Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 21(1): 92–146.
DOI: 10.3934/nhm.2026005
Received: 09 July 2025
Revised: 30 December 2025
Accepted: 05 January 2026
Published: 20 January 2026

Research article

On the initial-boundary value problem for a Kuramoto-Velarde equation

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Abstract: The Kuramoto-Velarde equation describes the spatio-temporal evolution of step morphology on crystal surfaces, as well as the dynamics of spinodal decomposition in phase-separating systems subjected to an external field. In this paper, we prove the well-posedness of the solutions for the initial-boundary value problem for this equation, under several possible boundary conditions.

Keywords: existence; uniqueness; stability; Kuramoto-Velarde equation; initial-boundary value problem

1. Introduction

In this paper, we investigate the well-posedness of the solutions for the equation:

$$\partial_t u + \partial_x f(u) + q_1\partial_x u^3 + \kappa(\partial_x u)^2 + \nu\partial_x^2 u + \delta\partial_x^3 u + \beta^2\partial_x^4 u + \gamma u\partial_x^2 u = 0, \tag{1.1}$$

with q, κ, ν, δ, β, $\gamma \in \mathbb{R}$, such that

$$\beta \neq 0, \quad \gamma = \frac{\kappa}{2}. \tag{1.2}$$

We are interested in the initial-boundary value problem for this equation. More precisely, we consider the following boundary conditions:

$$\begin{cases} u(t,0) = g(t), & t > 0,\\ \partial_x u(t,0) = h(t), & t > 0, \end{cases} \qquad g,\ h \in W^{1,\infty}(0,\infty), \quad g(0) = u_0(0), \quad q_1 = 0, \tag{1.3}$$

$$\begin{cases} u(t,0) = g(t), & t > 0,\\ \partial_x^2 u(t,0) = 0, & t > 0, \end{cases} \qquad g \in W^{1,\infty}(0,\infty), \quad g(0) = u_0(0), \quad q_1 = 0, \tag{1.4}$$

$$\begin{cases}\partial_x^2 u(t,0) = 0, & t > 0,\\ \partial_x^3 u(t,0) = 0, & t > 0.\end{cases} \qquad q_1 = -\tau^2 \neq 0, \tag{1.5}$$

$$\begin{cases}\partial_x u(t,0) = 0, & t > 0,\\ \partial_x^3 u(t,0) = 0, & t > 0,\end{cases} \qquad q_1 = -\tau^2 \neq 0, \quad \delta = 0, \tag{1.6}$$

$$\begin{cases}u(t,0) = 0, & t > 0,\\ \partial_x u(t,0) = 0, & t > 0.\end{cases} \tag{1.7}$$

Moreover, we augment Eq (1.1) with the following initial datum:

$$u(0,x) = u_0(x), \quad x > 0, \tag{1.8}$$

for which we assume:

$$u_0 \in H^2(0,\infty). \tag{1.9}$$

Equation (1.1) has been developed in order to model several physical phenomena like that:

- Spinodal decomposition of phase separating systems in an external field [1–3];
- Spatio-temporal evolution of the morphology of steps on crystal surfaces [4–6];
- Growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension where the function u represents is the surface slope, while the constants q and q_1 are the growth driving forces proportional to the difference between the bulk chemical potentials of the solid and fluid phases, respectively [7–11].

The Kuramoto-Velarde equation

$$\partial_t u + q\partial_x u^2 + \kappa(\partial_x u)^2 + \nu\partial_x^2 u + \delta\partial_x^3 u + \beta^2\partial_x^4 u + \gamma u\partial_x^2 u = 0, \tag{1.10}$$

can be obtained from Eq (1.1) taking

$$f(u) = qu^2, \quad q \in \mathbb{R}, \quad q_1 = 0. \tag{1.11}$$

Equation (1.10) has been deduced in order to describe slow space-time variations of disturbances at interfaces, diffusion-reaction fronts, and plasma instability fronts [12–14]. It was very useful in the modelization of Benard–Marangoni cells, that occur when there is large surface tension on the interface [15–17] in a microgravity environment. This situation arises in crystal growth experiments aboard an orbiting space station, although the free interface is metastable with respect to small perturbations. The nonlinearities $\gamma u\partial_x^2 u$ and $\kappa(\partial_x u)^2$ model pressure destabilization effects striving to rupture the interface. Equation (1.10) is deduced in [18] to describe the long waves on a viscous fluid owing down an inclined plane, and in [19] to model the drift waves in a plasma.

The mathematical results on Eq (1.10) can be resumed as follows: The exact solutions are studied in [20, 21], the existence of the solitons in [12, 22], the existence of traveling wave solutions in [23], the existence of periodic solutions in [24], and the well-posedness of the Cauchy problem in [25–27].

Taking $\kappa = \gamma = 0$ in Eq (1.10), we have

$$\partial_t u + q\partial_x u^2 + \nu\partial_x^2 u + \delta\partial_x^3 u + \beta^2\partial_x^4 u = 0. \tag{1.12}$$

It was also independently deduced by Kuramoto [28–30] to describe the phase turbulence in reaction–diffusion systems, and by Sivashinsky [31], to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (1.12) can be used to study incipient instabilities in several physical and chemical systems [32–34]. Moreover, Eq (1.12), which is also known as the Benney–Lin equation [35, 36], were derived by Kuramoto in the study of phase turbulence in Belousov–Zhabotinsky reactions [37].

The dynamical properties and the existence of exact solutions for Eq (1.12) have been investigated in [5, 38–42]. Control problems for Eq (1.12) are studied in [43–47], the global exponential stabilization in [48]. In [6], the existence of solitonic solutions for Eq (1.12) is proven. In [49–54] and [55–57], the well–posedness of the Cauchy problem for Eq (1.12) is proven. Numerical results on Eq (1.12) can be found in [58–63]. Finally, in [64], the convergence of the solution of Eq (1.12) to the unique entropy one of the Burgers equation is proven.

Key results on boundary value problems for evolutive equations can be found in [65–69].

The main result of this paper is the following theorem.

Theorem 1.1. *Fix $T > 0$ and assume Eqs* (1.2) *and* (1.9)*. Eq* (1.1) *augmented with one the boundary conditions* (1.3)–(1.7) *and the initial datum* (1.8) *admit a unique solution*

$$\begin{aligned} &u \in H^1((0,T)\times(0,\infty)) \cap L^\infty(0,T;H^2(0,\infty)) \cap L^4(0,T;W^{2,4}(\mathbb{R}_+)),\\ &\partial_x^4 u \in L^2((0,T)\times(0,\infty)). \end{aligned} \tag{1.13}$$

Moreover, if u_1 and u_2 are two solutions of the same initial-boundary value problem for Eq (1.1)*, we have*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,\infty)} \le e^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(0,\infty)}, \quad 0 \le t \le T, \tag{1.14}$$

for some suitable $C(T) > 0$ depending only on T, $\left\|u_{1,0}\right\|_{H^2(0,\infty)}$, and $\left\|u_{2,0}\right\|_{H^2(0,\infty)}$.

Hence, under Assumption (1.2), Theorem 1.1 gives the well-posedness of the initial-boundary value problems (1.1), (1.3)–(1.7). The proof of Theorem 1.1 relies on deriving suitable a priori estimates together with an application of the Cauchy-Kovalevskaya Theorem [70]. The regularity assumption (1.9) on the initial datum is essential in order to derive our a priori estimates. We do not think that the stability estimate (1.14) holds with a weaker assumption because the coefficient in the exponent depends on the H^2 norm of the initial data.

The paper is organized as follows. In Sections 2–6, we prove Theorem 1.1 for Eqs (1.1)-(1.3)-(1.8), Eqs (1.1)-(1.4)-(1.8), Eqs (1.1)-(1.5)-(1.8), Eqs (1.1)-(1.6)-(1.8), Eqs (1.1)-(1.7)-(1.8), respectively. In Section 7, we collected some long and technical proofs of some lemmas. We present the conclusions of the paper in Section 8.

2. Proof of the Theorem 1.1 for Eqs (1.1)-(1.3)-(1.8)

In this section, we prove Theorem 1.1 for Eqs (1.1)-(1.3)-(1.8).

Let us prove some a priori estimates on u, denoting with C_0 the constants, that depend only on the data, and with $C(T)$, the constants that depend also on T.

Following [71], we introduce the auxiliary variable:

$$v(t,x) = u(t,x) - g(t)e^{-x} - \left[g(t) + h(t)\right] xe^{-x}. \tag{2.1}$$

Observe that

$$\begin{aligned}
\partial_t v(t,x) =& \partial_t u(t,x) - g'(t)e^{-x} - \left[g'(t) + h'(t)\right] xe^{-x},\\
\partial_x v(t,x) =& \partial_x u(t,x) - h(t)e^{-x} + \left[g(t) + h(t)\right] xe^{-x},\\
\partial_x^2 v(t,x) =& \partial_x^2 u(t,x) + 2h(t)e^{-x} + g(t)e^{-x} - \left[g(t) + h(t)\right] xe^{-x},\\
\partial_x^3 v(t,x) =& \partial_x^3 u(t,x) - 3h(t)e^{-x} - 2g(t)e^{-x} + \left[g(t) + h(t)\right] xe^{-x},\\
\partial_x^4 v(t,x) =& \partial_x^4 u(t,x) + 4h(t)e^{-x} + 3g(t)e^{-x} - \left[g(t) + h(t)\right] xe^{-x}.
\end{aligned} \tag{2.2}$$

In particular, thanks to Eqs (1.1), (1.3), (2.1), and (2.2),

$$v(t,0) = u(t,0) - g(t) = 0, \quad \partial_x v(t,0) = \partial_x u(t,0) - h(t) = 0. \tag{2.3}$$

Moreover, thanks to Eqs (1.3) and (2.2), we have that

$$\|v_0\|^2_{L^2(0,\infty)} \le \|u_0\|^2_{L^2(0,\infty)}. \tag{2.4}$$

Again, with Eqs (1.1), (1.3) and (1.8), we have the following equation for v.

$$\begin{aligned}
&\partial_t v + 2qv\partial_x v + \kappa(\partial_x v)^2 + \nu\partial_x^2 u + \delta\partial_x^3 v + \beta^2\partial_x^4 v + \gamma v\partial_x^2 v\\
&\quad = -g'(t)e^{-x} - \left[g'(t) + h'(t)\right] xe^{-x} - 2qh(t)e^{-x}v - 2q(g(t) + h(t)xe^{-x}v\\
&\qquad - 2qg(t)\partial_x v - 2qh(t)g(t)e^{-2x} - 2q[g(t) + h(t)]g(t)xe^{-2x}\\
&\qquad + 2q[g(t) + h(t)xe^{-x}\partial_x v - 2q[g(t) - h(t)]h(t)xe^{-2x}\\
&\qquad + 2q(g^2(t) - h^2(t))x^2e^{-2x} - \left[g'(t) + h'(t)\right] xe^{-x}\\
&\qquad - 2\kappa h(t)e^{-x}\partial_x v + 2\kappa\left[g(t) + h(t)\right] xe^{-x}\partial_x v - \kappa h^2(t)e^{-2x}\\
&\qquad - \kappa\left[g(t) + h(t)\right]^2 x^2e^{-2x} + 2\kappa g(t)\left[g(t) + h(t)\right] xe^{-2x}\\
&\qquad + 2\nu h(t)e^{-x} + \nu g(t)e^{-x} - \nu\left[g(t) + h(t)\right] xe^{-x}\\
&\qquad - 3\delta h(t)e^{-x} - 2\delta g(t)e^{-x} + \delta\left[g(t) + h(t)\right] xe^{-x} + 4\beta^2 h(t)e^{-x}\\
&\qquad + 3\beta^2 g(t)e^{-x} - \beta^2\left[g(t) + h(t)\right] xe^{-x} + 2\gamma h(t)e^{-x}v\\
&\qquad + \gamma g(t)e^{-x}v - 2\gamma\left[g(t) + h(t)\right] xe^{-x}v - \gamma g(t)e^{-x}\partial_x^2 v\\
&\qquad + 2\gamma g(t)h(t)e^{-2x} + \gamma g^2(t)e^{-x} - \gamma g(t)\left[g(t) + h(t)\right] xe^{-2x}\\
&\qquad - \gamma\left[g(t) + h(t)\right] xe^{-x}\partial_x^2 v + 2\gamma h(t)\left[g(t) + h(t)\right] xe^{-2x}\\
&\qquad + \gamma\left[g(t) + h(t)\right] xe^{-x} - \gamma\left[g(t) + h(t)\right] x^2e^{-2x}.
\end{aligned} \tag{2.5}$$

Lemma 2.1. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$\begin{aligned}
\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le& C(T), \qquad (2.6)\\
\int_0^t \|\partial_x v(s,\cdot)\|^2_{L^2(0,\infty)} ds \le& C(T), \qquad (2.7)
\end{aligned}$$

for every $0 \le t \le T$.

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

Arguing as in [52, Lemmas 2.2 and 2.3], we have the following result.

Lemma 2.2. *Fix* $T > 0$ *and assume Eq* (1.2)*. There exists a constant* $C(T) > 0$*, such that*

$$\|u(t,\cdot)\|_{L^2(0,\infty)} \leq C(T), \tag{2.8}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.9}$$

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.10}$$

$$\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} ds \leq C(T), \tag{2.11}$$

for every $0 \leq t \leq T$*.*

Following [72, Lemma 2.3], we prove the following result.

Lemma 2.3. *Fix* $T > 0$ *and assume Eq* (1.2)*. There exists a constant* $C_0 > 0$*, such that*

$$\|\partial_x u(t,\cdot)\|^4_{L^4(0,\infty)} \leq C_0\left(\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 1\right), \tag{2.12}$$

for every $0 \leq t \leq T$*.*

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to Eq (1.3),

$$\begin{aligned}
\|\partial_x u(t,\cdot)\|^4_{L^4(0,\infty)} &= \int_0^\infty (\partial_x u)^3 \partial_x u dx \\
&= -u(t,0)(\partial_x u(t,0))^3 - 3\int_0^\infty u(\partial_x u)^2 \partial_x^2 u dx \\
&= -g(t)h^3(t) - 3\int_0^\infty u(\partial_x u)^2 \partial_x^2 u dx.
\end{aligned} \tag{2.13}$$

Due to the Young inequality,

$$\begin{aligned}
3\int_0^\infty |u|(\partial_x u)^2 |\partial_x^2 u| dx \leq& \frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(0,\infty)} + \frac{9}{2}\int_0^\infty u^2(\partial_x^2 u)^2 dx \\
\leq& \frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(0,\infty)} + \frac{9}{2}\|u\|^2_{L^\infty((0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

It follows from Eqs (1.3) and (2.13) that

$$\frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(0,\infty)} \leq C_0 + \frac{9}{2}\|u\|^2_{L^\infty((0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},$$

which gives Eq (2.12).

Lemma 2.4. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that*

$$\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} \leq C(T)\sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}\right)}, \tag{2.14}$$

for every $0 \leq t \leq T$. Moreover,

$$\|u(t,\cdot)\|_{L^\infty(0,\infty)} \leq C(T)\sqrt{\left(1 + \sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}\right)}\right)}, \tag{2.15}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by proving Eq (2.14). Thanks to Eqs (1.3), (2.8) and the Hölder inequality,

$$\begin{aligned}
\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} =& \int_0^\infty \partial_x u \partial_x u dx = -u(t,0)\partial_x u(t,0) - \int_0^\infty u\partial_x^2 u dx \\
=& -g(t)h(t) - \int_0^\infty u\partial_x^2 u dx \leq C_0 + \int_0^\infty |u||\partial_x^2 u| dx \\
\leq& C_0 + \|u(t,\cdot)\|_{L^2(0,\infty)} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)} \\
\leq& C(T)\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}\right),
\end{aligned}$$

which gives Eq (2.14).

Finally, we prove Eq (2.15). Due to Eqs (1.3), (2.8) and the Hölder inequality,

$$\begin{aligned}
u^2(t,x) =& 2\int_0^x u\partial_x u dx + 2g^2(t) \leq 2\int_0^\infty |u||\partial_x u| dx + C_0 \\
\leq& 2\|u(t,\cdot)\|_{L^2(0,\infty)} \|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} + C_0 \leq C(T)\left(1 + \|\partial_x u(t,\cdot)\|_{L^2(0,\infty)}\right).
\end{aligned}$$

Therefore, by Eq (2.14),

$$\|u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T)\left(1 + \sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}\right)}\right),$$

which gives Eq (2.15).

Following [73, Lemma 2.2], we prove the following result.

Lemma 2.5. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that*

$$\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(0,\infty))} \leq C(T), \tag{2.16}$$

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{42}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|_{L^2(0,\infty)} ds \leq C(T), \tag{2.17}$$

$$\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} \leq C(T), \tag{2.18}$$

$$\|u\|_{L^\infty((0,T)\times(0,\infty))} \leq C(T), \tag{2.19}$$

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.20}$$

$$\int_0^t (\partial_x^2 u(s,0))^2 dx \le C(T), \tag{2.21}$$

$$\int_0^t (\partial_x^3 u(s,0))^2 dx \le C(T), \tag{2.22}$$

$$\|\partial_x u\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.23}$$

for every $0 \le t \le T$.

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

Lemma 2.6. *Fix* $T > 0$ *and assume Eq* (1.2)*. There exists a constant* $C(T) > 0$*, such that*

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} ds \le C(T), \tag{2.24}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$, we begin by observing that, thanks to Eq (1.3),

$$\begin{aligned}
\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(0,\infty)} &= \int_0^\infty \partial_x^2 u(\partial_x^2 u)^3 dx \\
&= -\partial_x u(t,0)(\partial_x^2 u(t,0))^3 - 3\int_0^\infty \partial_x u(\partial_x^2 u)^2 \partial_x^3 u dx \\
&= -h(t)(\partial_x^2 u(t,0))^3 - 3\int_0^\infty \partial_x u(\partial_x^2 u)^2 \partial_x^3 u dx.
\end{aligned} \tag{2.25}$$

Observe that

$$(\partial_x^2 u(t,0))^3 = -3\int_0^\infty (\partial_x^2 u)^2 \partial_x^3 u dx.$$

Therefore, by Eq (2.25), we have that

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(0,\infty)} = 3h(t)\int_0^\infty (\partial_x^2 u)^2 \partial_x^3 u dx - 3\int_0^\infty \partial_x u(\partial_x^2 u)^2 \partial_x^3 u dx. \tag{2.26}$$

Thanks to Eqs (1.3), (2.23), and the Young inequality,

$$\begin{aligned}
3|h(t)|\int_0^\infty (\partial_x^2 u)^2 |\partial_x^3 u| dx &\le C_0 \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^3 u| dx = 2\int_0^\infty \left|\frac{(\partial_x^2 u)^2}{\sqrt{3}}\right| \left|\frac{\sqrt{3} C_0 \partial_x^3 u}{2}\right| dx \\
&\le \frac{1}{3}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + C_0 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\
3\int_0^\infty |\partial_x u|(\partial_x^2 u)^2 |\partial_x^3 u| dx &\le 3\|\partial_x u\|_{L^\infty((0,T)\times(0,\infty))} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^3 u| dx \\
&\le C(T)\int_0^\infty (\partial_x^2 u)^2 |\partial_x^4 u| dx \le \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

It follows from Eq (2.26) that

$$\frac{1}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(0,\infty)} \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

An integration on $(0, t)$ and Eq (2.20) give Eq (2.24).

Lemma 2.7. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that*

$$\int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.27}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying Eqs (1.1)-(1.3)-(1.8) by $2\partial_t u$, an integration on $(0,\infty)$ gives

$$\begin{aligned}
2\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} =& -2\kappa \int_0^\infty (\partial_x u)^2 \partial_t u dx - 2\nu \int_0^\infty \partial_x^2 u \partial_t u dx \\
&- 2\delta \int_0^\infty \partial_x^3 u \partial_t u dx - 2\beta^2 \int_0^\infty \partial_x^4 u \partial_t u dx - 2\gamma \int_0^\infty u \partial_x^2 u \partial_t u \\
&- 4q \int_0^\infty u \partial_x u \partial_t u dx.
\end{aligned} \tag{2.28}$$

Due to Eqs (2.17), (2.19), (2.20), (2.23) and the Young inequality,

$$\begin{aligned}
2|\kappa| \int_0^\infty (\partial_x u)^2 |\partial_t u| dx \leq& 2|\kappa| \|\partial_x u\|_{L^\infty((0,T)\times(0,\infty))} \int_0^\infty |\partial_x u||\partial_t u| dx \\
\leq& 2C(T) \int_0^\infty |\partial_x u||\partial_t u| dx = 2\int_0^\infty \left|\frac{C(T)\partial_x u}{\sqrt{D_5}}\right| \left|\sqrt{D_5}\partial_t u\right| dx \\
\leq& \frac{C(T)}{D_5} \|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \\
\leq& \frac{C(T)}{D_5} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\
2|\nu| \int_0^\infty |\partial_x^2 u||\partial_t u| dx =& 2\int_0^\infty \left|\frac{\nu \partial_x^2 u}{\sqrt{D_5}}\right| \left|\sqrt{D_5}\partial_t u\right| dx \\
\leq& \frac{\nu^2}{D_5} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \\
\leq& \frac{C(T)}{D_5} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\
2|\delta| \int_0^\infty |\partial_x^3 u||\partial_t u| dx =& 2\int_0^\infty \left|\frac{\delta \partial_x^3 u}{\sqrt{D_5}}\right| \left|\sqrt{D_5}\partial_t u\right| dx \\
\leq& \frac{\delta^2}{D_5} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\
2\beta^2 \int_0^\infty |\partial_x^4 u||\partial_t u| dx =& 2\int_0^\infty \left|\frac{\beta^2 \partial_x^4 u}{\sqrt{D_5}}\right| \left|\sqrt{D_5}\partial_t u\right| dx \\
\leq& \frac{\beta^4}{D_5} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_5 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\
2|\gamma| \int_0^\infty |u||\partial_x^2 u||\partial_t u| dx \leq& 2|\gamma| \|u\|_{L^\infty((0,T)\times(0,\infty))} \int_0^\infty |\partial_x^2 u||\partial_t u| dx \\
\leq& 2C(T) \int_0^\infty |\partial_x^2 u||\partial_t u| dx = 2\int_0^\infty \left|\frac{C(T)\partial_x^2 u}{\sqrt{D_5}}\right| \left|\sqrt{D_5}\partial_t u\right| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C(T)}{D_5}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_5} + \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
4|q|\int_0^\infty |u||\partial_x u||\partial_t u|dx &\leq 4|q|\left\|u\right\|_{L^\infty((0,T)\times(0,\infty))}\int_0^\infty |\partial_x u||\partial_t u|dx\\
&\leq 2C(T)\int_0^\infty |\partial_x u||\partial_t u|dx = 2\int_0^\infty \left|\frac{C(T)\partial_x u}{\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_t u\right|dx\\
&\leq \frac{C(T)}{D_5}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\leq \frac{C(T)}{D_5} + D_5\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

where D_5 is a positive constant, which will be specified later. Consequently, by Eq (2.28),

$$
2(1-3D_5)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq \frac{C(T)}{D_5} + \frac{\delta^2}{D_5}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^4}{D_5}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

Choosing $D_5 = \frac{1}{6}$, we have that

$$
\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq C(T) + 5\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 5\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

Integrating on $(0,t)$, by Eqs (2.17) and (2.20), we get

$$
\begin{aligned}
\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq& C(T)t + 5\delta^2\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\
&+ 5\beta^4\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T),
\end{aligned}
$$

which gives Eq (2.27).

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $T > 0$. Thanks to Lemmas 2.2, 2.5–2.7 and the Cauchy-Kovalevskaya Theorem [70], we have that u is solution of Eqs (1.1)-(1.3)-(1.8) and (1.13) holds.

We prove Eq (1.14). Let u_1 and u_2 be two solutions of Eqs (1.1)-(1.3)-(1.8), which verify Eq (1.13), that is

$$
\begin{cases}
\partial_t u_i + 2qu_i\partial_x u_i + \kappa(\partial_x u_i)^2 + \nu\partial_x^2 u_i & \\
\qquad +\delta\partial_x^3 u_i + \beta^2\partial_x^4 u_i + \gamma u_i\partial_x^2 u_i = 0, & t>0, \quad x>0,\\
u_i(t,0) = g(t), & t>0, \qquad\qquad i=1,2.\\
\partial_x u_i(t,0) = h(t), & t>0,\\
u_i(0,x) = u_{i,0}(x), & x>0,
\end{cases}
$$

Then, the function

$$
\omega = u_1 - u_2 \tag{2.29}
$$

solves the following initial-boundary value problem:

$$\begin{cases}\partial_t\omega + 2q(u_1\partial_x u_1 - u_2\partial_x u_2) + \kappa\left[(\partial_x u_1)^2 - (\partial_x u_2)^2\right) + \nu\partial_x^2\omega \\ \qquad +\delta\partial_x^3\omega + \beta^2\partial_x^4\omega + \gamma\left(u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2\right) = 0, & t>0, \quad x>0, \\ \omega(t,0) = 0, & t>0, \\ \partial_x\omega(t,0) = 0, & t>0, \\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x>0. \end{cases} \tag{2.30}$$

Observe that, thanks to Eq (2.29),

$$\begin{aligned} u_1\partial_x u_1 - u_2\partial_x u_2 &= u_1\partial_x u_1 - u_2\partial_x u_1 + u_2\partial_x u_1 - u_2\partial_x u_2 = \partial_x u_1\omega + u_2\partial_x\omega, \\ (\partial_x u_1)^2 - (\partial_x u_2)^2 &= (\partial_x u_1 + \partial_x u_2)(\partial_x u_1 - \partial_x u_2) = (\partial_x u_1 + \partial_x u_2)\,\partial_x\omega, \\ u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2 &= u_1\partial_x^2 u_1 - u_1\partial_x^2 u_2 + u_1\partial_x^2 u_2 - u_2\partial_x^2 u_2 = u_1\partial_x^2\omega + \partial_x^2 u_2\omega. \end{aligned}$$

Therefore, Eq (2.30) is equivalent to the following equation:

$$\begin{aligned} &\partial_t\omega + 2q\partial_x u_1\omega + 2qu_2\partial_x\omega + \kappa(\partial_x u_1 + \partial_x u_2)\,\partial_x\omega + \nu\partial_x^2\omega \\ &\quad + \delta\partial_x^3\omega + \beta^2\partial_x^4\omega + \gamma u_1\partial_x^2\omega + \gamma\partial_x^2 u_2\omega = 0. \end{aligned} \tag{2.31}$$

Moreover, since $u_1,\ u_2 \in L^\infty(0,T;H^2(0,\infty))$, we have that

$$\begin{aligned} &\|\partial_x u_1\|_{L^\infty((0,T)\times(0,\infty))}, \ \|\partial_x u_2\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \\ &\|u_1\|_{L^\infty((0,T)\times(0,\infty))}, \ \|\partial_x u_2(t,\cdot)\|_{L^2(0,\infty)} \le C(T). \end{aligned} \tag{2.32}$$

Observe again that, thanks to Eq (2.30),

$$\begin{aligned} 4q\int_0^\infty u_2\omega\partial_x\omega dx &= -2q\int_0^\infty \partial_x u_2\omega^2 dx, \\ 2\delta\int_0^\infty \omega\partial_x^3\omega dx &= -2\delta\int_0^\infty \partial_x\omega\partial_x^2\omega dx = 0, \\ 2\beta^2\int_0^\infty \omega\partial_x^4\omega dx &= -2\beta^2\int_0^\infty \partial_x\omega\partial_x^3\omega dx = 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{2.33}$$

Therefore, multiplying Eq (2.31) by 2ω, thanks to Eq (2.33), an integration on $(0,\infty)$ gives

$$\begin{aligned} &\frac{d}{dt}\|\partial_x\omega(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\quad = -2\kappa\int_0^\infty (\partial_x u_1 + \partial_x u_2)\,\omega\partial_x\omega dx - 2\nu\int_0^\infty \omega\partial_x^2\omega dx \\ &\qquad - 2\gamma\int_0^\infty u_1\omega\partial_x^2\omega dx - 2\gamma\int_0^\infty \partial_x^2 u_2\omega^2 dx \\ &\qquad - 4q\int_0^\infty \partial_x u_1\omega^2 dx + 2q\int_0^\infty \partial_x u_2\omega^2 dx. \end{aligned} \tag{2.34}$$

Due to Eq (2.32) and the Young inequality,

$$2|\kappa|\int_0^\infty |\partial_x u_1 + \partial_x u_2|\,\omega^2 dx \le C(T)\,\|\omega(t,\cdot)\|^2_{L^2(0,\infty)},$$

$$
\begin{aligned}
2|\nu|\int_0^\infty |\omega||\partial_x^2\omega|dx =& \int_0^\infty \left|\frac{2\nu\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx\\
\le&\frac{2\nu^2}{\beta^2}\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|\gamma|\int_0^\infty |u_1||\omega||\partial_x^2\omega|dx \le& C(T)\int_0^\infty |\omega||\partial_x^2\omega|dx\\
=&\int_0^\infty \left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx\\
\le& C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|\gamma|\int_0^\infty |\partial_x^2u_2|\omega^2dx \le& \gamma^2\int_0^\infty (\partial_x^2u_2)^2\omega^2dx+\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
\le& \gamma^2\left\|\omega(t,\cdot)\right\|^2_{L^\infty(0,\infty)}\left\|\partial_x^2u_2(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
\le& C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(0,\infty)}+\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
4|q|\int_0^\infty |\partial_xu_1|\omega^2dx \le& C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|q|\int_0^\infty \partial_xu_2\omega^2dx \le& C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

Therefore, by Eq (2.34),

$$
\begin{aligned}
&\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad\le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(0,\infty)}.
\end{aligned}
\tag{2.35}
$$

Observe that, thanks to Eq (2.30) and the Hölder inequality,

$$
\begin{aligned}
\omega^2(t,x) =&2\int_0^x \omega\partial_x\omega dy\le 2\int_0^\infty |\omega||\partial_x\omega|dx\\
\le& 2\left\|\omega(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_x\omega(t,\cdot)\right\|_{L^2(0,\infty)}.
\end{aligned}
$$

Therefore, by the Young inequality,

$$
\left\|\omega(t,\cdot)\right\|^2_{L^\infty(0,\infty)}\le\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

It follows from Eq (2.36) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad\le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}+C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
\tag{2.36}
$$

Observe that, thanks to Eq (2.30),

$$
C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}=C(T)\int_0^\infty \partial_x\omega\partial_x\omega dx=-C(T)\int_0^\infty \omega\partial_x^2\omega dx.
$$

Therefore, by the Young inequality,

$$\begin{aligned}C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& \int_0^\infty \left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right| dx\\ \le& C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Consequently, by Eq (2.36),

$$\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

The Gronwall Lemma and Eq (2.30) gives

$$\begin{aligned}\left\|\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} +& \frac{\beta^2 e^{C(T)t}}{2}\int_0^t e^{-C(T)s}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ \le& e^{C(T)t}\left\|u_{1,0}-u_{2,0}\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{2.37}$$

Equation (1.14) follows from Eqs (2.29) and (2.37).

3. Proof of the Theorem 1.1 for Eqs (1.1)-(1.4)-(1.8)

In this section, we prove Theorem 1.1 for Eq (1.1)-(1.4)-(1.8).

Inspired by [52], we consider the following function:

$$v(t,x) = u(t,x) - g(t)e^{-x}. \tag{3.1}$$

Observe that

$$\begin{aligned}\partial_t v(t,x) =& \partial_t u(t,x) - g'(t)e^{-x},\\ \partial_x v(t,x) =& \partial_x u(t,x) + g(t)e^{-x},\\ \partial_x^2 v(t,x) =& \partial_x^2 u(t,x) - g(t)e^{-x},\\ \partial_x^3 v(t,x) =& \partial_x^3 u(t,x) + g(t)e^{-x},\\ \partial_x^4 v(t,x) =& \partial_x^4 u - g(t)e^{-x}.\end{aligned} \tag{3.2}$$

By Eqs (1.1)-(1.4)-(1.8) and (3.2),

$$v(t,0) = u(t,0) - g(t) = 0, \quad \partial_x^2 v(t,0) = -g(t), \tag{3.3}$$

while, by Eqs (1.4) and (3.2), we have Eq (2.4).

Again by Eqs (1.1)-(1.4)-(1.8) and (3.2), we have the following equation for v.

$$\begin{aligned}&\partial_t v + 2qv\partial_x v + k(\partial_x v)^2 + \nu\partial_x^2 v + \delta\partial_x^3 v + \beta^2\partial_x^4 v + \gamma v\partial_x^2 v\\ &= -2qg(t)e^{-x}v - 2qg(t)e^{-x}\partial_x v + (2q-\kappa-\gamma)g^2(t)e^{-2x}\\ &\quad + 2\kappa g(t)e^{-x}\partial_x v - \nu g(t)e^{-x} + \delta g(t)e^{-x} - \beta^2 g(t)e^{-x} - \gamma g(t)e^{-x}v\\ &\quad - \gamma g(t)e^{-x}\partial_x^2 v + g'(t)e^{-x}.\end{aligned} \tag{3.4}$$

We prove the following result.

Lemma 3.1. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that*

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{6} \int_0^t e^{-C_0 s} \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{3.5}$$

$$\int_0^t (\partial_x v(s,0))^2 ds \leq C(T), \tag{3.6}$$

and Eq (2.7) *hold for every $0 \leq t \leq T$.*

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

Lemma 3.2. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that*

$$\int_0^t (\partial_x u(t,0))^2 ds \leq C(T), \tag{3.7}$$

and Eqs (2.8)–(2.10) *hold for every $0 \leq t \leq T$.*

Proof. Arguing as in [52, Lemma 3.2], the proof is concluded.

Lemma 3.3. *Fix $T > 0$ and assume Eq* (1.2)*. Then,*

$$\|\partial_x u(t,\cdot)\|_{L^4(\mathbb{R}_+)} \leq C_0 \left(1 + \|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right) \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \tag{3.8}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to Eq (1.4),

$$\begin{aligned} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} &= \int_0^\infty \partial_x u (\partial_x u)^3 dx \\ &= -u(t,0)(\partial_x u(t,0))^3 - 3\int_0^\infty u(\partial_x u)^2 \partial_x^2 u dx \\ &= -g(t)(\partial_x u(t,0))^3 - 3\int_0^\infty u(\partial_x u)^2 \partial_x^2 u dx. \end{aligned} \tag{3.9}$$

Observe that

$$(\partial_x u(t,0))^3 = -3\int_0^\infty (\partial_x u)^2 \partial_x^2 u dx.$$

Therefore, by Eq (3.9),

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} = 3g(t)\int_0^\infty (\partial_x u)^2 \partial_x^2 u dx - 3\int_0^\infty u(\partial_x u)^2 \partial_x^2 u dx. \tag{3.10}$$

Due to Eq (1.4) and the Young inequality,

$$\begin{aligned} 3|g(t)|\int_0^\infty (\partial_x u)^2 |\partial_x^2 u| dx &\leq C_0 \int_0^\infty (\partial_x u)^2 |\partial_x^2 u| dx \\ &\leq \frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \end{aligned}$$

$$
\begin{aligned}
3\int_0^\infty |u|(\partial_x u)^2|\partial_x^2 u|dx &= 2\int_0^\infty \left|\frac{(\partial_x u)^2}{3\sqrt{3}}\right|\left|\frac{3\sqrt{3}u\partial_x^2 u}{2}\right|dx\\
&\le \frac{1}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{27}{4}\int_0^\infty u^2(\partial_x^2 u)^2 dx\\
&\le \frac{1}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{27}{4}\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

It follows from Eq (3.10) that

$$
\frac{1}{6}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} \le C_0\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
$$

which gives Eq (3.8).

Lemma 3.4. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$
\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds \le C(T), \tag{3.11}
$$

$$
\int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R}_+)} ds \le C(T), \tag{3.12}
$$

and Eq (2.19) hold for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that, since $u(t,0) = g(t)$, we have that $\partial_t u(t,0) = g'(t)$. Therefore, by Eq (1.4)

$$
\begin{aligned}
-2\int_0^\infty \partial_x^2 u\partial_t u dx &= 2\partial_x u(t,0)g'(t) + \frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)},\\
-2\delta\int_0^\infty \partial_x^2 u\partial_x^3 u dx &= 0,\\
-2\beta^2\int_0^\infty \partial_x^2 u\partial_x^4 u dx &= 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
\tag{3.13}
$$

Consequently, thanks to Eq (3.13), multiplying Eq (1.1) by $-2\partial_x^2 u$, an integration on $(0,\infty)$ gives

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad= -2\partial_x u(t,0)g'(t) + 4q\int_0^\infty u\partial_x u\partial_x^2 u + 2\kappa\int_0^\infty (\partial_x u)^2\partial_x^2 u dx\\
&\qquad+ 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2\gamma\int_0^\infty u(\partial_x^2 u)^2 dx.
\end{aligned}
\tag{3.14}
$$

Thanks to Eqs (1.4), (3.8) and the Young inequality,

$$
\begin{aligned}
&2|\partial_x u(t,0)||g'(t)|dx \le 2C_0|\partial_x u(t,0)| \le C_0 + (\partial_x u(t,0))^2,\\
&4|q|\int_0^\infty |u\partial_x u||\partial_x^2 u|dx \le q^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

$$
\begin{aligned}
2|\kappa|\int_0^\infty (\partial_x u)^2|\partial_x^2 u|dx \leq& \kappa^2 \left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
\leq& C_0\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
2|\gamma|\int_0^\infty |u|(\partial_x^2 u)^2 dx \leq& \gamma^2\int_0^\infty u^2(\partial_x^2 u)^2 dx + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
\leq& \gamma^2\|u\|^2_{L^\infty((0,\infty)\times\mathbb{R}_+)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

It follows from Eq (3.14) that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad\leq C_0\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + C_0\\
&\qquad\quad + C_0\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + (\partial_x u(t,0))^2.
\end{aligned}
$$

An integration on $(0,\,t)$, Eqs (1.9), (2.10), (2.9) and (3.7) give

$$
\begin{aligned}
&\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds\\
&\quad\leq C_0 + C_0\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds + C_0 t\\
&\qquad + C_0\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} ds + \int_0^t(\partial_x u(s,0))^2 ds\\
&\quad\leq C(T)\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right).
\end{aligned}
\tag{3.15}
$$

We prove Eq (2.19). Thanks to Eqs (1.4), (2.8), (3.15) and the Hölder inequality,

$$
\begin{aligned}
u^2(t,x) =& 2\int_0^\infty u\partial_x u dy + g^2(t) \leq \int_0^\infty |u\partial_x u|dx + C_0\\
\leq& 2\|u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)} + C_0\\
\leq& C(T)\sqrt{\left(1+\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+}\right)}.
\end{aligned}
$$

Hence,

$$
\|u\|^4_{L^\infty(0,T)\times\mathbb{R}_+} - C(T)\|u\|^2_{L^\infty(0,T)\times\mathbb{R}_+} - C(T) \leq 0,
$$

which gives Eq (2.19).

Equation (3.11) follows from Eqs (2.19) and (3.15).

Finally, we prove Eq (3.12). Thanks to Eqs (2.19) and (3.8), we have that

$$
\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} \leq C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
$$

An integration on $(0,\,t)$ and Eq (2.10) gives Eq (3.12).

Lemma 3.5. *Fix $T>0$ and assume Eq (1.2). There exists a constant $C(T)>0$, such that*

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2}\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds \leq C(T),
\tag{3.16}
$$

Eqs (2.22)–(2.24) hold for every $0\leq t\leq T$.

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

4. Proof of the Theorem 1.1 for Eqs (1.1)-(1.5)-(1.8)

In this section, we prove Theorem 1.1 for Eqs (1.1)-(1.5)-(1.8).

We begin by proving the following result

Lemma 4.1. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds & \\ + \frac{\tau^2 e^{C_0 t}}{20}\int_0^t e^{-C_0 s} u^4(t,0)ds \le C(T)&, \end{aligned} \tag{4.1}$$

$$\int_0^t u^2(s,0)ds \le C(T), \tag{4.2}$$

Eqs (2.9), (3.7) and (2.10) hold for every $0 \le t \le T$.

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

Lemma 4.2. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)} \le C(T)\sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right)}, \tag{4.3}$$

$$\|u(t,\cdot)\|_{L^\infty(0,\infty)} \le C(T)\sqrt{\sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right)}}, \tag{4.4}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4((0,\infty)} ds \le C(T)\sqrt{\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)}, \tag{4.5}$$

for every $0 \le t \le T$.

The proof of this lemma is quite long and technical. Therefore, in order to improve the readability of the paper, we postponed it in Section 7.

Lemma 4.3. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds \le C(T), \tag{4.6}$$

Eqs (2.16), (2.18)–(2.20), (2.23) and (2.24) hold for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Observe that, thanks to Eq (1.5),

$$\begin{aligned} 2\int_0^\infty \partial_x^4 u \partial_t u dx &= \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\ 2\delta\int_0^\infty \partial_x^3 u \partial_x^4 u dx &= 0. \end{aligned} \tag{4.7}$$

Therefore, thanks to Eq (4.7), multiplying Eq (1.1) by $2\partial_x^4 u$, an integration on $(0, \infty)$ gives

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad = -4q\int_0^\infty u\partial_x u\partial_x^4 u dx + 6\tau^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx - 2\kappa\int_0^\infty (\partial_x u)^2\partial_x^4 u dx\\
&\qquad - 2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx - 2\gamma\int_0^\infty u\partial_x^2 u\partial_x^4 u dx.
\end{aligned} \tag{4.8}$$

Thanks to the Young inequality,

$$\begin{aligned}
&4|q|\int_0^\infty |u\partial_x u||\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{2qu\partial_x u}{\beta\sqrt{D_9}}\right|\left|\beta\sqrt{D_9}\partial_x^4 u\right|dx\\
&\le \frac{4q^2}{D_9}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_9\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
&6\tau^2\int_0^\infty u^2|\partial_x u||\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{3\tau^2 u^2\partial_x u}{\beta\sqrt{D_9}}\right|\left|\beta\sqrt{D_9}\partial_x^4 u\right|dx\\
&\quad\le \frac{9\tau^4}{\beta^2 D_9}\int_0^\infty u^4(\partial_x u)^2 dx + \beta^2 D_9\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le \frac{9\tau^4}{\beta^2 D_9}\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_9\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
&2|\kappa|\int_0^\infty (\partial_x u)^2|\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{\kappa(\partial_x u)^2}{\beta\sqrt{D_9}}\right|\left|\beta\sqrt{D_9}\partial_x^4 u\right|dx\\
&\quad\le \frac{\kappa^2}{\beta^2 D_9}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \beta^2 D_9\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
&2|\nu|\int_0^\infty |\partial_x^2 u||\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{\nu\partial_x^2 u}{\beta\sqrt{D_9}}\right|\left|\beta\sqrt{D_9}\partial_x^4 u\right|dx\\
&\quad\le \frac{\nu^2}{\beta^2 D_9}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_9\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
&2|\gamma|\int_0^\infty |u\partial_x^2 u||\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{\gamma u\partial_x^2 u}{\beta\sqrt{D_9}}\right|\left|\beta\sqrt{D_9}\partial_x^4 u\right|dx\\
&\quad\le \frac{\gamma}{\beta^2 D_9}\int_0^\infty u^2(\partial_x^2 u)^2 dx + \beta^2 D_9\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le \frac{\gamma}{\beta^2 D_9}\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_9\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}$$

where D_9 is a positive constant, which will be specified later. It follows from Eq (4.8) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2(2-5D_9)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le \frac{C_0}{D_9}\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + \frac{C_0}{D_9}\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}
\end{aligned}$$

$$+\frac{C_0}{D_9}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)}.$$

Taking $D_9 = 1/5$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)}.
\end{aligned}\tag{4.9}$$

Integrating on $(0, t)$, by Eqs (1.9), (2.9) and (4.1),

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\quad\le C_0 + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\qquad + C_0\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R}_+)}ds\\
&\quad\le C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right) + C_0\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R}_+)}ds.
\end{aligned}$$

Thanks to Eqs (4.5) and (7.42), we have that

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\qquad\le C(T)\sqrt{\left(1+\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)}.
\end{aligned}\tag{4.10}$$

Thanks to Eq (4.10), we get

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^\infty(0,T;L^2(\mathbb{R}_+))} - C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))} - C(T)\le 0,$$

which gives Eq (2.16).

Equation (4.6) follows from Eqs (2.16) and (4.10), while Eqs (4.3), (4.4) and (4.6) give Eqs (2.18) and (2.19), respectively.

We prove Eq (2.20). Observe that, thanks to Eq (1.5),

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty \partial_x^3 u\partial_x^3 u dx = -\int_0^\infty \partial_x^2 u\partial_x^4 u dx \tag{4.11}$$

Due to Eq (4.6) and the Young inequality,

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \le \int_0^\infty |\partial_x^2 u||\partial_x^4 u|dx$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\leq C(T) + \frac{1}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

An integration on (0, t) and Eq (4.6) give Eq (2.20).

We prove Eq (2.23). Thanks to Eqs (2.18), (4.6) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x u(t,x))^2 =& 2\int_0^x \partial_x u \partial_x^2 u dx + (\partial_x u(t,0))^2 \leq 2\int_0^\infty |\partial_x u||\partial_x^2 u| dx - 2\int_0^\infty \partial_x u \partial_x^2 u dx\\
\leq& 4\int_0^\infty |\partial_x u||\partial_x^2 u| dx \leq 4\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)} \leq C(T).
\end{aligned}
$$

Hence,

$$
\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)} \leq C(T),
$$

which gives Eq (2.23).

Finally, we prove Eq (2.24). We begin by observing that, thanks to Eq (1.5),

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} = \int_0^\infty \partial_x^2 u(\partial_x^2 u)^3 dx = -3\int_0^\infty \partial_x u(\partial_x^2 u)^2 \partial_x^3 u dx. \tag{4.12}
$$

Arguing as in Lemma 2.6, we have Eq (2.24).

Lemma 4.4. *Fix $T > 0$ and assume Eq* (1.2)*. There exists a constant $C(T) > 0$, such that Eq* (2.27) *holds.*

Proof. Let $0 \leq t \leq T$. Arguing as in Lemma 2.7, we have

$$
\begin{aligned}
\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq& C(T) + 5\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+ 5\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 6\tau^2\int_0^\infty u^2|\partial_x u||\partial_t u| dx.
\end{aligned}
\tag{4.13}
$$

Thanks to Eqs (2.18), (2.19) and the Young inequality,

$$
\begin{aligned}
6\tau^2\int_0^\infty u^2|\partial_x u||\partial_t u| dx =& 6\tau^2\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\int_0^\infty |\partial_x u||\partial_t u| dx\\
\leq& C(T)\int_0^\infty |\partial_x u||\partial_t u| dx \leq C(T)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
\leq& C(T) + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

It follows from Eq (4.13) that

$$
\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + 5\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 5\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

An integration on (0, t), Eqs (2.20) and (4.6) give Eq (2.27).

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $T > 0$. Thanks to Lemmas 4.1, 4.3, 4.4 and the Cauchy-Kovalevskaya Theorem [70], we have that u is solution of Eqs (1.1)-(1.5)-(1.8) and (1.13) holds.

Arguing as in Section 2, we have Eq (1.14).

5. Proof of the Theorem 1.1 for Eqs (1.1)-(1.6)-(1.8)

In this section, we prove Theorem 1.1 for Eqs (1.1)-(1.6)-(1.8).
We begin by proving the following result.

Lemma 5.1. *Fix $T > 0$ and assume Eq* (1.2). *There exists a constant $C(T) > 0$, such that*

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &+ \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds \\ &+ \frac{2\tau^2 e^{C_0 t}}{3}\int_0^t e^{-C_0 s} u^4(t,0)ds \le C(T), \end{aligned} \tag{5.1}$$

Eq (2.9) *holds for every $0 \le t \le T$.*

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to Eq (1.6), we have

$$\begin{aligned} \frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} =&2\int_0^\infty u\partial_t u dx, \\ -6\tau^2\int_0^\infty u^3\partial_x u dx =&\frac{3\tau^2}{2}u^4(t,0), \\ 2\beta^2\int_0^\infty u\partial_x^4 u dx =&2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\ 2\gamma\int_0^\infty u^2\partial_x^2 u dx =& -4\gamma\int_0^\infty u(\partial_x u)^2 dx. \end{aligned} \tag{5.2}$$

Thanks to Eq (5.2), multiplying Eq (1.1) by $2u$, an integration on $(0, \infty)$ gives

$$\begin{aligned} \frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &+ 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{3\tau^2}{2}u^4(t,0) \\ &= 2(2\gamma-\kappa)\int_0^\infty u(\partial_x u)^2 dx - 2\nu\int_0^\infty u\partial_x^2 u dx + \frac{2q}{3}u^3(t,0). \end{aligned}$$

By Eq (1.2), we have that

$$\begin{aligned} \frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &+ 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{3\tau^2}{2}u^4(t,0) \\ &= -2\nu\int_0^\infty u\partial_x^2 u dx + \frac{2q}{3}u^3(t,0). \end{aligned} \tag{5.3}$$

Thanks to the Young inequality,

$$\begin{aligned} 2|\nu|\int_0^\infty |u||\partial_x^2 u|dx &= 2\int_0^\infty \left|\frac{\nu u}{\beta}\right|\left|\beta\partial_x^2 u\right| dx \\ &\le \frac{\nu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\ \frac{2|q|}{3}|u(t,0)|^3 dx = \left|\frac{2qu(t,0)}{3\tau}\right| |\tau| u^2(t,0) &\le \frac{2q^2}{9}u^2(t,0) + \frac{\tau^2}{2}u^4(t,0) \end{aligned}$$

$$= 2\frac{q^2\sqrt{3}}{9|\tau|}\frac{|\tau|}{\sqrt{3}}u^2(t,0) + \frac{\tau^2}{2}u^4(t,0) \le \frac{q^4}{9\tau^2} + \frac{5\tau^2}{6}u^4(t,0).$$

It follows from Eq (5.3) that

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{2\tau^2}{3}u^4(t,0)\\ \le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0.\end{aligned}$$

By the Gronwall Lemma and Eq (1.9), we get

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds + \frac{2\tau^2 e^{C_0 t}}{3}\int_0^t e^{-C_0 s}u^4(t,0)ds\\ \le C_0 e^{C_0 t} \le C(T),\end{aligned}$$

which gives Eq (5.1).

We prove Eq (2.9). We begin by observing that, thanks to Eq (1.6),

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty \partial_x u \partial_x u dx = -\int_0^\infty u\partial_x^2 u dx.$$

Thanks to Eq (5.1) and the Hölder inequality,

$$\begin{aligned}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &\le \int_0^\infty |u||\partial_x^2 u|dx\\ &\le \|u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)} \le C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)}.\end{aligned} \tag{5.4}$$

Therefore, by the Young inequality

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} \le C(T) + \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.$$

An integration on $(0, t)$ and Eq (5.1) give Eq (2.9).

Finally, arguing as in Lemma 4.1 we have Eq (2.9).

Lemma 5.2. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that Eqs (2.16), (4.6), (2.18)–(2.21), (2.23) and (2.24) hold.*

Proof. Let $0 \le t \le T$. Observe that, thanks to Eq (1.6), $\partial_t\partial_x u(t,0) = 0$. Therefore, again by Eq (1.6),

$$2\int_0^\infty \partial_x^4\partial_t u dx = \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}. \tag{5.5}$$

Therefore, thanks to Eq (5.5), multiplying Eq (1.1) by $2\partial_x^4 u$, an integration on $(0, \infty)$ gives

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\quad = -4q\int_0^\infty u\partial_x u\partial_x^4 u dx + 6\tau^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx - 2\kappa\int_0^\infty (\partial_x u)^2\partial_x^4 u dx\end{aligned} \tag{5.6}$$

$$-2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx - 2\gamma\int_0^\infty u\partial_x^2 u\partial_x^4 u dx.$$

Arguing as in Lemma 4.3, we have Eq (4.9). Observe that, thanks Eq (1.6),

$$\|\partial_x u(t,\cdot)\|^4_{L^\infty(0,\infty)} = \int_0^\infty \partial_x u(\partial_x u)^3 dx = -3\int_0^\infty u(\partial_x u)^2\partial_x^2 u dx. \tag{5.7}$$

Due to the Young inequality,

$$\begin{aligned}3\int_0^\infty |u|(\partial_x u)^2|\partial_x^2 u|dx =& \int_0^\infty (\partial_x^2 u)^2|3u\partial_x^2 u|dx\\
\le&\frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{9}{2}\int_0^\infty u^2(\partial_x^2 u)^2 dx\\
\le&\frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{9}{2}\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Therefore, by Eq (5.7),

$$\|\partial_x u(t,\cdot)\|^4_{L^\infty(0,\infty)} \le 9\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}. \tag{5.8}$$

It follows from Eqs (4.9) and (5.8) that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Integrating on $(0,\ t)$, by Eqs (1.9), (2.9) and (5.1), we have

$$\begin{aligned}&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds\\
&\le C_0 + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} ds\\
&\quad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds\\
&\le C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right).\end{aligned} \tag{5.9}$$

Observe that, thanks to Eqs (5.1), (5.4) and the Hölder inequality,

$$\begin{aligned}u^2(t,x) =&2\int_0^x u\partial_x u dx + 2u^2(t,0) \le 2\int_0^\infty |u||\partial_x u|dx - 2\int_0^\infty u\partial_x u dx\\
\le&4\int_0^\infty |u||\partial_x u|dx \le 4\|u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\\
\le&C(T)\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}.\end{aligned}$$

Hence,

$$\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)} \le C(T)\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}. \tag{5.10}$$

Therefore, by Eqs (5.9) and (5.10),

$$\begin{aligned}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds&\\ \le C(T)\left(1+\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)&.\end{aligned} \tag{5.11}$$

Arguing as in Lemma 2.5, we have Eq (2.16).

Equation (4.6) follows from Eqs (2.16) and (5.11). Equation (2.18) gives Eq (4.6), while Eqs (2.16) and (5.10) give Eq (2.19). Arguing as in Lemma 4.3, we have Eqs (2.20) and (2.23), while arguing as in Lemma 2.5, we have Eq (2.21).

Finally, we prove Eq (2.24). Observe that, thanks to Eq (1.6), we have Eq (4.12). Therefore, arguing as in Lemma 4.3, we have Eq (2.24).

Arguing as in Section 4, we have Lemma 4.4.

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $T > 0$. Thanks to Lemmas 4.4, 5.1, 5.2 and the Cauchy-Kovalevskaya Theorem [70], we have that u is a solution of Eqs (1.1)-(1.6)-(1.8) and (1.13) holds.

Arguing as in Section 2, we have Eq (1.14).

6. Proof of the Theorem 1.1 for Eqs (1.1)-(1.7)-(1.8)

In this section, we prove Theorem 1.1 for Eqs (1.1)-(1.7)-(1.8).

We begin by proving the following result.

Lemma 6.1. *Fix $T > 0$ and assume Eq (1.2). There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds \le C(T), \tag{6.1}$$

Eqs (2.9) and (5.4) hold for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Observe that, thanks to Eq (1.7),

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} =&2\int_0^\infty u\partial_t u dx,\\ 4q\int_0^\infty u^2\partial_x u dx =&0,\\ 6q_1\int_0^\infty u^3\partial_x u =&0,\\ 2\delta\int_0^\infty u\partial_x^3 u dx =&-2\int_0^\infty \partial_x u\partial_x^2 u dx = 0,\\ 2\beta^2\int_0^\infty u\partial_x^2 u dx =&2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\end{aligned} \tag{6.2}$$

$$2\gamma\int_0^\infty u^2\partial_x^2 u dx = -4\gamma\int_0^\infty u(\partial_x u)^2 dx.$$

Therefore, thanks to Eq (6.2), multiplying Eq (1.1) by $2u$, thanks to Eq (6.2), an integration on $(0,\ t)$ gives

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &+ 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\ &= 2(2\gamma-\kappa)\int_0^\infty u(\partial_x u)^2 dx - 2\nu\int_0^\infty u\partial_x^2 u dx.\end{aligned}$$

Thanks to Eq (1.2), we have

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} = -2\nu\int_0^\infty u\partial_x^2 u dx.$$

Arguing as in Lemma 5.1, we have Eq (6.1). Finally, arguing as in Lemma 5.1, we have Eqs (2.9) and (5.4).

Lemma 6.2. *Fix $T > 0$ and assume Eq* (1.2). *There exists a constant $C(T) > 0$, such that Eqs* (2.16)–(2.24) *hold.*

Proof. Let $0 \le t \le T$. Observe that, by Eq (1.7), $\partial_t u(t,0) = \partial_t\partial_x u(t,0) = 0$. Consequentially,

$$\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} = 2\int_0^\infty \partial_x^4 u\partial_t u dx. \tag{6.3}$$

Therefore, thanks to Eq (6.3) and arguing as in Lemma 4.3, we have

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} &+ \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\ &\quad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\quad + C_0\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + 2|\delta|\int_0^\infty |\partial_x^3 u||\partial_x^4 u| dx.\end{aligned} \tag{6.4}$$

Thanks to the Young inequality,

$$\begin{aligned}2|\delta|\int_0^\infty |\partial_x^3 u||\partial_x^4 u| dx &= \int_0^\infty \left|\frac{2\delta\partial_x^3 u}{\beta}\right|\beta\partial_x^4 u| dx\\ &\le \frac{2\delta^2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

It follows from Eq (6.4) that

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} &+ \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\ &\le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\end{aligned} \tag{6.5}$$

$$
\begin{aligned}
&+ C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&+ C_0\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Arguing as in Lemma 2.5, we have that

$$
\begin{aligned}
\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& \left(\frac{3}{D_{11}} + \frac{3}{2D_{10}}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+ \left(3D_{11} + \frac{9D_{10}^2}{2}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}
\tag{6.6}
$$

where D_{10}, D_{11} are two positive constant, which will be specified later. Therefore, by Eq (6.5),

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \left(\frac{\beta^2}{2} - C_0D_{11} - \frac{C_0D_{10}^2}{2}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\le C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C_0\left(1 + \frac{1}{D_{11}} + \frac{1}{D_{10}} + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad + C_0\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}.
\end{aligned}
$$

Taking

$$
D_{11} = \frac{\beta^2}{3C_0}, \qquad D_{10} = \frac{\sqrt{2}|\beta|}{\sqrt{7}C_0},
$$

we have that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{42}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}.
\end{aligned}
\tag{6.7}
$$

Thanks to Eq (1.7), we have Eq (5.7). Hence, by Eqs (5.8) and (6.7),

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{42}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\le C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad + C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Integrating on $(0, t)$, by Eqs (1.9), (2.9) and (6.1), we get

$$
\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{42}\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\quad\le C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R}_+)}ds
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
&+ C_0\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds\\
\le\ & C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right).
\end{aligned}
$$

Arguing as in Lemma 5.2, we have Eq (5.10). Therefore, by Eqs (5.10) and (6.8), we get

$$
\begin{aligned}
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{42}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} ds&\\
\le C(T)\left(1+\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right).&
\end{aligned}
\tag{6.9}
$$

Arguing as in Lemma 2.5, we have Eqs (2.16) and (2.17). Equation (2.18) follows from Eqs (2.17) and (5.4), while Eqs (2.16) and (5.10) give Eq (2.19). Arguing as in Lemma 2.5, we have Eqs (2.20)–(2.23).

Finally, we prove Eq (2.24). Observe that, by Eqs (1.7) and (2.25),

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} = -3\int_0^\infty \partial_x u(\partial_x^2 u)^2\partial_x^3 u dx. \tag{6.10}
$$

Thanks to Eq (2.23) and the Young inequality,

$$
\begin{aligned}
3\int_0^\infty |\partial_x u|(\partial_x^2 u)^2|\partial_x^3 u| dx \le& \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + \frac{9}{2}\int_0^\infty (\partial_x u)^2(\partial_x^3 u)^2 dx\\
\le& \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\\
\le& \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Hence, by Eq (6.10),

$$
\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)} \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
$$

An integration on $(0, t)$ and Eq (2.20) give Eq (2.24).

Arguing as in Section 4, we have Lemma 4.4.
Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $T > 0$. Thanks to Lemmas 4.4, 6.1, 6.2 and the Cauchy-Kovalevskaya Theorem [70], we have that u is a solution of Eqs (1.1)-(1.7)-(1.8) and (1.13) holds.
Arguing as in Section 2, we have Eq (1.14).

7. Technical section

In this section, we collect the proof of Lemmas 2.1, 2.5, 3.1, 3.5, 4.1, and 4.2.

Proof of Lemma 2.1. Let $0 \le t \le T$. We begin by observing that, thanks to Eq (2.3),

$$
2\int_0^\infty v\partial_t v dx = \frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)},
$$

$$
\begin{aligned}
4q\int_0^\infty v\partial_x v dx =&0,\\
2\delta\int_0^\infty v\partial_x^3 v dx =&-2\delta\int_0^\infty \partial_x v\partial_x^2 v dx = 0,\\
2\beta\int_0^\infty v\partial_x^4 v dx =&-2\beta^2\int_0^\infty \partial_x v\partial_x^3 v dx = 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2\gamma\int_0^\infty v^2\partial_x^2 v dx =&-4\gamma\int_0^\infty v(\partial_x v)^2 dx.
\end{aligned}
\tag{7.1}
$$

Therefore, by Eq (7.1), multiplying Eq (2.5) by $2v$, an integration on $(0,\infty)$ gives

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad= 2\,(2\gamma-\kappa)\int_0^\infty v(\partial_x v)^2 dx - 2\nu\int_0^\infty v\partial_x^2 v dx - 2g'(t)\int_0^\infty e^{-x}v dx\\
&\qquad - 2\left[g'(t)+h'(t)\right]\int_0^\infty xe^{-x}v dx - 4\kappa h(t)\int_0^\infty e^{-x}v\partial_x v dx\\
&\qquad - 2\kappa h^2(t)\int_0^\infty e^{-2x}v dx - 2\kappa\left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}v dx\\
&\qquad + 4\kappa g(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx + 4\nu h(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 2\nu g(t)\int_0^\infty e^{-x}v dx - 2\nu\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx\\
&\qquad - 6\delta h(t)\int_0^\infty e^{-x}v dx - 4\delta g(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 2\delta\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx + 8\beta^2 h(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 6\beta^2 g(t)\int_0^\infty e^{-x}v dx - 2\beta^2\int_0^\infty\left[g(t)+h(t)\right]xe^{-x}v dx\\
&\qquad + 4\gamma h(t)\int_0^\infty e^{-x}v^2 dx + 2\gamma g(t)\int_0^\infty e^{-x}v^2 dx\\
&\qquad - 4\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v^2 dx - 2\gamma g(t)\int_0^\infty e^{-x}v\partial_x^2 v dx\\
&\qquad + 4\gamma g(t)h(t)\int_0^\infty e^{-2x}v dx + 2\gamma g^2(t)\int_0^\infty e^{-x}v dx\\
&\qquad - 2\gamma g(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx - 2\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v\partial_x^2 v dx\\
&\qquad + 4\gamma h(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx + 2\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx\\
&\qquad - 2\gamma\left[g(t)+h(t)\right]\int_0^\infty x^2e^{-2x}v dx - 2qh(t)\int_0^\infty e^{-x}v^2 dx - 2q(g(t)+h(t)\int_0^\infty xe^{-x}v^2 dx\\
&\qquad + 4q[g(t)+h(t)\int_0^\infty xe^{-x}v\partial_x v dx - 4q[g(t)-h(t)]h(t)\int_0^\infty xe^{-2x}v.
\end{aligned}
$$

Thanks to Eq (1.2), we have that

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad= -2\nu\int_0^\infty v\partial_x^2 v dx - 2g'(t)\int_0^\infty e^{-x}v dx\\
&\qquad - 2\left[g'(t)+h'(t)\right]\int_0^\infty xe^{-x}v dx - 4\kappa h(t)\int_0^\infty e^{-x}v\partial_x v dx\\
&\qquad - 2\kappa h^2(t)\int_0^\infty e^{-2x}v dx - 2\kappa\left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}v dx\\
&\qquad + 4\kappa g(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx + 4\nu h(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 2\nu g(t)\int_0^\infty e^{-x}v dx - 2\nu\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx\\
&\qquad - 6\delta h(t)\int_0^\infty e^{-x}v dx - 4\delta g(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 2\delta\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx + 8\beta^2 h(t)\int_0^\infty e^{-x}v dx\\
&\qquad + 6\beta^2 g(t)\int_0^\infty e^{-x}v dx - 2\beta^2\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx\\
&\qquad + 4\gamma h(t)\int_0^\infty e^{-x}v^2 dx + 2\gamma g(t)\int_0^\infty e^{-x}v^2 dx\\
&\qquad - 4\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v^2 dx - 2\gamma g(t)\int_0^\infty e^{-x}v\partial_x^2 v dx\\
&\qquad + 4\gamma g(t)h(t)\int_0^\infty e^{-2x}v dx + 2\gamma g^2(t)\int_0^\infty e^{-x}v dx\\
&\qquad - 2\gamma g(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx - 2\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v\partial_x^2 v dx\\
&\qquad + 4\gamma h(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}v dx + 2\gamma\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}v dx\\
&\qquad - 2\gamma\left[g(t)+h(t)\right]\int_0^\infty x^2e^{-2x}v dx - 2qh(t)\int_0^\infty e^{-x}v^2 dx\\
&\qquad - 2q(g(t)+h(t))\int_0^\infty xe^{-x}v^2 dx + 4q(g(t)+h(t))\int_0^\infty xe^{-x}v\partial_x v dx\\
&\qquad - 4q(g(t)-h(t))h(t)\int_0^\infty xe^{-2x}v dx.
\end{aligned}
\tag{7.2}
$$

Observe that, for each $x \in (0,\infty)$,

$$
\begin{gathered}
e^{-x}\le 1,\quad xe^{-x}\le e,\quad \int_0^\infty e^{-2x}dx = \frac{1}{2}\\
\int_0^\infty x^2e^{-4x}dx = \frac{1}{32},\quad \int_0^\infty x^2e^{-2x}dx = \frac{1}{4},\quad \int_0^\infty x^4e^{-4x}dx = \frac{3}{128}.
\end{gathered}
\tag{7.3}
$$

Due Eqs (1.3), (7.3) and the Young inequality,

$$
\begin{aligned}
2|\nu|\int_0^\infty |v||\partial_x^2 v|dx &= 2\int_0^\infty \left|\frac{\nu v}{\beta\sqrt{D_1}}\right|\left|\beta\sqrt{D_1}\partial_x^2 v\right|dx\\
&\le \frac{\nu^2}{\beta^2 D_1}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 D_1\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|g'(t)|\int_0^\infty e^{-x}|v|dx &\le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2\left|g'(t)+h'(t)\right|\int_0^\infty xe^{-x}|v|dx &\le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
4|\kappa||h(t)|\int_0^\infty e^{-x}|v||\partial_x v|dx &\le 2C_0\int_0^\infty |v||\partial_x v|dx\\
&\le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2|\kappa|h^2(t)\int_0^\infty e^{-2x}|v|dx &\le 2C_0\int_0^\infty e^{-2x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2|\kappa|\left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}|v|dx &\le 2C_0\int_0^\infty x^2e^{-2x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
4|\kappa||g(t)|\left|g(t)+h(t)\right|\int_0^\infty xe^{-2x}|v|dx &\le 2C_0\int_0^\infty e^{-2x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
4|\nu||h(t)|\int_0^\infty e^{-x}|v|dx &\le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2|\nu||g(t)|\int_0^\infty e^{-x}|v|dx &\le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2|\nu|\left|g(t)+h(t)\right|\int_0^\infty xe^{-x}|v|dx &\le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
6|\delta||h(t)|\int_0^\infty e^{-x}|v|dx &\le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
4|\delta||g(t)|\int_0^\infty e^{-x}|v|dx &\le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},
\end{aligned}
$$

$$
\begin{aligned}
&2|\delta|\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&8\beta^2|h(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&6\beta^2|g(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2\beta^2\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|\gamma||h(t)|\int_0^\infty e^{-x}v^2dx \le C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma||g(t)|\int_0^\infty e^{-x}v^2dx \le C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|\gamma|\,|g(t)+h(t)|\int_0^\infty xe^{-x}v^2dx \le C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma||g(t)|\int_0^\infty e^{-x}|v||\partial_x^2v|dx \le 2C_0\int_0^\infty |v||\partial_x^2v|dx\\
&\qquad = 2\int_0^\infty \left|\frac{C_0v}{\beta\sqrt{D_1}}\right|\left|\beta\sqrt{D_1}\partial_x^2v\right|dx\\
&\qquad \le \frac{C_0}{D_1}\,\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2D_1\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&4|\gamma||g(t)h(t)|\int_0^\infty e^{-2x}|v|dx \le 2C_0\int_0^\infty e^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma|g^2(t)\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma||g(t)|\,|g(t)+h(t)|\int_0^\infty xe^{-2x}|v|dx \le 2C_0\int_0^\infty xe^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma|\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v||\partial_x^2v|dx \le 2C_0\int_0^\infty |v||\partial_x^2v|dx\\
&\qquad = 2\int_0^\infty \left|\frac{C_0v}{\beta\sqrt{D_1}}\right|\left|\beta\sqrt{D_1}\partial_x^2v\right|dx\\
&\qquad \le \frac{C_0}{D_1}\,\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2D_1\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}
$$

$$
\begin{aligned}
&4|\gamma||h(t)|\,|g(t)+h(t)|\int_0^\infty xe^{-2x}|v|dx \le 2C_0\int_0^\infty xe^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma|\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|\gamma|\,|g(t)+h(t)|\int_0^\infty x^2e^{-2x}|v|dx \le 2C_0\int_0^\infty x^2e^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|q||h(t)|\int_0^\infty e^{-x}v^2dx \le C_0\,\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)},\\
&2|q||(g(t)+h(t)|\int_0^\infty xe^{-x}v^2dx \le C_0\,\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)},\\
&4|q||(g(t)+h(t)|\int_0^\infty xe^{-x}|v||\partial_x v|dx \le C_0\int_0^\infty |v||\partial_x v|dx\\
&\qquad\quad \le C_0\,\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)},\\
&4|q||g(t)-h(t)||h(t)|\int_0^\infty xe^{-2x}|v|dx \le C_0\int_0^\infty xe^{-2x}|v|dx\\
&\qquad \le C_0\int_0^\infty x^2e^{-4x}dx + \|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\qquad \le C_0 + \|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

where D_1 is a positive constant, which will be specified later. It follows from Eq (7.2) that

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2(2-3D_1)\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad \le C_0\left(1+\frac{1}{D_1}\right)\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0.
\end{aligned}
$$

Taking $D_1 = \frac{1}{3}$, we have that

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad \le C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0.
\end{aligned}
\tag{7.4}
$$

Thanks to Eq (2.3),

$$
C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} = C_0\int_0^\infty \partial_x v\partial_x v dx = -C_0\int_0^\infty v\partial_x^2 v dx.
$$

Therefore, by the Young inequality,

$$
C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \le \int_0^\infty \left|\frac{C_0 v}{\beta}\right|\left|\beta\partial_x^2 v\right| dx
$$

$$\leq C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Consequently, by Eq (7.4),

$$\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0.$$

By the Gronwall Lemma and Eq (2.4), we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{L^2(0,\infty)} &+ \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\ &\leq C_0 e^{C_0 t} + C_0 e^{C_0 t}\int_0^t e^{-C_0 s} ds \leq C(T),\end{aligned}$$

which gives Eq (2.6).

Finally, we prove Eq (2.7). By Eq (2.3),

$$\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} = \int_0^\infty \partial_x v \partial_x v dx = -\int_0^\infty v \partial_x^2 v dx.$$

Due to Eq (2.6) and the Young inequality,

$$\begin{aligned}\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \leq &\int_0^\infty |v||\partial_x^2 v| dx \\ \leq &\frac{1}{2}\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{2}\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ \leq &C(T) + \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Integrating on $(0, t)$, by Eq (2.6), we have Eq (2.7).

Proof of Lemma 2.5. Let $0 \leq t \leq T$. We begin by observing that, since from Eq (1.3) $\partial_t u(t,0) = g'(t)$ and $\partial_t \partial_x u(t,0) = h'(t)$. Therefore,

$$\begin{aligned}2\int_0^\infty \partial_x^4 u \partial_t u dx =& -2\partial_x^3 u(t,0)\partial_t u(t,0) - 2\int_0^\infty \partial_x^3 u \partial_t \partial_x u dx \\ =& -2\partial_x^3 u(t,0)\partial_t u(t,0) + 2\partial_x^2 u(t,0)\partial_t\partial_x u(t,0) \\ &+ \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ =& -2g'(t)\partial_x^3 u(t,0) + 2h'(t)\partial_x^2 u(t,0) \\ &+ \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{7.5}$$

Consequently, thanks to Eq (7.5), multiplying Eq (1.1) by $2\partial_x^4 u$, an integration on $(0,\infty)$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)} \\ &= 2g'(t)\partial_x^3 u(t,0) - 2h'(t)\partial_x^2 u(t,0) - 2\kappa\int_0^\infty (\partial_x u)^2 \partial_x^4 u dx\end{aligned} \tag{7.6}$$

$$
\begin{aligned}
&- 2\nu \int_0^\infty \partial_x^2 u \partial_x^4 u dx - 2\delta \int_0^\infty \partial_x^3 u \partial_x^4 u dx - 2\gamma \int_0^\infty u \partial_x^2 u \partial_x^4 u dx \\
&- 4q \int_{\mathbb{R}} u \partial_x u \partial_x^4 u dx.
\end{aligned}
$$

Due to Eqs (1.3), (2.12) and the Young inequality,

$$
\begin{aligned}
&2|g'(t)||\partial_x^3 u(t,0)| \le 2C_0|\partial_x^3 u(t,0)| \le C_0 + (\partial_x^3 u(t,0))^2, \\
&2|h'(t)||\partial_x^2 u(t,0)| \le 2C_0|\partial_x^2 u(t,0)| \le C_0 + (\partial_x^2 u(t,0))^2, \\
&2|\kappa| \int_0^\infty (\partial_x u)^2 |\partial_x^4 u| dx = 2\int_0^\infty \left|\frac{\kappa(\partial_x u)^2}{\beta\sqrt{D_2}}\right| \left|\beta\sqrt{D_2}\partial_x^4 u\right| dx \\
&\quad \le \frac{\kappa^2}{\beta^2 D_2} \|\partial_x u(t,\cdot)\|_{L^4(0,\infty)}^4 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 \\
&\quad \le \frac{C_0}{D_2} \|u\|_{L^\infty((0,T)\times(0,\infty))}^2 \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \frac{C_0}{D_2}, \\
&2|\nu| \int_0^\infty |\partial_x^2 u||\partial_x^4 u| dx = 2\int_0^\infty \left|\frac{\nu\partial_x^2 u}{\beta\sqrt{D_2}}\right| \left|\beta\sqrt{D_2}\partial_x^4 u\right| dx \\
&\quad \le \frac{\nu^2}{\beta^2 D_2} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2, \\
&2|\delta| \int_0^\infty |\partial_x^3 u||\partial_x^4 u| dx = 2\int_0^\infty \left|\frac{\delta\partial_x^3 u}{\beta\sqrt{D_2}}\right| \left|\beta\sqrt{D_2}\partial_x^4 u\right| dx \\
&\quad \le \frac{\delta^2}{\beta^2 D_2} \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2, \\
&2|\gamma| \int_0^\infty |u\partial_x^2 u||\partial_x^4 u| dx = 2\int_0^\infty \left|\frac{\gamma u\partial_x^2 u}{\beta\sqrt{D_2}}\right| \left|\beta\sqrt{D_2}\partial_x^4 u\right| dx \\
&\quad \le \frac{\gamma^2}{\beta^2 D_2} \int_0^\infty u^2 \partial_x^2 u^2 dx + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 \\
&\quad \le \frac{\gamma^2}{\beta^2 D_2} \|u\|_{L^\infty((0,T)\times(0,\infty))}^2 \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}^2, \\
&4|q| \int_{\mathbb{R}} |u\partial_x u||\partial_x^4 u| dx = 2\int_{\mathbb{R}} \left|\frac{2qu\partial_x u}{\beta\sqrt{D_2}}\right| \left|\beta\sqrt{D_2}\partial_x^4 u\right| dx \\
&\quad \le \frac{2q^2}{\beta^2 D_2} \|u(t,\cdot)\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 + \beta^2 D_2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,
\end{aligned}
$$

where, D_2 is a positive constants, which will be specified later. Therefore, by Eq (7.6),

$$
\begin{aligned}
&\frac{d}{dt} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \beta^2 (2 - 5D_2) \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)} \\
&\quad \le C_0 \left(1 + \frac{1}{D_2}\right) + (\partial_x^3 u(t,0))^2 + (\partial_x^2 u(t,0))^2 + \frac{C_0}{D_2} \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 \\
&\qquad + \frac{C_0}{D_2} \left(1 + \|u\|_{L^\infty((0,T)\times(0,\infty))}^2\right) \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}^2 + \frac{C_0}{D_2} \|u(t,\cdot)\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2.
\end{aligned}
$$

Taking $D_2 = \frac{1}{5}$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}\\
&\quad\le C_0 + (\partial_x^3 u(t,0))^2 + (\partial_x^2 u(t,0))^2 + C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}\tag{7.7}$$

Observe that, by the Young inequality,

$$\begin{aligned}
(\partial_x^3 u(t,0))^2 =& -2\int_0^\infty \partial_x^3 u\partial_x^4 u dx \le 2\int_0^\infty |\partial_x^3 u||\partial_x^4 u| dx\\
=&\int_0^\infty \left|\frac{2\partial_x^3 u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx \le \frac{2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
(\partial_x^2 u(t,0))^2 =& -2\int_0^\infty \partial_x^2 u\partial_x^3 u dx \le 2\int_0^\infty |\partial_x^2 u||\partial_x^3 u| dx\\
\le&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}\tag{7.8}$$

Consequently, by Eq (7.7),

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}\\
&\quad\le C_0 + C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}\tag{7.9}$$

Observe that

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} = \int_0^\infty \partial_x^3 u\partial_x^3 u dx = -\partial_x^2 u(t,0)\partial_x^3 u(t,0) - \int_0^\infty \partial_x^2 u\partial_x^4 u dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}
\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le& |\partial_x^2 u(t,0)||\partial_x^3 u(t,0)| + \int_0^\infty |\partial_x^2 u||\partial_x^4 u| dx\\
\le&\frac{1}{2D_3}(\partial_x^2 u(t,0))^2 + \frac{D_3}{2}(\partial_x^3 u(t,0))^2 + \int_0^\infty \left|\frac{\partial_x^2 u}{\sqrt{D_4}}\right|\left|\sqrt{D_4}\partial_x^4 u\right| dx\\
\le&\frac{1}{2D_3}(\partial_x^2 u(t,0))^2 + \frac{D_3}{2}(\partial_x^3 u(t,0))^2\\
&+\frac{1}{2D_4}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{D_4}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}\tag{7.10}$$

where D_3, D_4 are two positive constants, which will be specified later. Thanks to the Young inequality,

$$\frac{1}{2D_3}(\partial_x^2 u(t,0))^2 = \frac{1}{D_3}\int_0^\infty \partial_x^2 u\partial_x^3 u dx = 2\int_0^\infty \left|\frac{\partial_x^2 u}{2D_3}\right|\left|\partial_x^3 u\right| dx$$

$$
\begin{aligned}
&\leq \frac{1}{4D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
\frac{D_3}{2}(\partial_x^3 u(t,0))^2 &= D_3\int_0^\infty \partial_x^3 u \partial_x^4 u dx = 2\int_0^\infty \left|\frac{\partial_x^3 u}{\sqrt{3}}\right|\left|\frac{\sqrt{3}D_3\partial_x^4 u}{2}\right| dx\\
&\leq \frac{1}{3}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{3D_3^2}{4}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

It follows from Eq (7.10) that

$$
\begin{aligned}
\frac{1}{6}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq &\left(\frac{1}{2D_4} + \frac{1}{4D_3}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+ \left(\frac{D_4}{2} + \frac{3D_3^2}{4}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}
$$

that is

$$
\begin{aligned}
\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq &\left(\frac{3}{D_4} + \frac{3}{2D_3}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+ \left(3D_4 + \frac{9D_3^2}{2}\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
\tag{7.11}
$$

Therefore, by Eq (7.9),

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(\frac{\beta^2}{2} - C_0 D_4 - C_0 D_3^2\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq C_0 + C_0\left(1 + \frac{1}{D_4} + \frac{1}{D_3} + \|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad + C_0\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Taking

$$
D_4 = \frac{\beta^2}{3C_0}, \quad D_3 = \frac{|\beta|}{\sqrt{7C_0}},
\tag{7.12}
$$

we have that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{42}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(0,\infty)}\\
&\quad\leq C_0 + C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

Integrating on $(0,t)$, by Eqs (1.9), (2.10), (2.9) and (2.15),

$$
\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{42}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|_{L^2(0,\infty)} ds\\
&\quad\leq C_0 + C_0 t + C_0\left(1 + \|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds
\end{aligned}
\tag{7.13}
$$

$$
\begin{aligned}
&+ C_0 \int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} ds \\
&\le C(T)\left(1 + \|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right) \\
&\le C(T)\left(1 + \sqrt{\left(1 + \left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(0,\infty))}\right)}\right).
\end{aligned}
$$

We prove Eq (2.16). Thanks to Eq (7.13), we have that

$$
\left\|\partial_x^2 u\right\|^2_{L^\infty(0,T;L^2(0,\infty))} \le C(T)\left(1 + \sqrt{\left(1 + \left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(0,\infty))}\right)}\right).
$$

Hence,

$$
\left\|\partial_x^2 u\right\|^4_{L^\infty(0,T;L^2(0,\infty))} - C(T)\left\|\partial_x^2 u\right\|_{L^\infty(0,T;L^2(0,\infty))} - C(T) \le 0.
$$

Arguing as in [74, Lemma 2.4], we have Eq (2.16).

Equation (2.17) follows from Eqs (2.16) and (7.13). Moreover, Eqs (2.14) and (2.16) give Eq (2.18), while Eq (2.19) follows from Eqs (2.15) and (2.16).

We prove Eq (2.20). Thanks to Eqs (7.11) and (7.12), we have that

$$
\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C_0\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right).
$$

Integrating on $(0,\infty)$, by Eqs (2.10) and (2.17), we have Eq (2.20).

Equations (2.21) and (2.22) follow from Eqs (2.10), (2.17), (2.20), (7.8) and an integration on $(0,t)$.

Finally, we prove Eq (2.23). Thanks to Eqs (1.3), (2.17), (2.18) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x u(t,x))^2 =& 2\int_0^x \partial_x u \partial_x^2 u dy + h^2(t) \le 2\int_0^\infty |\partial_x u||\partial_x^2 u| dx + C_0 \\
\le& 2\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)} + C_0 \le C(T).
\end{aligned}
$$

Hence,

$$
\|\partial_x u\|^2_{L^\infty((0,T)\times(0,\infty))} \le C(T),
$$

which gives Eq (2.23).

Proof of Lemma 3.1. Let $0 \le t \le T$. We begin by observing that, thanks to Eq (3.3),

$$
\begin{aligned}
2\int_0^\infty v\partial_t v dx =& \frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
4q\int_0^\infty v^2\partial_x v dx =& 0, \\
2\delta\int_0^\infty v\partial_x^3 v dx =& -2\delta\int_0^\infty \partial_x v \partial_x^2 v dx \\
2\beta^2\int_0^\infty v\partial_x^4 v dx =& -2\beta^2\int_0^\infty \partial_x v\partial_x^3 v dx \\
=& 2\beta^2\partial_x v(t,0)\partial_x^2 v(t,0) + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
=& -2\beta^2\partial_x v(t,0)g(t) + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}
\tag{7.14}
$$

$$2\gamma\int_0^\infty v^2\partial_x^2 v dx = -4\gamma\int_0^\infty v(\partial_x v)^2 dx.$$

Consequently, multiplying Eq (3.4) by $2v$, thanks to Eq (7.14), an integration on $(0,\infty)$ gives

$$\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2(\kappa-2\gamma)\int_0^\infty v(\partial_x u)^2 dx\\
&= -4qg(t)\int_0^\infty e^{-x}v^2 dx - 4qg(t)\int_0^\infty e^{-x}v\partial_x v dx + 2(2q-\kappa-\gamma)g^2(t)\int_0^\infty e^{-2x}v dx\\
&\quad + 4\kappa g(t)\int_0^\infty e^{-x}v\partial_x v dx - 2\nu g(t)\int_0^\infty e^{-x}v dx + 2\delta g(t)\int_0^\infty e^{-x}v dx\\
&\quad - 2\beta^2 g(t)\int_0^\infty e^{-x}v dx - 2\gamma g(t)\int_0^\infty e^{-x}v^2 ds - 2\gamma g(t)\int_0^\infty e^{-x}v\partial_x^2 v dx\\
&\quad + 2g'(t)\int_0^\infty e^{-x}v dx - 2\delta\int_0^\infty \partial_x v\partial_x^2 v dx - 2\beta^2\partial_x v(t,0)g(t).
\end{aligned}$$

Thanks to Eq (1.2), we have that

$$\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad = -4qg(t)\int_0^\infty e^{-x}v^2 dx - 4qg(t)\int_0^\infty e^{-x}v\partial_x v dx\\
&\quad + 2(2q-\kappa-\gamma)g^2(t)\int_0^\infty e^{-2x}v dx + 4\kappa g(t)\int_0^\infty e^{-x}v\partial_x v dx\\
&\quad - 2\nu g(t)\int_0^\infty e^{-x}v dx + 2\delta g(t)\int_0^\infty e^{-x}v dx\\
&\quad - 2\beta^2 g(t)\int_0^\infty e^{-x}v dx - 2\gamma g(t)\int_0^\infty e^{-x}v^2 ds\\
&\quad - 2\gamma g(t)\int_0^\infty e^{-x}v\partial_x^2 v dx + 2g'(t)\int_0^\infty e^{-x}v dx\\
&\quad - 2\delta\int_0^\infty \partial_x v\partial_x^2 v dx - 2\beta^2\partial_x v(t,0)g(t).
\end{aligned} \tag{7.15}$$

Since

$$\int_0^\infty e^{-4x}dx = \frac{1}{4}, \tag{7.16}$$

thanks to Eqs (1.4), (7.3), (7.16) and the Young inequality,

$$\begin{aligned}
&4|q||g(t)|\int_0^\infty e^{-x}v^2 dx \le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|q||g(t)|\int_0^\infty e^{-x}|v||\partial_x v|dx \le 2C_0\int_0^\infty |v||\partial_x v|dx\\
&\qquad \le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|2q-\kappa-\gamma|g^2(t)\int_0^\infty e^{-2x}|v|dx \le 2C_0\int_0^\infty e^{-2x}|v|dx
\end{aligned}$$

$$
\begin{aligned}
&\le C_0 \int_0^\infty e^{-4x} dx + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} \\
&\le C_0 + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
4|\kappa||g(t)| \int_0^\infty e^{-x} |v||\partial_x v| dx &\le 2C_0 \int_0^\infty |v||\partial_x v| dx \\
&\le C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} + C_0 \left\| \partial_x v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\nu||g(t)| \int_0^\infty e^{-x} |v| dx &\le 2C_0 \int_0^\infty e^{-x} |v| dx \\
&\le C_0 \int_0^\infty e^{-2x} dx + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} \\
&\le C_0 + +C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\delta||g(t)| \int_0^\infty e^{-x} |v| dx &\le 2C_0 \int_0^\infty e^{-x} |v| dx \\
&\le C_0 \int_0^\infty e^{-2x} dx + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} \\
&\le C_0 + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2\beta^2 |g(t)| \int_0^\infty e^{-x} |v| dx &\le 2C_0 \int_0^\infty e^{-x} |v| dx \\
&\le C_0 \int_0^\infty e^{-2x} dx + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} \\
&\le C_0 + \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\gamma||g(t)| \int_0^\infty e^{-x} v^2 dx &\le C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\gamma||g(t)| \int_0^\infty e^{-x} |v||\partial_x^2 v| dx &\le C_0 \int_0^\infty |v||\partial_x^2 v| dx \\
&= \int_0^\infty \left| \frac{C_0 v}{\beta} \right| \left| \beta \partial_x^2 v \right| dx \\
&\le C_0 \left\| v(t,\cdot) \right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\| \partial_x^2 v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|g'(t)| \int_0^\infty e^{-x} |v| dx &\le 2C_0 \int_0^\infty e^{-x} |v| dx \\
&\le C_0 \int_0^\infty e^{-2x} dx + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} \\
&\le C_0 + C_0 \left\| v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\delta| \int_0^\infty |\partial_x v||\partial_x^2 v| dx &= \int_0^\infty \left| \frac{2\delta \partial_x v}{\beta} \right| \left| \beta \partial_x^2 v \right| dx \\
&\le \frac{2\delta^2}{\beta^2} \left\| \partial_x v(t,\cdot) \right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2} \left\| \partial_x^2 v(t,\cdot) \right\|^2_{L^2(0,\infty)}, \\
2\beta^2 |\partial_x v(t,0)||g(t)| dx &\le 2C_0 |\partial_x v(t,0)| \le C_0 + (\partial_x v(t,0))^2.
\end{aligned}
$$

It follows from Eq (7.15) that

$$\begin{aligned}\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)}+\beta^2\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)}&\\ \le C_0\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0\|\partial_xv(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+(\partial_xv(t,0))^2+C_0.&\end{aligned}\tag{7.17}$$

Observe that

$$(\partial_xv(t,0))^2=-2\int_0^\infty\partial_xv\partial_x^2vdx\le2\int_0^\infty|\partial_xv||\partial_x^2v|dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}(\partial_xv(t,0))^2\le&\int_0^\infty\left|\frac{2\partial_xv}{\beta}\right|\left|\beta\partial_x^2v\right|dx\\ \le&\frac{2}{\beta^2}\|\partial_xv(t,\cdot)\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}\tag{7.18}$$

Consequently, by Eq (7.17),

$$\begin{aligned}\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)}&\\ \le C_0\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0\|\partial_xv(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0.&\end{aligned}\tag{7.19}$$

Observe that, thanks to Eq (3.3), we have that

$$C_0\|\partial_xv(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}=C_0\int_0^\infty\partial_xv\partial_xvdx=-C_0\int_0^\infty v\partial_x^2vdx.$$

Thanks to the Young inequality,

$$\begin{aligned}C_0\|\partial_xv(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}=&2\int_0^\infty\left|\frac{\sqrt{3}C_0v}{2\beta}\right|\left|\frac{\beta\partial_x^2v}{\sqrt{3}}\right|dx\\ \le&C_0\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\frac{\beta^2}{3}\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

If follows from Eq (7.19)

$$\begin{aligned}\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)}+\frac{\beta^2}{6}\left\|\partial_x^2v(t,\cdot)\right\|^2_{L^2(0,\infty)}&\\ \le C_0\|v(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0.&\end{aligned}$$

By the Gronwall Lemma and Eq (2.4), we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{L^2(0,\infty)}+\frac{\beta^2e^{C_0t}}{3}\int_0^te^{-C_0s}\left\|\partial_x^2v(s,\cdot)\right\|^2_{L^2(0,\infty)}ds&\\ \le C_0e^{C_0t}+C_0e^{C_0t}\int_0^te^{-C_0s}ds\le C(T),&\end{aligned}$$

which gives Eq (3.5).

Arguing as in Lemma 2.1, we have Eq (2.7).

Finally, we prove Eq (3.6). Integrating Eq (7.18) on $(0, t)$, we have that

$$\int_0^\infty (\partial_x v(s,0))^2 ds \le C_0 \int_0^t \|v(s,\cdot)\|^2_{L^2(\mathbb{R}_+)} ds + \frac{\beta^2}{3}\int_0^t \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds.$$

Equation (3.6) follows from Eqs (2.7) and (3.5).

Proof of Lemma 3.5. Let $0 \le t \le T$. We begin by observing that, thanks Eq (1.4), $\partial_t u(t,0) = g'(t)$. Consequentially, by Eq (1.4), we have that

$$\begin{aligned} 2\int_0^\infty \partial_x^4 u \partial_t u dx &= -2\partial_x^3 u(t,0)\partial_t u(t,0) - 2\int_0^\infty \partial_x^3 u \partial_t \partial_x u dx \\ &= -2\partial_x^3 u(t,0)\partial_t u(t,0) + \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\ &= -2\partial_x^3 u(t,0) g'(t) + \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{7.20}$$

Therefore, thanks to Eq (7.20), an integration of Eq (1.1) on $(0, \infty)$ gives

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\quad = -4q\int_0^\infty u\partial_x u \partial_x^4 u dx - 2\kappa\int_0^\infty (\partial_x u)^2 \partial_x^4 u dx - 2\nu\int_0^\infty \partial_x^2 u \partial_x^4 u dx \\ &\quad - 2\delta\int_0^\infty \partial_x^3 u \partial_x^4 u dx - 2\gamma\int_0^\infty u\partial_x^2 u \partial_x^4 u dx + 2\partial_x^3 u(t,0) g'(t). \end{aligned} \tag{7.21}$$

Due to Eqs (1.4), (2.19), (3.11) and the Young inequality,

$$\begin{aligned} 4|q|\int_0^\infty |u||\partial_x u||\partial_x^4 u| dx &\le 4|q|\,\|u\|_{L^\infty((0,T)\times\mathbb{R}_+)}\int_0^\infty |\partial_x u||\partial_x^4 u| dx \\ &\le 2C(T)\int_0^\infty |\partial_x u||\partial_x^4 u| dx = 2\int_0^\infty \left|\frac{C(T)\partial_x u}{\beta\sqrt{D_6}}\right|\left|\beta\sqrt{D_6}\partial_x^4 u\right| dx \\ &\le \frac{C(T)}{D_6}\|\partial_x u(t,\cdot)\|^2_{L^2((0,\infty))} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\le \frac{C(T)}{D_6} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\ 2|\kappa|\int_0^\infty (\partial_x u)^2 |\partial_x^4 u| dx &= 2\int_0^\infty \left|\frac{\kappa(\partial_x u)^2}{\beta\sqrt{D_6}}\right|\left|\beta\sqrt{D_6}\partial_x^4 u\right| dx \\ &\le \frac{\kappa^2}{\beta^2 D_6}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\ 2|\nu|\int_0^\infty |\partial_x^2 u||\partial_x^4 u| dx &= 2\int_0^\infty \left|\frac{\nu\partial_x^2 u}{\beta\sqrt{D_6}}\right|\left|\beta\sqrt{D_6}\partial_x^4 u\right| dx \\ &\le \frac{\nu^2}{\beta^2 D_6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \end{aligned}$$

$$
\begin{aligned}
2|\delta|\int_0^\infty |\partial_x^3 u||\partial_x^4 u|dx &= 2\int_0^\infty \left|\frac{\delta\partial_x^3 u}{\beta\sqrt{D_6}}\right|\left|\beta\sqrt{D_6}\partial_x^4 u\right|dx \\
&\le \frac{\delta^2}{\beta^2 D_6}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\gamma|\int_0^\infty |u||\partial_x^2 u||\partial_x^4 u|dx &\le 2|\gamma|\,\|u\|_{L^\infty((0,T)\times\mathbb{R}_+)}\int_0^\infty |\partial_x^2 u||\partial_x^4 u|dx \\
&\le 2C(T)\int_0^\infty |\partial_x^2 u||\partial_x^4 u|dx = 2\int_0^\infty \left|\frac{C(T)\partial_x^2 u}{\beta\sqrt{D_6}}\right|\left|\beta\sqrt{D_6}\partial_x^4 u\right|dx \\
&\le \frac{C(T)}{D_6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2 D_6\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \\
2|\partial_x^3 u(t,0)||g'(t)| &\le 2C_0|\partial_x^3 u(t,0)|| \le C_0 + (\partial_x^3 u(t,0))^2,
\end{aligned}
$$

where D_6 is a positive constant, which will be specified later. It follows from Eq (7.21) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2(2-5D_6)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
&\quad \le \frac{C(T)}{D_6} + \frac{\kappa^2}{\beta^2 D_6}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{C(T)}{D_6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
&\qquad + \frac{\delta^2}{\beta^2 D_6}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + C_0 + (\partial_x^3 u(t,0))^2.
\end{aligned}
$$

Taking $D_6 = 1/5$, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
&\quad \le C(T) + \frac{5\kappa^2}{\beta^2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
&\qquad + \frac{5\delta^2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + C_0 + (\partial_x^3 u(t,0))^2.
\end{aligned}
\tag{7.22}
$$

Observe that

$$
(\partial_x^3 u(t,0))^2 = -2\int_0^\infty \partial_x^3 u\partial_x^4 u dx \le 2\int_0^\infty |\partial_x^3 u||\partial_x^4 u|dx.
$$

Therefore, by the Young inequality,

$$
\begin{aligned}
(\partial_x^3 u(t,0))^2 &\le \int_0^\infty \left|\frac{2\partial_x^3 u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx \\
&\le \frac{2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
\tag{7.23}
$$

Consequentially, by Eq (7.22), we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} \\
&\quad \le C(T) + \frac{5\kappa^2}{\beta^2}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}
\end{aligned}
$$

$$+\frac{5\delta^2+2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+C_0.$$

Integrating on $(0, t)$, by Eqs (1.9), (2.10), (3.11) and (3.12), we get

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+\frac{\beta^2}{2}\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\quad\le C_0+C(T)t+\frac{5\kappa^2}{\beta^2}\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R}_+)}ds+C(T)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds\\
&\quad\quad+\frac{5\delta^2+2}{\beta^2}\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds+C_0 t\le C(T),
\end{aligned}$$

which gives Eq (3.16).

Equation (2.22), follows from Eqs (3.11), (3.16), (7.23) and an integration on $(0, t)$.

We prove Eq (2.23). Thanks to Eqs (3.11), (3.16) and the Hölder inequality,

$$\begin{aligned}
(\partial_x u(t,x))^2 =&2\int_0^x\partial_x u\partial_x^2 u dy+2\int_0^\infty\partial_x u\partial_x^2 u dx\le 4\int_0^\infty|\partial_x u||\partial_x^2 u|dx\\
\le&4\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\left\|\partial_x^2 u(t,\ \cdot)\right\|_{L^2(\mathbb{R}_+)}\le C(T).
\end{aligned}$$

Hence,

$$\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\le C(T),$$

which gives Eq (2.23).

Finally, we prove Eq (2.24). We begin by observing that, thanks to

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}=\int_0^\infty\partial_x^2 u(\partial_x^2 u)^3 dx=-3\int_0^\infty\partial_x u(\partial_x^2 u)^2\partial_x^3 u dx. \tag{7.24}$$

Thanks to Eq (2.23) and the Young inequality,

$$\begin{aligned}
3\int_0^\infty&|\partial_x u|(\partial_x^2 u)^2|\partial_x^3 u|dx=\int_0^\infty\left|3\partial_x u\partial_x^3 u\right|(\partial_x^2 u)^2 dx\\
&\le\frac{9}{2}\int_0^\infty(\partial_x u)^2(\partial_x^3 u)^2 dx+\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}\\
&\le\frac{9}{2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(0\,\infty)}\\
&\le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}.
\end{aligned}$$

It follows from Eq (7.24) that

$$\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R}_+)}\le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.$$

An integration on $(0, t)$ and Eq (3.11) give Eq (2.24).

Arguing as in Section 2, we have Lemma 2.7.

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $T > 0$. Thanks to Lemmas 2.7, 3.2, 3.4, 3.5 and the Cauchy-Kovalevskaya Theorem [70], we have that u is solution of Eqs (1.1)-(1.4)-(1.8) and (1.13) holds.

Arguing as in Section 2, we have Eq (1.14).

Proof of Lemma 4.1. Let $0 \leq t \leq T$. Multiplying Eq (1.1) by $2u$, thanks to Eqs (1.2) and (1.5), an integration on $(0, \infty)$ gives

$$\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} =&2\int_0^\infty u\partial_t u dx\\
=&-4q\int_0^\infty u^2\partial_x u dx + 6\tau^2\int_0^t u^3\partial_x u dx - 2\kappa\int_0^\infty u(\partial_x u)^2 dx\\
&-2\nu\int_0^\infty u\partial_x^2 u dx - 2\delta\int_0^\infty u\partial_x^3 u dx - 2\beta^2\int_0^\infty u\partial_x^4 u dx\\
&-2\gamma\int_0^\infty u^2\partial_x^2 u dx\\
=&\frac{4q}{3}u^3(t,0) - \frac{3\tau^2}{2}u^4(t,0) + 2(2\gamma-\kappa)\int_0^\infty u(\partial_x u)^2 dx\\
&-2\nu\int_0^\infty u\partial_x^2 u dx + 2\delta\int_0^\infty \partial_x u\partial_x^2 u dx + 2\beta^2\int_0^\infty \partial_x u\partial_x^3 u dx\\
&+2\gamma u^2(t,0)\partial_x u(t,0)\\
=&\frac{4q}{3}u^3(t,0) - \frac{3\tau^2}{2}u^4(t,0) - 2\nu\int_0^\infty u\partial_x^2 u dx\\
&+2\delta\int_0^\infty \partial_x u\partial_x^2 u dx - 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2\gamma u^2(t,0)\partial_x u(t,0).
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{3\tau^2}{2}u^4(t,0)\\
&\quad= \frac{4q}{3}u^3(t,0) - 2\nu\int_0^\infty u\partial_x^2 u dx + 2\delta\int_0^\infty \partial_x u\partial_x^2 u dx\\
&\qquad+ 2\gamma u^2(t,0)\partial_x u(t,0).
\end{aligned} \tag{7.25}$$

Due to the Young inequality,

$$\begin{aligned}
\left|\frac{4q}{3}\right||u(t,0)|^3 =& 2\left|\frac{2qu(t,0)}{\tau\sqrt{D_7}}\right|\left|\tau\sqrt{D_3}u^2(t,0)\right| dx\\
\leq& \frac{4q^2}{\tau^2 D_7}u^2(t,0) + \tau^2 D_7 u^4(t,0) = 2\frac{q^2}{\tau^2|\tau|D_7\sqrt{D_7}}|\tau|\sqrt{D_7}u^2(t,0) + \tau^2 D_7 u^4(t,0)\\
\leq& \frac{q^4}{\tau^6 D_7^3} + 2\tau^2 D_7 u^4(t,0),\\
2\nu\int_0^\infty |u||\partial_x^2 u| dx =& \int_0^\infty \left|\frac{2\nu u}{\beta}\right|\left|\beta\partial_x^2 u\right| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2\nu^2}{\beta^2} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
2|\delta| \int_0^\infty |\partial_x u||\partial_x^2 u| dx &= \int_0^\infty \left|\frac{2\delta\partial_x u}{\beta}\right| \left|\beta\partial_x^2 u\right| dx\\
&\leq \frac{2\delta^2}{\beta^2} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},\\
2|\gamma| u^2(t,0)|\partial_x u(t,0)| &= 2|\tau \sqrt{D_7} u^2(t,0) \frac{|\gamma\partial_x u(t,0)}{|\tau| \sqrt{D_7}}\\
&\leq \tau^2 D_7 u^4(t,0) + \frac{\gamma^2}{\tau^2 D_7} (\partial_x u(t,0))^2,
\end{aligned}
$$

where D_7 is a positive constant, which will be specified later. It follows from Eq (7.25) that

$$
\begin{aligned}
&\frac{d}{dt} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 3\tau^2 \left(\frac{1}{4} - D_7\right) u^4(t,0)\\
&\qquad \leq C_0 \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{C_0}{D_7^3} + \frac{C_0}{D_7} (\partial_x u(t,0))^2.
\end{aligned}
$$

Taking $D_7 = 1/20$, we have that

$$
\begin{aligned}
&\frac{d}{dt} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{3\tau^2}{20} u^4(t,0)\\
&\qquad \leq C_0 \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0(\partial_x u(t,0))^2 + C_0\\
&\qquad \leq C_0 \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + 2C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0(\partial_x u(t,0))^2 + C_0.
\end{aligned}
\tag{7.26}
$$

Observe that

$$
\begin{aligned}
C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0(\partial_x u(t,0))^2 &= C_0 \int_0^\infty \partial_x u \partial_x u dx - 2C_0 \int_0^\infty \partial_x u \partial_x^2 u dx\\
&= -C_0 u(t,0)\partial_x u(t,0) - C_0 \int_0^\infty u\partial_x^2 u dx - 2C_0 \int_0^\infty \partial_x u \partial_x^2 u dx.
\end{aligned}
\tag{7.27}
$$

Due to the Young inequality,

$$
\begin{aligned}
2C_0 \int_0^\infty |\partial_x u||\partial_x^2 u| dx &= 2\int_0^\infty \left|\frac{\partial_x u}{\sqrt{D_8}}\right| \left|C_0 \sqrt{D_8} \partial_x^2 u\right| dx\\
&\leq \frac{1}{D_8} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0 D_8 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&= -\frac{1}{D_8} u(t,0)\partial_x u(t,0) - \frac{1}{D_8} \int_0^\infty u\partial_x^2 u dx + D_8 C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

where D_8 is a positive constant, which will be specified later. It follows from Eq (7.27) that

$$
\begin{aligned}
&C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0(\partial_x u(t,0))^2\\
&\qquad \leq -\left(C_0 + \frac{1}{D_8}\right) u(t,0)\partial_x u(t,0) - \left(C_0 + \frac{1}{D_8}\right) \int_0^\infty u\partial_x^2 u dx
\end{aligned}
\tag{7.28}
$$

$$+ D_8C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.$$

Due to the Young inequality,

$$\begin{aligned}
\left(C_0+\frac{1}{D_8}\right)|u(t,0)||\partial_x u(t,0)| &= \frac{\left(C_0+\frac{1}{D_8}\right)}{C_0}|u(t,0)|C_0|\partial_x u(t,0)|\\
&\le \left(\frac{\left(C_0+\frac{1}{D_8}\right)}{C_0}\right)^2 u^2(t,0)+\frac{C_0}{2}(\partial_x u(t,0))^2,\\
\left(C_0+\frac{1}{D_8}\right)\int_0^\infty |u||\partial_x^2 u|dx &= 2\int_0^\infty \left|\frac{\left(C_0+\frac{1}{D_8}\right)u}{2\sqrt{D_8}}\right|\left|\sqrt{D_8}\partial_x^4 u\right|dx\\
&\frac{\left(C_0+\frac{1}{D_8}\right)^2}{4D_8}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + D_8\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Therefore, by Eq (7.28), we have that

$$\begin{aligned}
&C_0\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\frac{C_0}{2}(\partial_x u(t,0))^2\\
&\quad\le \left(\frac{\left(C_0+\frac{1}{D_8}\right)}{C_0}\right)^2 u^2(t,0)+\frac{\left(C_0+\frac{1}{D_8}\right)^2}{4D_8}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\quad + D_8(C_0+1)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}$$

which gives

$$\begin{aligned}
&2C_0\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0(\partial_x u(t,0))^2\\
&\quad\le \left(\frac{2\left(C_0+\frac{1}{D_8}\right)}{C_0}\right)^2 u^2(t,0)+\frac{2\left(C_0+\frac{1}{D_8}\right)^2}{4D_8}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\\
&\quad\quad + 2D_8(C_0+1)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}\tag{7.29}$$

It follows from Eqs (7.26) and (7.29) that

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\left(\beta^2-2D_8(1+C_0)\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+\frac{3\tau^2}{20}u^4(t,0)\\
&\quad\le \left(C_0+\frac{2\left(C_0+\frac{1}{D_8}\right)^2}{4D_8}\right)\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\left(\frac{2\left(C_0+\frac{1}{D_8}\right)}{C_0}\right)^2 u^2(t,0)+C_0.
\end{aligned}$$

Taking

$$D_8=\frac{\beta^2}{4(1+C_0)},\tag{7.30}$$

we have that

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}+\frac{3\tau^2}{20}u^4(t,0)\\
&\quad\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}+C_0u^2(t,0)+C_0.
\end{aligned}\tag{7.31}$$

Thanks to the Young inequality

$$C_0u^2(t,0) = 2\frac{C_0\sqrt{20}}{2\sqrt{2}|\tau|}\frac{\sqrt{2}|\tau|u^2(t,0)}{20} \le C_0 + \frac{2\tau^2}{20}u^4(t,0). \tag{7.32}$$

It follows from Eq (7.31) that

$$\begin{aligned}&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2}{2}\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + \frac{\tau^2}{20}u^4(t,0)\\ &\quad\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + C_0.\end{aligned}$$

By the Gronwall Lemma and Eq (1.9), we get

$$\begin{aligned}&\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{\beta^2e^{C_0t}}{2}\int_0^t e^{-C_0s}\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}ds + \frac{\tau^2e^{C_0t}}{20}\int_0^t e^{-C_0s}u^4(t,0)ds\\ &\le C_0e^{C_0t} \le C(T),\end{aligned}$$

which gives Eq (4.1).

Equation (4.2) follows Eqs (4.1), (7.32) and an integration on (0, t), while, Eqs (4.1) (4.2), (7.29), (7.30) and an integration on (0, t) give Eqs (2.9) and (3.7).

Finally, we prove Eq (2.9). We begin by proving

$$\int_0^t \|\partial_xu(s,\cdot)\|^2_{L^\infty(0,\infty)}ds \le C(T). \tag{7.33}$$

By the Young inequality,

$$\begin{aligned}(\partial_xu(t,x))^2 =&2\int_0^x \partial_xu\partial_x^2u dy + 2(\partial_xu(t,0))^2 \le 2\int_0^\infty \partial_xu\partial_x^2u dx + 2(\partial_xu(t,0))^2\\ \le&2\int_0^\infty |\partial_xu||\partial_x^2u|dx + 2(\partial_xu(t,0))^2\\ \le&\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2(\partial_xu(t,0))^2.\end{aligned}$$

Hence,

$$\|\partial_xu(t,\cdot)\|^2_{L^\infty(0,\infty)} \le \|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + 2(\partial_xu(t,0))^2.$$

Equation (7.33) follows from Eqs (2.9), (3.7), (4.1) and an integration on (0, t). We prove

$$\int_0^t \|u(s,\cdot)\partial_xu(s,\cdot)\|^2_{L^2(\mathbb{R}_+)}ds \le C(T). \tag{7.34}$$

Observe that, by Eq (4.1),

$$\int_0^\infty u^2(\partial_xu)^2dx \le \|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}\|\partial_xu(t,\cdot)\|^2_{L^\infty(0,\infty)} \le C(T)\|\partial_xu(t,\cdot)\|^2_{L^\infty(0,\infty)}.$$

An integration on (0, t) gives Eq (7.34).

Therefore, the proof is concluded.

Proof of Lemma 4.2. Let $0 \le t \le T$. We begin by proving Eq (4.3). Observe that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty \partial_x u \partial_x u = -u(t,0)\partial_x u(t,0) - \int_0^\infty u\partial_x^2 u dx. \tag{7.35}$$

Thanks to Eq (4.1) and the Hölder and the Young inequalities,

$$|u(t,0)||\partial_x u(t,0)| \le \frac{1}{2}u^2(t,0) + \frac{1}{2}(\partial_x u(t,0))^2 \le u^2(t,0) + (\partial_x u(t,0))^2$$
$$\int_0^\infty |u||\partial_x^2 u| \le \|u(t,\cdot)\|_{L^2(\mathbb{R}_+)} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)} \le C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)}.$$

Therefore, by Eq (7.35), we have that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} \le u^2(t,0) + (\partial_x u(t,0))^2 + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)}. \tag{7.36}$$

Observe that

$$u^2(t,0) + (\partial_x u(t,0))^2 = -2\int_0^\infty u\partial_x u dx - 2\int_0^\infty \partial_x u\partial_x^2 u dx. \tag{7.37}$$

Due to Eq (4.1) and the Hölder and the Young inequalities,

$$\begin{aligned}
2\int_0^\infty |u||\partial_x u| dx &= 2\int_0^\infty \left|\frac{u}{\sqrt{D_8}}\right|\left|\sqrt{D_8}\partial_x u\right| dx \\
&\le \frac{1}{D_8}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + D_8\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} \\
&\le \frac{C(T)}{D_8} + D_8\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)}, \\
2\int_0^\infty |\partial_x u||\partial_x^2 u| dx &= 2\int_0^\infty \left|\sqrt{D_8}\partial_x u\right|\left|\frac{\partial_x^2 u}{\sqrt{D_8}}\right| dx \\
&\le D_8\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{D_8}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}$$

where D_8 is a positive constant, which will be specified later. It follows from Eq (7.36) that

$$\begin{aligned}
&u^2(t,0) + (\partial_x u(t,0))^2 \\
&\quad \le \frac{C(T)}{D_8} + 2D_8\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{D_8}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned} \tag{7.38}$$

Therefore, by Eqs (7.36) and (7.38)

$$(1 - 2D_8)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} \le \frac{C(T)}{D_8} + \frac{1}{D_8}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)}.$$

Taking $D_8 = 1/4$, thanks to the Young inequality, we have that

$$\begin{aligned}
\frac{1}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R}_+)} &\le C(T) + 4\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)} + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}_+)} \\
&\le C(T)\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right),
\end{aligned}$$

which gives Eq (4.3).

We prove Eq (4.4). Thanks to Eqs (4.1), (4.3) and the Hölder inequality,

$$\begin{aligned} u^2(t,x) =&2\int_0^x u\partial_x u dy + u^2(t,0) \le 2\int_0^\infty u\partial_x u dx - 2\int_0^\infty u\partial_x u dx \le 4\int_0^\infty |u|\partial_x u| dx \\ \le& 4\,\|u(t,\cdot)\|_{L^2(\mathbb{R}_+)}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R}_+)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right)}. \end{aligned}$$

Hence,

$$\|u(t,\cdot)\|^2_{L^\infty(0,\infty)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right)},$$

which gives Eq (4.4).

Finally, we prove Eq (4.5). We begin by proving that

$$\|u(\cdot,0)\|^2_{L^\infty(0,T)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)}. \tag{7.39}$$

Observe that, thanks Eqs (4.1), (4.3) and the Hölder inequality,

$$\begin{aligned} u^2(t,0) =& -2\int_0^\infty u\partial_x u dx \le 2\int_0^\infty |u||\partial_x u| dx \\ \le& 2\,\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}\right)}. \end{aligned}$$

Hence,

$$\|u(\cdot,\ 0)\|^2_{L^\infty(0,T)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)},$$

which gives Eq (7.39).

Observe that

$$\begin{aligned} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} =& \int_0^\infty \partial_x u(\partial_x^2 u)^3 dx \\ =& -u(t,0)(\partial_x u(t,0))^3 - 3\int_0^\infty u(\partial_x u)^2\partial_x^2 u dx \\ =& 3u(t,0)\int_0^\infty (\partial_x u)^2\partial_x^2 u dx - 3\int_0^\infty u(\partial_x u)^2\partial_x^2 u dx. \end{aligned} \tag{7.40}$$

Due to the Young inequality,

$$\begin{aligned} 3|u(t,0)|\int_0^\infty (\partial_x u)^2|\partial_x^2 u| dx =& \int_0^\infty (\partial_x u)^2|3u(t,0)\partial_x^2 u| dx \\ \le& \frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{(0,\infty)} + \frac{9}{2}\int_0^\infty u^2(t,0)(\partial_x^2 u)^2 dx \\ \le& \frac{1}{2}\|\partial_x u(t,\cdot)\|^4_{(0,\infty)} + \frac{9}{2}\|u(\cdot,0)\|^2_{L^\infty(0,T)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}, \end{aligned}$$

$$
\begin{aligned}
3\int_0^\infty |u|(\partial_x u)^2|\partial_x^2 u|dx &= 2\int_0^\infty \left|\frac{(\partial_x u)^2}{\sqrt{3}}\right|\left|\frac{3\sqrt{3}u\partial_x^2 u}{2}\right|dx\\
&\le \frac{1}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{27}{4}\int_0^\infty u^2(\partial_x^2 u)^2 dx\\
&\le \frac{1}{3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} + \frac{27}{4}\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
$$

It follows from Eq (7.40) that

$$
\begin{aligned}
&\frac{1}{6}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)}\\
&\quad\le \frac{9}{2}\left(\|u(\cdot,0)\|^2_{L^\infty(0,T)} + \frac{3}{2}\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
\end{aligned}
\tag{7.41}
$$

By Eq (4.4), we have that

$$
\|u\|^2_{L^\infty((0,T)\times\mathbb{R}_+)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)}. \tag{7.42}
$$

Therefore, by Eqs (7.39), (7.41) and (7.42),

$$
\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R}_+)} \le C(T)\sqrt{\left(1+\left\|\partial_x^2 u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}_+))}\right)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R}_+)}.
$$

An integration on $(0, t)$ and Eq (4.1) give Eq (4.5)

8. Conclusions

In this paper, we have investigated the initial-boundary value problem associated with the Kuramoto-Velarde equation on the half-line. Under suitable structural assumptions on the coefficients and assuming initial data in $H^2(0,\infty)$, we established the well-posedness of the problem for a wide class of boundary conditions, including Dirichlet, mixed, and higher-order boundary constraints. The main result guarantees existence, uniqueness, and continuous dependence on the initial data of solutions in appropriate Sobolev spaces with precise stability estimates. The analysis relies on the derivation of delicate a priori estimates, which are obtained through energy methods tailored to the high-order and strongly nonlinear structure of the equation. A crucial role is played by the compatibility condition between the nonlinear terms and the fourth-order dissipation, which allows us to control the nonlinear effects and close the estimates. The use of suitable auxiliary functions further enables the treatment of nonhomogeneous boundary conditions. From a mathematical perspective, these results extend the theory for the Kuramoto-Velarde and related equations, which has so far been mainly focused on the Cauchy problem, by providing a comprehensive well-posedness theory in the presence of boundaries. From a physical viewpoint, the analysis supports the mathematical consistency of models arising in crystal growth, spinodal decomposition, and interface dynamics when boundary effects are taken into account. However, several questions remain open. Possible directions for future research include the study of global-in-time behavior, the existence and stability of stationary or traveling wave solutions under boundary constraints, and the extension of the present analysis to weaker initial regularity or to multidimensional settings. Another interesting perspective concerns the investigation of control and stabilization problems for the Kuramoto-Velarde equation in bounded or semi-bounded domains.

Author contributions

Giuseppe Maria Coclite and Lorenzo di Ruvo equally contributed to the methodologies, typesetting, and the development of the paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

GMC is member of the Gruppo Nazionale per lOAnalisi Matematica, la Probabilita e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). GMC has been partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.4 -Call for tender No. 3138 of 16/12/2021 of Italian Ministry of University and Research funded by the European Union -NextGenerationEUoAward Number: CN000023, Concession Decree No. 1033 of 17/06/2022 adopted by the Italian Ministry of University and Research, CUP: D93C22000410001, Centro Nazionale per la Mobilità Sostenibile, the Italian Ministry of Education, University and Research under the Programme Department of Excellence Legge 232/2016 (Grant No. CUP - D93C23000100001), and the Research Project of National Relevance "Evolution problems involving interacting scales" granted by the Italian Ministry of Education, University and Research (MIUR Prin 2022, project code 2022M9BKBC, Grant No. CUP D53D23005880006).

Conflict of interest

Giuseppe Maria Coclite is an editorial board member for NHM and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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