31} and B_{31} for v_7, and A_{21} for v_{13} in Eq (5.5) using the Relation (1.8) and $b_{ij} = \overline{a_{ij}} = A_{ij} + \mathbf{i} B_{ij} \, (ij = 30, 21, 12, 31)$. Thus, we first determine the perturbed parameters n, A_{30}, and B_{30} from the equations $v_{15} = v_{17} = v_{19} = 0$, where $v_{2k+1} = \mathbf{i}\mu_k, k = 7, 8, 9$, and μ_ks are given in **Case 2.1** of Lemma 3.1. Without a loss of generality, we assume that

$$\begin{aligned} A_{30} &= {A_{30}}^* + k_{11}\epsilon_0 + k_{12}\epsilon_0^2 + k_{13}\epsilon_0^3, \\ B_{30} &= {B_{30}}^* + k_{21}\epsilon_0 + k_{22}\epsilon_0^2 + k_{23}\epsilon_0^3, \\ n &= n^* + k_{31}\epsilon_0 + k_{32}\epsilon_0^2 + k_{33}\epsilon_0^3, \end{aligned} \tag{5.6}$$

where n^*, ${A_{30}}^*$, and ${B_{30}}^*$, which are critical values such that $v_{15} = v_{17} = v_{19} = 0$ are given in Eq (5.1). Then, we substitute Eq (5.6) into v_{15}, v_{17}, and v_{19}, and obtain that by expanding them in the Taylor series up to the ϵ_0^3-order as follows:

$$v_{19} = e_{10} + e_{11}\epsilon_0 + o(\epsilon_0), \quad v_{17} = e_{20} + e_{21}\epsilon_0 + e_{22}\epsilon_0^2 + o(\epsilon_0^2), \quad v_{15} = e_{30} + e_{31}\epsilon_0 + e_{32}\epsilon_0^2 + e_{33}\epsilon_0^3 + o(\epsilon_0^3),$$

where e_{ij} are functions of n, A_{30}, and B_{30}, as well as functions of $k_{ij}, i, j = 1, 2, 3$. From Eq (5.5), we equate the coefficients of corresponding powers of ϵ_0, yielding the following:

$$\begin{aligned} &e_{10}(n, A_{30}, B_{30}) = e_{20}(n, A_{30}, B_{30}) = e_{21}(n, A_{30}, B_{30}) = 0, \\ &e_{30}(n, A_{30}, B_{30}) = e_{31}(n, A_{30}, B_{30}) = e_{32}(n, A_{30}, B_{30}) = 0, \\ &e_{11}(n, A_{30}, B_{30}) = K_{10} = 3.7397917147 \times 10^{6}, \\ &e_{22}(n, A_{30}, B_{30}) = K_9 = -8.9755001153 \times 10^{7}, \\ &e_{33}(n, A_{30}, B_{30}) = K_8 = 1.2341312659 \times 10^{9}. \end{aligned} \tag{5.7}$$

Then, substituting Eq (5.6) into Eq (5.7), and then solving them for k_{ij}, we have

$$\begin{aligned}
&k_{12} = k_{13} = k_{23} = 0, \quad && k_{11} = -\,509.2252922536, \quad && k_{21} = 141.2368933207,\\
&k_{22} = 38917.8125355654, \quad && k_{31} = 1.0715693315\times 10^{-98}, &&\\
&k_{32} = 20498.7276794495, \quad && k_{33} = -\,4.4430981781\times 10^{6}. &&
\end{aligned}$$

Therefore, the perturbed values for parameters n, A_{30}, and B_{30} are derived from Eq (5.6) as follows

$$\begin{aligned}
n &= -\,2.4792533315 + 1.0715693314\times 10^{-98}\epsilon_0\\
&\quad + 20498.7276794495\epsilon_0^2 - 4.4430981781\times 10^{6}\epsilon_0^3,\\
A_{30} &= -\,0.5462851325 - 509.2252922536\epsilon_0,\\
B_{30} &= -\,1.9696143106 + 141.2368933207\epsilon_0 + 38917.8125355654\epsilon_0^2.
\end{aligned} \tag{5.8}$$

Now, we turn to determine the perturbed value for parameter A_{21} from $v_{13} = \mathbf{i}\mu_6$ and $b_{ij} = \overline{a_{ij}} = \overline{A_{ij} + \mathbf{i}B_{ij}}\,(ij = 30, 21, 12, 31)$, where μ_6 is given in Eq (3.1). Then, we solve $v_{13} = 0$ for the critical value, denoted by ${A_{21}}^*$, of parameter A_{21} as follows

$${A_{21}}^* = -3.6208506663.$$

Therefore, we assume that

$$A_{21} = {A_{21}}^* + k_{41}\epsilon_0 + k_{42}\epsilon_0^2 + k_{43}\epsilon_0^3 + k_{44}\epsilon_0^4. \tag{5.9}$$

By substituting Eq (5.9) into v_{13} and expanding it in the Taylor series up to the ϵ_0^4-order, we have

$$v_{13} = e_{40} + e_{41}\epsilon_0 + e_{42}\epsilon_0^2 + e_{43}\epsilon_0^3 + e_{44}\epsilon_0^4 + o(\epsilon_0^4),$$

where e_{ij} are functions of A_{21}, as well as functions of k_{4j}, $j = 1, 2, 3, 4$. From Eq (5.5), we equate the coefficients of corresponding powers of ϵ_0, yielding

$$e_{40} = e_{41} = e_{42} = e_{43} = 0, \quad e_{44} = K_7 = -1.0727966513\times 10^{10}.$$

Substituting Eq (5.9) into the above equations, and then solving for k_{4j}, we have

$$k_{41} = k_{42} = k_{43} = 0, \quad k_{44} = 7.3232245633\times 10^{7}.$$

Thus, A_{21} can be written as

$$\begin{aligned}
A_{21} &= -\,3.6208506663 + 2.1431386629\times 10^{-99}\epsilon_0 + 4099.7455358899\epsilon_0^2\\
&\quad - 888619.6356280381\epsilon_0^3 + 7.3232245633\times 10^{7}\epsilon_0^4.
\end{aligned} \tag{5.10}$$

Similarly, we can determine the perturbed values for the remaining coefficients of System (1.7) as follows,

$$\begin{aligned}
\delta =& -3.9270631957\times 10^{10}\epsilon_0^{10}, \quad B_{21} = -1.1502243431\times 10^{11}\epsilon_0^{9},\\
B_{12} =& 0.3939228621 - 28.2473786641\epsilon_0 - 7783.5625071131\epsilon_0^2\\
& - 2.1111730955\times 10^{8}\epsilon_0^5 + 1.5138777560\times 10^{10}\epsilon_0^6\\
& + 4.1714887184\times 10^{12}\epsilon_0^7 + 2.5264892267\times 10^{11}\epsilon_0^8,\\
A_{12} =& -0.1092570265 - 101.8450584507\epsilon_0\\
& + 5.8554736737\times 10^{7}\epsilon_0^5 + 5.4582398737\times 10^{10}\epsilon_0^6,\\
B_{31} =& 5.9088429317 - 423.7106799620\epsilon_0 - 116753.4376066961\epsilon_0^2\\
& + 2.6389663694\times 10^{9}\epsilon_0^5 + 2.2021375112\times 10^{11}\epsilon_0^6\\
& - 5.0088760458\times 10^{13}\epsilon_0^7,\\
A_{31} =& -1.6388553976 - 1527.6758767607\epsilon_0 - 7.3193420921\times 10^{8}\epsilon_0^5\\
& - 6.7368775156\times 10^{11}\epsilon_0^6 + 8.0093378406\times 10^{12}\epsilon_0^7.
\end{aligned} \tag{5.11}$$

By using the Relation (1.8), and the perturbed values of Parameters (5.8), (5.10), and (5.11), we can derive the perturbed focal quantities of System (1.7) at the origin as follows:

$$\begin{aligned}
&v_1 - 1 = -2.4674465771\times 10^{11}\epsilon_0^{10} + o(\epsilon_0^{10}),\\
&v_3 = 7.2270726928\times 10^{11}\epsilon_0^9 + o(\epsilon_0^9), && v_5 = -8.6719487014\times 10^{11}\epsilon_0^8 + o(\epsilon_0^8),\\
&v_7 = 5.7181415317\times 10^{11}\epsilon_0^7 + o(\epsilon_0^7), && v_9 = -2.3233830007\times 10^{11}\epsilon_0^6 + o(\epsilon_0^6),\\
&v_{11} = 6.1336323913\times 10^{10}\epsilon_0^5 + o(\epsilon_0^5), && v_{13} = -1.0727966513\times 10^{10}\epsilon_0^4 + o(\epsilon_0^4),\\
&v_{15} = 1.2341312659\times 10^{9}\epsilon_0^3 + o(\epsilon_0^3), && v_{17} = -8.9755001153\times 10^{7}\epsilon_0^2 + o(\epsilon_0^2),\\
&v_{19} = 3.7397917147\times 10^{6}\epsilon_0 + o(\epsilon_0), && v_{21} = -67996.2129945366 + o(1).
\end{aligned} \tag{5.12}$$

It follows that the maximum number of independent sign changes in the displacement function near this weak focus is ten. Therefore, the positive zeros of the displacement function $d(\epsilon h)$ in Eq (5.3) are sufficiently close to $\sqrt{k}$, $k = 1, 2, \cdots, 10$. Then, we can conclude that System (1.7) has ten small-amplitude limit cycles near the circles $x^2 + y^2 = k\epsilon^2$. Thus, we complete the proof of Theorem 1.3.

Finally, in accordance with Theorem 1.3, we present a numerical example of System (1.7) that has exactly ten limit cycles bifurcating from the 10th-order weak focus. Let $\epsilon = \frac{1}{20000\sqrt{5}}$ (i.e., $\epsilon_0 = \frac{1}{2000000000}$). Then, the perturbed values of parameters for $\delta, A_{30}, B_{30}, A_{12}, B_{12}, A_{21}, B_{21}, A_{31}, B_{31}$, and n can be derived by substituting $\epsilon = \frac{1}{20000\sqrt{5}}$ into their expressions in Theorem 1.3; we omit them here for simplicity. By substituting $\epsilon_0 = \frac{1}{2000000000}$ into Eq (5.12), we obtain the perturbed focal quantities as follows:

$$\begin{aligned}
&v_1 - 1 = -2.4674465771\times 10^{-82},\\
&v_3 = 1.4115376353\times 10^{-72}, && v_5 = -3.3874799614\times 10^{-63},\\
&v_7 = 4.4672980716\times 10^{-54}, && v_9 = -3.6302859385\times 10^{-45},\\
&v_{11} = 1.9167601222\times 10^{-36}, && v_{13} = -6.70497907\times 10^{-28},\\
&v_{15} = 1.54266408\times 10^{-19}, && v_{17} = -2.2438750288\times 10^{-11},\\
&v_{19} = 0.0018698958573, && v_{21} = -67996.2129945366,
\end{aligned}$$

for which Eq (5.3) has ten positive zeros as follows:

$h_1 \approx 0.000022361$, $h_2 \approx 0.000031623$, $h_3 \approx 0.000038730$, $h_4 \approx 0.000044723$, $h_5 \approx 0.000049985$,
$h_6 \approx 0.000054863$, $h_7 \approx 0.000058917$, $h_8 \approx 0.000063642$, $h_9 \approx 0.000066767$, $h_{10} \approx 0.000070796$,

as expected.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Methodology, Z.H. and J.L.; software, Z.H.; validation, J.L.; formal analysis, Z.H.; investigation, Z.H. and J.L.; writing-original draft preparation, Z.H.; writing-review and editing, J.L.; supervision, J.L.; funding acquisition, Z.H. All authors have read and agreed to the published version of the manuscript.

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