Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(1): 147–169.
DOI: 10.3934/nhm.2026006
Received: 06 October 2025
Revised: 30 December 2025
Accepted: 05 January 2026
Published: 22 January 2026

Research article

A penalty-free *hp*-version mixed discontinuous Galerkin method for the biharmonic equation

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Abstract: This paper focused on an *hp*-version mixed discontinuous Galerkin method without penalty terms for the biharmonic equation with Navier boundary conditions. By introducing the auxiliary variable $v = u_{xx}$, we reduced the fourth-order problem to a second-order system and derived its penalty-free variational formulation. The analysis replaces the standard coercivity condition with a polynomial-degree-dependent inf-sup condition for the bilinear form $B(\cdot,\cdot)$. While h-convergence rates for both u_h and v_h were optimal, the p-convergence exhibited contrasting behavior: suboptimal in L^2-norm but optimal in the energy norm, regardless of the p^2 scaling in the inf-sup condition. Numerical results revealed that p-convergence order-doubling for boundary-aligned singularities significantly enhanced the efficacy for singular solutions. Furthermore, the method was shown to extend to nonlinear biharmonic equations, while the treatment of Dirichlet boundary conditions necessitated the introduction of penalty terms.

Keywords: *hp*-version error estimate; mixed discontinuous Galerkin method; biharmonic equation; inf-sup condition; without penalty term

1. Introduction

Consider *hp*-version mixed discontinuous Galerkin method without penalty term to solve the following one-dimensional biharmonic equation,

$$-u_{xxxx} = f(x), \quad \text{in}\,\Omega, \tag{1.1}$$

$$u(a) = \alpha_1, \qquad u(b) = \alpha_2, \tag{1.2}$$

$$u_{xx}(a) = \beta_1, \qquad u_{xx}(b) = \beta_2, \tag{1.3}$$

where $f(x)$ is a given function in $L^2(\Omega)$, $\Omega = \{x|a < x < b; a, b \in \mathbb{R}\}$, α_i, $\beta_i \in \mathbb{R}$, $i = 1, 2$. The systems (1.1)–(1.3) have been widely used in physics, biology, dynamic system, and other fields, such as the modeling of thin beams and thin plates, strain gradient elasticity, heat convection, and phase separation in binary mixtures [1–3].

In recent years, the discontinuous Galerkin (DG) method has attracted significant attention for its ability to handle discontinuous solutions and its inherent adaptivity (see, e.g., [4, 5]). For biharmonic problems, DG approaches generally fall into two main categories. The first is the interior penalty DG method, which operates directly on the original equation [6–9]. This method employs discontinuous finite element spaces and enforces both C^0- and C^1-continuity conditions weakly through interior penalties. A nonsymmetric variant was introduced in [6]. Later work [8] analyzed symmetric, nonsymmetric, and hybrid formulations while deriving a priori error estimates.

Due to the structure of its boundary conditions, the biharmonic problem can be fully decomposed into two Poisson equations. This observation has motivated the development of mixed methods, such as mixed finite element methods [10] and mixed DG methods [11–13]. However, a coercive bilinear form in the variational formulation of the mixed DG approach typically requires the introduction of penalty terms. While penalty coefficients must be chosen sufficiently large to satisfy theoretical conditions, their practical selection is often nontrivial and poses implementation challenges. In contrast, mixed finite element methods for the biharmonic problem with Navier boundary conditions do not require penalty terms [10].

To address this issue for one-dimensional Poisson equations, Burman et al. [14] proposed a symmetric DG method without penalty terms. Rather than relying on coercivity, they established well-posedness and derived error estimates via an inf-sup condition. Riviere and Sardar [15] applied a penalty-free DG method to incompressible Navier-Stokes equations. Gao et al. [16] improved a conforming DG method without interior penalty terms, enabling it to handle nonhomogeneous Dirichlet boundary conditions. Jaśkowiec and Sukumar [17] introduced a new high-order DG method for Poisson problem that requires neither penalty nor stabilization parameters. Liu and Yin [18, 19] developed a mixed discontinuous Galerkin method without interior penalty for time-dependent fourth-order problems, extending it to nonlinear Swift-Hohenberg equations [19] and a class of fourth-order gradient fows [20]. Wang and Zhang [21] proposed an ultraweak-local discontinuous Galerkin method for nonlinear biharmonic Schrödinger equations, demonstrating unconditional stability without any penalty terms. In Section 4.1 of [20], a mixed DG method without interior penalty was proposed for fourth-order elliptic partial differential equations (PDEs), and the authors proved the existence of a unique solution to the discrete problem, though error estimates were not provided. Inspired by these works, we extend this penalty-free mixed DG approach to biharmonic equations.

Recent years have also seen growing interest in *hp*-version analyses of existing DG methods for second-order linear elliptic PDEs [9, 22–27]. The ability to handle discontinuous finite element functions with locally varying approximation order seven on irregular meshes with hanging nodes offers notable flexibility and computational convenience. For smooth problems with local singularities, for instance, *hp*-adaptive spaces can be tailored to the solution behavior, yielding high-order algebraic or even exponential convergence rates [23]. Inspired by these works, we extend this penalty-free mixed DG approach to biharmonic equations and further establish *hp*-version error estimates.

In this paper, we use penalty-free mixed DG approach to biharmonic equations and further establish *hp*-version error estimates. A key point of these estimates is inf-sup condition based on mesh size h

and degree p of polynomial, i.e.,

$$|||w_h||| \le C_t \hat{p}^2 \sup_{0 \neq q_h \in V_h^{\mathbf{p}}} \frac{B(w_h, q_h)}{|||q_h|||}, \quad \forall w_h \in V_h^{\mathbf{p}}.$$

Referring to [14], we present a rigorous proof. Due to the application of the trace theorem, it is impossible to eliminate the factor of $\hat{p}^2$ related to the degree of the polynomial in the coefficients. While the h-convergence rates for both u_h and v_h are optimal in both the L^2-norm and the energy norm, the p-convergence is only optimal in the energy norm; it is suboptimal in the L^2-norm. We also apply the method numerically to a fourth-order problem featuring a nonlinear reaction term. The results show convergence rates consistent without the reaction term case. It is also observed that under Dirichlet boundary conditions, the discrete scheme must be supplemented with a penalty term on the boundary.

The structure of this paper is as follows: In Section 2, the variational problem of the mixed DG method without penalty terms is obtained, inf-sup condition dependent on the polynomial degree p are presented, and the well-posedness of the variational problem is proved. In Section 3, the error estimates under the energy norm and L^2 norm concerned with the mesh size h and polynomial degree p are analyzed. Finally, in Section 4, numerical examples are provided.

2. Mixed DG method without penalty terms and its variation

We apply an hp-version mixed DG method without penalty terms for the biharmonic equations (1.1)–(1.3). Rewrite the problems (1.1)–(1.3) into the following second-order system by introducing an auxiliary variable $v = u_{xx}$, i.e.,

$$-u_{xx} + v = 0, \qquad \text{in } \Omega, \tag{2.1}$$

$$-v_{xx} = f(x), \qquad \text{in } \Omega, \tag{2.2}$$

$$u(a) = \alpha_1, \quad u(b) = \alpha_2, \tag{2.3}$$

$$v(a) = \beta_1, \quad v(b) = \beta_2, \tag{2.4}$$

where $\Omega = \{x | a < x < b\}$. If $f(x) \in L^2(\Omega)$, there exists a unique solution $u \in H^4(\Omega)$ to Eqs (1.1)–(1.3) and then $v \in H^2(\Omega)$ (see [28]).

2.1. Finite element spaces

Let $a = x_0 < x_1 < \cdots < x_N = b$ be a partition of the interval (a, b). Denote $I_n = (x_{n-1}, x_n), h_n = x_n - x_{n-1}, h = \max_{1 \le n \le N} h_n$. Assume that there exist two positive constants κ_1, κ_2 such that

$$\kappa_2 \le h_i / h_n \le \kappa_1, \quad i, n = 1, 2, \cdots, N.$$

Denote the set of all intervals $I_n, n = 1, 2, \cdots, N$ by $\mathcal{T}_h$, the set of all nodes $x_n, n = 0, 1, \cdots, N$ by $\mathcal{N}_h$, and the set of all interior nodes $x_n, n = 1, 2, \cdots, N-1$ by $\mathcal{N}_h^i$. Define the broken Sobolev space

$$H^s(\Omega, \mathcal{T}_h) = \{w \in L^2(\Omega) : w|_{I_n} \in H^s(I_n), \ \forall I_n \in \mathcal{T}_h\},$$

and the discontinuous finite element space

$$V_h^{\mathbf{p}} = \{w \in L^2(\Omega) : w|_{I_n} \in P_{p_n}(I_n), \ \forall I_n \in \mathcal{T}_h\},$$

where $P_{p_n}(I_n)$ denotes the polynomial space of degree at most p_n on element I_n. Set $\mathbf{p} = (p_n : I_n \in \mathcal{T}_h)$, $\hat{p} = \max_{1\le n\le N} p_n$, $\check{p} = \min_{1\le n\le N} p_n$. Assume that there exist positive constants σ_1, σ_2 such that

$$\sigma_2 \le p_i/p_n \le \sigma_1, \quad i, n = 1, 2, \cdots, N.$$

In order to facilitate numerical analysis, during this article, we denote $\|\cdot\|_E$ as the L^2 norm and $(\cdot,\cdot)_E$ as its inner product. We set $\|\cdot\|_{s,E}$ and $|\cdot|_{s,E}$ as the norm and seminorm of classical Sobolev space $H^s(E)$, $s > 0$, respectively. For any function $w \in H^s(\Omega, \mathcal{T}_h)$, $s \ge 1$, define its jump and average at interior nodes

$$[\![w]\!]_n = w(x_n^-) - w(x_n^+), \quad \{\!\{w\}\!\}_n = \frac{1}{2}\left(w(x_n^-) + w(x_n^+)\right).$$

On boundary nodes, we set $[\![w]\!]_0 = -w(a)$, $\{\!\{w\}\!\}_0 = w(a)$, $[\![w]\!]_N = w(b)$, $\{\!\{w\}\!\}_N = w(b)$. For any $w \in H^s(\Omega, \mathcal{T}_h)$, define

$$|||w|||_h := \left(\sum_{I_n\in\mathcal{T}_h} \|w'\|_{I_n}^2 + \sum_{x_n\in\mathcal{N}_h} \frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n} [\![w]\!]_n^2 \right)^{1/2}, \tag{2.5}$$

where $\{\!\{h\}\!\}_n = (h_n + h_{n+1})/2$ and $\{\!\{p\}\!\}_n = (p_n + p_{n+1})/2$. It can be proven that this is a norm in $H^1(\Omega)$.

2.2. Weak variational form of the model

Multiplying Eqs (2.1) and (2.2) by any sufficiently smooth functions w, q, respectively, integrating over the interval I_n, using integration by parts, and summing up over all intervals I_n, we obtain

$$\sum_{I_n\in\mathcal{T}_h} \int_{I_n} u_x w_x dx - \sum_{x_n\in\mathcal{N}_h} \{\!\{u_x\}\!\}_n [\![w]\!]_n - \sum_{x_n\in\mathcal{N}_h^i} [\![u_x]\!]_n \{\!\{w\}\!\}_n + (v, w)_\Omega = 0, \tag{2.6}$$

$$\sum_{I_n\in\mathcal{T}_h} \int_{I_n} v_x q_x dx - \sum_{x_n\in\mathcal{N}_h} \{\!\{v_x\}\!\}_n [\![q]\!]_n - \sum_{x_n\in\mathcal{N}_h^i} [\![v_x]\!]_n \{\!\{q\}\!\}_n = (f, q)_\Omega. \tag{2.7}$$

If $u, v \in H^2(\Omega)$, then

$$[\![u]\!]_n = 0, [\![u_x]\!]_n = 0, [\![v]\!]_n = 0, [\![v_x]\!]_n = 0, \quad \forall x_n \in \mathcal{N}_h^i. \tag{2.8}$$

In order to keep the symmetry of the elliptic equation, adding two items $-\sum_{x_n\in\mathcal{N}_h} [\![u]\!]_n \{\!\{w_x\}\!\}_n$ and $-\sum_{x_n\in\mathcal{N}_h} [\![v]\!]_n \{\!\{q_x\}\!\}_n$ to Eqs (2.6) and (2.7), respectively, then using Eq (2.8) and boundary conditions (2.3) and (2.4), we have the following problem, i.e., find $u, v \in H^s(\Omega, \mathcal{T}_h)$ such that

$$B(w, u) + (v, w)_\Omega = L_1(w), \quad \forall w \in H^s(\Omega, \mathcal{T}_h), \tag{2.9}$$

$$B(v, q) = L_2(q), \quad \forall q \in H^s(\Omega, \mathcal{T}_h), \tag{2.10}$$

where bilinear form $B(w, q)$ and $L_1(w), L_2(q)$ are given by

$$B(w, q) = \sum_{I_n\in\mathcal{T}_h} \int_{I_n} w_x q_x dx - \sum_{x_n\in\mathcal{N}_h} \left(\{\!\{w_x\}\!\}_n [\![q]\!]_n + [\![w]\!]_n \{\!\{q_x\}\!\}_n \right), \tag{2.11}$$

and

$$L_1(w) = \alpha_1 w_x(a) - \alpha_2 w_x(b), \quad L_2(q) = \int_\Omega fq dx + \beta_1 q_x(a) - \beta_2 q_x(b). \tag{2.12}$$

Hence, the discrete variational form of the original problem, i.e., find $u_h, v_h \in V_h^{\mathbf{p}}$ such that

$$\begin{aligned} B(w_h, u_h) + (v_h, w_h)_\Omega &= L_1(w_h), \quad \forall w_h \in V_h^{\mathbf{p}}, &(2.13)\\ B(v_h, q_h) &= L_2(q_h), \quad \forall q_h \in V_h^{\mathbf{p}}. &(2.14)\end{aligned}$$

Following the inference above, we may obtain the following orthogonality of the solution (u, v) and (u_h, v_h).

Theorem 2.1. *Assume that u is the exact solution of the problems (1.1)–(1.3) such that $u \in H^s(\Omega)$, $s \geq 4$. Let $v = u_{xx}$. Assume that $(u_h, v_h) \in V_h^{\mathbf{p}} \times V_h^{\mathbf{p}}$ is the solution of the problems (2.13) and (2.14). We have the following orthogonality equations*

$$\begin{aligned} B(w_h, u - u_h) + (v - v_h, w_h)_\Omega &= 0, \quad \forall w_h \in V_h^{\mathbf{p}}, &(2.15)\\ B(v - v_h, q_h) &= 0, \quad \forall q_h \in V_h^{\mathbf{p}}. &(2.16)\end{aligned}$$

2.3. Well-posedness of the discrete variational problem

In this section, we will present the well-posedness of the discrete variational problems (2.13) and (2.14). We first list the trace inequality and inverse inequality of the space $V_h^{\mathbf{p}}$ and $H^s(\Omega, \mathcal{T}_h)$ referred to [29, 30], then prove the continuity and inf-sup condition of the bilinear form $B(\cdot, \cdot)$, and finally provide well-posedness of the discrete variational problem.

Lemma 2.1. *For $w \in H^s(\Omega, \mathcal{T}_h)$, $s \geq 1$, there exists a positive constant C_a independent of h_n and w such that*

$$|w(x)|^2 \leq C_a \left(\frac{1}{h_n} \|w\|_{I_n}^2 + \|w\|_{I_n} \|w_x\|_{I_n} \right), \quad \forall x \in \{x_{n-1}, x_n\}.$$

Lemma 2.2. *For $w \in P_{p_n}(I_n)$, there exist positive constants C_b, C_d independent of h_n and p_n such that*

$$|w(x)| \leq C_b p_n h_n^{-1/2} \|w\|_{I_n}, \quad \forall x \in \{x_{n-1}, x_n\},$$

$$\|w_x\|_{I_n} \leq C_d p_n^2 h_n^{-1} \|w\|_{I_n}.$$

In the following, we give the inf-sup condition of the bilinear form $B(\cdot, \cdot)$. The inf-sup condition is crucial for the well-posed of the discrete variational problem and error estimates. However, different from the inf-sup condition [14], the condition we provide is related to the degree $\hat{p}$ of polynomials. The idea of the proof comes from [14].

Lemma 2.3. *Assumed that $p_n \geq 2$. There exists a positive constant C_t independent of all h_n and $p_n, n = 1, 2, \cdots, N$ such that*

$$|||w_h||| \leq C_t \hat{p}^2 \sup_{0 \neq q_h \in V_h^{\mathbf{p}}} \frac{B(w_h, q_h)}{|||q_h|||}, \quad \forall w_h \in V_h^{\mathbf{p}}. \tag{2.17}$$

Proof. The proof proceeds by constructing, for any given w_h, a function of the form $q_h = w_h - \lambda y_h$ such that

$$\frac{B(w_h, q_h)}{|||q_h|||} \geq C \, ||| \, w_h \, ||| \, . \tag{2.18}$$

We first construct a suitable y_h satisfying $|||y_h||| \leq C \, ||| \, w_h|||$, and then select an appropriate constant λ to ensure that inequality $|||q_h||| \leq C \, ||| \, w_h||| $ holds.

Assume that $y_h \in V_h^{\mathbf{p}}$ satisfies

$$\begin{cases} (y_h, z_h)_\Omega = 0, & \forall z_h \in V_h^{\mathbf{p}-2}, \\ \{\!\{(y_h)_x\}\!\}_n = \frac{\{\!\{p\}\!\}_n}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n, & \forall n \in \{0, 1, \cdots, N\}, \\ \{\!\{y_h\}\!\}_n = 0, & \forall n \in \{1, \cdots, N-1\}, \end{cases} \tag{2.19}$$

where $\mathbf{p} - 2 = (p_n - 2 : n = 1, 2, \cdots, N)$. Now we prove that Eq (2.19) has a unique solution y_h such that $|||y_h||| \leq C \, ||| \, w_h|||$. Because the problem (2.19) is actually a linear system of equations with a square coefficient matrix, the existence of its solution is equivalent to uniqueness. Letting $z_h = 1, x \in I_n; z_h = 0, x \notin I_n$ in the first equation of Eq (2.19), we have $\int_{I_n} y_h dx = 0$, i.e., y_h has zero mean in every interval I_n. Hence, y_h has at least one zero point in I_n and denote one of the zeros as ξ_n. Then, the Cauchy-Schwarz inequality implies that

$$|y_h(x)|^2 = |y_h(x) - y_h(\xi_n)|^2 = \left| \int_{\xi_n}^{x} (y_h)_x dt \right|^2 \leq h_n \|(y_h)_x\|_{I_n}^2, \quad \forall x \in I_n.$$

Thus, we have

$$\sum_{x_n \in \mathcal{N}_h} \frac{1}{\{\!\{h\}\!\}_n} [\![y_h]\!]_n^2 \leq 8 \sum_{I_n \in \mathcal{T}_h} \|(y_h)_x\|_{I_n}^2. \tag{2.20}$$

Integration by parts and Eq (2.19) satisfies

$$\begin{aligned} \sum_{I_n \in \mathcal{T}_h} \|(y_h)_x\|_{I_n}^2 &= -\sum_{I_n \in \mathcal{T}_h} (y_h, (y_h)_{xx})_{I_n} + \sum_{x_n \in \mathcal{N}_h} \{\!\{(y_h)_x\}\!\}_n [\![y_h]\!]_n + \sum_{x_n \in \mathcal{N}_h^i} \{\!\{y_h\}\!\}_n [\![(y_h)_x]\!]_n \\ &= \sum_{x_n \in \mathcal{N}_h} \frac{\{\!\{p\}\!\}_n}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n [\![(y_h)_x]\!]_n \leq \left(\sum_{x_n \in \mathcal{N}_h} \frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n^2 \sum_{x_n \in \mathcal{N}_h} \frac{1}{\{\!\{h\}\!\}_n} [\![y_h]\!]_n^2 \right)^{1/2} \\ &\leq 2\sqrt{2} \left(\sum_{x_n \in \mathcal{N}_h} \frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n^2 \sum_{I_n \in \mathcal{T}_h} \|(y_h)_x\|_{I_n}^2 \right)^{1/2}, \end{aligned}$$

i.e.,

$$\sum_{I_n \in \mathcal{T}_h} \|(y_h)_x\|_{I_n}^2 \leq 8 \sum_{x_n \in \mathcal{N}_h} \frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n^2. \tag{2.21}$$

From Eqs (2.21) and (2.20), we know $y_h \equiv 0$ if $[\![w_h]\!]_n = 0, n = 0, 1, \cdots, N$, i.e., the solution of the problem (2.19) is unique. Therefore, we obtain the existence of the solution y_h in Eq (2.19). Combining Eqs (2.21) and (2.20), then we can obtain

$$|||y_h|||^2 \leq (8\hat{p}^2 + 1) \sum_{I_n \in \mathcal{T}_h} \|(y_h)_x\|_{I_n}^2 \leq 8(8\hat{p}^2 + 1) \sum_{x_n \in \mathcal{N}_h} \frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n} [\![w_h]\!]_n^2 \leq 8(8\hat{p}^2 + 1) \, ||| \, w_h \, |||^2 \, . \tag{2.22}$$

Similarly, we have

$$\sum_{I_n\in\mathcal{T}_h}\int_{I_n}(y_h)_x(w_h)_x dx \;=\; \sum_{x_n\in\mathcal{N}_h}\{\!\{(w_h)_x\}\!\}_n[\![y_h]\!]_n.$$

Thus, we can obtain

$$B(w_h,-y_h)=\sum_{x_n\in\mathcal{N}_h}\{\!\{(y_h)_x\}\!\}_n[\![w_h]\!]_n=\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n}{\{\!\{h\}\!\}_n}[\![w_h]\!]_n^2.$$

According to bilinear form Eq (2.11), it can be concluded that

$$B(w_h,w_h)=\sum_{I_n\in\mathcal{T}_h}\int_{I_n}(w_h)_x{}^2dx-2\sum_{x_n\in\mathcal{N}_h}\left(\{\!\{(w_h)_x\}\!\}_n[\![w_h]\!]_n\right).$$

Next, we will estimate the last term of the above equation. Using Lemma 2.2, we obtain

$$\frac{\sqrt{\{\!\{h\}\!\}_n}}{\{\!\{p\}\!\}_n}|\{\!\{(w_h)_x\}\!\}_n|\le C_b\sqrt{\kappa_1}\left(\|(w_h)_x\|_{I_n}+\|(w_h)_x\|_{I_{n+1}}\right)\le C_b\sqrt{2\kappa_1}\|(w_h)_x\|_{I_n\cup I_{n+1}}. \tag{2.23}$$

Young's inequality satisfies

$$\sum_{x_n\in\mathcal{N}_h}[\![w_h]\!]_n\{\!\{(w_h)_x\}\!\}_n \;\le\; \varepsilon C_b^2\kappa_1\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n}[\![w_h]\!]_n^2+\frac{1}{\varepsilon}\sum_{I_n\in\mathcal{T}_h}\|(w_h)_x\|_{I_n}^2.$$

Set $\varepsilon=2$ and $C_0=4C_b^2\kappa_1$, then it follows:

$$B(w_h,w_h)\ge\frac{1}{2}\sum_{I_n\in\mathcal{T}_h}\|(w_h)_x\|_{I_n}^2-C_0\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n}[\![w_h]\!]_n^2.$$

Using the above inequality, choosing $\lambda=(\frac{1}{2}+C_0)\hat{p}$, we obtain

$$B(w_h,w_h-\lambda y_h)\ge\frac{1}{2}\;|\!|\!|\;w_h\;|\!|\!|^2.$$

Triangle inequality and Eq (2.22) implies that

$$|\!|\!|w_h-\lambda y_h|\!|\!|\le(1+\lambda(8\hat{p}+3))\;|\!|\!|\;w_h|\!|\!|\le 6(1+C_0)\hat{p}^2\;|\!|\!|\;w_h\;|\!|\!|.$$

Therefore, we have

$$\sup_{0\ne q_h\in V_h^{\mathbf{p}}}\frac{B(w_h,q_h)}{|\!|\!|q_h|\!|\!|}\ge\frac{B(w_h,w_h-\lambda y_h)}{|\!|\!|w_h-\lambda y_h|\!|\!|}\ge\frac{1}{C_t\hat{p}^2}\;|\!|\!|\;w_h|\!|\!|,$$

where $C_t=12(1+C_0)$. It completes the proof of Lemma 2.3.

Remark. In Eq (2.17), the coefficient $\hat{p}^2$ arises from applying the trace theorem in Eq (2.23). Notably, the polynomial of degree p_n does not appear in the estimate given in Eq (2.20).

Theorem 2.2. *Under the assumption that $p_n \geq 2$, $n = 1, \cdots, N$, problems (2.13) and (2.14) possess a unique solution $(u_h, w_h) \in V_h^{\mathbf{p}}$.*

Proof. Because the existence of the solution of the problems (2.13) and (2.14) is equivalent to uniqueness, we only prove that the following homogeneous equation has only the trivial solution

$$B(w_h, u_h) + (v_h, w_h)_\Omega = 0, \quad \forall w_h \in V_h^{\mathbf{p}}, \tag{2.24}$$
$$B(v_h, q_h) = 0, \quad \forall q_h \in V_h^{\mathbf{p}}. \tag{2.25}$$

From Eqs (2.25) and (2.17), we can obtain $v_h \equiv 0$. Substituting v_h into Eq (2.25) and combining Eq (2.17) and the symmetry of the bilinear form $B(\cdot,\cdot)$, we have $u_h \equiv 0$.

3. *hp*-version prior error estimates

In this section, we carry out error estimates of the approximate solution (u_h, v_h) under the norm $L^2(\Omega)$ and the norm $|||\cdot|||$. We first list approximation properties of piecewise polynomials in the space V_h^k according to [31], then establish error estimates under all the norm.

Lemma 3.1. *For $w \in H^s(\Omega, \mathcal{T}_h)$ and $s \geq 1$, there exist a polynomial $\Pi_h^n w \in P_{p_n}(I_n)$ and a positive constant C_I independent of h_n and p_n such that*

$$\|w - \Pi_h^n w\|_{i,I_n} \leq C_I \frac{h_n^{\mu_n - i}}{p_n^{s-i}} |w|_{s,I_n}, \quad for\ 0 \leq i \leq s, \tag{3.1}$$

where $\mu_n = \min\{p_n + 1, s\}$.

As $w \in H^s(\Omega)$, define $\Pi_h : H^s(\Omega) \to V_h^{\mathbf{p}}, \Pi_h w|_{I_n} = \Pi_h^n w$.

Lemma 3.2. *Assume that $w \in H^s(\Omega)$, $s \geq 1$, then there exists a positive constant C_h independent of h_n and p_n such that*

$$|||w - \Pi_h w||| \leq C_h \left(\sum_{I_n \in \mathcal{T}_h} \frac{h_n^{2\mu_n - 2}}{p_n^{2s-3}} |w|_{s,I_n}^2 \right)^{1/2}, \tag{3.2}$$

with $\mu_n = \min\{p_n + 1, s\}$.

Proof. Applying Lemmas 2.1 and 3.1, we have

$$|w - \Pi_h w|_{1,I_n}^2 \leq C_I^2 \frac{h_n^{2\mu_n - 2}}{p_n^{2s-2}} |w|_{s,I_n}^2,$$
$$[\![w - \Pi_h w]\!]_n^2 \leq 4C_a C_I^2 \frac{h_n^{2\mu_n - 1}}{p_n^{2s-1}} |w|_{s,I_n}^2 + 4C_a C_I^2 \frac{h_{n+1}^{2\mu_{n+1} - 1}}{p_{n+1}^{2s-1}} |w|_{s,I_{n+1}}^2,$$

where $\mu_n = min\{p_n + 1, s\}$. Summing up over all elements, we have

$$\sum_{I_n \in \mathcal{T}_h} \|(w - \Pi_h w)_x\|_{I_n}^2 \leq C_I^2 \sum_{I_n \in \mathcal{T}_h} \frac{h_n^{2\mu_n - 2}}{p_n^{2s-2}} |w|_{s,I_n}^2, \tag{3.3}$$

$$\begin{aligned}\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n}[\![w-\Pi_h w]\!]_n^2 &\le \sum_{x_n\in\mathcal{N}_h}4C_aC_I^2\left(\frac{h_n^{2\mu_n-2}}{p_n^{2s-1}}|w|_{s,I_n}^2+\frac{h_{n+1}^{2\mu_{n+1}-2}}{p_{n+1}^{2s-1}}|w|_{s,I_{n+1}}^2\right)\{\!\{p\}\!\}_n^2\\ &\le 16C_aC_I^2\sigma_1^2\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|w|_{s,I_n}^2.\end{aligned} \tag{3.4}$$

Taking $C_h = C_I\max\{1, 4\sqrt{C_a}\sigma_1\}$, we obtain the result (3.2).

Lemma 3.3. *Assume that $w \in H^s(\Omega)$, $s \ge 2$ and $q_h \in V_h^{\mathbf{p}}$. Then, there exists a positive constant C_f independent of h_n and p_n such that*

$$|B(\Pi_h w - w, q_h)| \le C_f\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|w|_{s,I_n}^2\right)^{1/2} |||\, q_h\, ||| . \tag{3.5}$$

Proof. Let $\chi = \Pi_h w - w$. Based on the definition of $B(\cdot,\cdot)$, Cauchy-Schwarz inequality, and Young's inequality, we have

$$\left|\sum_{I_n\in\mathcal{T}_h}(\chi_x,(q_h)_x)_{I_n}\right| \le \left(\sum_{I_n\in\mathcal{T}_h}\|\chi_x\|_{I_n}^2\right)^{1/2}\left(\sum_{I_n\in\mathcal{T}_h}\|(q_h)_x\|_{I_n}^2\right)^{1/2},$$

$$\left|\sum_{x_n\in\mathcal{N}_h}[\![\chi]\!]_n\{\!\{(q_h)_x\}\!\}_n\right| \le \left(\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n}[\![\chi]\!]_n^2\right)^{1/2}\left(\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{h\}\!\}_n}{\{\!\{p\}\!\}_n^2}\{\!\{(q_h)_x\}\!\}_n^2\right)^{1/2},$$

$$\left|\sum_{x_n\in\mathcal{N}_h}\{\!\{\chi_x\}\!\}_n[\![q_h]\!]_n\right| \le \left(\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{h\}\!\}_n}{\{\!\{p\}\!\}_n^2}\{\!\{\chi_x\}\!\}_n^2\right)^{1/2}\left(\sum_{x_n\in\mathcal{N}_h}\frac{\{\!\{p\}\!\}_n^2}{\{\!\{h\}\!\}_n}[\![q_h]\!]_n^2\right)^{1/2}.$$

According to the trace inequality of Lemmas 2.1 and 2.2, and the interpolation polynomial approximation of Lemma 3.1, it can be obtained that

$$\begin{aligned}\{\!\{(q_h)_x\}\!\}_n^2 &\le \frac{1}{2}C_b^2p_n^2h_n^{-1}\|(q_h)_x\|_{I_n}^2+\frac{1}{2}C_b^2p_{n+1}^2h_{n+1}^{-1}\|(q_h)_x\|_{I_{n+1}}^2,\\ \{\!\{\chi_x\}\!\}_n^2 &\le C_aC_I^2\left(\frac{h_n^{2\mu_n-3}}{p_n^{2s-3}}|w|_{s,I_n}^2+\frac{h_{n+1}^{2\mu_{n+1}-3}}{p_{n+1}^{2s-3}}|w|_{s,I_{n+1}}^2\right).\end{aligned}$$

Based on the above estimates and Eqs (3.3) and (3.4), we have

$$\begin{aligned}|B(\chi,q_h)| &= \left|\sum_{I_n\in\mathcal{T}_h}(\chi_x,(q_h)_x)_{I_n}-\sum_{x_n\in\mathcal{N}_h}\left([\![\chi]\!]_n\{\!\{(q_h)_x\}\!\}_n+\{\!\{\chi_x\}\!\}_n[\![q_h]\!]_n\right)\right|\\ &\le C_f\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|w|_{s,I_n}^2\right)^{1/2}|||\, q_h|||,\end{aligned}$$

where $C_f = C_I\max\{1+8\sqrt{C_a\kappa_1}C_b\sigma_1, 2\sqrt{2C_a\kappa_1}\}$.

Next, we derive a priori error estimates.

Theorem 3.1. *Assume that u is the exact solution of the problems (1.1)–(1.3) such that $u \in H^s(\Omega)$, $s \geq 4$. Let $v = u_{xx}$. Assume that $(u_h, v_h) \in V_h^{\mathbf{p}} \times V_h^{\mathbf{p}}$ is the solution of the problems (2.13) and (2.14) and $p_n \geq 2$ for $n = 1, 2, \cdots, N$. Then, there is a positive constant C_g independent of h_n and p_n such that*

$$|||v - v_h||| \leq C_g \hat{p}^2 \left(\sum_{I_n \in \mathcal{T}_h} \frac{h_n^{2\bar{\mu}_n - 2}}{p_n^{2\bar{s}-3}} |v|^2_{\bar{s}, I_n} \right)^{1/2}, \tag{3.6}$$

where $\bar{\mu}_n = \min\{p_n + 1, \bar{s}\}$, $\bar{s} = s - 2$.

Proof. Let $v - v_h = (v - \Pi_h v) + (\Pi_h v - v_h)$. First, consider the estimate of $|||\Pi_h v - v_h|||$. According to Lemma 2.3 and orthogonality (2.16), it can be obtained that

$$\begin{aligned} |||\Pi_h v - v_h||| &\leq C_t \hat{p}^2 \sup_{0 \neq w_h \in V_h^{\mathbf{p}}} \frac{B(\Pi_h v - v_h, w_h)}{|||w_h|||} \\ &\leq C_t \hat{p}^2 \sup_{0 \neq w_h \in V_h^{\mathbf{p}}} \frac{B(\Pi_h v - v, w_h) + B(v - v_h, w_h)}{|||w_h|||} \\ &\leq C_t \hat{p}^2 \sup_{0 \neq w_h \in V_h^{\mathbf{p}}} \frac{B(\Pi_h v - v, w_h)}{|||w_h|||}. \end{aligned} \tag{3.7}$$

With Eqs (3.5) and (3.2) and the triangle inequality, we obtain the estimate Eq (3.6) with $C_g = 2\max\{1, C_t C_f\}$.

Next, we carry out the optimal L^2 norm error estimate of v with the standard duality argument. We define auxiliary function φ to be the solution of the following adjoint problem:

$$-\varphi_{xx} = v - v_h \doteq e_v, \quad \text{in } \Omega, \tag{3.8}$$

$$\varphi(a) = 0, \qquad \varphi(b) = 0. \tag{3.9}$$

According to solution regularity, $\varphi \in H^2(\Omega)$ such that

$$\|\varphi\|_{2,\Omega} \leq C_g \|v - v_h\|_{\Omega}. \tag{3.10}$$

Theorem 3.2. *Under the same assumption as Theorem 3.1, there is a positive constant C_i independent of h_n and p_n such that*

$$\|v - v_h\|_{\Omega} \leq C_i h \left(\sum_{I_n \in \mathcal{T}_h} \frac{h_n^{2\bar{\mu}_n - 2}}{p_n^{2\bar{s}-2}} |v|^2_{\bar{s}, I_n} \right)^{1/2}, \tag{3.11}$$

where $\bar{\mu}_n = \min\{p_n + 1, \bar{s}\}$, $\bar{s} = s - 2$.

Proof. Let φ be the solution of Eqs (3.8) and (3.9) and $\Pi_h \varphi \in V_h^{\mathbf{p}}$ be such that the estimate Eq (3.1). Because $\varphi \in H^2(\Omega)$, then we have $[\![\varphi_x]\!]_n = 0$ for $x_n \in \mathcal{N}_h^i$ and $[\![\varphi]\!]_n = 0$ for $x_n \in \mathcal{N}_h$. Now multiplying the adjoint problem (3.8) with $v - v_h$ and integrating over Ω, using integration by parts and Eqs (3.9) and (2.16), we obtain

$$\|v - v_h\|^2_{\Omega} = \sum_{I_n \in \mathcal{T}_h} \int_{I_n} \varphi_x (v - v_h)_x dx - \sum_{x_n \in \mathcal{N}_h} \{\!\{\varphi_x\}\!\}_n [\![v - v_h]\!]_n$$

$$= \sum_{I_n\in\mathcal{T}_h}\int_{I_n}\varphi_x(v-v_h)_x dx - \sum_{x_n\in\mathcal{N}_h}\left(\{\!\{\varphi_x\}\!\}_n[\![v-v_h]\!]_n + [\![\varphi]\!]_n\{\!\{(v-v_h)_x\}\!\}_n\right)$$
$$= B(\varphi, v-v_h) = B(\varphi-\Pi_h\varphi, v-v_h).$$

Applying the similar inferences in Lemma 3.3, we have

$$\sum_{I_n\in\mathcal{T}_h}\int_{I_n}(\varphi-\Pi_h\varphi)_x(v-v_h)_x dx - \sum_{x_n\in\mathcal{N}_h}\{\!\{(\varphi-\Pi_h\varphi)_x\}\!\}_n[\![v-v_h]\!]_n \le C_f\frac{h}{\check{p}}|\varphi|_{2,\Omega}\,|||\, v-v_h\,|||\,.$$

$$\left|\sum_{x_n\in\mathcal{N}_h}[\![\varphi-\Pi_h\varphi]\!]_n\{\!\{(v-\Pi_h v)_x\}\!\}_n\right| \le 4C_a^2C_I^2\frac{h^{3/2}}{\check{p}^{3/2}}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-3}}{p_n^{2\bar{s}-3}}|v|^2_{\bar{s},I_n}\right)^{1/2}|\varphi|_{2,\Omega},$$

and

$$\left|\sum_{x_n\in\mathcal{N}_h}[\![\varphi-\Pi_h\varphi]\!]_n\{\!\{(v_h-\Pi_h v)_x\}\!\}_n\right| \le 4C_bC_aC_I\kappa_1\sigma_1\frac{h}{\sqrt{\check{p}}}|\varphi|_{2,\Omega}\|(v_h-\Pi_h v)_x\|_{\Omega}.$$

Also, since

$$[\![\varphi-\Pi_h\varphi]\!]_n\{\!\{(v-v_h)_x\}\!\}_n = [\![\varphi-\Pi_h\varphi]\!]_n\{\!\{(v-\Pi_h v)_x + (\Pi_h v - v_h)_x\}\!\}_n,$$

thus, we have

$$\begin{aligned} B(\varphi-\Pi_h\varphi, v-v_h) &\le C_s\frac{h}{\check{p}^{1/2}}\,|||\, v-\Pi_h v\,|||\,|\varphi|_{2,\Omega} + 4C_a^2C_I^2\frac{h^{3/2}}{\check{p}^{3/2}}|\varphi|_{2,\Omega}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-3}}{p_n^{2\bar{s}-3}}|v|^2_{\bar{s},I_n}\right)^{1/2} \\ &\le 2\max\{C_sC_h\sqrt{\sigma_1}, 4C_a^2C_I^2\}h\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|v|^2_{\bar{s},I_n}\right)^{1/2}|\varphi|_{2,\Omega}, \end{aligned} \tag{3.12}$$

where $C_s = \max\{C_f, 4C_b^2C_a^2C_I^2\kappa_1\sigma_1\}$. Taking $C_i = 2C_g\max\{C_sC_h\sqrt{\sigma_1}, 4C_a^2C_I^2\}$ and using Eq (3.10), we obtain the estimate Eq (3.11).

Further, for the convenience of proving the next theorem, we will proceed to estimate $(v-v_h, w_h)$. Multiplying the problem (3.8) by w_h, then integrating over Ω and using Eq (3.9), we can obtain

$$(v-v_h, w_h)_\Omega = \sum_{I_n\in\mathcal{T}_h}\int_{I_n}\varphi_x(w_h)_x dx - \sum_{x_n\in\mathcal{N}_h}\{\!\{\varphi_x\}\!\}_n[\![w_h]\!]_n. \tag{3.13}$$

Using Cauchy-Schwarz inequality and Lemma 2.1, we have

$$|(v-v_h, w_h)| \le C_D\|v-v_h\|_\Omega\,|||\, w_h|||, \tag{3.14}$$

where $C_D = 1 + C_a\sqrt{\kappa_1}$.

Theorem 3.3. *Under the same assumption as Theorem 3.1, there is a positive constant C_m independent of h_n and p_n such that*

$$|||u-u_h||| \le C_m\hat{p}^2\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|u|^2_{s,I_n}\right)^{1/2} + C_m h\hat{p}^2\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|u|^2_{\bar{s}+2,I_n}\right)^{1/2}, \tag{3.15}$$

where $\mu_n = \min\{p_n+1, s\}, \bar{\mu}_n = \min\{p_n+1, \bar{s}\}, \bar{s} = s-2.$

Proof. First we consider $|||\Pi_h u - u_h|||$. According to Lemma 2.3 and orthogonality (2.16), it can be obtained that

$$|||\Pi_h u - u_h||| \quad \le \quad C_t\hat{p}^2 \sup_{0\neq w_h\in V_h^{\mathbf{p}}}\frac{B(\Pi_h u - u, w_h) - (v - v_h, w_h)}{|||w_h|||}. \tag{3.16}$$

The estimate (3.5) implies that

$$|B(\Pi_h u - u, w_h)| \le C_f\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|u|^2_{s,I_n}\right)^{1/2} |||\, w_h \,|||\,. \tag{3.17}$$

Inserting Eqs (3.14) and (3.17) into Eq (3.16), we obtain

$$|||\Pi_h u - u_h||| \quad \le \quad C_t\hat{p}^2\left(C_f\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-3}}|u|^2_{s,I_n}\right)^{1/2} + C_D\|v-v_h\|_\Omega\right). \tag{3.18}$$

Taking $C_m = \max\{C_tC_DC_i, C_h + C_tC_f\}$, combining the L^2 norm result of $(v - v_h)$ in Theorem 3.2, and the triangle inequality, we may obtain the result (3.15).

Next, we carry out the optimal L^2 norm error estimate of u with the standard duality argument. We define auxiliary function ϕ and ψ to be the solution of the following dual problem,

$$-\phi_{xx} + \psi = 0, \qquad \text{in } \Omega, \tag{3.19}$$

$$-\psi_{xx} = u - u_h, \qquad \text{in } \Omega, \tag{3.20}$$

$$\phi(a) = 0, \quad \phi(b) = 0, \tag{3.21}$$

$$\psi(a) = 0, \quad \psi(b) = 0. \tag{3.22}$$

Letting (ϕ,ψ) be the solution of the problems (3.19)–(3.22), we have $\phi \in H^4(\Omega)$, $\psi \in H^2(\Omega)$ such that

$$\|\phi\|_{4,\Omega} + \|\psi\|_{2,\Omega} \le C_H\|u-u_h\|_\Omega. \tag{3.23}$$

Theorem 3.4. *Under the same assumption as Theorem 3.1 and $p_n \ge 3, n = 1, 2, \cdots, N$, there is a positive constant C_M independent of h_n and p_n such that*

$$\|u-u_h\|_\Omega \quad \le \quad C_M h\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-2}}|u|^2_{s,I_n}\right)^{1/2} + C_M\frac{h^3}{\check{p}^2}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|u|^2_{\bar{s}+2,I_n}\right)^{1/2}, \tag{3.24}$$

where $\mu = \min\{p_n+1, s\}, \bar{\mu} = \min\{p_n+1, \bar{s}\}, \bar{s} = s-2.$

Proof. Let ϕ, ψ be the exact solution of the dual problems (3.19)–(3.22), and let $\Pi_h\psi, \Pi_h\phi \in V_h^{\mathbf{p}}$ satisfy the estimate (3.1). Since $\psi \in H^2(\Omega), \phi \in H^4(\Omega)$, we have $[\![\psi]\!]_n = 0, [\![\phi]\!]_n = 0$ for $x_n \in \mathcal{N}_h$. Multiplying two-sided of Eq (3.20) by $u - u_h$ and integrating over Ω, integrating by parts over each element I_n, we obtain

$$\begin{aligned}
\|u-u_h\|_\Omega^2 &= \sum_{I_n\in\mathcal{T}_h}\int_{I_n}\psi_x(u-u_h)_x dx - \sum_{x_n\in\mathcal{N}_h}\{\!\{\psi_x\}\!\}_n[\![(u-u_h)]\!]_n \\
&= \sum_{I_n\in\mathcal{T}_h}\int_{I_n}\psi_x(u-u_h)_x dx - \sum_{x_n\in\mathcal{N}_h}\left(\{\!\{\psi_x\}\!\}_n[\![(u-u_h)]\!]_n + [\![\psi]\!]_n\{\!\{(u-u_h)_x\}\!\}_n\right) \\
&= B(\psi, u-u_h).
\end{aligned}$$

Similarly, we obtain

$$B(\phi, v - v_h) + (\psi, v - v_h)_\Omega = 0. \tag{3.25}$$

Using the two equations above and the orthogonality of Eqs (2.15) and (2.16), we obtain

$$\|u-u_h\|_\Omega^2 = B(\psi - \Pi_h\psi, u-u_h) + (\psi - \Pi_h\psi, v - v_h)_\Omega + B(\phi - \Pi_h\phi, v - v_h). \tag{3.26}$$

With the estimate (3.12), we have

$$|B(\psi - \Pi_h\psi, u-u_h)| \le C_i h\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-2}}|u|_{s,I_n}^2\right)^{1/2}|\psi|_{2,\Omega}, \tag{3.27}$$

$$|B(\phi - \Pi_h\phi, v-v_h)| \le C_i \frac{h^3}{\check{p}^2}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|v|_{\bar{s},I_n}^2\right)^{1/2}|\phi|_{4,\Omega}, \tag{3.28}$$

and

$$|(\psi - \Pi_h\psi, v-v_h)_\Omega| \le C_I C_i \frac{h^3}{\check{p}^2}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|v|_{\bar{s},I_n}^2\right)^{1/2}|\psi|_{2,\Omega}. \tag{3.29}$$

Finally inserting Eqs (3.27)–(3.29) into Eq (3.26), using the estimate (3.23), and taking $C_M = C_H C_I C_i$, we obtain the estimate (3.24).

Remark. When $p_n = 2, n = 1, 2, \cdots, N$, the estimate (3.28) becomes

$$B(\phi - \Pi_h\phi, v-v_h) \le C_i \frac{h^2}{\check{p}}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|v|_{\bar{s},I_n}^2\right)^{1/2}|\phi|_{3,\Omega}.$$

Then, the estimate (3.24) will be

$$\|u-u_h\|_\Omega \le C_M h\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\mu_n-2}}{p_n^{2s-2}}|u|_{s,I_n}^2\right)^{1/2} + C_M\frac{h^2}{\check{p}}\left(\sum_{I_n\in\mathcal{T}_h}\frac{h_n^{2\bar{\mu}_n-2}}{p_n^{2\bar{s}-2}}|u|_{\bar{s}+2,I_n}^2\right)^{1/2}. \tag{3.30}$$

4. Numerical examples

The following examples are divided into two parts. Examples 4.1 and 4.2 serve as primary verification of the proposed method and the theoretical convergence rates established in Section 3. Furthermore, in response to suggestions received during the review process, we include Examples 4.3 and 4.4 as preliminary numerical investigations. These examples explore the method's performance with a nonlinear reaction term and under mixed boundary conditions, respectively. A comprehensive theoretical analysis for these extended cases is reserved for future work.

Noticing that for all problems presented in this section we have an analytical solution available such that the righthand function f and boundary conditions can be calculated. Let the polynomial degree $p_n = p, n = 1, 2, \cdots, N$. We use the following notations to denote the errors between exact solution and approximate solution

$$||e_u||_\infty := ||u - u_h||_\infty, \quad ||e_u||_\Omega := ||u - u_h||_\Omega, \quad |||e_u||| := |||u - u_h|||,$$
$$||e_v||_\infty := ||v - v_h||_\infty, \quad ||e_v||_\Omega := ||v - v_h||_\Omega, \quad |||e_v||| := |||v - v_h|||,$$

and use *order* to denote the convergent order of the mesh size h. Let $r(p)$ denote the convergent order of the polynomial degree and let $r(p)$ be defined by

$$r(p) = -\frac{\log(e_{p+1}) - \log(e_p)}{\log(p+1) - \log(p)},$$

where e_p is three norms of e_u or e_v for the polynomial degree p.

Example 4.1. *We consider problems (1.1)–(1.3) with* $\Omega = (0, 1)$. *Boundary conditions and the right-hand function* f *are given from the solution* $u(x) = \sin(12x)exp(1.5x)$. *The numerical results are shown in Tables 1 and 2.*

In Example 4.1, the order of convergence with respect to the mesh size h is checked. Tables 1 and 2 show the convergence orders of the error of u_h and v_h under the L^2 norm and energy norm with respect to the mesh size h are all $p + 1$ and p, respectively.

Table 1. Errors and orders of e_u for given polynomial degree p in Example 4.1.

p	h	$\|\|e_u\|\|_\infty$	order	$\|\|e_u\|\|_\Omega$	order	$\|\|\|e_u\|\|\|$	order
2	1/40	3.9425e-03		1.2802e-03		3.2743e-01	
	1/80	5.0031e-04	2.98	1.6532e-04	2.95	8.5788e-02	1.93
	1/160	6.2343e-05	3.00	2.0857e-05	2.98	2.1753e-02	1.98
3	1/40	3.3250e-05		5.8485e-06		2.1214e-03	
	1/80	2.0417e-06	4.02	3.4465e-07	4.08	2.5518e-04	3.05
	1/160	1.2642e-07	4.01	2.0947e-08	4.04	3.1389e-05	3.02
4	1/40	4.2878e-07		1.3791e-07		8.4598e-05	
	1/80	1.3660e-08	4.97	4.3698e-09	4.98	5.3781e-06	3.97
	1/160	4.2889e-10	4.99	1.3706e-10	4.99	3.3782e-07	3.99

Table 2. Errors and orders of e_v for given polynomial degree p in Example 4.1.

p	h	$\|\|e_v\|\|_\infty$	order	$\|\|e_v\|\|_\Omega$	order	$\|\|\|e_v\|\|\|$	order
2	1/40	5.6419e-01		1.8925e-01		4.9496e+01	
	1/80	7.3404e-02	2.94	2.4543e-02	2.94	1.2831e+01	1.95
	1/160	9.2698e-03	2.98	3.0993e-03	2.98	3.2418e+00	1.98
3	1/40	5.2497e-03		8.5629e-04		3.0852e-01	
	1/80	3.2609e-04	4.01	5.0060e-05	4.09	3.6871e-02	3.06
	1/160	2.0317e-05	4.00	3.0257e-06	4.05	4.5196e-03	3.03
4	1/40	6.4624e-05		2.0794e-05		1.2786e-02	
	1/80	2.0585e-06	4.97	6.5766e-07	4.98	8.1062e-04	3.98
	1/160	6.4732e-08	4.99	2.0618e-08	4.99	5.0860e-05	3.99

Example 4.2. *Assuming that the exact solution is $u(x) = x^{9/2}\cos(3x)$ and $\Omega = (0, 1)$. Then, the function $f(x)$ and boundary conditions can be calculated from $u(x)$. Since the solution $u(x) \in H^{5-\epsilon}(\Omega), \epsilon > 0$, we use the solution to check the convergence orders of the polynomial degree p under three norms. The corresponding results are put in Tables 3–5 and Figures 1 and 2.*

In Example 4.2, the exact solution is $u(x) \in H^s(\Omega)$, $s = 5 - \epsilon$, and then the singular $x = 0$ is the end point of an element. Tables 3 and 4 demonstrate that, for a given polynomial degree p, the approximate solution u_h exhibits convergence orders of $\min\{p+1, 5\}$ and $\min\{p+1, 5\}-1$ in the L^2-norm and energy norm, respectively, under h-refinement. Similarly, for v_h, the corresponding convergence orders are $\min\{p + 1, 3\}$ and $\min\{p + 1, 3\} - 1$. Table 5 shows the convergence orders of polynomial degree p for u_h under the L^2 norm and energy norm are more than 9 and 8, where $h = 1/40$, and those about p for v_h under the L^2 norm and energy norm are 5 and 4, respectively. However, if we ignore p^2 in inf-sup condition in theory, the convergence orders of p for u_h under the L^2 norm and energy norm are approximately 4 and 3.5, and those for v_h are approximately 2 and 1.5, respectively. The numerical results with respect to p have the order-doubling phenomenon [32].

Table 3. Errors and orders of e_u for given polynomial degree p in Example 4.2.

p	h	$\|\|e_u\|\|_\infty$	order	$\|\|e_u\|\|_\Omega$	order	$\|\|\|e_u\|\|\|$	order
2	1/40	3.3733e-05		1.0581e-05		2.6900e-03	
	1/80	4.7357e-06	2.83	1.3438e-06	2.98	6.9241e-04	1.96
	1/160	6.2636e-07	2.92	1.6874e-07	2.99	1.7520e-04	1.98
3	1/40	2.6358e-07		3.5828e-08		1.2556e-05	
	1/80	1.6668e-08	3.98	2.0643e-09	4.12	1.4948e-06	3.07
	1/160	1.0478e-09	3.99	1.2310e-10	4.07	1.8210e-07	3.04
4	1/40	1.2160e-09		4.2805e-10		2.6111e-07	
	1/80	3.8138e-11	4.99	1.3392e-11	4.99	1.6436e-08	3.98
	1/160	1.1933e-12	5.00	4.1867e-13	5.00	1.0304e-09	3.99
5	1/40	6.9358e-12		9.5536e-13		5.8920e-10	
	1/80	2.9660e-13	4.55	1.9870e-14	5.58	2.5028e-11	4.56
	1/160	1.2999e-14	4.51	5.3576e-16	5.21	1.3245e-12	4.24

Table 4. Errors and orders of e_v for given polynomial degree p in Example 4.2.

p	h	$\|\|e_v\|\|_\infty$	order	$\|\|e_v\|\|_\Omega$	order	$\|\|\|e_v\|\|\|$	order
2	1/40	1.6370e-03		6.0461e-04		1.5607e-01	
	1/80	2.0617e-04	2.99	7.5828e-05	2.99	3.9434e-02	1.98
	1/160	2.5829e-05	3.00	9.4879e-06	3.00	9.9013e-03	1.99
3	1/40	9.6014e-06		1.0333e-06		3.5785e-04	
	1/80	1.6538e-06	2.54	1.0347e-07	3.32	6.9566e-05	2.36
	1/160	2.9043e-07	2.51	1.2050e-08	3.10	1.5966e-05	2.12
4	1/40	1.7747e-06		1.3222e-07		6.3330e-05	
	1/80	3.1650e-07	2.48	1.6668e-08	2.98	1.5946e-05	1.98
	1/160	5.6073e-08	2.49	2.0895e-09	2.99	3.9984e-06	1.99
5	1/40	5.2183e-07		3.6445e-08		2.2842e-05	
	1/80	9.2363e-08	2.49	4.5631e-09	2.99	5.7191e-06	1.99
	1/160	1.6333e-08	2.50	5.7062e-10	3.00	1.4303e-06	2.00

Table 5. Errors and orders about polynomial degree p for given $h = 1/40$ in Example 4.2.

p	$\|\|e_u\|\|_\Omega$	$r(p)$	$\|\|\|e_u\|\|\|$	$r(p)$	$\|\|e_v\|\|_\Omega$	$r(p)$	$\|\|\|e_v\|\|\|$	$r(p)$
2	1.0581e-05		2.6900e-03		6.0461e-04		1.5607e-01	
3	3.5828e-08	14.03	1.2556e-05	13.23	1.0333e-06	15.71	3.5785e-04	14.99
4	4.2805e-10	15.39	2.6111e-07	13.46	1.3222e-07	7.15	6.3330e-05	6.02
5	9.5536e-13	27.36	5.8920e-10	27.31	3.6445e-08	5.77	2.2842e-05	4.57
6	6.0288e-14	15.15	5.1156e-11	13.40	1.3117e-08	5.60	1.0328e-05	4.35
7	1.1813e-14	10.57	1.2178e-11	9.31	5.5870e-09	5.54	5.3351e-06	4.28
8	3.1966e-15	9.79	3.8648e-12	8.59	2.6851e-09	5.49	3.0268e-06	4.24
9	1.1028e-15	9.03	1.4634e-12	8.24	1.4104e-09	5.46	1.8594e-06	4.13
10	-	-	-	-	7.9554e-10	5.43	1.1965e-06	4.18
11	-	-	-	-	4.7701e-10	5.36	8.0752e-07	4.12

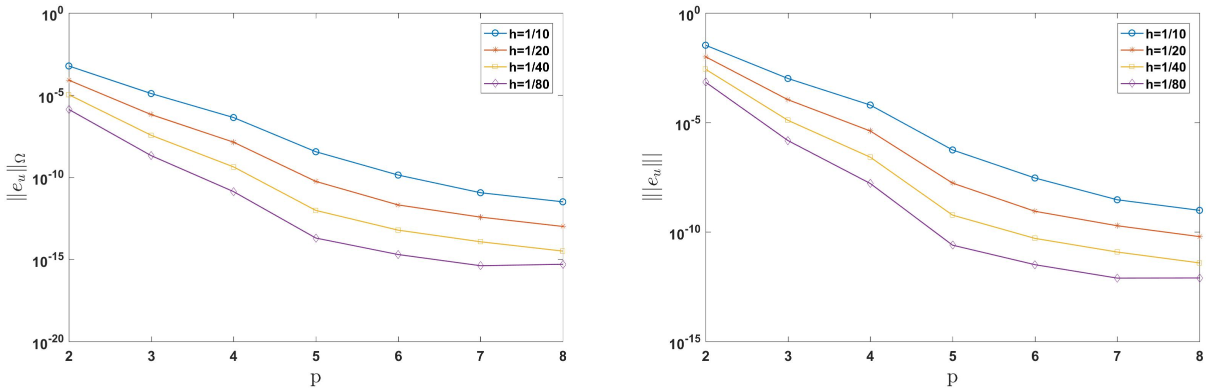

Figure 1. Error for $||e_u||_\Omega$ (left) and $|||e_u|||$ (right) with respect to p in Example 4.2.

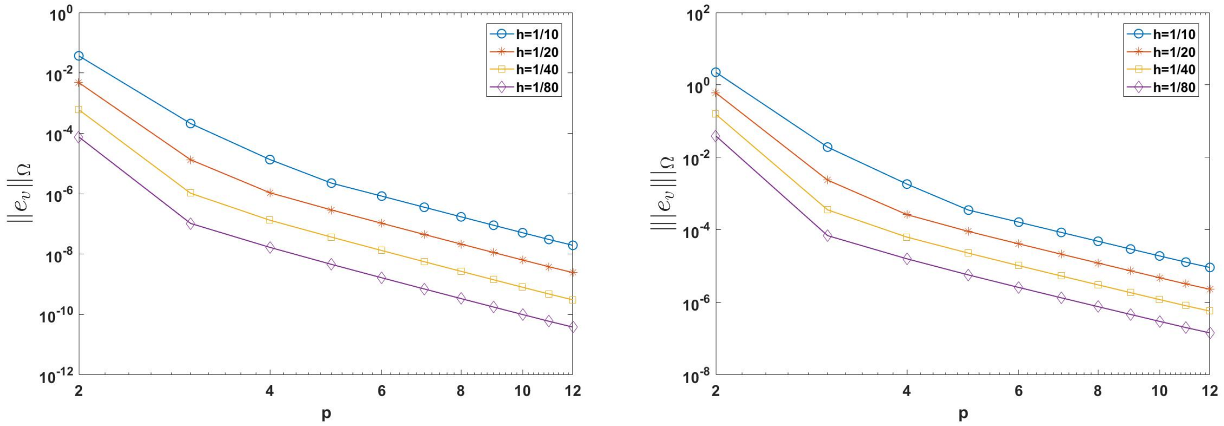

Figure 2. Error for $\|e_v\|_\Omega$ (left) and $|||e_v|||$ (right) with respect to p in Example 4.2.

Example 4.3. With a nonlinear reaction term. *Assuming that the equation is*

$$-u_{xxxx} - u - u^3 = f(x),$$

and the exact solution is $u(x) = \sin(\pi x)exp(x)$ *and* $\Omega = (0, 1)$. *Then, the function* $f(x)$ *and boundary conditions can be calculated from* $u(x)$. *The corresponding results are put in Tables 6 and 7 and Figures 3 and 4.*

In Example 4.3, we check the performance of our method for solving the equation with a nonlinear reaction term. For the resulting discrete nonlinear system, we employ Newton-Raphson iteration method to obtain the solution.

From Tables 6 and 7, we know that the approximate solution u_h, v_h produced by our method for the fourth order equation with a nonlinear reaction term exhibits optimal-order convergence with respect to the L^2-norm, L^∞-norm, and energy norm. Figure 3 and Figure 4 show that the approximate solution u_h, v_h both achieve the exponential convergence with respect to polynomial degree in L^2-norm and energy norm.

Table 6. Errors and orders of e_u for given polynomial degree p in Example 4.3.

p	h	$\|e_u\|_\infty$	order	$\|e_u\|_\Omega$	order	$|||e_u|||$	order
2	1/40	3.4206e-05		1.7042e-05		4.4102e-03	
	1/80	4.5237e-06	2.91	2.1406e-06	2.99	1.1138e-03	1.98
	1/160	5.8103e-07	2.96	2.6792e-07	3.00	2.7963e-04	1.99
3	1/40	1.1612e-07		2.3539e-08		8.5850e-06	
	1/80	7.2553e-09	4.00	1.4137e-09	4.05	1.0505e-06	3.03
	1/160	4.5338e-10	4.00	8.6525e-11	4.03	1.2991e-07	3.01
4	1/40	3.2174e-10		1.4865e-10		9.1680e-08	
	1/80	1.0076e-11	4.99	4.6495e-12	4.99	5.7362e-09	3.99
	1/160	3.1545e-13	4.99	1.4533e-13	5.00	3.5862e-10	4.00

Table 7. Errors and orders of e_v for given polynomial degree p in Example 4.3.

p	h	$\|\|e_v\|\|_\infty$	order	$\|\|e_v\|\|_\Omega$	order	$\|\|\|e_v\|\|\|$	order
2	1/40	4.3331e-04		2.1012e-04		5.5003e-02	
	1/80	5.4358e-05	2.99	2.6330e-05	2.99	1.3784e-02	1.99
	1/160	6.7989e-06	3.00	3.2934e-06	3.00	3.4483e-03	2.00
3	1/40	1.2671e-06		2.2623e-07		8.1529e-05	
	1/80	7.9774e-08	3.98	1.3420e-08	4.07	9.8983e-06	3.04
	1/160	5.0039e-09	3.99	8.1525e-10	4.04	1.2190e-06	3.02
4	1/40	4.2565e-09		1.9003e-09		1.1658e-06	
	1/80	1.3325e-10	4.99	5.9412e-11	4.99	7.3109e-08	3.99
	1/160	4.1772e-12	4.99	1.8569e-12	5.00	4.5761e-09	3.99

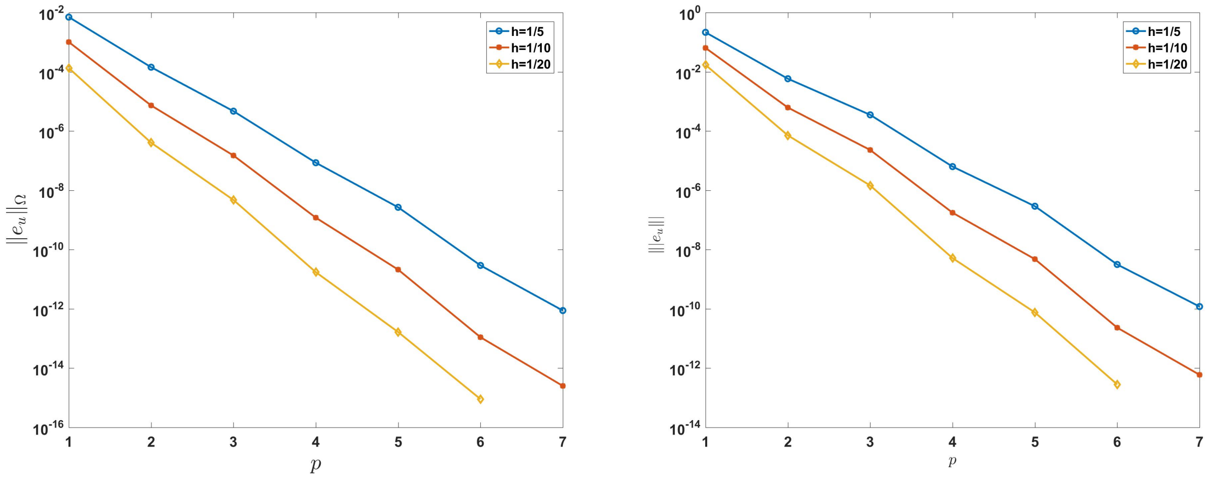

Figure 3. Error for $||e_u||_\Omega$ (left) and $|||e_u|||$ (right) with respect to p in Example 4.3.

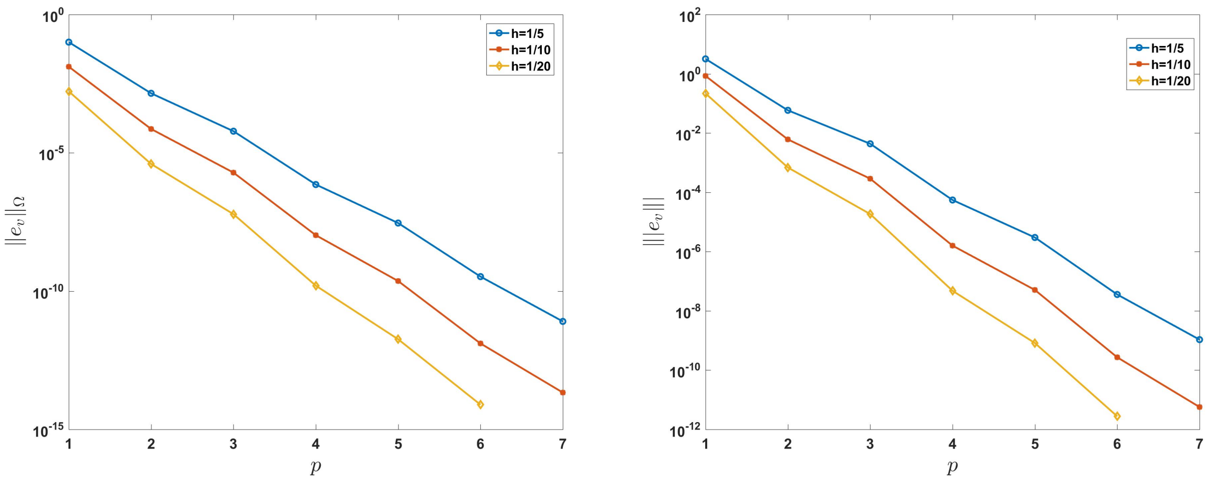

Figure 4. Error for $||e_v||_\Omega$ (left) and $|||e_v|||$ (right) with respect to p in Example 4.3.

Example 4.4. Dirichlet boundary condition. *Assuming that the equation is*

$$\begin{aligned} -u_{xxxx} &= f(x), \quad in\,\Omega, \\ u(a) &= \alpha_1, \quad u(b) = \alpha_2, \\ u_x(a) &= \gamma_1, \quad u_x(b) = \gamma_2. \end{aligned} \tag{4.1}$$

Let the exact solution be $u(x) = \sin(\pi x)exp(x)$ *and* $\Omega = (0, 1)$*. Then, the function* $f(x)$ *and boundary conditions can be calculated from* $u(x)$*.*

Let

$$\tilde{B}(w, q) = \sum_{I_n \in \mathcal{T}_h} \int_{I_n} w_x q_x dx - \sum_{x_n \in \mathcal{N}_h} \{\!\{w_x\}\!\}_n [\![q]\!]_n - \sum_{x_n \in \mathcal{N}_h^i} [\![w]\!]_n \{\!\{q_x\}\!\}_n. \tag{4.2}$$

The discrete form of the problem (4.1) is as follows: find $u_h, v_h \in V_h^{\mathbf{p}}$ such that

$$\tilde{B}(w_h, u_h) + (v_h, w_h)_\Omega = \tilde{L}_1(w_h), \quad \forall w_h \in V_h^{\mathbf{p}}, \tag{4.3}$$

$$\tilde{B}(v_h, q_h) = \tilde{L}_2(q_h), \quad \forall q_h \in V_h^{\mathbf{p}}, \tag{4.4}$$

where $\tilde{L}_1(\cdot)$ and $\tilde{L}_2(\cdot)$ are defined

$$L_1(w) = \alpha_1 w_x(a) - \alpha_2 w_x(b) + \gamma_2 w(b) - \gamma_1 w(a), \quad L_2(q) = \int_\Omega f q dx.$$

We can prove that if a solution to the discrete problems (4.3) and (4.4) exists, then the component v_h is uniquely determined whereas u_h is not. These findings are supported by the numerical results presented in Table 8.

Table 8. Errors and orders of e_u for given polynomial degree p in Example 4.4.

p	h	$\|e_u\|_\infty$	$\|e_u\|_\Omega$	$\|\|\|e_u\|\|\|$	$\|e_v\|_\infty$	$\|e_v\|_\Omega$	$\|\|\|e_v\|\|\|$
2	1/40	2.8762e+13	1.9903e+12	4.1479e+14	2.0114e+00	8.8837e-01	3.0486e+00
	1/80	9.8131e+12	4.6767e+11	1.9493e+14	9.5204e-02	4.3703e-02	1.6692e-01
	1/160	6.9612e+12	2.3459e+11	1.9556e+14	1.6416e-02	8.9685e-03	2.9953e-02
3	1/40	5.8346e+09	3.0323e+08	1.2117e+11	2.3970e-04	5.2132e-05	7.4077e-04
	1/80	1.2690e+16	4.6629e+14	3.7266e+17	1.4546e+02	8.3452e+01	2.8915e+02
	1/160	1.2533e+15	3.2564e+13	5.2050e+16	1.3679e+01	7.4783e+00	2.5756e+01
4	1/40	8.4842e+09	3.8917e+08	2.4347e+11	2.9533e-04	1.5720e-04	3.6463e-04
	1/80	2.4965e+05	8.0974e+03	1.0131e+07	5.3561e-07	2.6803e-07	8.0955e-07
	1/160	6.9471e+03	1.5933e+02	3.9871e+05	2.3919e-08	1.2063e-08	3.6875e-08

If we introduce a penalty term into Eq (4.4), the discrete problems (4.3) and (4.4) become as follows: Find $u_h, v_h \in V_h^{\mathbf{p}}$ such that

$$\begin{aligned} \tilde{B}(w_h, u_h) + (v_h, w_h)_\Omega &= \tilde{L}_1(w_h), \quad \forall w_h \in V_h^{\mathbf{p}}, \\ \tilde{B}(v_h, q_h) - u_h(a)q_h(a)/h - u_h(b)q_h(b)/h &= \tilde{L}_2(q_h) - \alpha_1 q_h(a)/h - \alpha_2 q_h(b)/h, \quad \forall q_h \in V_h^{\mathbf{p}}. \end{aligned}$$

Tables 9 and 10 demonstrate that the numerical solutions of the above discrete problem with a penalty term on the boundary achieve the optimal convergence order, even without tuning the penalty parameter. This observation aligns with the findings reported in [18, Section 4.2] for time-dependent fourth-order problems.

Table 9. Errors and orders of e_u for given polynomial degree p in Example 4.4.

p	h	$\|\|e_u\|\|_\infty$	order	$\|\|e_u\|\|_\Omega$	order	$\|\|\|e_u\|\|\|$	order
2	1/40	4.0845e-05		1.7054e-05		4.4116e-03	
	1/80	4.1138e-06	3.31	2.1051e-06	3.01	1.1022e-03	2.00
	1/160	5.2231e-07	298	2.6501e-07	2.99	2.7774e-04	1.99
3	1/40	5.0689e-06		2.7900e-07		1.1299e-04	
	1/80	3.1912e-07	3.99	1.2436e-08	4.48	1.0068e-05	3.49
	1/160	2.0017e-08	3.99	5.5434e-10	4.48	8.9696e-07	3.49
4	1/40	2.2554e-08		1.0986e-09		6.8712e-07	
	1/80	6.8758e-10	5.03	2.4069e-11	5.51	3.0092e-08	4.51
	1/160	2.1186e-11	5.02	5.3639e-13	5.48	1.3404e-09	4.49

Table 10. Errors and orders of e_v for given polynomial degree p in Example 4.4.

p	h	$\|\|e_v\|\|_\infty$	order	$\|\|e_v\|\|_\Omega$	order	$\|\|\|e_v\|\|\|$	order
2	1/40	4.3465e-04		2.1001e-04		5.4985e-02	
	1/80	5.4431e-05	2.99	2.6327e-05	2.99	1.3784e-02	1.99
	1.160	6.8035e-06	3.00	3.2933e-06	3.99	3.4483e-03	2.00
3	1/40	6.2681e-07		2.0282e-07		7.7008e-05	
	1/80	3.9447e-08	3.99	1.2661e-08	4.00	9.6105e-06	3.00
	1/160	2.4741e-09	3.99	7.9108e-10	4.00	1.2008e-06	3.00
4	1/40	4.2565e-09		1.8816e-09		1.1590e-06	
	1/80	1.3328e-10	4.99	5.9132e-11	4.99	7.2908e-08	3.99
	1/160	4.4032e-12	4.92	1.8547e-12	4.99	4.5712e-09	3.99

5. Conclusions

This paper presents an analysis of the hp-version DG method without penalty for solving one-dimensional biharmonic equations. The proposed method replaces the standard coercivity requirement of the bilinear form $B(\cdot,\cdot)$ with a polynomial-degree-dependent inf-sup condition. We establish a priori error estimates in both the energy norm and L^2-norm, explicitly tracking their dependence on the mesh size h and polynomial degree p. Theoretical analysis and numerical experiments confirm that the approximate solutions u_h and v_h achieve optimal convergence rates with respect to h in both norms.

For p-convergence, we prove suboptimal rates in the L^2-norm and a two-order reduction in the energy norm due to the p^2-dependence of the inf-sup condition. However, for solutions with boundary singularities that coincide with element endpoints, our numerical results demonstrate an unexpected order-doubling phenomenon in the p-convergence rates, consistent with prior observations in [32].

A key finding of this work is that adding a penalty term on the boundary is necessary to guarantee stability and optimal convergence for this equation with Dirichlet boundary conditions, which aligns with the earlier results for time-dependent equations in [18, Section 4.2]. The method is also validated on fourth-order equations with a nonlinear reaction term, where it achieves optimal-order convergence in the L^2-, L^∞-, and energy norms. The theoretical analysis of these extended scenarios is reserved for future study.

Author Contributions

Hongying Huang: Conceptualization, methodology, investigation, data curation, writing-original draft, review and editing, visualization, project administration, funding acquisition, supervision. Jingjing Yin: Writing-original draft, investigation, formal analysis, programming. Lin Zhang: Investigation, methodology, writing, review, revision and editing, visualization, validation. All authors read and approved the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions, which greatly improved the manuscript. We also thank Peimeng Yin for his insightful ideas and contributions to this research. Huang's work was supported by the National Natural Science Foundation of China (Grants No. 11771398 and No. 11471195).

Conflict of interest

The authors declare no conflicts of interest.

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