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Research article

Determinant approach of the (p, q)-Hermite-Appell polynomials and some of their components

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Abstract: In this work, we offer the novel class of (p, q)-Hermite-Appell polynomials. Some attributes of this class are constructed, along with the generating function, series definition, (p, q)-derivative properties, (p, q)-integral representation, summation formulas, and determinate representation. Additionally, we consider a few components for the (p, q)-Hermite-Appell polynomials and infer certain elements of their traits. The generating function and series expansions of some classes of two-dimensional (p, q)-Hermite-Appell polynomials are provided. Moreover, we acquire a (p, q)-differential operator formula for (p, q)-Hermite-Appell polynomials. Finally, the Wolfram Mathematica software is used to plot the graphical diagrams of select components of (p, q)-Hermite-Appell, along with two-dimensional (p, q)-Hermite-Appell polynomials.

Keywords: (p, q)-polynomials; (p, q)-Appell polynomials; (p, q)-Hermite polynomials; determinant approach

1. Introduction

Recently, (p, q)-calculus and its applications in a variety of mathematical, physics, and engineering domains have garnered increased attention. The theory of (p, q)-calculus was created by numerous mathematicians and physicists as a supplement for q-calculus. Currently, (p, q)-calculus is commonly utilized in numerous domains calculus, differential equations, quantum theory, number theory, and approximation theory, *etc.* [1–4]. (p, q)-analogues for multiple ordinary special functions, as well as polynomials as Beta, Gamma, Euler, Bernoulli, and other polynomials, have been explored and investigated [5–8].

Let $0 < q < p \leq 1$ and $\theta \in \mathbb{N}$. The (p,q)-number $[\theta]_{p,q}$ corresponds to the following outline [9–14]:

$$[\theta]_{p,q} = \frac{q^{\theta} - p^{\theta}}{q - p}.$$

Let $\theta \geq 1$. The description of the (p,q)-factorial was stated as follows [9–14]:

$$[\theta]_{p,q}! = \prod_{\upsilon=0}^{\theta} [\upsilon]_{p,q},$$

with $[0]_{p,q}! = 1$. The next determines the (p,q)-binomial coefficient are as follows [2, 3]:

$$\begin{bmatrix} \upsilon \\ \theta \end{bmatrix}_{p,q} = \frac{[\upsilon]_{p,q}!}{[\theta]_{p,q}![\upsilon - \theta]_{p,q}!}.$$

The (p,q)-derivative for an expression f alongside regards to o, expressed as $D_{p,q;o}f(o)$, is stated as follows [15]:

$$D_{p,q;o}f(o) := D_{p,q}f(o) = \frac{f(qo) - f(po)}{(q-p)\,o} \qquad \left(D_{p,q}f(o) \text{ when } o \neq 0; f'(0) \text{ when } o = 0\right), \tag{1.1}$$

which satisfies the next rule

$$D_{p,q}(f(o)\,g(o)) = f(po)\,D_{p,q}g(o) + g(qo)\,D_{p,q}f(o).$$

According to reference [2], each of the (p,q)-exponential expressions, symbolized through $e_{p,q}(o)$ and $E_{p,q}(o)$, are outlined below:

$$e_{p,q}(o) = \sum_{\theta=0}^{\infty} \frac{p^{\binom{\theta}{2}} o^{\theta}}{[\theta]_{p,q}!} \tag{1.2}$$

and

$$E_{p,q}(o) = \sum_{\theta=0}^{\infty} \frac{q^{\binom{\theta}{2}} o^{\theta}}{[\theta]_{p,q}!}. \tag{1.3}$$

From Eqs (1.1)–(1.3), it can be inferred that

$$D_{p,q}e_{p,q}(o) = e_{p,q}(po) \text{ and } D_{p,q}E_{p,q}(o) = E_{p,q}(qo). \tag{1.4}$$

Considering a function f, the value for the (p,q)-definite integral is established through the following:

$$\int_0^a f(o)\,d_{p,q}o = (p-q)\,a \sum_{\theta=0}^{\infty} \frac{p^{\theta}}{q^{\theta+1}} f\left(\frac{p^{\theta}}{q^{\theta+1}} a\right). \tag{1.5}$$

Duran *et al.* [4] created and categorized what are known as (p,q)-Hermite polynomials $H_{\theta,p,q}(o)$ using a specific generating relation:

$$e_{p,q}([2]_{p,q}o\zeta)\, e_{p,q}(-\zeta^2) = \sum_{\theta=0}^{\infty} H_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}. \tag{1.6}$$

Sadjang [7] offered a generating formula that generates (p,q)-Appell polynomials class $\{\mathcal{A}_{\theta,p,q}(o)\}_{\theta=0}^{\infty}$:

$$e_{p,q}(o\zeta)\mathcal{A}_{p,q}(\zeta) = \sum_{\theta=0}^{\infty} \frac{\mathcal{A}_{\theta,p,q}(o)\zeta^\theta}{[\theta]_{p,q}!}, \qquad \mathcal{A}_{p,q}(\zeta) \neq 0, \;\; \mathcal{A}_{0,p,q} = 1, \tag{1.7}$$

where $\mathcal{A}_{\theta,p,q} := \mathcal{A}_{\theta,p,q}(0)$ signifies the (p,q)-Appell-numbers as well as

$$\mathcal{A}_{p,q}(\zeta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}\frac{\zeta^\theta}{[\theta]_{p,q}!}, \qquad \mathcal{A}_{0,p,q} \neq 0. \tag{1.8}$$

According to reference [7], the series formula for the (p,q)AP $\mathcal{A}_{\theta,p,q}(o)$ was provided as follows [7]:

$$\mathcal{A}_{\theta,p,q}(o) = \sum_{\upsilon=0}^{\theta} p^{\binom{\theta-\upsilon}{2}} o^{\theta-\upsilon}\mathcal{A}_{\upsilon,p,q}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}, \qquad \mathcal{A}_{0,p,q} \neq 0.$$

Table 1 summarizes the components of the (p,q)-Appell polynomial classes.

Table 1. A few known components of $\mathcal{A}_{\theta,p,q}(o)$.

S.No.	$\mathcal{A}_{p,q}(\zeta)$	Generating function	Polynomials
I	$\mathcal{A}_{p,q}(\zeta) = \frac{\zeta}{e_{p,q}(\zeta)-1}$	$\frac{\zeta}{e_{p,q}(\zeta)-1}\, e_{p,q}(o\zeta) = \sum_{\theta=0}^{\infty} \mathcal{B}_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	(p,q)-Bernoulli polynomials [5,7]
II	$\mathcal{A}_{p,q}(\zeta) = \frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}\, e_{p,q}(o\zeta) = \sum_{\theta=0}^{\infty} \mathcal{E}_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	(p,q)-Euler polynomials [5]
III	$\mathcal{A}_{p,q}(\zeta) = \frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}\, e_{p,q}(o\zeta) = \sum_{\theta=0}^{\infty} \mathcal{G}_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	(p,q)-Genocchi polynomials [5]

The generating function for the two-dimensional $(p,q)AP$ $\mathcal{A}_{\theta,p,q}(o,\mu)$ is provided by the following [8]:

$$e_{p,q}(o\zeta)\, E_{p,q}(\mu\zeta)\mathcal{A}_{p,q}(\zeta) = \sum_{\theta=0}^{\infty} \frac{\mathcal{A}_{\theta,p,q}(o,\mu)\zeta^\theta}{[\theta]_{p,q}!}, \qquad \mathcal{A}_{\theta,p,q} = \mathcal{A}_{\theta,p,q}(0,0). \tag{1.9}$$

Some components of the two-dimensional $(p,q)AP$ $\mathcal{A}_{\theta,p,q}(o,\mu)$ are listed under their conditions in Table 2.

Table 2. Some categories of $\mathcal{A}_{\theta,p,q}(o,\mu)$.

S.No.	$\mathcal{A}_{p,q}(\zeta)$	Generating function	Polynomials
I	$\mathcal{A}_{p,q}(\zeta) = \frac{\zeta}{e_{p,q}(\zeta)-1}$	$\frac{\zeta}{e_{p,q}(\zeta)-1}\, e_{p,q}(o\zeta)\, E_{p,q}(\mu\zeta) = \sum_{\theta=0}^{\infty} \mathcal{B}_{\theta,p,q}(o,\mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	Two-dimensional (p,q)-Bernoulli polynomials [8]
II	$\mathcal{A}_{p,q}(\zeta) = \frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}\, e_{p,q}(o\zeta)\, E_{p,q}(\mu\zeta) = \sum_{\theta=0}^{\infty} \mathcal{E}_{\theta,p,q}(o,\mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	Two-dimensional (p,q)-Euler polynomials [8]
III	$\mathcal{A}_{p,q}(\zeta) = \frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}\, e_{p,q}(o\zeta)\, E_{p,q}(\mu\zeta) = \sum_{\theta=0}^{\infty} \mathcal{G}_{\theta,p,q}(o,\mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}$	Two-dimensional (p,q)-Genocchi polynomials [8]

This paper is motivated by the work of Duran *et al.* [4] on (p,q)-Hermite polynomials and the discovery that both ordinary Hermite polynomials and (p,q)-Hermite polynomials have applications in various branches of mathematics and science, such as quantum harmonic oscillators and quantum physics, signal processing, and combinatorics. Additional motivation comes from (p,q)-calculus's versatility in many mathematical and scientific domains. Furthermore, this work is motivated by the usefulness of determinant approaches for some special polynomials and their generalization as a linear interpolation problem [16]. The work in this paper is organized in the following manner: in Section 2, the generating function that generates (p,q)-Hermite-Appell polynomials ${}_H\mathcal{A}_{\theta,p,q}(o)$ and some of their certain properties are presented and explored; in Section 3, some determinant approaches for (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ are provided; within Section 4, certain components that generate (p,q)-Hermite-Appell polynomials, including (p,q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$, (p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$, and (p,q)HGP ${}_H\mathcal{G}_{\theta,p,q}(o)$ are investigated and their generating function, series definition and determinant approaches are discussed; the generating function and series definition for the categories of two-dimensional (p,q)-Hermite-Appell polynomials are discussed and examined in Section 5; and within Section 6, a few graphical diagrams for some classes of (p,q)-Hermite-Appell polynomials are shown for adequately indexed quantities.

2. (p,q)-Hermite-Appell polynomials

Section 2 discusses and analyzes the generating function and series definition for the categories of two-dimensional (p,q)-Hermite-Appell polynomials ${}_HA_{\theta,p,q}(o)$.

We observe from the left part of Eq (1.7) that by expanding the primary exponential function $e_{p,q}(o\zeta)$ and then replacing the power of o, (i.e., $o^0, o^1, po^2, \ldots, p^{\binom{\theta}{2}}o^\theta$) by substituting the associated polynomials $H_{0,p,q}(o), H_{1,p,q}(o), H_{2,p,q}(o), \ldots, H_{\theta,p,q}(o)$ at the left part and $H_{1,p,q}(o)$ in the right part within the final formula, we receive the following:

$$\mathcal{A}_{p,q}(\zeta)\left(1 + H_{1,p,q}(o)\frac{\zeta}{[1]_{p,q}!} + H_{2,p,q}(o)\frac{\zeta^2}{[2]_{p,q}!} + \cdots + H_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}\right) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}\Big(H_{1,p,q}(o)\Big)\frac{\zeta^\theta}{[\theta]_{p,q}!}.$$

Furthermore, we sum up the expansion in the left part, then employ formula (1.6) within the resultant formula to generate

$$e_{p,q}([2]_{p,q}o\zeta)e_{p,q}(-\zeta^2)\mathcal{A}_{p,q}(\zeta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}\Big(H_{1,p,q}(o)\Big)\frac{\zeta^\theta}{[\theta]_{p,q}!}.$$

Lastly, we write the outcome of (p,q)HAP on the right part of the previous equation as follows:

$$\mathcal{A}_{\theta,p,q}\Big(H_{1,p,q}(o)\Big) = {}_H\mathcal{A}_{\theta,p,q}(o),$$

which provides the following definition.

Definition 1. We generate the function that produces the (p,q)HAP ${}_HA_{\theta,p,q}(o)$ as follows:

$$e_{p,q}([2]_{p,q}o\zeta)e_{p,q}(-\zeta^2)\mathcal{A}_{p,q}(\zeta) = \sum_{\theta=0}^{\infty} \frac{{}_H\mathcal{A}_{\theta,p,q}(o)\zeta^\theta}{[\theta]_{p,q}!}, \tag{2.1}$$

whereby $H_{\theta,p,q}(o)$ and $\mathcal{A}_{p,q}(\zeta)$ are provided through Eqs (1.6) and (1.8), respectively.

Now, we examine several properties of ${}_{H}\mathcal{A}_{\theta,p,q}(o)$ by the following consecutive theorems with their proofs.

Theorem 1. The subsequent series definition for the (p,q)HAP ${}_{H}A_{\theta,p,q}(o)$ can be expanded as:

$$ {}_{H}\mathcal{A}_{\theta,p,q}(o)=\sum_{\upsilon=0}^{\theta} A_{\upsilon,p,q}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} H_{\theta-\upsilon,p,q}(o), \qquad A_{0,p,q}\neq 0. \tag{2.2}$$

Proof. Applying Eqs (1.6) and (1.8) to the left portion for expression (2.1) yields the following:

$$\sum_{\theta=0}^{\infty} H_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}\sum_{\upsilon=0}^{\infty} A_{\upsilon,p,q}\frac{\zeta^{\upsilon}}{[\upsilon]_{p,q}!}=\sum_{\theta=0}^{\infty} {}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}.$$

Through the application of the Cauchy product formulation, we can achieve the following:

$$\sum_{\theta=0}^{\infty} {}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}=\sum_{\theta=0}^{\infty}\sum_{\upsilon=0}^{\theta}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}\mathcal{A}_{\upsilon,p,q}\, H_{\theta-\upsilon,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}.$$

Statement (2.2) is generated by corresponding coefficients of similar powers of ζ at every part of the previously provided equation.

Theorem 2. The subsequent standard summation expression for (p,q)HAP ${}_{H}A_{\theta,p,q}(o)$ can be given by the following:

$$ {}_{H}\mathcal{A}_{\theta,p,q}(o)=\sum_{\upsilon=0}^{\lfloor\frac{\theta}{2}\rfloor}\frac{(-1)^{\upsilon}\,[\theta]_{p,q}!\mathcal{A}_{\theta-2\upsilon,p,q}(o)}{[\upsilon]_{p,q}!\,[\theta-2\upsilon]_{p,q}!}, \tag{2.3}$$

where $\lfloor\cdot\rfloor$ marks the highest integer function.

Proof. Using (cf. [4])

$$\sum_{\theta=0}^{\infty}\sum_{\upsilon=0}^{\infty} A(\upsilon,\theta)=\sum_{\theta=0}^{\infty}\sum_{\upsilon=0}^{\lfloor\frac{\theta}{2}\rfloor} A(\upsilon,\theta-2\upsilon), \tag{2.4}$$

it can be acquired by Eqs (1.7) and (2.1) that

$$\begin{aligned}\sum_{\theta=0}^{\infty} {}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!} &= \mathcal{A}_{p,q}(\zeta)\, e_{p,q}\left([2]_{p,q}\, o\zeta\right) e_{p,q}\left(-\zeta^{2}\right)\\ &= \left(\sum_{\theta=0}^{\infty}(-1)^{\theta}\frac{\zeta^{2\theta}}{[\theta]_{p,q}!}\right)\left(\sum_{\theta=0}^{\infty}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}\right)\\ &= \sum_{\theta=0}^{\infty}\left([\theta]_{p,q}!\sum_{\upsilon=0}^{\lfloor\frac{\theta}{2}\rfloor}\frac{(-1)^{\upsilon}\mathcal{A}_{\theta-2\upsilon,p,q}(o)}{[\theta-2\upsilon]_{p,q}!\,[\upsilon]_{p,q}!}\right)\frac{\zeta^{\theta}}{[\theta]_{p,q}!};\end{aligned}$$

by comparing the coefficients ζ^{θ} of the two portions above, we yield the stated formula (2.3).

Theorem 3. An addition formula for ${}_H\mathcal{A}_{\theta,p,q}(o)$ is given below:

$$ {}_H\mathcal{A}_{\theta,p,q}\left(o_1 \oplus_{p,q} o_2\right) = \sum_{\upsilon=0}^{\theta} p^{\binom{\upsilon}{2}}\ {}_H\mathcal{A}_{\theta-\upsilon,p,q}(o_1)\left([2]_{p,q}\, o_2\right)^{\upsilon} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}, \tag{2.5} $$

where (*cf.* [17])

$$ \left(o_1 \oplus_{p,q} o_2\right)^{\theta} = \sum_{\upsilon=0}^{\theta} o_1^k o_2^{n-k} p^{\upsilon(\upsilon-\theta)} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}. $$

Proof. It is readily seen from Eq (2.1) that

$$ \begin{aligned} \sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}\left(o_1 \oplus_{p,q} o_2\right)\frac{\zeta^{\theta}}{[\theta]_{p,q}!} &= \mathcal{A}_{p,q}(\zeta)\, e_{p,q}\left([2]_{p,q}(o_1+o_2)\zeta\right) e_{p,q}\left(-\zeta^2\right) \\ &= \mathcal{A}_{p,q}(\zeta)\, e_{p,q}\left([2]_{p,q}\, o_1\zeta\right) e_{p,q}\left(-\zeta^2\right) e_{p,q}\left([2]_{p,q}\, o_2\zeta\right) \\ &= \sum_{\theta=0}^{\infty} p^{\binom{\theta}{2}}\left([2]_{p,q}\, o_2\right)^{\theta}\frac{\zeta^{\theta}}{[\theta]_{p,q}!}\sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}(o_1)\frac{\zeta^{\theta}}{[\theta]_{p,q}!} \\ &= \sum_{\theta=0}^{\infty}\sum_{\upsilon=0}^{\theta} {}_H\mathcal{A}_{\theta-\upsilon,p,q}(o_1)\, p^{\binom{\upsilon}{2}} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}\left([2]_{p,q}\, o_2\right)^{\upsilon}\frac{\zeta^{\theta}}{[\theta]_{p,q}!}, \end{aligned} $$

which gives the alleged result Eq (2.5).

Two special cases of Theorem 3 are given as follows.

Corollary 1. *The (p,q)HAP ${}_HA_{\theta,p,q}(o)$ satisfies the subsequent summation formulas:*

$$ {}_H\mathcal{A}_{\theta,p,q}\left(o \oplus_{p,q} 1\right) = \sum_{\upsilon=0}^{\theta} [2]_{p,q}^{\upsilon}\ {}_H\mathcal{A}_{\theta-\upsilon,p,q}(o) \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} p^{\binom{\upsilon}{2}} $$

and

$$ {}_H\mathcal{A}_{\theta,p,q}(o) = \sum_{\upsilon=0}^{\theta} {}_H\mathcal{A}_{\theta-\upsilon,p,q}\, p^{\binom{\upsilon}{2}} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}\left([2]_{p,q}\, o\right)^{\upsilon}. $$

We research the (p,q)-derivative and (p,q)-integral expressions of ${}_H\mathcal{A}_{\theta,p,q}(o)$ as follows.

Theorem 4. (p,q)-derivative property of ${}_H\mathcal{A}_{\theta,p,q}(o)$

$$ D_{p,q;o}\left[{}_H\mathcal{A}_{\theta,p,q}(o)\right] = [\theta]_{p,q}[2]_{p,q}\ {}_H\mathcal{A}_{\theta-1,p,q}(po) \tag{2.6} $$

as well as the (p,q)-integral representation for ${}_H\mathcal{A}_{\theta,p,q}(o)$

$$ \int_a^b {}_H\mathcal{A}_{\theta,p,q}(o)\, d_{p,q}o = \frac{{}_H\mathcal{A}_{\theta+1,p,q}\left(\frac{b}{p}\right) - {}_H\mathcal{A}_{\theta+1,p,q}\left(\frac{a}{p}\right)}{[2]_{p,q}[\theta+1]_{p,q}} $$

hold for θ being a positive integer.

Proof. Applying the derivative operator $D_{p,q;o}$ Eq (1.1) on every side of Eq (2.1) along with regard to ζ and employing Eq (1.4), we receive the following:

$$\begin{aligned}\sum_{\theta=0}^{\infty} D_{p,q;o}\left[{}_{H}\mathcal{A}_{\theta,p,q}(o)\right]\frac{\zeta^{\theta}}{[\theta]_{p,q}!} &= D_{p,q;o}\left[\mathcal{A}_{p,q}(\zeta)\,e_{p,q}\left([2]_{p,q}\,o\zeta\right)e_{p,q}\left(-\zeta^{2}\right)\right]\\ &= D_{p,q;o}\left[e_{p,q}\left([2]_{p,q}\,o\zeta\right)\right]\mathcal{A}_{p,q}(\zeta)\,e_{p,q}\left(-\zeta^{2}\right)\\ &= [2]_{p,q}\,\zeta\mathcal{A}_{p,q}(\zeta)\,e_{p,q}\left([2]_{p,q}\,po\zeta\right)e_{p,q}\left(-\zeta^{2}\right)\\ &= [2]_{p,q}\sum_{\theta=0}^{\infty}{}_{H}\mathcal{A}_{\theta,p,q}(po)\frac{\zeta^{\theta+1}}{[\theta]_{p,q}!};\end{aligned}$$

additionally using Eq (1.5), we obtain the following:

$$\begin{aligned}\int_{a}^{b}{}_{H}\mathcal{A}_{\theta,p,q}(o)\,d_{p,q}o &= \frac{1}{[2]_{p,q}[\theta+1]_{p,q}}\int_{a}^{b}D_{p,q;o}\left[{}_{H}\mathcal{A}_{\theta,p,q}(o)\right]d_{p,q}o\\ &= \frac{{}_{H}\mathcal{A}_{\theta+1,p,q}\left(\frac{b}{p}\right)-{}_{H}\mathcal{A}_{\theta+1,p,q}\left(\frac{a}{p}\right)}{[2]_{p,q}[\theta+1]_{p,q}},\end{aligned}$$

which completes the proofs.

The immediate result of Eq (2.6) is given for $\upsilon<\theta$ as follows:

$$D_{p,q;\eta}^{(\upsilon)}\left[{}_{H}\mathcal{A}_{\theta,p,q}(o)\right]=\frac{[2]_{p,q}^{\upsilon}[\theta]_{p,q}!\,p^{\binom{\upsilon}{2}}}{[\theta-\upsilon]_{p,q}!}\,{}_{H}\mathcal{A}_{\theta-\upsilon,p,q}(p^{\upsilon}o), \tag{2.7}$$

where $D_{p,q;o}^{(\upsilon)}$ denotes the (p,q)-derivative operator of order υ in regard to o as $D_{p,q;o}^{(\upsilon)}=D_{p,q;o}^{(\upsilon-1)}D_{p,q;o}$.

We consider the (p,q)-differential operator for $\upsilon\in\mathbb{N}_0$ as follows:

$$\left(oD_{p,q;o}\right)^{(\upsilon)}=\left(oD_{p,q;o}\right)^{(\upsilon-1)}\left(oD_{p,q;o}\right). \tag{2.8}$$

We observe from Eq (2.8) that for $1\leq\theta$,

$$\left(oD_{p,q;o}\right)^{(\upsilon)}o^{\theta}=\left(oD_{p,q;o}\right)\left(oD_{p,q;o}\right)\ldots\left(oD_{p,q;o}\right)o^{\theta}=[\theta]_{p,q}^{\upsilon}\,p^{\binom{\upsilon}{2}}(p^{\upsilon}o)^{\theta}. \tag{2.9}$$

Hence, we obtain from Eq (2.9) that

$$\left(oD_{p,q;o}\right)^{(\upsilon)}f_{p,q}(o)=\sum_{\theta=0}^{\infty}a_{\theta}\left(oD_{p,q;o}\right)^{(\upsilon)}o^{\theta}=\sum_{\theta=0}^{\infty}a_{\theta}[\theta]_{p,q}^{\upsilon}\,p^{\binom{\upsilon}{2}}(p^{\upsilon}o)^{\theta},$$

in which $f_{p,q}(o)=\sum_{\theta=0}^{\infty}a_{\theta}o^{\theta}$ is a formal power series.

Theorem 5. The subsequent (p,q)-operator formula of the (p,q)HAP ${}_{H}A_{\theta,p,q}(o)$ holds true:

$$\left(o_2D_{p,q;o_2}\right)^{(\upsilon)}{}_{H}\mathcal{A}_{\theta,p,q}\left(o_1\oplus_{p,q}o_2\right)=\sum_{l=0H}^{\theta}\mathcal{A}_{\theta-l,p,q}(o_1)\,[2]_{p,q}^{l}(p^{\upsilon}o_2)^{l}p^{\binom{l}{2}+\binom{\upsilon}{2}}[l]_{p,q}^{\upsilon}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}, \tag{2.10}$$

which holds for $\theta\in\mathbb{N}$ and $\upsilon\in\mathbb{N}_0$.

Proof. It is readily seen from Eq (2.1) that

$$\left({}_{o_2}D_{p,q;o_2}\right)^{(\upsilon)} {}_H\mathcal{A}_{\theta,p,q}\left(o_1 \oplus_{p,q} o_2\right) = \sum_{l=0}^{\theta} \begin{bmatrix}\theta \\ \upsilon\end{bmatrix}_{p,q} {}_H\mathcal{A}_{\theta-l,p,q}(o_1)\, p^{\binom{l}{2}}\, [2]_{p,q}^{l} \left({}_{o_2}D_{p,q;o_2}\right)^{(\upsilon)} o_2^{l}$$
$$= \sum_{l=0}^{\theta} \begin{bmatrix}\theta \\ \upsilon\end{bmatrix}_{p,q} {}_H\mathcal{A}_{\theta-l,p,q}(o_1)\, p^{\binom{l}{2}}\, [2]_{p,q}^{l}\, [l]_{p,q}^{\upsilon}\, p^{\binom{\upsilon}{2}} (p^{\upsilon} o_2)^{l},$$

which means the desired consequence Eq (2.10).

A special case of Theorem 5 is given as follows.

Corollary 2. *(p,q)HAP ${}_HA_{\theta,p,q}(o)$ satisfies the subsequent (p,q)-operator formula:*

$$\left({}_oD_{p,q;o}\right)^{(\upsilon)} {}_H\mathcal{A}_{\theta,p,q}(o) = \sum_{l=0}^{\theta} [2]_{p,q}^{l}\, [l]_{p,q}^{\upsilon}\, {}_H\mathcal{A}_{\theta-l,p,q}\, p^{\binom{l}{2}} \begin{bmatrix}\theta \\ \upsilon\end{bmatrix}_{p,q} p^{\binom{\upsilon}{2}} (p^{\upsilon} o)^{l}.$$

3. Determinant approach for (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$

The determinant approximations of the special polynomials are important for numerical computations, as well as for solving linear interpolation difficulties. The determinant approach of Bernoulli, Appell, Sheffer and Bessel polynomial sequences were studied by Costabile *et al.* in [18, 19]. The hunt for versions of known special polynomials and numbers yielded many useful identities and characteristics in mathematics. The determinant form of special polynomials can be used to address many problems in mathematics. A growing number of determinant forms of special polynomials and their various variations have been studied in recent years, according to their importance and wide range of applications. Plenty of researchers are investigating the q and (p,q)-determinant expressions for multiple classically, and composite special polynomials (see [6, 8, 20]). Principally motivated by these studies, this section introduces the determinant forms of (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$. By selecting proper choices for a known formula $A_{\theta,p,q}(\zeta)$, the components that contain the class for (p,q)HAP ${}_HA_{\theta,p,q}(o)$ are found.

Here, the determinant approach for (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ is constructed.

Theorem 6. The (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ that have degree o holds the subsequent determinant approach:

$${}_H\mathcal{A}_{0,p,q}(o) = \frac{1}{\mathfrak{B}_{0,p,q}}, \tag{3.1}$$

$$
{}_H\mathcal{A}_{\theta,p,q}(o) = \frac{(-1)^\theta}{(\mathfrak{B}_{0,p,q})^{\theta+1}} \times \begin{vmatrix} 1 & H_{1,p,q}(o) & H_{2,p,q}(o) & \cdots & H_{\theta-1,p,q}(o) & H_{\theta,p,q}(o) \\ \mathfrak{B}_{0,p,q} & \mathfrak{B}_{1,p,q} & \mathfrak{B}_{2,p,q} & \cdots & \mathfrak{B}_{\theta-1,p,q} & \mathfrak{B}_{\theta,p,q} \\ 0 & \mathfrak{B}_{0,p,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} & \cdots & \begin{bmatrix} \theta-1 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} & \begin{bmatrix} \theta \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-1,p,q} \\ 0 & 0 & \mathfrak{B}_{0,p,q} & \cdots & \begin{bmatrix} \theta-1 \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-3,p,q} & \begin{bmatrix} \theta \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathfrak{B}_{0,p,q} & \begin{bmatrix} \theta \\ \theta-1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} \end{vmatrix}, \tag{3.2}
$$

where

$$
\mathfrak{B}_{\theta,p,q} = -\frac{1}{\mathcal{A}_{\theta,p,q}}\Big(\sum_{\upsilon=0}^{\theta} \begin{bmatrix} \theta \\ \upsilon \end{bmatrix}_{p,q} \mathcal{A}_{\upsilon,p,q} \ \mathfrak{B}_{\theta-\upsilon,p,q}\Big), \qquad \theta = 1, 2, 3, \ldots,
$$

with $\mathfrak{B}_{0,p,q} \neq 0$, $\mathfrak{B}_{0,p,q}, \mathfrak{B}_{1,p,q}, \ldots \mathfrak{B}_{\theta,p,q} \in \mathbb{R}$, and $\mathfrak{B}_{0,p,q} = \frac{1}{\mathcal{A}_{0,p,q}}$. Additionally, here $H_{\theta,p,q}(\upsilon)$ are the (p,q)-Hermite polynomials that have degree θ.

Proof. Assume that ${}_H\mathcal{A}_{\theta,p,q}(o)$ is the sequence that includes (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ stated in Eq (2.1), whereas $\mathcal{A}_{\theta,p,q}, \mathfrak{B}_{\theta,p,q}$ represent the sequences that are numerically available in such a way that:

$$
\mathcal{A}_{p,q}(\zeta) = \mathcal{A}_{0,p,q} + \mathcal{A}_{1,p,q}\frac{\zeta}{[1]_{p,q}!} + \mathcal{A}_{2,p,q}\frac{\zeta^2}{[2]_{p,q}!} + \cdots + \mathcal{A}_{\theta,p,q}\frac{\zeta^\theta}{[\theta]_{p,q}!} + \ldots \ , \mathcal{A}_{0,p,q} \neq 0, \tag{3.3}
$$

$$
\hat{\mathcal{A}}_{p,q}(\zeta) = \mathfrak{B}_{0,p,q} + \mathfrak{B}_{1,p,q}\frac{\zeta}{[1]_{p,q}!} + \mathfrak{B}_{2,p,q}\frac{\zeta^2}{[2]_{p,q}!} + \cdots + \mathfrak{B}_{\theta,p,q}\frac{\zeta^\theta}{[\theta]_{p,q}!} + \ldots, \ \mathfrak{B}_{0,p,q} \neq 0, \tag{3.4}
$$

which fulfills

$$
\hat{\mathcal{A}}_{p,q}(\zeta)\mathcal{A}_{p,q}(\zeta) = 1. \tag{3.5}
$$

Hence, implementing the Cauchy product method on the previous expression leads to the following:

$$
\hat{\mathcal{A}}_{p,q}(\zeta)\mathcal{A}_{p,q}(\zeta) = \sum_{\upsilon=0}^{\infty} \mathfrak{B}_{\upsilon,p,q}\frac{\zeta^\upsilon}{[\upsilon]_{p,q}!} \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}\frac{\zeta^\theta}{[\theta]_{p,q}!}
$$

$$= \sum_{\theta=0}^{\infty}\sum_{\upsilon=0}^{\theta} \mathcal{A}_{\upsilon,p,q}\ \mathfrak{B}_{\theta-\upsilon,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} \frac{\zeta^{\theta}}{[\theta]_{p,q}!}; \tag{3.6}$$

accordingly,

$$\sum_{\upsilon=0}^{\theta} A_{\upsilon,p,q}\ \mathfrak{B}_{\theta-\upsilon,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} = \begin{cases}1, & if\ \theta = 0,\\ 0, & if\ \theta > 0.\end{cases} \tag{3.7}$$

Particularly,

$$\begin{cases}\mathfrak{B}_{0,p,q} = \frac{1}{\mathcal{A}_{0,p,q}},\\ \mathfrak{B}_{\theta,p,q} = -\frac{1}{\mathcal{A}_{0,p,q}}\Big(\sum_{\upsilon=0}^{\theta} \mathcal{A}_{\upsilon,p,q}\ \mathfrak{B}_{\theta-\upsilon,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q}\Big), & \theta = 1, 2, \ldots\end{cases} \tag{3.8}$$

By multiplying each side of formula (2.1) by $\hat{\mathcal{A}}_{p,q}(\zeta)$, we receive the following:

$$e_{p,q}([2]_{p,q} o\zeta) e_{p,q}(-\zeta^2)\mathcal{A}_{p,q}(\zeta)\hat{\mathcal{A}}_{p,q}(\zeta) = \hat{\mathcal{A}}_{p,q}(\zeta)\sum_{\theta=0}^{\infty} {}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}. \tag{3.9}$$

Considering formulas (2.1), (3.4), and (3.7), the previously mentioned formula generates the following:

$$\sum_{\theta=0}^{\infty} H_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!} = \sum_{\theta=0}^{\infty} \mathfrak{B}_{\theta,p,q}\frac{\zeta^{\theta}}{[\theta]_{p,q}!}\sum_{\theta=0}^{\infty} {}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^{\theta}}{[\theta]_{p,q}!}. \tag{3.10}$$

Employing the Cauchy composition method over the two series on the right side with Eq (3.10) generates a corresponding infinite structure with the unidentified ${}_{H}\mathcal{A}_{\theta,p,q}(o)$:

$$\begin{cases}{}_{H}\mathcal{A}_{0,p,q}(o)\mathfrak{B}_{0,p,q} = 1,\\ {}_{H}\mathcal{A}_{0,p,q}(o)\mathfrak{B}_{1,p,q} + {}_{H}\mathcal{A}_{1,p,q}(o)\mathfrak{B}_{0,p,q} = H_{1,p,q}(o) & ,\\ {}_{H}\mathcal{A}_{0,p,q}(o)\mathfrak{B}_{2,p,q} + \begin{bmatrix}2\\ 1\end{bmatrix}_{p,q} {}_{H}\mathcal{A}_{1,p,q}(o)\mathfrak{B}_{1,p,q} + \mathfrak{B}_{0,p,q}\ {}_{H}\mathcal{A}_{2,p,q}(o) = H_{2,p,q}(o),\\ \vdots\\ {}_{H}\mathcal{A}_{0,p,q}(o)\mathfrak{B}_{\theta-1,p,q} + \begin{bmatrix}\theta-1\\ 1\end{bmatrix}_{p,q} {}_{H}\mathcal{A}_{1,p,q}(o)\mathfrak{B}_{\theta-2,p,q} + \cdots + {}_{H}\mathcal{A}_{\theta,p,q}(o)\mathfrak{B}_{0,p,q} = H_{\theta-1,p,q}(o),\\ {}_{H}\mathcal{A}_{0,p,q}(o)\mathfrak{B}_{\theta,p,q} + \begin{bmatrix}\theta\\ 1\end{bmatrix}_{p,q} {}_{H}\mathcal{A}_{1,p,q}(o)\mathfrak{B}_{\theta-1,p,q} + \cdots + {}_{H}\mathcal{A}_{1,p,q}(o)\mathfrak{B}_{0,p,q} = H_{\theta,p,q}(o),\\ \vdots\end{cases} \tag{3.11}$$

It is evident that the very first statement (3.1) can be determined through the first formula within the system (3.11).

Employing the Cramer technique on the primary $\theta + 1$ formula (3.11) makes it possible to extract the unidentified ${}_{H}\mathcal{A}_{\theta,p,q}(o)$ from the bottom triangular value matrix inside structure (3.1). Thus, we can draw a conclusion.

$$
{}_{H}\mathcal{A}_{\theta,p,q}(o) = \frac{\begin{vmatrix} \mathfrak{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1 \\ \mathfrak{B}_{1,p,q} & \mathfrak{B}_{0,p,q} & 0 & \cdots & 0 & H_{1,p,q}(o) \\ \mathfrak{B}_{2,p,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} & \mathfrak{B}_{0,p,q} & \cdots & 0 & H_{2,p,q}(o) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{B}_{\theta-1,p,q} & \begin{bmatrix} \theta-1 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} & \begin{bmatrix} \theta-1 \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-3,p,q} & \cdots & \mathfrak{B}_{0,p,q} & H_{\theta-1,p,q}(o) \\ \mathfrak{B}_{\theta,p,q} & \begin{bmatrix} \theta \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-1,p,q} & \begin{bmatrix} \theta \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} & \cdots & \begin{bmatrix} \theta \\ \theta-1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} & H_{\theta,p,q}(o) \end{vmatrix}}{\begin{vmatrix} \mathfrak{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1 \\ \mathfrak{B}_{1,p,q} & \mathfrak{B}_{0,p,q} & 0 & \cdots & 0 & 0 \\ \mathfrak{B}_{2,p,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} & \mathfrak{B}_{0,p,q} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{B}_{\theta-1,p,q} & \begin{bmatrix} \theta-1 \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} & \begin{bmatrix} \theta-1 \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-3,p,q} & \cdots & \mathfrak{B}_{0,p,q} & 0 \\ \mathfrak{B}_{\theta,p,q} & \begin{bmatrix} \theta \\ 1 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-1,p,q} & \begin{bmatrix} \theta \\ 2 \end{bmatrix}_{p,q} \mathfrak{B}_{\theta-2,p,q} & \cdots & \begin{bmatrix} \theta \\ \theta-1 \end{bmatrix}_{p,q} \mathfrak{B}_{1,p,q} & \mathfrak{B}_{0,p,q} \end{vmatrix}},
$$

whenever $\theta = 1, 2, 3 \ldots$.

Stretching the determinant and employing the transposition associated with the determinant within the numerator, produces the following:

$$
{}_{H}\mathcal{A}_{\theta,p,q}(o)=\frac{1}{(\mathfrak{B}_{0,p,q})^{\theta+1}}\times\begin{vmatrix}
\mathfrak{B}_{0,p,q} & \mathfrak{B}_{1,p,q} & \mathfrak{B}_{2,p,q} & \cdots & \mathfrak{B}_{\theta-1,p,q} & \mathfrak{B}_{\theta,p,q}\\
0 & \mathfrak{B}_{0,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathfrak{B}_{1,p,q} & \cdots & \begin{bmatrix}\theta-1\\1\end{bmatrix}_{p,q}\mathfrak{B}_{\theta-2,p,q} & \begin{bmatrix}\theta\\1\end{bmatrix}_{p,q}\mathfrak{B}_{\theta-1,p,q}\\
0 & 0 & \mathfrak{B}_{0,p,q} & \cdots & \begin{bmatrix}\theta-1\\2\end{bmatrix}_{p,q}\mathfrak{B}_{\theta-3,p,q} & \begin{bmatrix}\theta\\2\end{bmatrix}_{p,q}\mathfrak{B}_{\theta-2,p,q}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \mathfrak{B}_{0,p,q} & \begin{bmatrix}\theta\\\theta-1\end{bmatrix}_{p,q}\mathfrak{B}_{1,p,q}\\
1 & H_{1,p,q}(o) & H_{2,p,q}(o) & \cdots & H_{\theta-1,p,q}(o) & H_{\theta,p,q}(o)
\end{vmatrix} \tag{3.12}
$$

After changing the υ^{th} row to the $(\upsilon+1)^{th}$ location for $\upsilon=1,2,3,\ldots,\theta-1$, the assertion (3.2) is reached.

Theorem 7. This identity is valid for (p,q)HAP ${}_{H}\mathcal{A}_{\theta,p,q}(o)$:

$$
{}_{H}\mathcal{A}_{\theta,p,q}(o)=\frac{1}{\mathfrak{B}_{0,p,q}}\left(H_{\theta,p,q}(o)-\sum_{\upsilon=0}^{\theta-1}\begin{bmatrix}\theta\\\upsilon\end{bmatrix}_{p,q}\mathfrak{B}_{\theta-\upsilon,p,q}\ {}_{H}\mathcal{A}_{\theta,p,q}(o)\right),\qquad \theta=1,2,3,\ldots. \tag{3.13}
$$

Proof. To achieve our needed findings, we expand the value of determinant (3.2) relative to the $(\upsilon+1)^{th}$ row by employing the same procedure outlined in reference [20].

4. Certain components for the (p,q)-Hermite-Appell polynomials

Throughout this part, we introduce multiple components of the (p,q)-Hermite-Appell classes by picking suitable values over the function $\mathcal{A}_{\theta,p,q}(o)$.

4.1. The (p,q)-Hermite-Bernoulli polynomials

With $\mathcal{A}_{p,q}(\zeta)=\frac{\zeta}{e_{p,q}(\zeta)-1}$, (p,q)Ap $\mathcal{A}_{\theta,p,q}(o)$ simplify into (p,q)P $\mathcal{B}_{\theta,p,q}(o)$ (Table 1(I)). For the selected $\mathcal{B}_{p,q}(\zeta)$, (p,q) HAP ${}_{H}\mathcal{A}_{\theta,p,q}(o)$ simplify to (p,q) HBP ${}_{H}\mathcal{B}_{\theta,p,q}(o)$, which are determined by the subsequent generating function:

$$
\frac{\zeta e_{p,q}([2]_{p,q}o\zeta)\ e_{p,q}(-\zeta^2)}{e_{p,q}(\zeta)-1}=\sum_{\theta=0}^{\infty}{}_{H}\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}. \tag{4.1}
$$

(p,q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$ of degree θ are determined by the following series:

$$ {}_H\mathcal{B}_{\theta,p,q}(o) = \sum_{\upsilon=0}^{\theta} \mathcal{B}_{\upsilon,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} H_{\theta-\upsilon,p,q}(o), \qquad \mathcal{B}_{0,p,q} \neq 0. \tag{4.2}$$

(p,q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$ has a particular identity:

$$ {}_H\mathcal{B}_{\theta,p,q}(o) = \frac{H_{\theta,p,q}(o)}{\mathcal{B}_{0,p,q}} - \frac{1}{\mathcal{B}_{0,p,q}} \sum_{\upsilon=0}^{\theta-1} \mathfrak{B}_{\theta-\upsilon,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} {}_H\mathcal{B}_{\theta,p,q}(o), \qquad \theta = 1,2,3,\ldots. \tag{4.3}$$

Employing $\mathfrak{B}_{0,p,q} = 1$ and $\mathfrak{B}_{\theta,p,q} = \frac{1}{[\theta+1]_{p,q}}$, θ=1,2,3..., allows us to produce the determinant formula of (p,q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$ in Eqs (3.1) and (3.2).

Definition 2. (p,q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$ with class θ can be identified by the following:

$$ {}_H\mathcal{B}_{\theta,p,q}(o) = (-1)^{\theta} \begin{vmatrix} 1 & H_{1,p,q}(o) & H_{2,p,q}(o) & \cdots & H_{\theta-1,p,q}(o) & H_{\theta,p,q}(o) \\ 1 & \frac{1}{[2]_{p,q}} & \frac{1}{[3]_{p,q}} & \cdots & \frac{1}{[\theta]_{p,q}} & \frac{1}{[\theta+1]_{p,q}} \\ 0 & 1 & \frac{\begin{bmatrix}2\\1\end{bmatrix}_{p,q}}{[2]_{p,q}} & \cdots & \frac{\begin{bmatrix}\theta-1\\1\end{bmatrix}_{p,q}}{[\theta-1]_{p,q}} & \frac{\begin{bmatrix}\theta\\1\end{bmatrix}_{p,q}}{[\theta]_{p,q}} \\ 0 & 0 & 1 & \cdots & \frac{\begin{bmatrix}\theta-1\\2\end{bmatrix}_{p,q}}{[\theta-2]_{p,q}} & \frac{\begin{bmatrix}\theta\\2\end{bmatrix}_{p,q}}{[\theta-1]_{p,q}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{\begin{bmatrix}\theta\\ \theta-1\end{bmatrix}_{p,q}}{[2]_{p,q}} \end{vmatrix}, $$

with

$$ {}_H\mathcal{B}_{0,p,q}(o) = 1, \tag{4.4}$$

where $H_{\theta,p,q}(o)$ are the θ^{th} (p,q)-Hermite polynomials ($\theta = 0,1,2,\ldots$).

4.2. The (p,q)-Hermite-Euler polynomials

With $\mathcal{A}_{p,q}(\zeta)=\frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}$, (p,q)Ap $\mathcal{A}_{\theta,p,q}(o)$ simplify into (p,q)EP $\mathcal{E}_{\theta,p,q}(o)$ (Table 1 (II)). By selecting $\mathcal{A}_{p,q}(\zeta)$, (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ simplify to (p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$, which are determined by the subsequent generating function:

$$ \frac{[2]_{p,q} e_{p,q}([2]_{p,q} o\zeta)\ e_{p,q}(-\zeta^2)}{1+e_{p,q}(\zeta)} = \sum_{\theta=0}^{\infty} {}_H\mathcal{E}_{\theta,p,q}(o) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}. \tag{4.5}$$

(p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$ of degree θ are received by the following series:

$$ {}_H\mathcal{E}_{\theta,p,q}(o) = \sum_{\upsilon=0}^{\theta} \mathcal{E}_{\theta,p,q} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} H_{\theta-\upsilon,p,q}(o),\ \mathcal{E}_{0,p,q} \neq 0. \tag{4.6}$$

The next formula to (p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$ holds true:

$$ {}_H\mathcal{E}_{\theta,p,q}(o) = \frac{H_{\theta,p,q}(o)}{\mathcal{E}_{0,p,q}} - \frac{1}{\mathcal{E}_{0,p,q}} \sum_{\upsilon=0}^{\theta-1} \mathfrak{B}_{\theta-\upsilon,p,q} \begin{bmatrix} \theta \\ \upsilon \end{bmatrix}_{p,q} {}_H\mathcal{E}_{\theta,p,q}(o), \qquad \theta = 1,2,3,\ldots. \tag{4.7} $$

By selecting $\mathfrak{E}_{0,p,q} = 1$ and $\mathfrak{E}_{\theta,p,q} = \frac{1}{[2]_{p,q}}$, $\theta = 1,2,3\ldots$, at Eqs (3.1) and (3.2), we receive the determinant formula that corresponds to (p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$.

Definition 3. (p,q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$ of degree θ is supplied as follows:

$$ {}_H\mathcal{E}_{0,p,q}(o) = 1 \tag{4.8} $$

$$ {}_H\mathcal{E}_{\theta,p,q}(o) = (-1)^{\theta} \begin{vmatrix} 1 & H_{1,p,q}(o) & H_{2,p,q}(o) & \cdots & H_{\theta-1,p,q}(o) & H_{\theta,p,q}(o) \\ 1 & \frac{1}{[2]_{p,q}} & \frac{1}{[2]_{p,q}} & \cdots & \frac{1}{[2]_{p,q}} & \frac{1}{[2]_{p,q}} \\ 0 & 1 & \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q}}{[2]_{p,q}} & \cdots & \frac{\begin{bmatrix} \theta-1 \\ 1 \end{bmatrix}_{p,q}}{[2]_{p,q}} & \frac{\begin{bmatrix} \theta \\ 1 \end{bmatrix}_{p,q}}{[2]_{p,q}} \\ 0 & 0 & 1 & \cdots & \frac{\begin{bmatrix} \theta-1 \\ 2 \end{bmatrix}_{p,q}}{[2]_{p,q}} & \frac{\begin{bmatrix} \theta \\ 2 \end{bmatrix}_{p,q}}{[2]_{p,q}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{\begin{bmatrix} \theta \\ \theta-1 \end{bmatrix}_{p,q}}{[2]_{p,q}} \end{vmatrix}, $$

where $H_{\theta,p,q}(o)$ are the θ^{th} (p,q)-Hermite polynomials ($\theta = 0,1,2,\ldots$).

4.3. The (p,q)-Hermite-Genocchi polynomials

With $\mathcal{A}_{p,q}(\zeta)=\frac{[2]_{p,q}\zeta}{e_{p,q}(\zeta)+1}$, (p,q)Ap $\mathcal{A}_{\theta,p,q}(o)$ simplify to (p,q)GP $\mathcal{G}_{\theta,p,q}(o)$ (Table 1 (III)). By selecting $\mathcal{A}_{p,q}(\zeta)$, (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o)$ reduces to (p,q)HGP ${}_H\mathcal{G}_{\theta,p,q}(o)$, which are determined by the subsequent generating function:

$$ \frac{[2]_{p,q}\zeta e_{p,q}(o\zeta)\, e_{p,q}(-\zeta^2)}{1+e_{p,q}(\zeta)} = \sum_{\theta=0}^{\infty} {}_H\mathcal{G}_{\theta,p,q}(o) \frac{\zeta^{\theta}}{[\theta]_{p,q}!}. \tag{4.9} $$

(p,q)HGP ${}_H\mathcal{G}_{\theta,p,q}(o)$ of degree θ are provided by the following series:

$$ {}_H\mathcal{G}_{\theta,p,q}(o) = \sum_{\upsilon=0}^{\theta} \mathcal{G}_{\upsilon,p,q} \begin{bmatrix} \theta \\ \upsilon \end{bmatrix}_{p,q} H_{\theta-\upsilon,p,q}(o), \qquad \mathcal{G}_{0,p,q} \neq 0. \tag{4.10} $$

The next identity of (p,q)HGP ${}_H\mathcal{G}_{\theta,p,q}(o)$ holds true:

$$ {}_H\mathcal{G}_{\theta,p,q}(o) = \frac{H_{\theta,p,q}(o)}{\mathcal{G}_{0,p,q}} - \frac{1}{\mathcal{G}_{0,p,q}} \begin{bmatrix} \theta \\ \upsilon \end{bmatrix}_{p,q} {}_H\mathcal{G}_{\theta,p,q}(o), \qquad \theta = 1,2,3,\ldots. \tag{4.11} $$

5. Two-dimensional (p, q)-Hermite-Appell polynomials

The introduction for two-dimensional (p, q)-Appell polynomials, which correspond to the 2-variable extension related to (p, q)-Appell polynomials designated as two-dimensional (p, q)-Appell polynomials $\mathcal{A}_{\theta,p,q}(o, \mu)$, are provided. The strategy employed in the preceding section is utilized to provide the two-dimensional (p, q)-Hermite-Appell polynomials $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$, alongside emphasizing on deriving their generating functions and series expressions.

The next findings are demonstrated for the purpose of determining the generating function for $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$.

Theorem 8. The $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$ maintain their generating function for the following:

$$e_{p,q}([2]_{p,q} o\zeta)\ e_{p,q}(-\zeta^2) E_{p,q}(\mu\zeta) \mathcal{A}_{p,q}(\zeta) \ = \sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}(o, \mu) \frac{\zeta^\theta}{[\theta]_{p,q}!}. \tag{5.1}$$

Proof. We can see that in the left part of Eq (1.9), we expanded the primary exponential function $e_{p,q}(o\zeta)$ and then replaced the power of o, (i.e., $o^0, o^1, po^2, \ldots, p^{\binom{\theta}{2}} o^\theta$) with the corresponding polynomials $H_{0,p,q}(o), H_{1,p,q}(o), H_{2,p,q}(o), \ldots, H_{\theta,p,q}(o)$ at the left part and o with $H_{1,p,q}(o)$ into the right part of the resultant formula, we receive

$$\mathcal{A}_{p,q}(\zeta)\left(1+H_{1,p,q}(o)\frac{\zeta}{[1]_{p,q}!}+H_{2,p,q}(o)\frac{\zeta^2}{[2]_{p,q}!}+\cdots+H_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}\right)E_{p,q}(\mu\zeta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}(H_{1,p,q}(o), \mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}. \tag{5.2}$$

Further, we sum up the expansion in the left part, then employ the Eq (1.6) within the resultant formula to generate

$$\mathcal{A}_{p,q}(\zeta)\ e_{p,q}([2]_{p,q} o\zeta) e_{p,q}(-\zeta^2) E_{p,q}(\mu\zeta) = \sum_{\theta=0}^{\infty} \mathcal{A}_{\theta,p,q}(H_{1,p,q}(o), \mu) \frac{\zeta^\theta}{[\theta]_{p,q}!}. \tag{5.3}$$

Finally, we write the outcome of $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$ on the right part of the previous equation as

$$\mathcal{A}_{\theta,p,q}\Big(H_{1,p,q}(o, \mu)\Big) = {}_H\mathcal{A}_{\theta,p,q}(o, \mu).$$

Thus, the assertion Eq (5.1) is proved.

We can acquire the series description for $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$ through proving the subsequent statement.

Theorem 9. The $2D(p, q)$HAP ${}_H\mathcal{A}_{\theta,p,q}(o, \mu)$ exhibit through the series description shown below:

$${}_H\mathcal{A}_{\theta,p,q}(o, \mu) = \sum_{\upsilon=0}^{\theta} {}_H\mathcal{A}_{\theta-\upsilon,p,q}(o)\ q^{\binom{\upsilon}{2}} \begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} \mu^\upsilon. \tag{5.4}$$

Proof. Expanding the left part of formula (5.1) using formulas (1.3) and (2.1) , we gain

$$\sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!}\sum_{\upsilon=0}^{\infty} q^{\binom{\upsilon}{2}}\,\mu^\upsilon\frac{\zeta^\upsilon}{[\upsilon]_{p,q}!} = \sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}(o,\mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}.$$

Consequently, the application of the Cauchy product rule yields

$$\sum_{\upsilon=0}^{\infty}\sum_{\theta=0}^{\infty}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} q^{\binom{\upsilon}{2}}\,\mu^\upsilon {}_H\mathcal{A}_{\theta-\upsilon,p,q}(o)\frac{\zeta^\theta}{[\theta]_{p,q}!} = \sum_{\theta=0}^{\infty} {}_H\mathcal{A}_{\theta,p,q}(o,\mu)\frac{\zeta^\theta}{[\theta]_{p,q}!}.$$

The assertion Eq (5.4) can be obtained by comparing the coefficients of similar powers of ζ within each side of the formula.

Table 3 enumerates several components of the two-dimensional (p,q)-Appell class. Each component of the two-dimensional (p,q)-Appell class is associated with the corresponding special polynomials that exist within the two-dimensional (p,q)-Hermite-Appell class. By selecting appropriate values for the function $\mathcal{A}_{p,q}$ in Eqs (5.1) and (5.4), the generating functions and series expressions for the associated components of the two-dimensional (p,q)-Hermite Appell class can be derived. The next table lists the resulting components of the two-dimensional (p,q)-Hermite Appell class together with their generating functions and series expressions.

Table 3. Some known two-dimensional (p,q)-Hermite-type Apostol polynomials ${}_H\mathcal{A}_{\theta,p,q}(o,\mu)$.

S.No.	$\mathcal{A}_{p,q}(\zeta)$	Generating function and relation	Polynomials
I	$\mathcal{A}_{p,q}(\zeta)=\frac{\zeta}{e_{p,q}(\zeta)-1}$	$\frac{\zeta}{e_{p,q}(\zeta)-1}\,e_{p,q}(o\zeta)\,e_{p,q}(-\zeta^2)\,E_{p,q}(\mu\zeta)=\sum_{\theta=0}^{\infty}{}_H\mathcal{B}_{\theta,p,q}(o,\mu)\,\frac{\zeta^\theta}{[\theta]_{p,q}!}$	
		${}_H\mathcal{B}_{\theta,p,q}(o,\mu)=\sum_{\upsilon=0}^{\theta}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} q^{\binom{\upsilon}{2}}\,\mu^\upsilon\,{}_H\mathcal{B}_{\theta-\upsilon,p,q}(o)$	Two-dimensional (p,q)-Hermite–Bernoulli polynomials
II	$\mathcal{A}_{p,q}(\zeta)=\frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}}{e_{p,q}(\zeta)+1}\,e_{p,q}(o\zeta)\,E_{p,q}(\mu\zeta)=\sum_{\theta=0}^{\infty}{}_H\mathcal{E}_{\theta,p,q}(o,\mu)\,\frac{\zeta^\theta}{[\theta]_{p,q}!}$	
		${}_H\mathcal{E}_{\theta,p,q}(o,\mu)=\sum_{\upsilon=0}^{\theta}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_{p,q} q^{\binom{\upsilon}{2}}\,\mu^\upsilon\,{}_H\mathcal{E}_{\theta-\upsilon,p,q}(o)$	Two-dimensional (p,q)-Hermite–Euler polynomials
III	$\mathcal{A}_{p,q}(\zeta)=\frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}$	$\frac{[2]_{p,q}\,\zeta}{e_{p,q}(\zeta)+1}\,e_{p,q}(o\zeta)\,e_{p,q}(-\zeta^2)\,E_{p,q}(\mu\zeta)=\sum_{\theta=0}^{\infty}{}_H\mathcal{G}_{\theta,p,q}(o,\mu)\,\frac{\zeta^\theta}{[\theta]_{p,q}!}$	Two-dimensional (p,q)-Hermite–Genocchi polynomials

Remark 1. For $p=1$, the series formula (5.4) of (p,q)HAP ${}_H\mathcal{A}_{\theta,p,q}(o,\mu)$, provides the series definition of (qHAP) ${}_H\mathcal{A}_{\theta,q}(o,\mu)$ as follows:

$${}_H\mathcal{A}_{\theta,q}(\zeta,\mu)=\sum_{\upsilon=0}^{\theta}{}_H\mathcal{A}_{\theta-\upsilon,q}(o)\,q^{\binom{\upsilon}{2}}\begin{bmatrix}\theta\\ \upsilon\end{bmatrix}_q\mu^\upsilon. \tag{5.5}$$

This is the two-dimensional q-Hermite Appell polynomials' series definition.

6. Graphical diagrams

This section includes graphical diagrams of the novel class of (p,q)-Hermite-Appell polynomials and $2D$ (p,q)-Hermite-Appell polynomials. Numerical research confirms theoretical findings in the complex plane, thus providing insight into their analytical structure.

We used the Mathematica software to create graphs of (p, q)HBP ${}_H\mathcal{B}_{\theta,p,q}(o)$, (p, q)HEP ${}_H\mathcal{E}_{\theta,p,q}(o)$, 2D$(p, q)$HBP ${}_H\mathcal{B}_{\theta,p,q}(o, \mu)$, and 2D$(p, q)$HEP ${}_H\mathcal{E}_{\theta,p,q}(o, \mu)$. To draw the graphs of these polynomials, by taking $p = \frac{1}{2}$, $q = \frac{1}{3}$ into the determinant formulations (4.4) and (4.8), we obtain the outcomes displayed in Table 4 for $\theta = 1$ and 2.

Table 4. The expressions of ${}_H\mathcal{B}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$ and ${}_H\mathcal{E}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$ for $\theta = 1, 2$.

Polynomial	$\theta = 1$	$\theta = 2$
${}_H\mathcal{B}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$	$-\frac{6}{5} + \frac{5}{6}o$	$\frac{5}{6}o^2 - \frac{5}{6}o - \frac{871}{570}$
${}_H\mathcal{E}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$	$-\frac{6}{5} + \frac{5}{6}o$	$\frac{5}{6}o^2 - \frac{5}{6}o - \frac{5}{6}$

Now, with the help of Mathematica software and using Eqs (4.4) and (4.8) and the expressions of ${}_H\mathcal{B}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$ and ${}_H\mathcal{E}_{\theta,\frac{1}{2},\frac{1}{3}}(o)$ from Table 5, we get the graphs in diagrams 1 and 2 which demonstrate the continued existence and structure of (p, q)-polynomial familial relationships.

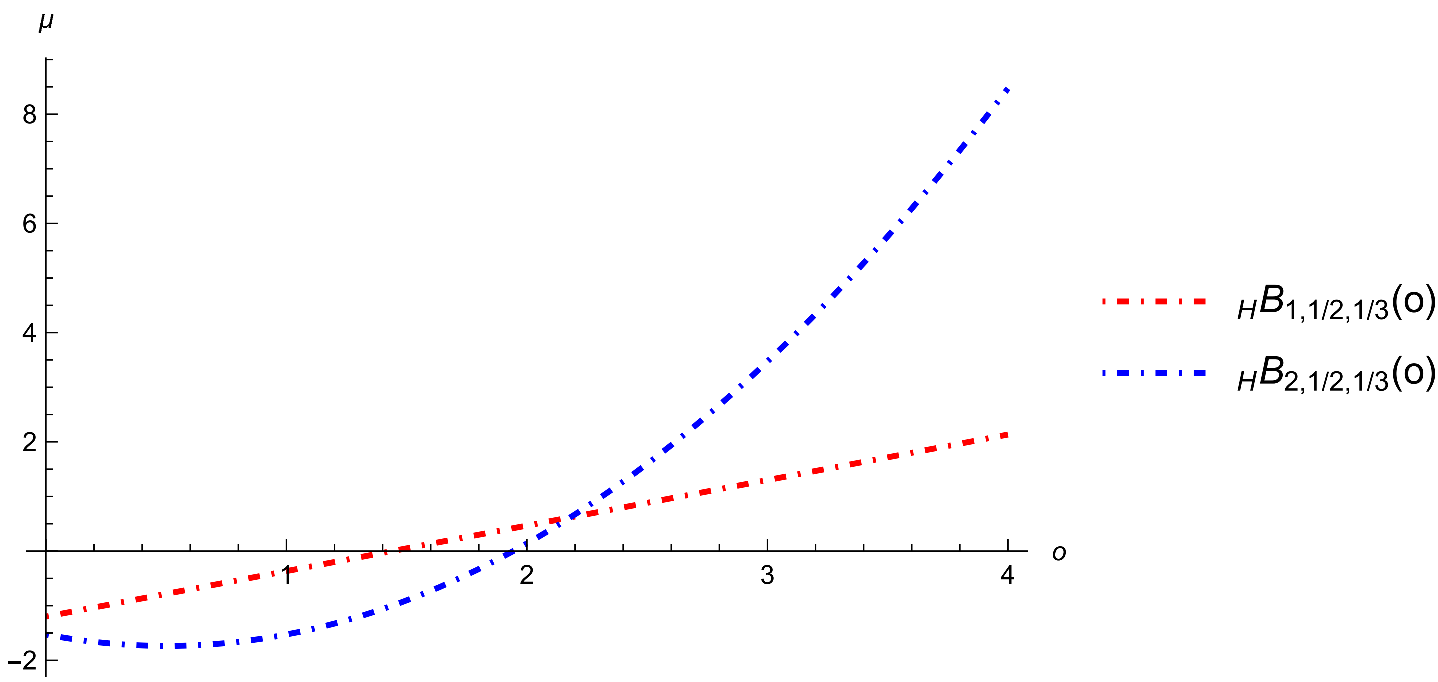

Figure 1. Graphs of ${}_H\mathcal{B}_{1,1/2,1/3}(o)$, ${}_H\mathcal{B}_{2,1/2,1/3}(o)$.

Figure 1 depicts these polynomials over set $p = 1/2$ and $q = 1/3$, thus emphasizing their behavior for $\theta = 1, 2$.

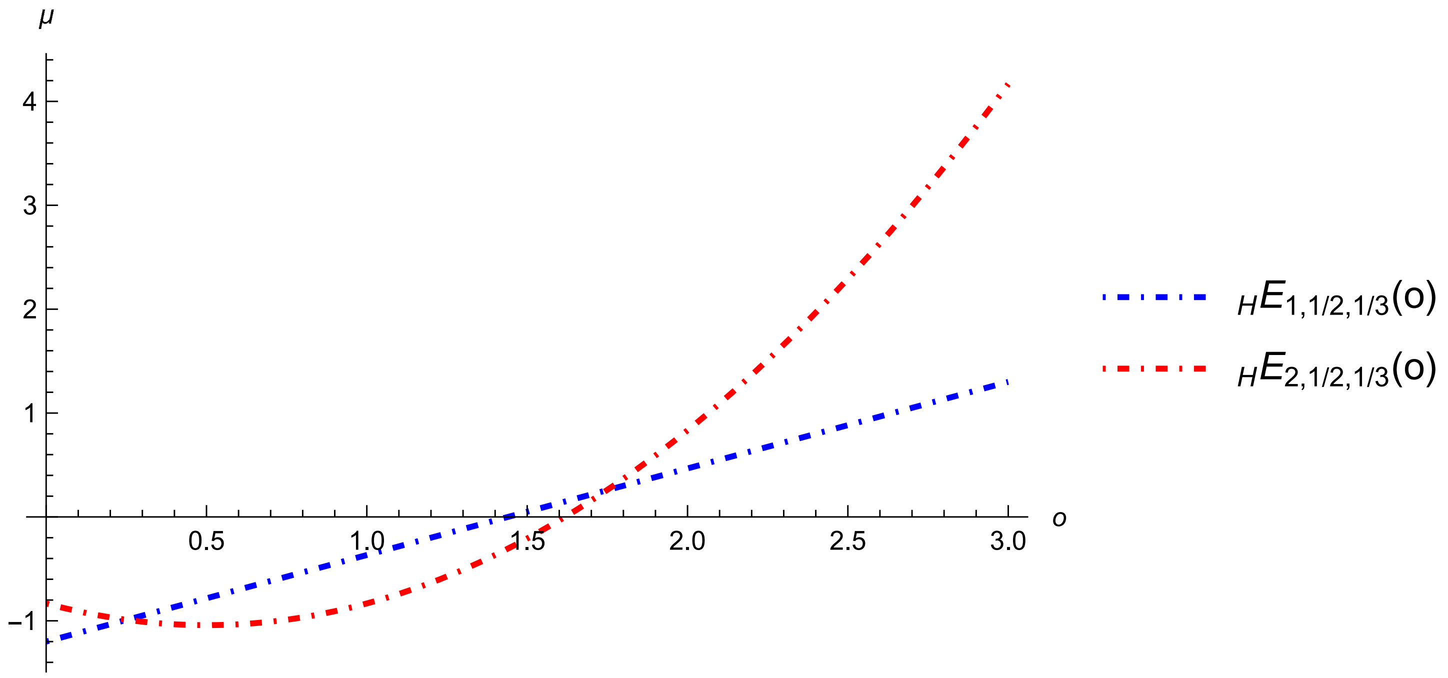

Figure 2. Graphs of ${}_H\mathcal{E}_{1,1/2,1/3}(o)$, ${}_H\mathcal{E}_{2,1/2,1/3}(o)$.

In a comparable manner, Figure 2 portrays their behavior for $\theta = 1, 2$ and varied values of $p = 1/2$ and $q = 1/3$, thus offering an extensive description of their dependence on the parameters involved.

To draw the graphs of $2D$ polynomials, we take $p = \frac{1}{2}$, $q = \frac{1}{3}$ into the determinant expressions (4.4) and (4.8); then, we obtain the outcomes indicated in Table 5 for $\theta = 2$.

Table 5. The expressions of ${}_H\mathcal{B}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$ and ${}_H\mathcal{E}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$.

Polynomial	Expression for $\theta = 2$
${}_H\mathcal{B}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$	$\frac{5}{6}o^2 + \frac{1}{3}\mu^2 - \frac{5}{6}o - \mu + \frac{25}{36}o\mu - \frac{871}{570}$
${}_H\mathcal{E}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$	$\frac{5}{6}o^2 + \frac{1}{3}\mu^2 - \frac{5}{6}o - \mu + \frac{25}{36}o\mu - \frac{5}{6}$

Now, with the help of the Mathematica software and using Eqs (4.4) and (4.8) and the expressions of $2D(1/2, 1/3)$HBP ${}_H\mathcal{B}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$ and $2D(1/2, 1/3)$HEP ${}_H\mathcal{E}_{2,\frac{1}{2},\frac{1}{3}}(o,\mu)$ via Table 5, we obtain the graphs of Figures 3 and 4.

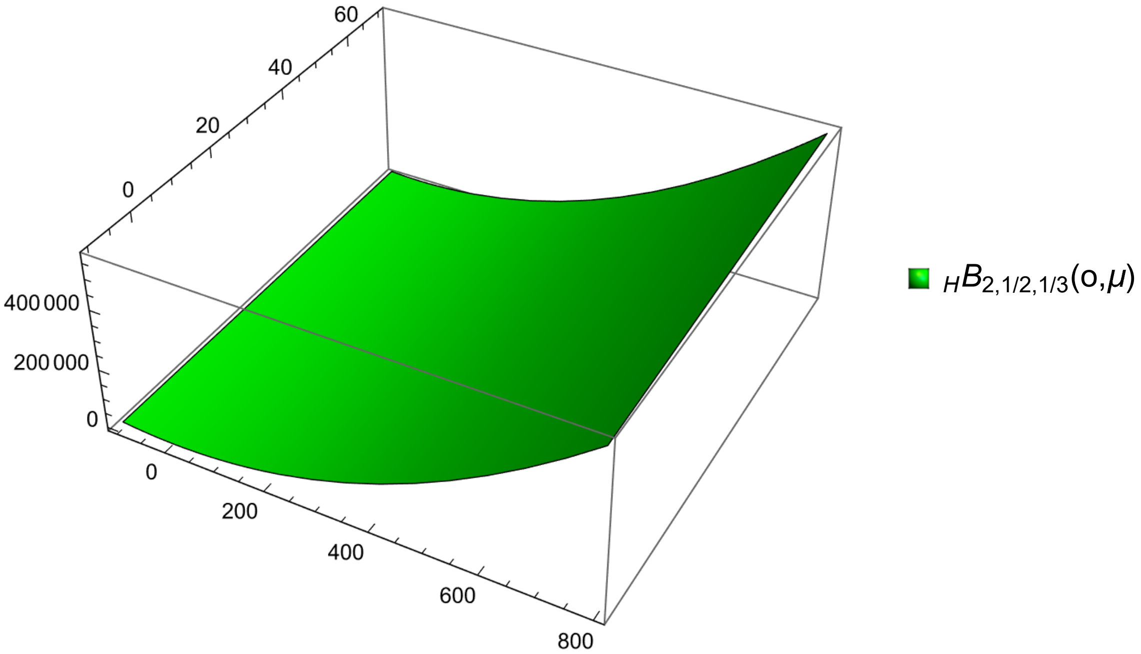

Figure 3. Surface plot for $_{H}\mathcal{B}_{2,1/2,1/3}(o,\mu)$.

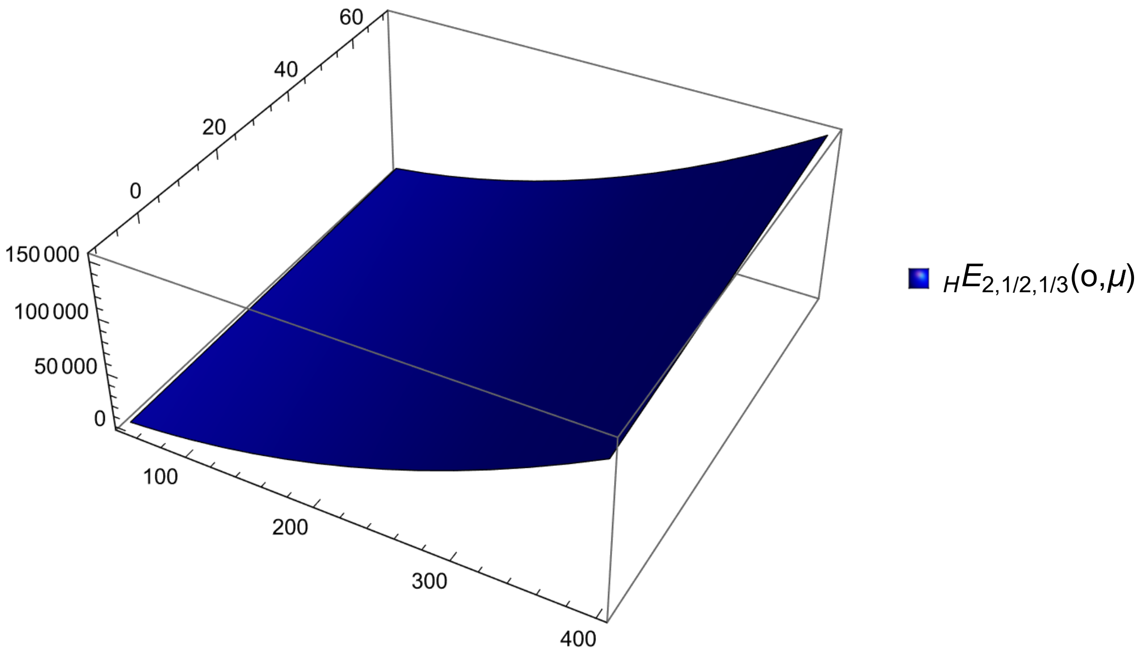

Figure 4. Surface plot for $_{H}\mathcal{B}_{1,1/2,1/3}(o,\mu)$.

The surface plots depict how a dependent response variable (Z-axis) changes in $3D$ across two independent variables (X and Y axes), thereby revealing patterns such as peaks (maxima), valleys (minima), curvature, and interactions. It aids in the discovery of optimal settings for processes such as maximizing the performance or minimizing defects by revealing complex relationships beyond simple linear models.

7. Conclusions

In this article, we created a new class of (p, q)-Hermite-Appell polynomials by combining (p, q)-Hermite and (p, q)-Appell polynomials. We provided this class's qualities, including the generating function, series definition, derivative properties, integral representation, summation formulas, and determinant representation. We analyzed a few components of (p, q)-Hermite-Appell polynomials, such as (p, q)-Hermite-Bernoulli polynomials, (p, q)-Hermite-Euler polynomials, and (p, q)-Hermite-Genocchi polynomials; then, we inferred some of their features. Moreover, we acquired a (p, q)-differential operator formula for (p, q)-Hermite-Appell polynomials and we used the Wolfram Mathematica software to plot the graphical diagrams of select components of (p, q)-Hermite-Appell along with two-dimensional (p, q)-Hermite-Appell polynomials. The (p, q)-Hermite-Appell polynomials and their generalizations will be used in a variety of real-world applications, thus serving as a strong mathematical basis to tackle intricate problems in physics, engineering, and economics. Our research advances our comprehension of (p, q)-series and (p, q)-special functions, which may find applications in a variety of disciplines, including mathematical physics, number theory, combinatorics, financial mathematics, quantum mechanics, probability theory, image processing, engineering, and complex systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Clemente Cesarano is an editorial board member for Networks and Heterogeneous Media and was not involved in the editorial review or the decision to publish this article. All authors declare that they have no competing interests.

Author contributions

Mohammed Fadel: Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Ugur Duran:** Conceptualization; data curation; formal analysis; investigation; methodology; project administration; software; resources; supervision; validation; visualization; writing-original draft; writing – review and editing. **Clemente Cesarano:** Conceptualization; formal analysis; methodology; project administration; funding acquisition;

supervision; validation; visualization; writing review and editing. **William Ramírez:** Conceptualization; formal analysis; methodology; project administration; funding acquisition; supervision; validation; visualization; writing review and editing.

References

1. A. Benahmadi, A. Ghanmi, Post-quantum complex Hermite polynomials, *Bol. Soc. Mat. Mex.*, **30** (2024). https://doi.org/10.1007/s40590-023-00586-0
2. R. Chakrabarti, R. Jagannathan, A (p,q)-oscillator realization of two-parameter quantum algebras, *J. Phys. A*, **24** (1991), L711−L718. https://doi.org/10.1088/0305-4470/24/13/002
3. R. B. Corcino, On (p,q)-binomial coefficients, *Electron. J. Combin. Number Theory*, **8** (2008).
4. U. Duran, M. Acikgoz, A. Esi, S. Araci, A note on the (p,q)-Hermite polynomials, *J. Appl. Math. Inf. Sci.*, **12** (2018), 227−231. https://doi.org/10.18576/amis/120122
5. U. Duran, M. Acikgoz, S. Araci, On higher order (p,q)-Frobenius-Euler polynomials, *TWMS J. Pure Appl. Math.*, **8** (2017), 198−208.
6. N. Raza, M. Fadel, W. S. Du, New summation and integral representations for 2-variable (p,q)-Hermite polynomials, *Axioms*, **13** (2024), 196. https://doi.org/10.3390/axioms13030196
7. P. N. Sadjang, On (p,q)-Appell polynomials, *Anal. Math.*, **45** (2019), 583–598. https://doi.org/10.1007/s10476-019-0826-z
8. G. Yasmin, A. Muhyi, On a family of (p,q)-hybrid polynomials, *Kragujevac J. Math.,* **45** (2019), 409−426. https://doi.org/10.46793/kgjmat2103.409y
9. N. Raza, M. Fadel, S. Khan, On monomiality property of q-Gould-Hopper-Appell polynomials, *Arab. J. Basic Appl. Sci.*, **32** (2025), 21−29. https://doi.org/10.1080/25765299.2025.2457206
10. W. Ramírez, D. Bedoya, A. Urieles, C. Cesarano, M. Ortega, New U–Bernoulli, U–Euler and U–Genocchi polynomials and their matrices, *Carpathian Math.*, **15** (2023), 449−467. https://doi.org/10.15330/cmp.15.2.449-467
11. W. Ramírez, C. Kızılateş, D. Bedoya, C. Cesarano, C. S. Ryoo, On certain properties of three parametric kinds of Apostol–type unified Bernoulli–Euler polynomials, *AIMS Math.*, **10** (2025), 137–158. https://doi.org/10.3934/math.2025008
12. A. Urieles, M. Ortega, W. Ramírez, D. Bedoya, Fourier expansion and integral representation for generalized Apostol–type Frobenius–Euler polynomials, *Adv. Differ. Equ.*, **534** (2020), 1−14. https://doi.org/10.1186/s13662-020-02988-0
13. N. Raza, M. Fadel, C. Cesarano, On 2-variable q-Legendre polynomials: The view point of the q-operational technique, *Carpathian Math. Publ.*, **17** (2025), 14−26. https://doi.org/10.15330/cmp.17.1.14-26
14. M. Fadel, N. Raza, A. Al-Gonah, U. Duran, Bivariate q-Laguerre-Appell polynomials and their applications, *Appl. Math. Sci. Eng.*, **32** (2024). https://doi.org/10.1080/27690911.2024.2412545
15. I. M. Burban, A. U. Klimyk, (p,q)-Differentiation, (p,q)-Integration, and (p,q)-Hypergeometric functions related to quantum groups, *Integr. Transforms Spec. Funct.* **2** (1994), 15−36. https://doi.org/10.1080/10652469408819035

16. F. A. Costabile, E. Longo, $\Delta h-$Appell sequences and related interpolation problem, *Numer. Algorithms*, **63** (2013), 165−186. https://doi.org/10.1007/s11075-012-9619-1

17. P. N. Sadjang, U. Duran, On two bivariate kinds of (p,q)-Bernoulli polynomials, *Miskolc Math. Notes*, **20** (2019), 1185−1199. https://doi.org/10.18514/mmn.2019.2587

18. F. A. Costabile, E. Longo, A determinantal approach to Appell polynomials, *J. Comput. Appl. Math.*, **234** (2010), 1528−1542. https://doi.org/10.1016/j.cam.2010.02.033

19. F. A. Costabile, E. Longo, An algebraic approach to Sheffer polynomial sequences, *Integr. Transforms Spec. Funct.*, **25** (2014), 295−311. https://doi.org/10.1080/10652469.2013.842234

20. M. E. Keleshteri, N. I. Mahmudov, A study on q-Appell polynomials from determinantal point of view, *Appl. Math. Comput.*, **260** (2015), 351−369. https://doi.org/10.1016/j.amc.2015.03.017

Appendix: Graphic's codes

Code Figure 1:

Plot$[-6/5 + 5/6o, 5/6o^2 2 - 5/6o - 871/570, \{o, 0, 4\},$
PlotStyle → {{AbsoluteThickness[2.5], DotDashed,Red},
AbsoluteThickness[2.5],DotDashed,Blue}}, PlotLegends → {"$_H\mathcal{B}_{1,1/2,1/3}(o)$", "$_H\mathcal{B}_{2,1/2,1/3}(o)$"},
AxesLabel→ $\{o, \mu\}$]}

Code Figure 2:

Plot$[-6/5 + 5/6o, 5/6o^2 - 5/6o - 5/6, \{o, 0, 3\},$
PlotStyle→{{AbsoluteThickness[2.5],DotDashed,Blue}, AbsoluteThickness[2.5], DotDashed,Red}},
PlotLegends → {"$_H\mathcal{E}_{1,1/2,1/3}(o)$", "$_H\mathcal{E}_{2,1/2,1/3}(o)$"}, AxesLabel → $\{o, \mu\}$]

Code Figure 3:

Plot3D$[5/6o^2 + 1/3\mu^2 - 5/6o - \mu + 25/36o * \mu - 871/370, \{o, -100, 800\}, \{\mu, -10, 70\}$, PlotLegends → {"$_H\mathcal{B}_{2,1/2,1/3}(o,\mu)$"}, FillingStyle → White, Mesh → None, PlotStyle → Directive[Blue, Specularity[White, 50], Opacity[3]], ExclusionsStyle → {Green,GreenFunction}]

Code Figure 4:

Plot3D$[5/6o^2 + 1/3\mu^2 - 5/6o - \mu + 25/36o * \mu - 5/6, \{o, -50, 400\}, \{\mu, -10, 70\}$, PlotLegends→ {"$_H\mathcal{E}_{2,1/2,1/3}(o,\mu)$"}, FillingStyle→White, Mesh→ None, PlotStyle→ Directive[Green,Specularity[Yellow,20], ExclusionsStyle→ {Yellow, YellowFunction}]