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Research article

An Oleinik inequality for a class of heterogeneous balance laws

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Abstract: Although an Oleinik inequality is not expected for general heterogeneous scalar balance laws, there is a favorable case, when the source term $g(x)$ is smooth and nondecreasing. In this setting, we are able to obtain an Oleinik-type inequality and, consequently, a smoothing effect.

Keywords: balance laws; entropy solution; strictly convex flux; Oleinik inequality; wave interaction; generalized Riemann problem; regularizing effect

1. Introduction

The Oleinik one-sided inequality [1] is a fundamental tool in the analysis of scalar conservation laws with strictly convex fluxes (i.e., f' is strictly increasing, although f'' may vanish at some points). Introduced in the 1950s, it provides a precise description of the nonlinear smoothing effect and is pivotal to establish the well-posedness of entropy solutions, as well as in the study of the large-time asymptotics of solutions [2] and in control theory [3]. Beyond the scalar case, see [4] for Oleinik-type one-sided estimates for particular hyperbolic systems of conservation laws, [5] for genuinely nonlinear 2×2 systems, and [6, 7] for balance laws systems with small data.

In this work, we extend the classical Oleinik one-sided inequality to a broader setting of scalar conservation laws by allowing a nontrivial source term $g(x)$ in the equation. A partial explanation for why an Oleinik-type estimate with source appeared only decades after Oleinik's seminal result is that, when the source is spatial dependent (i.e., $g = g(x)$), such an estimate can fail: In reference [8], even for uniformly convex fluxes, namely $f''(u) \geq c > 0$ for some constant c, one can construct entropy solutions whose spatial profiles have an infinite total variation. This precludes any global Oleinik bound.

However, the inviscid Burgers equation admits global smooth solutions for nondecreasing initial data. The same remains true in the presence of a smooth nondecreasing spatial source term: one still obtains global smooth solutions. This motivates us to focus on *spatially nondecreasing* sources, namely $g = g(x)$ with $g'(x) \geq 0$. In this regime, an Oleinik inequality is available. For this purpose, we introduce a time and space-dependent monotone reparametrization $H = H(t, x, u)$ such that for every fixed (t, x), the map $u \mapsto H(t, x, u)$ is increasing. With this choice, we recover a one-sided Oleinik-type control in the H-variables: for a.e. $t > 0$ and all $x < y$,

$$H(t, y, u(t, y)) - H(t, x, u(t, x)) \ \le \ y - x,$$

where H is a function that only depends on the flux f and the source term g, but not on the solution itself.

The one-dimensional hyperbolic balance law studied in this paper is as follows:

$$\begin{cases} \partial_t u + \partial_x (f(u)) = g(x), \\ u(0, x) = u_0(x), \end{cases} \tag{1.1}$$

where the unknown $u = u(t, x)$ depends on time and space variables $(t, x) \in [0, \infty) \times \mathbb{R}$ and the initial data $u_0 \in L^\infty(\mathbb{R})$. The flux function f is of class C^2 on $\mathbb{R}$, and the source term g is Lipschitz, bounded, and of class C^1 on $\mathbb{R}$ that satisfies $g'(x) \ge 0$.

Let u be the unique entropy solution of Eq (1.1). Since $v(t, x) = H(t, x, u(t, x))$ is bounded on any bounded strip in time due to the maximum principle for u and v is one sided-Lipschitz by the Oleinik inequality, v is in BV_{loc} in space. It is important that the function H we construct is strictly increasing with respect to u; this property is crucial to control the variation of u by the variation of v.

A distinctive feature of the balance law with spatial source $g = g(x)$ is revealed by the method of characteristics: The resulting characteristic relations form a *coupled* system for the pair $(X(t, x_0), u(t, X(t, x_0)))$. In contrast, when the source only depends on the unknown, $g = g(u)$ [9], the evolution of u along characteristics satisfies a scalar ordinary differential equation (ODE) independent of the spatial position, so the system is decoupled. In the present $g(x)$ setting, the position and the state influence each other along characteristic curves and the dynamics are naturally described by a two-dimensional *vectorial flow* in the (x, u)-plane, see Section 2.1.

It is crucial that the source term $g(x)$ be a C^1 nondecreasing function. The C^1 regularity is useful because g directly appears in the characteristic system, and the construction of these characteristics requires g to be sufficiently smooth. The monotonicity assumption $g'(x) \ge 0$ is equally important: It ensures a global-in-time well-posedness theory for the generalized Riemann problem introduced in Section 4, and it allows us to control the wave interactions that arise from such data.

However, when g is a decreasing source term, the situation becomes substantially more complex. It is well known that the inviscid Burgers equation cannot admit global smooth solutions for decreasing initial data: Shocks necessarily form in finite time and possibly multiple shocks. In our setting, such behavior may prevent the existence of a smooth global solution, which, in turn, breaks the construction based on generalized Riemann problems. This scenario is discussed in Section 6, where we present two examples that illustrate how and where the breakdown of global smoothness may occur.

Classical results of Oleinik and Peter, [1, 2] show that for scalar conservation laws with no source ($g \equiv 0$) and *uniformly* convex fluxes, entropy solutions exhibit instantaneous spatial smoothing: For every $t > 0$, one has $u(t, \cdot) \in BV_{\mathrm{loc}}(\mathbb{R})$. This BV-regularization fails once uniform convexity is lost. Still, in the strictly convex case with *polynomial* degeneracy (for instance $f(u) = |u|^3$ or $f(u) = u^4$), a weaker yet quantitative form of regularity survives: the solution belongs to a fractional bounded-variation class $BV^s_{\mathrm{loc}}(\mathbb{R})$ for some $s \in (0,1)$ [10, 11] or more generalized BV spaces [12, 13].

A result of this paper is to show that even when the flux is only strictly convex, we can still recover *full* BV_{loc} regularity under an additional assumption on the spatial source $g(x)$. This mirrors the well–known mechanism, where the source acts as an effective damping to restore BV regularity; see [14] for a stochastic source and [9] for the autonomous source. In the systems setting, the BV regularizing effects is rare; see [15–17] for 2×2 nonlinear systems. This is the reason why the initial data are already BV for balance laws system [18].

For the proof of the one-sided Oleinik inequality, a key methodological point is that we *do not* use the classical wave–front tracking algorithm [5, 19]. In fact, we do not need to restrict ourselves to piecewise-constant *solutions*. Nevertheless, we will borrow the piecewise-constant approximation of the initial data from wave-front tracking (WFT) to reduce the dynamics to a finite number of Riemann problems, track and control wave interactions, and verify Oleinik's one-sided estimate on each piece. By the additivity of the H-increment across interfaces,

$$\big[H(t, x_{i+1}, u(t, x_{i+1})) - H(t, x_i, u(t, x_i))\big] \;\leq\; x_{i+1} - x_i,$$

and summing over a partition yields the global bound $\Delta H \leq \Delta x$. This avoids classical front tracking and results in a simpler proof mechanism. Finally, passing to the limit in the initial–data approximation and using the stability and uniqueness of entropy solutions, the inequality persists in the limit, so the generalized Oleinik bound holds.

The paper is organized as follows. In Section 2, we introduce the vectorial flow generated by the characteristic system of Eq (1.1) and record its key properties; then, we use this flow to prove Oleinik's equality in the setting of strong solutions. Section 3 focuses on our main theorem and its key consequences for the regularizing effects of entropy solutions. In Section 4, we study the generalized Riemann problem associated with the balance law (1.1) and provide semi explicit formulas for the solution. Section 5 is devoted to the proof of Oleinik's one-sided inequality in the framework of entropy solutions using the exact WFT (eWFT) detailed in Subsection 5.1. We analyze interactions between consecutive Riemann problems and show that the inequality holds; moreover, we establish the regularizing effects enjoyed by the entropy solution. Finally, Section 6 shows that when g is decreasing, the situation is considerably more complicated, and we illustrate this with two examples for Burgers equation.

2. Oleinik equality for smooth solutions

This section is devoted to the proof of Oleinik's equality for strong solutions of the balance law (1.1). For such solutions, the one-sided Oleinik Lipschitz condition holds with equality. By contrast, for discontinuous solutions (e.g., across shocks), the corresponding statement becomes an inequality.

2.1. Characteristics system for heterogenous balance laws

Using the method of characteristics, the solution u satisfies the following system of ordinary differential equations along each characteristic:

$$\frac{dX(t, x_0)}{dt} = f'(u(t, X(t, x_0))), \qquad \frac{du}{dt} = g(X(t, x_0)). \tag{2.1}$$

We introduce the vector $U = \begin{pmatrix} X \\ u \end{pmatrix} \in \mathbb{R}^2$, so that the system of characteristics can be compactly written as follows:

$$\frac{dU}{dt} = F(U),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the vector field

$$F(x, u) := \begin{pmatrix} f'(u) \\ g(x) \end{pmatrix},$$

with $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

Vectorial flow. Let $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow generated by the following:

$$\frac{d}{dt}\varphi_t(x_0, v) = F(\varphi_t(x_0, v)), \qquad \varphi_0(x_0, v) = (x_0, v). \tag{2.2}$$

We write its components as follows:

$$\varphi_t(x_0, v) = (\varphi_t^1(x_0, v),\ \varphi_t^2(x_0, v)). \tag{2.3}$$

Restriction to the initial profile. Given $u_0 : \mathbb{R} \to \mathbb{R}$, evaluate the flow along the curve $v = u_0(x_0)$:

$$\varphi_t(x_0, u_0(x_0)) = (X(t, x_0),\ u(t, X(t, x_0))).$$

We set

$$X(t, x_0) := \varphi_t^1(x_0, u_0(x_0)), \qquad \varphi_t^2(x_0, u_0(x_0)) = u(t, X(t, x_0)).$$

where:

- $\varphi_t^1(x_0, u_0(x_0))$ is the position at time t along the characteristic curve issued from x_0,
- $\varphi_t^2(x_0, u_0(x_0))$ is the value of the solution evaluated along this characteristic.

Hyperbolicity. It is important to have hyperbolicity, which means a finite speed for the propagation of the information.

For this purpose, we assume that

$$u_0 \in L^\infty(\mathbb{R}), \qquad \|u_0\|_{L^\infty(\mathbb{R})} \le M_0,$$

and the source term is bounded,

$$g \in L^\infty(\mathbb{R}), \qquad \|g\|_{L^\infty(\mathbb{R})} \le M_g.$$

Fix $T > 0$. Then, for every $(t, x) \in [0, T] \times \mathbb{R}$, we have the L^∞–estimate as follows:

$$|u(t, x)| \le M_0 + TM_g.$$

In particular,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le M_0 + TM_g, \qquad t \in [0,T].$$

Since f' is continuous, we can define the following:

$$\Lambda(T) \ := \ \sup\{\,|f'(v)| : |v| \le M_0 + TM_g\} < \infty.$$

It follows that

$$|f'(u(t,x))| \le \Lambda(T), \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Then, the hyperbolic propagation of information with finite speed is clear on every strip $[0,T] \times \mathbb{R}$, with a maximal possible speed $\Lambda(T)$.

Fix $x_0 < x_1$ and define the trapezoid as follows:

$$\mathcal{K}_T := \{(t,x) \in [0,T] \times \mathbb{R} : x_0 - \Lambda(T)\,(T-t) \le x \le x_1 + \Lambda(T)\,(T-t)\}.$$

By finite speed of propagation, the solution u in $\mathcal{K}_T$ only depends on the initial data and the source term $g(x)$ on the bounded interval

$$[x_0 - \Lambda(T)T,\ x_1 + \Lambda(T)T].$$

Since, u is bounded in $[0,T] \times \mathbb{R}$ for all $T > 0$ and g is Lipschitz, the phase–space flow φ_t is globally defined for all $t > 0$.

Invariant. Let $E(x,u) := f(u) - G(x)$ with $G'(x) = g(x)$. Along the characteristic curves,

$$\frac{d}{dt}\,E(\varphi_t^1(x_0,u_0), \varphi_t^2(x_0,u_0)) = f'(\varphi_t^2)\,\partial_t\varphi_t^2 - G'(\varphi_t^1)\,\partial_t\varphi_t^1 = f'(\varphi_t^2)\,g(\varphi_t^1) - g(\varphi_t^1)\,f'(\varphi_t^2) = 0.$$

Hence, E is conserved for smooth solutions: $E(\varphi_t(x_0,u_0)) \equiv E(x_0,u_0)$ for all t.

Proposition 2.1. *[Monotonicity of the flow in each initial variable] Let $F(x,u) = (f'(u), g(x))$ with $f \in C^2$, $g \in C^1$, and assume $f'' \ge 0$, $g' \ge 0$. Let $\varphi_s = (\varphi_s^1, \varphi_s^2)$ be the flow of F. Fix $(x_0,u_0) \in \mathbb{R}^2$ and $s \ge 0$. Define the following four variational quantities:*

$$A(s) := \partial_{x_0}\varphi_s^1(x_0,u_0), \quad B(s) := \partial_{x_0}\varphi_s^2(x_0,u_0), \quad C(s) := \partial_{u_0}\varphi_s^1(x_0,u_0), \quad D(s) := \partial_{u_0}\varphi_s^2(x_0,u_0).$$

Then,

$$A(s) \ge 1, \qquad B(s) \ge 0, \qquad C(s) \ge 0, \qquad D(s) \ge 1. \tag{2.4}$$

Proof. We consider the following characteristic system:

$$\begin{cases} \dfrac{d}{ds}\varphi_s^1(x_0,u_0) = f'(\varphi_s^2(x_0,u_0)), \\ \dfrac{d}{ds}\varphi_s^2(x_0,u_0) = g(\varphi_s^1(x_0,u_0)), \\ \varphi_0^1(x_0,u_0) = x_0, \quad \varphi_0^2(x_0,u_0) = u_0. \end{cases} \tag{2.5}$$

By standard ODE theory and the regularity of f and g, the flow

$$(x_0, u_0) \longmapsto \varphi_s(x_0, u_0)$$

is of class C^1 for every fixed $s \geq 0$. In particular, the partial derivatives

$$A(s) := \partial_{x_0}\varphi_s^1(x_0, u_0), \quad B(s) := \partial_{x_0}\varphi_s^2(x_0, u_0), \quad C(s) := \partial_{u_0}\varphi_s^1(x_0, u_0), \quad D(s) := \partial_{u_0}\varphi_s^2(x_0, u_0)$$

are well-defined and continuous in s.

Step 1. ODEs for (A, B) **and** (C, D)**.** Differentiating Eq (2.5) with respect to x_0 and using the chain rule, we obtain

$$\frac{d}{ds}A(s) = \partial_{x_0}\Big(f'(\varphi_s^2(x_0, u_0))\Big) = f''(\varphi_s^2(x_0, u_0))\,\partial_{x_0}\varphi_s^2(x_0, u_0) = f''(\varphi_s^2)\,B(s),$$

and

$$\frac{d}{ds}B(s) = \partial_{x_0}\Big(g(\varphi_s^1(x_0, u_0))\Big) = g'(\varphi_s^1(x_0, u_0))\,\partial_{x_0}\varphi_s^1(x_0, u_0) = g'(\varphi_s^1)\,A(s).$$

From the initial conditions in Eq (2.5), we obtain the following:

$$A(0) = \partial_{x_0}\varphi_0^1(x_0, u_0) = 1, \qquad B(0) = \partial_{x_0}\varphi_0^2(x_0, u_0) = 0.$$

Hence, (A, B) solves the following system:

$$\begin{cases} A'(s) = f''(\varphi_s^2(x_0, u_0))\,B(s), \\ B'(s) = g'(\varphi_s^1(x_0, u_0))\,A(s), \\ A(0) = 1,\ B(0) = 0. \end{cases} \tag{2.6}$$

Similarly, differentiating Eq (2.5) with respect to u_0 yields

$$C'(s) = f''(\varphi_s^2)\,D(s), \qquad D'(s) = g'(\varphi_s^1)\,C(s),$$

with initial data

$$C(0) = \partial_{u_0}\varphi_0^1(x_0, u_0) = 0, \qquad D(0) = \partial_{u_0}\varphi_0^2(x_0, u_0) = 1.$$

Thus, (C, D) solves the following:

$$\begin{cases} C'(s) = f''(\varphi_s^2(x_0, u_0))\,D(s), \\ D'(s) = g'(\varphi_s^1(x_0, u_0))\,C(s), \\ C(0) = 0,\ D(0) = 1. \end{cases} \tag{2.7}$$

Step 2. Sign properties for (A, B)**.** Since f is convex, we have $f'' \geq 0$, and since g is nondecreasing, $g' \geq 0$. Define the following:

$$I := \{\, t \geq 0 :\ A(s) \geq 1,\ B(s) \geq 0 \text{ for all } s \in [0, t] \,\}.$$

The set I is nonempty: By continuity and the initial conditions $A(0) = 1$, $B(0) = 0$ we have $[0, \delta] \subset I$ for some $\delta > 0$. Set

$$T^* := \sup I \in (0, +\infty].$$

On the interval $[0, T^*)$, we have $A(s) \geq 1$ and $B(s) \geq 0$ by the definition of I; hence, from Eq (2.6) and the assumptions $f'' \geq 0$, $g' \geq 0$, we obtain the following:

$$A'(s) = f''(\varphi_s^2)\, B(s) \geq 0, \qquad B'(s) = g'(\varphi_s^1)\, A(s) \geq 0 \quad \text{for all } s \in [0, T^*).$$

Thus, A and B are nondecreasing on $[0, T^*)$. Using $A(0) = 1$ and $B(0) = 0$, it follows that

$$A(s) \geq 1, \quad B(s) \geq 0 \quad \text{for all } s \in [0, T^*).$$

Then, by continuity of A and B, we have

$$A(T^*) \geq 1, \qquad B(T^*) \geq 0$$

whenever $T^* < +\infty$.

Evaluating Eq (2.6) at $s = T^*$, we still have the following:

$$A'(T^*) = f''(\varphi_{T^*}^2)\, B(T^*) \geq 0, \qquad B'(T^*) = g'(\varphi_{T^*}^1)\, A(T^*) \geq 0.$$

Hence, there exists $\varepsilon > 0$ such that

$$A(s) \geq 1, \quad B(s) \geq 0 \quad \text{for all } s \in [T^*, T^* + \varepsilon),$$

which implies $[0, T^* + \varepsilon) \subset I$. This contradicts the definition of T^* as the supremum of I unless $T^* = +\infty$. Therefore,

$$A(s) \geq 1, \quad B(s) \geq 0 \quad \text{for all } s \geq 0.$$

Step 3. Sign properties for (C, D)**.** The argument for (C, D) is completely analogous. Define

$$J := \{\, t \geq 0 : \ C(s) \geq 0, \ D(s) \geq 1 \text{ for all } s \in [0, t] \,\}$$

and let $S^* := \sup J$. From the initial conditions $C(0) = 0$, $D(0) = 1$ and continuity, we have $J \neq \emptyset$ and $S^* > 0$.

On $[0, S^*)$, by the definition of J and Eq (2.7) together with $f'' \geq 0$, $g' \geq 0$, we obtain the following:

$$C'(s) = f''(\varphi_s^2)\, D(s) \geq 0, \qquad D'(s) = g'(\varphi_s^1)\, C(s) \geq 0.$$

Thus, C and D are nondecreasing on $[0, S^*)$, and using $C(0) = 0$, $D(0) = 1$, we obtain the following:

$$C(s) \geq 0, \quad D(s) \geq 1 \quad \text{for all } s \in [0, S^*).$$

By continuity, $C(S^*) \geq 0$ and $D(S^*) \geq 1$ (if $S^* < +\infty$), and evaluating Eq (2.7) at $s = S^*$ again yields $C'(S^*) \geq 0$, $D'(S^*) \geq 0$. Hence, we can extend the inequalities slightly beyond S^*, which contradicts the definition of S^* unless $S^* = +\infty$.

We conclude that

$$C(s) \geq 0, \quad D(s) \geq 1 \quad \text{for all } s \geq 0.$$

Combining the conclusions of Steps 2 and 3, we have proved that for every $s \geq 0$,

$$\partial_{x_0}\varphi_s^1(x_0, u_0) = A(s) \geq 1, \quad \partial_{x_0}\varphi_s^2(x_0, u_0) = B(s) \geq 0,$$

and

$$\partial_{u_0}\varphi_s^1(x_0, u_0) = C(s) \geq 0, \quad \partial_{u_0}\varphi_s^2(x_0, u_0) = D(s) \geq 1,$$

which is precisely the statement of the proposition.

Proposition 2.2. *Under the same hypotheses as Proposition 2.1, for every $s \geq 0$ and all $(x, v) \in \mathbb{R}^2$,*

$$\partial_v(\varphi^2_{-s}(x, v)) \;\geq\; 1; \tag{2.8}$$

hence, $v \mapsto \varphi^2_{-s}(x, v)$ is strictly increasing.

Proof. Fix $s \geq 0$ and an arbitrary (x, v) in the domain of φ_{-s}, and set

$$(x_0, u_0) := \varphi_{-s}(x, v) \qquad \Longrightarrow \qquad (x, v) = \varphi_s(x_0, u_0).$$

Consider the following:

$$M := D\varphi_{-s}(x, v).$$

(i) By definition of the Jacobian,

$$M = \begin{pmatrix} \partial_x\varphi^1_{-s} & \partial_v\varphi^1_{-s} \\ \partial_x\varphi^2_{-s} & \partial_v\varphi^2_{-s} \end{pmatrix}_{(x,v)}.$$

(ii) By the chain rule applied to $\varphi_{-s} \circ \varphi_s = \mathrm{Id}$ at (x_0, u_0),

$$D\varphi_{-s}(\varphi_s(x_0, u_0))\, D\varphi_s(x_0, u_0) = I \quad \Longrightarrow \quad M = (D\varphi_s(x_0, u_0))^{-1}.$$

Since $\operatorname{div} F = 0$, Liouville's formula gives $\det D\varphi_s \equiv 1$; hence,

$$(D\varphi_s(x_0, u_0))^{-1} = \begin{pmatrix} \partial_{u_0}\varphi^2_s & -\partial_{u_0}\varphi^1_s \\ -\partial_{x_0}\varphi^2_s & \partial_{x_0}\varphi^1_s \end{pmatrix}_{(x_0,u_0)}.$$

Comparing the $(2, 2)$ entries of the two expressions for M yields the following:

$$\partial_v\varphi^2_{-s}(x, v) = \partial_{x_0}\varphi^1_s(x_0, u_0).$$

Using Proposition 2.1, $\partial_{x_0}\varphi^1_s(x_0, u_0) \geq 1$, and since $(x_0, u_0) = \varphi_{-s}(x, v)$ was chosen from the given (x, v), we conclude the following:

$$\partial_v\varphi^2_{-s}(x, v) \;\geq\; 1.$$

As (x, v) was arbitrary in the domain of φ_{-s}, the claim follows.

Remark 2.3. Since f is strictly convex, f' is a strictly increasing function. Using the previous proposition, we obtain that $v \mapsto H(t, x, v)$ is strictly increasing, where H is defined in Eq (2.10). Indeed, for any $v_1 < v_2$ and any $s \in [0, t]$, by Proposition 2.2, the monotonicity of $v \mapsto \varphi^2_{-s}(x, v)$ implies the following:

$$\varphi^2_{-s}(x, v_1) < \varphi^2_{-s}(x, v_2);$$

hence, using the strict convexity of the flux, we obtain the following:

$$f'(\varphi^2_{-s}(x, v_1)) < f'(\varphi^2_{-s}(x, v_2)).$$

Integrating in s over $[0, t]$ yields $H(t, x, v_1) < H(t, x, v_2)$; thus, $v \mapsto H(t, x, v)$ is strictly increasing.

2.2. Oleinik equality

As a first step, we derive the Oleinik inequality in the setting of smooth solutions. This preliminary result clarifies that the one-sided regularity estimate controls the spatial spreading of characteristics. We give the following Theorem.

Theorem 2.4 (Oleinik equality for strong solutions)**.** *Assume that $f \in C^2(\mathbb{R})$ is strictly convex, and $g \in C^1(\mathbb{R})$ is Lipschitz, bounded, and satisfies $g'(x) \geq 0$ for all $x \in \mathbb{R}$. Let u be a strong solution of Eq* (1.1) *with a bounded initial data. Then, for all $t > 0$ and any two points $\bar{x} < x$, we have the following:*

$$H(t, x, u(t, x)) \;-\; H(t, \bar{x}, u(t, \bar{x})) \;=\; (x - \bar{x}) \;-\; (x_0 - \bar{x}_0), \tag{2.9}$$

where x_0 and $\bar{x}_0$ denote the footpoints of the two corresponding characteristics. The function H is defined on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ by the following:

$$H(t, x, v) = \int_0^t f'(\varphi^2_{-s}(x, v))\, ds, \tag{2.10}$$

where φ^2 is the second component of the characteristic flow introduced in Section 2.1.

Proof. Suppose that $u \in C^1([0, T] \times \mathbb{R})$ is a strong solution of the balance law (1.1).

By integrating the first equation of the ODE system (2.1) between 0 and t, we obtain the following:

$$x \;=\; x_0 + \int_0^t f'\Big(\varphi^2_s(x_0, u_0(x_0))\Big)\, ds.$$

Now, consider another characteristic issued from $\bar{x}_0$, and set $\Delta x_0 := x_0 - \bar{x}_0 \geq 0$. Since we are dealing with a strong solution (no shocks up to time t), characteristics do not intersect, so their ordering is preserved as follows:

$$X(t, x_0) - X(t, \bar{x}_0) \;\geq\; 0 \qquad \text{for all } t > 0.$$

Writing $\bar{x} := X(t, \bar{x}_0)$, the same computation gives the following:

$$\bar{x} \;=\; \bar{x}_0 + \int_0^t f'\Big(\varphi^2_s(\bar{x}_0, u_0(\bar{x}_0))\Big)\, ds.$$

Using the **semigroup property** of the flow,

$$\varphi^2_s(x_0, u_0(x_0)) = \varphi^2_{s-t}(\varphi_t(x_0, u_0(x_0))) = \varphi^2_{s-t}(x, u(t, x)),$$

we obtain

$$\Delta x = \Delta x_0 + \int_0^t f'\Big(\varphi^2_{s-t}(x, u(t, x))\Big)\, ds - \int_0^t f'\Big(\varphi^2_{s-t}(\bar{x}, u(t, \bar{x}))\Big)\, ds.$$

Applying the change of variables $s \mapsto t - s$ in each integral,

$$\Delta x = \Delta x_0 + \int_0^t f'\Big(\varphi^2_{-s}(x, u(t, x))\Big)\, ds - \int_0^t f'\Big(\varphi^2_{-s}(\bar{x}, u(t, \bar{x}))\Big)\, ds.$$

Hence,

$$\Delta x \;=\; \Delta x_0 \;+\; H(t, x, u(t, x)) \;-\; H(t, \bar{x}, u(t, \bar{x})) \;=:\; \Delta x_0 + \Delta H,$$

where

$$H(t, x, v) := \int_0^t f'\Big(\varphi^2_{-s}(x, v)\Big)\, ds;$$

thus, we obtain the following generalized *Oleinik identity* for smooth solutions as follows:

$$\Delta x = \Delta x_0 + \Delta H. \tag{2.11}$$

Remark 2.5. Since $\Delta x_0 \geq 0$, we can write the equality (2.11) as follows:

$$\Delta H \leq \Delta x. \tag{2.12}$$

3. Oleinik inequality for entropy solutions

In this section, we present our main theorem and outline some consequential implications. The central tool is the Oleinik one-sided Lipschitz inequality for entropy solutions, which is proven under the hypotheses stated below. As a consequence, we obtain quantitative regularizing effects for the unique entropy solution, which are made precise in the results that follow.

Theorem 3.1. *[Generalized Oleinik inequality] Assume that $f \in C^2(\mathbb{R})$ is strictly convex, and $g \in C^1(\mathbb{R})$ is bounded, Lipschitz that satisfies $g'(x) \geq 0, \forall x \in \mathbb{R}$. Let $u_0 \in L^\infty(\mathbb{R})$, and u be the unique entropy solution of Eq* (1.1)*. Then, for a.e. $t > 0$ and any two points $\bar{x} < x$, we have the following estimate:*

$$\Delta H \leq \Delta x, \tag{3.1}$$

where the function H is defined by the relation (2.10)*.*

This result is particularly significant. One-sided Oleinik inequalities have far-reaching implications in the theory of scalar conservation laws. In this paper, we focus on the regularizing effect on the unique entropy solution to Eq (1.1).

We begin with the classical case of a uniformly convex flux. In this setting, the Oleinik one-sided Lipschitz bound yields the following spatial regularity result.

Corollary 3.2. *[BV regularizing effect] Under the assumptions of Theorem 3.1 and f uniformly convex, the unique entropy solution $u(t, \cdot)$ of Eq* (1.1) *belongs to $BV_{\mathrm{loc}}(\mathbb{R})$ for every $t > 0$.*

When the assumption of uniform convexity is relaxed to strict convexity, the BV smoothing effect can fail. To quantify this loss, we introduce the notion of the *polynomial degeneracy* of f.

Definition 3.3. (Degeneracy). Let $f \in C^1(K, \mathbb{R})$, where $K \subset \mathbb{R}$ is a closed interval, and let $a(u) := f'(u)$. We say that the degeneracy of f on K is at least $p > 0$ if

$$\inf_{\substack{(u,v)\in K\times K\\ u\neq v}} \frac{|f'(u) - f'(v)|}{|u - v|^p} > 0. \tag{3.2}$$

The smallest such real number p (if it exists) is called the *degeneracy exponent* of f on K.

Next, we address the case where fluxes whose convexity may degenerate. In this setting, full *BV* smoothing fails. However, the entropy solution still enjoys a fractional *BV* regularization. The following result makes this precise.

Corollary 3.4. *[BV^s regularizing effect] Under the assumptions of Theorem 3.1 and f that satisfies Eq* (3.2)*, the unique entropy solution $u(t,\cdot)$ of* (1.1) *belongs to $BV^s_{\mathrm{loc}}(\mathbb{R})$ for every $t > 0$, where $s = 1/p$.*

Finally, we address the case where f is only strictly convex. An additional condition is added on the source term g to recover the full *BV* regularity.

Theorem 3.5. *Assume that $g(x) \neq 0$ for all $x \in \mathbb{R}$. Under the assumptions of Theorem 3.1, the unique entropy solution $u(t,\cdot)$ of Eq* (1.1) *belongs to $BV_{\mathrm{loc}}(\mathbb{R})$ for every $t > 0$.*

Generically, to get the *BV* regularity, the condition g never vanishes can be relaxed by $g' > 0$ or g vanishes only one time. Indeed, if g vanishes at exactly one point, the situation is more delicate: One cannot always expect a BV_{loc} regularization, see the discussion in Section 5.4. In particular, there is no transversal condition linking g and f'' as in the case $g = g(u)$ [9].

4. The Riemann problem

In this section, we study the Riemann problem associated with the balance law (1.1). This analysis is a key step in the proof of Oleinik's one-sided inequality for entropy solutions. Indeed, after discretizing the initial data by a piecewise-constant function, the argument reduces to verifying the inequality on a finite family of Riemann problems generated at the jump points. Because the source term is spatial, the solution is no longer self-similar and explicitly depends on the space variable. To handle this, we introduce a *generalized Riemann problem* in which the initial data consist of two nondecreasing smooth profiles on the left and right.

4.1. The generalized Riemann problem

As explained just above, we now consider the generalized Riemann problem associated with the balance law (1.1), given by the following:

$$\begin{cases} \partial_t u + \partial_x(f(u)) = g(x), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \begin{cases} U_L(x), & x < \alpha, \\ U_R(x), & x > \alpha, \end{cases} \end{cases} \tag{4.1}$$

where $U_L', U_R' \geq 0$, and $U_L, U_R \in C^1$.

Shock case Assume $U_L(\alpha) > U_R(\alpha)$. Let $(t, \gamma(t))$ parametrize the shock curve with $\gamma(0) = \alpha$. The *shock footpoints* $\xi_L(t)$ and $\xi_R(t)$ are the initial locations (at $t = 0$) of the characteristics that reach the shock at time t, which is implicitly defined by the following:

$$\varphi_t^1\big(\xi_L(t),\ U_L(\xi_L(t))\big) = \gamma(t), \qquad \varphi_t^1\big(\xi_R(t),\ U_R(\xi_R(t))\big) = \gamma(t), \qquad \xi_L(0) = \xi_R(0) = \alpha. \tag{4.2}$$

The *left/right traces* at the shock are the one-sided limits

$$U_-(t) := \lim_{x\uparrow\gamma(t)} u(t,x), \qquad U_+(t) := \lim_{x\downarrow\gamma(t)} u(t,x). \tag{4.3}$$

In terms of the characteristic flow, these traces are as follows:

$$U_-(t) = \varphi_t^2\big(\xi_L(t),\ U_L(\xi_L(t))\big), \qquad U_+(t) = \varphi_t^2\big(\xi_R(t),\ U_R(\xi_R(t))\big). \tag{4.4}$$

Then, the unique entropy solution is given by

$$u(t,x) = \begin{cases} \varphi_t^2(x_0,\ U_L(x_0)), & \text{if } x < \gamma(t) \text{ with } x_0 \text{ such that } \varphi_t^1(x_0,\ U_L(x_0)) = x, \\ \varphi_t^2(x_0,\ U_R(x_0)), & \text{if } x > \gamma(t) \text{ with } x_0 \text{ such that } \varphi_t^1(x_0,\ U_R(x_0)) = x, \end{cases} \tag{4.5}$$

and the shock speed satisfies the Rankine–Hugoniot condition expressed through the traces as follows:

$$\dot\gamma(t) = \frac{f(U_+(t)) - f(U_-(t))}{U_+(t) - U_-(t)}. \tag{4.6}$$

Proposition 4.1 (Entropy shock (generalized data at α))**.** *Assume $f \in C^2$ with $f'' \geq 0$, $g \in C^1$ with $g' \geq 0$, and $U_L(\alpha) > U_R(\alpha)$. Let u be defined by Eq* (4.5)*, with shock curve γ solving Eq* (4.6)*, traces $U_\pm$ given by Eq* (4.4)*, and footpoints $\xi_{L/R}$ as in Eq* (4.2)*. Then, u is the unique entropy solution of the generalized Riemann problem* (4.1) *at $x = \alpha$.*

Proof. Fix $t > 0$ to be small and define $\xi(t;u)$ by

$$\varphi_t^1(\xi(t;u), u) = \gamma(t),$$

and the trace map

$$\Psi_t(u) := \varphi_t^2(\xi(t;u), u).$$

Differentiating the defining relation with respect to u (chain rule, t fixed) gives the following:

$$\partial_{x_0}\varphi_t^1(\xi(t;u), u)\,\xi'(t;u) + \partial_{u_0}\varphi_t^1(\xi(t;u), u) = 0;$$

thus,

$$\xi'(t;u) = -\left.\frac{\partial_{u_0}\varphi_t^1}{\partial_{x_0}\varphi_t^1}\right|_{(x_0,u_0)=(\xi(t;u),u)}.$$

Consequently,

$$\Psi_t'(u) = \partial_{x_0}\varphi_t^2\,\xi'(t;u) + \partial_{u_0}\varphi_t^2 = \left.\frac{-\,\partial_{x_0}\varphi_t^2\,\partial_{u_0}\varphi_t^1 + \partial_{u_0}\varphi_t^2\,\partial_{x_0}\varphi_t^1}{\partial_{x_0}\varphi_t^1}\right|_{(\xi(t;u),u)}.$$

Since $\det D\varphi_t = \partial_{x_0}\varphi_t^1\,\partial_{u_0}\varphi_t^2 - \partial_{u_0}\varphi_t^1\,\partial_{x_0}\varphi_t^2 \equiv 1$ (Liouville; $\operatorname{div} F = 0$), we obtain the following:

$$\Psi_t'(u) = \frac{1}{\partial_{x_0}\varphi_t^1(\xi(t;u), u)};$$

thus, Ψ_t is strictly increasing. Let

$$u_L^{\text{in}}(t) := U_L(\xi_L(t)), \qquad u_R^{\text{in}}(t) := U_R(\xi_R(t)).$$

By continuity and $u_L^{\text{in}}(0) = U_L(\alpha) > U_R(\alpha) = u_R^{\text{in}}(0)$, there exists

$$T_1 := \inf\{\, t > 0 : \; u_L^{\text{in}}(t) = u_R^{\text{in}}(t) \,\} \in (0, \infty]$$

such that $u_L^{\text{in}}(t) > u_R^{\text{in}}(t)$ for all $t \in (0, T_1)$. Then, for $t \in (0, T_1)$,

$$U_-(t) = \Psi_t(u_L^{\text{in}}(t)) \; > \; \Psi_t(u_R^{\text{in}}(t)) = U_+(t),$$

and strict convexity yields the following Lax inequalities:

$$f'(U_-(t)) \; > \; \sigma(t) := \frac{f(U_-(t)) - f(U_+(t))}{U_-(t) - U_+(t)} \; > \; f'(U_+(t)).$$

Remark 4.2. (*No gaps in the characteristic covering*) For each fixed $t \in (0, T_1)$, the side maps

$$X_L(t, \xi) := \varphi_t^1(\xi, U_L(\xi)) \quad (\xi < \alpha), \qquad X_R(t, \xi) := \varphi_t^1(\xi, U_R(\xi)) \quad (\xi > \alpha)$$

are strictly increasing with $X_L(t, \xi_L(t)) = \gamma(t) = X_R(t, \xi_R(t))$ and tails $X_{L/R}(t, \xi) \to \pm\infty$ as $\xi \to \pm\infty$.

Hence,

$$\operatorname{Im} X_L(t, \cdot) = (-\infty, \gamma(t)], \qquad \operatorname{Im} X_R(t, \cdot) = [\gamma(t), \infty);$$

thus, their union is $\mathbb{R}$: there is *no uncovered spatial interval*.

Remark 4.3. (*Finite-time termination of the shock and continuation*) If $T_1 = \infty$, then $U_-(t) > U_+(t)$ for all t and the shock persists globally. If $T_1 < \infty$, then at $t = T_1$, the traces meet as follows:

$$U_-(T_1) = U_+(T_1) =: U_*.$$

The time slice $u(T_1, \cdot)$ is continuous at $x = \gamma(T_1)$ (left and right limits coincide), nondecreasing, but not necessarily C^1 at that point (the one-sided derivatives may differ). From $t = T_1$ onward, continue the evolution with the *continuous, nondecreasing* theory.

Proposition 4.4. *(Rarefaction wave solution) Consider the generalized Riemann problem at $x = \alpha$ with $U_L(\alpha) < U_R(\alpha)$. Define the fan-edge curves and their values by the following:*

$$\begin{aligned} \gamma_-(t) &:= \varphi_t^1(\alpha,\, U_L(\alpha)), \quad U_-(t) := \varphi_t^2(\alpha,\, U_L(\alpha)), \\ \gamma_+(t) &:= \varphi_t^1(\alpha,\, U_R(\alpha)), \quad U_+(t) := \varphi_t^2(\alpha,\, U_R(\alpha)). \end{aligned} \tag{4.7}$$

Then, the unique entropy solution u is given by the following:

$$u(t,x) = \begin{cases} \varphi_t^2(x_0,\, U_L(x_0)), & \text{if } x < H(t, \gamma_-(t), U_-(t)) \text{ with } x_0 \text{ s.t. } \varphi_t^1(x_0,\, U_L(x_0)) = x, \\ \Delta H(t,x,u) = \Delta x, & \text{if } H(t, \gamma_-(t), U_-(t)) \le x \le H(t, \gamma_+(t), U_+(t)), \\ \varphi_t^2(x_0,\, U_R(x_0)), & \text{if } x > H(t, \gamma_+(t), U_+(t)) \text{ with } x_0 \text{ s.t. } \varphi_t^1(x_0,\, U_R(x_0)) = x. \end{cases} \tag{4.8}$$

The proof of Proposition 4.4 is the same as in the autonomous case $g = g(u)$. For details, see [9].

Remark 4.5 (Constant-data reduction at $\alpha = 0$)**.** When $U_L(x) \equiv u_L$ and $U_R(x) \equiv u_R$ are constants with an interface at $\alpha = 0$, the generalized formulas reduce to the following classical Riemann solutions:

Shock case ($u_L > u_R$). Let $\xi_L(t), \xi_R(t)$ and $\gamma(t)$ be defined by the following:

$$\varphi_t^1(\xi_L(t), u_L) = \varphi_t^1(\xi_R(t), u_R) = \gamma(t), \qquad \xi_L(0) = \xi_R(0) = 0,$$

and set the traces

$$U_-(t) := \varphi_t^2(\xi_L(t), u_L), \qquad U_+(t) := \varphi_t^2(\xi_R(t), u_R).$$

Then, the unique entropy solution u is given by the following:

$$u(t,x) = \begin{cases} \varphi_t^2(x_0, u_L), & \text{if } x < \gamma(t) \text{ with } x_0 \text{ s.t. } \varphi_t^1(x_0, u_L) = x, \\ \varphi_t^2(x_0, u_R), & \text{if } x > \gamma(t) \text{ with } x_0 \text{ s.t. } \varphi_t^1(x_0, u_R) = x, \end{cases} \tag{4.9}$$

and the shock speed satisfies the Rankine–Hugoniot condition expressed through the traces as follows:

$$\dot{\gamma}(t) = \frac{f(U_+(t)) - f(U_-(t))}{U_+(t) - U_-(t)}. \tag{4.10}$$

Rarefaction case ($u_L < u_R$). Define the fan edges and their values by the following:

$$\gamma_-(t) := \varphi_t^1(0, u_L), \quad U_-(t) := \varphi_t^2(0, u_L), \qquad \gamma_+(t) := \varphi_t^1(0, u_R), \quad U_+(t) := \varphi_t^2(0, u_R).$$

Then, the unique entropy solution u is given by the following:

$$u(t,x) = \begin{cases} \varphi_t^2(x_0, u_L), & \text{if } x < H(t, \gamma_-(t), U_-(t)) \quad \text{with } x_0 \text{ s.t. } \varphi_t^1(x_0, u_L) = x, \\ \Delta H(t,x,u) = \Delta x, & \text{if } H(t, \gamma_-(t), U_-(t)) \le x \le H(t, \gamma_+(t), U_+(t)), \\ \varphi_t^2(x_0, u_R), & \text{if } x > H(t, \gamma_+(t), U_+(t)) \quad \text{with } x_0 \text{ s.t. } \varphi_t^1(x_0, u_R) = x. \end{cases} \tag{4.11}$$

4.2. Example: Burgers equation with a linear source term

Let $\lambda \ge 0$ be a real number. Consider the Burgers equation with a linear source:

$$\begin{cases} \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = \lambda x & \text{on } (0, +\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{4.12}$$

Characteristic system in matrix form.

$$\dot{X} = u, \qquad \dot{u} = \lambda X, \qquad (X, u)\big|_{t=0} = (x_0, u_0).$$

Set the state vector and matrix as follows:

$$U := \begin{pmatrix} X \\ u \end{pmatrix}, \qquad A := \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

Then,

$$\frac{dU}{dt} = A\,U, \qquad U_0 = \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}, \quad \Rightarrow \quad U(t) = e^{tA}\, U_0.$$

Closed form of the flow. Let $a := \sqrt{\lambda}$ and write

$$C := \cosh(at), \qquad S := \sinh(at).$$

A direct computation yields the following:

$$e^{tA} = \begin{pmatrix} C & \dfrac{S}{a} \\ a\,S & C \end{pmatrix} \quad \Big(\text{with } \tfrac{S}{a} \to t,\ aS \to 0 \text{ as } a \to 0\Big).$$

Therefore, the characteristic flows are as follows:

$$\varphi_t^1(x_0, u_0) = x_0\,C + \frac{u_0}{a}\,S, \qquad \varphi_t^2(x_0, u_0) = a\,x_0\,S + u_0\,C.$$

Now, consider the Riemann problem centered at the origin:

$$u(0, x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

Case 1: $u_L > u_R$. Then, the unique entropy solution is given by

$$u(t, x) = \begin{cases} a\,x\,\tanh(at) + \dfrac{u_L}{C}, & x < \gamma(t), \\ a\,x\,\tanh(at) + \dfrac{u_R}{C}, & x > \gamma(t), \end{cases} \tag{4.13}$$

with the shock position

$$\gamma(t) = \frac{u_L + u_R}{2a}\,S, \qquad \gamma(0) = 0. \tag{4.14}$$

Case 2: $u_L < u_R$. The unique entropy solution is given by the following:

$$u(t, x) = \begin{cases} a\,x\,\tanh(at) + \dfrac{u_L}{C}, & x < \dfrac{u_L}{a}\,S, \\ a\,x\coth(at), & \dfrac{u_L}{a}\,S \le x \le \dfrac{u_R}{a}\,S, \\ a\,x\,\tanh(at) + \dfrac{u_R}{C}, & x > \dfrac{u_R}{a}\,S. \end{cases} \tag{4.15}$$

5. Proof of the Oleinik inequality for entropy solutions

This section is devoted to the proof of Theorem 3.1, which establishes Oleinik's one-sided inequality in the setting of entropy solutions. Then, as direct consequences of this estimate, we derive the regularizing effects stated in Corollaries 3.2 and 3.4, and in Theorems 3.5 and 5.2. Additionally, we discuss the case of a constant source term in Subsection 5.5.

5.1. eWFT: An exact scheme

We introduce an eWFT, which inspired by the classical WFT and the method of characteristics. Unlike numerical front tracking, eWFT builds *exact entropy solutions* at every stage.

Description of the method.

1. *Piecewise-constant initialization.* Discretize the initial datum into a piecewise-constant profile. At $t = 0$, this yields a finite family of *classical* Riemann problems (left/right constants), as in WFT and Godunov's method.
2. *Exact local solvers.* Solve each Riemann problem exactly, and propagate the resulting waves via the characteristic flow. Because of the spatial source $g(x)$, the data at the interaction points are no longer constant; the ensuing local problems are *generalized* Riemann problems, which we also solve exactly.
3. *Interactions and first interaction time.* Wave interactions are handled as in WFT, but they are resolved *exactly* in eWFT. Let $T^* > 0$ denote the first interaction time between two distinct wave fronts generated by the piecewise constant initial datum. At time $t = 0$, any two neighboring discontinuities are separated by at least the mesh size Δx, while each front propagates with speed bounded in absolute value by $\Lambda(T)$ (as given by the finite-speed estimate). Hence, before two neighboring fronts can interact, they must close an initial gap with a size of at least Δx while moving towards each other with a relative speed of at most $2\Lambda(T)$. In particular, as long as

   $$2\,\Lambda(T)\,t < \Delta x,$$

   no interaction can occur, and we obtain the lower bound with a CFL condition as follows:

   $$T^* \;\geq\; \frac{\Delta x}{2\,\Lambda(T)}.$$

   For a strictly convex f, the interaction analysis shows that no new wave families are created; in particular, the number of fronts does not increase in time in the domains we consider.
4. *Oleinik along the evolution.* Oleinik's one-sided bound is verified on each elementary block (smooth monotone segments, shocks, rarefactions) and is *preserved across interactions* by the *additivity of H*. Hence, the exact entropy solution issued from the piecewise-constant datum satisfies Oleinik's inequality for all $t > 0$.
5. *Limit to the original datum.* Letting the mesh size tend to zero and invoking L^1-stability (Kružkov [20]), we pass to the limit and obtain Oleinik's inequality for the entropy solution associated with the original initial data.

5.2. Wave interactions

In this subsection, we study interactions between two adjacent Riemann problems. Since f is strictly convex, interactions decrease the number of waves; hence, the number of fronts does not increase, and the result extends (by induction) to any finite family of waves (see below).

It is important to verify that the structural properties of the data are preserved by the eWFT construction. More precisely, on each trapezoidal domain of dependence $\mathcal{K}_T$ under consideration, the solution consists of a finite number of singular curves (of SH– or SC–type), and between any two consecutive singular curves, the constructed solution is C^1 and nondecreasing with respect to the spatial variable x. In the analysis of wave interactions, we shall check that whenever a new singular curve is generated, it is again of an SH– or SC–type, and the above structural pattern of the solution is preserved in time.

Fix $T > 0$ and $x_0 < x_1$. By the finite speed of propagation, the solution u in $\mathcal{K}_T$ only depends on the initial data on the bounded interval

$$[x_0 - \Lambda(T)T,\ x_1 + \Lambda(T)T].$$

Since the initial datum will be discretized on a mesh with a size of Δx, only a finite number of Riemann problems influence $\mathcal{K}_T$.

On $\mathcal{K}_T$, the solution u is piecewise C^1, with a finite family of Lipschitz curves along which it is not C^1. We distinguish two types of such curves.

Definition 5.1 (Shock and singular curves)**.** A Lipschitz curve $\gamma : [0, T_\gamma] \to \mathbb{R}$ is called a *shock curve* (SH) if u has distinct left/right traces along it:

$$U^-(t) := \lim_{x\uparrow\gamma(t)} u(t,x), \qquad U^+(t) := \lim_{x\downarrow\gamma(t)} u(t,x),$$

with $U^-(t) \neq U^+(t)$ for a.e. $t \in (0, T_\gamma)$. The Rankine–Hugoniot relation holds as follows:

$$\dot\gamma(t) = \frac{f(U^+(t)) - f(U^-(t))}{U^+(t) - U^-(t)}.$$

A Lipschitz curve $\sigma : [0, T_\sigma] \to \mathbb{R}$ is called a *singular curve* (SC) if u is continuous but not C^1 across it, i.e.,

$$\lim_{x\uparrow\sigma(t)} u(t,x) = \lim_{x\downarrow\sigma(t)} u(t,x) =: U_{\mathrm{sing}}(t),$$

while the one-sided spatial derivatives do not coincide in general. There is no jump in u along such a curve, only a loss of C^1-regularity.

We denote the finite family of all shock and singular curves in $\mathcal{K}_T$ by $\mathcal{F}$.

SH-SH interaction Assume that both γ_1 and γ_2 are shock curves, issued from (t_1, a) and (t_1, b), respectively, with $a < b$. For $t < T^*$, along the two curves, we have

$$u_1(t, \gamma_1(t)^-) > u_2(t, \gamma_1(t)^+), \qquad u_2(t, \gamma_2(t)^-) > u_3(t, \gamma_2(t)^+),$$

i.e., Lax shocks separating the phases $u_1(t,\cdot), u_2(t,\cdot)$ and $u_2(t,\cdot), u_3(t,\cdot)$.

For $i = 1, 2$, denote the left/right traces along γ_i by

$$U_{-,i}(t) := \lim_{x\uparrow\gamma_i(t)} u(t,x), \qquad U_{+,i}(t) := \lim_{x\downarrow\gamma_i(t)} u(t,x),$$

so that

$$U_{-,1}(t) = u_1(t, \gamma_1(t)^-), \quad U_{+,1}(t) = u_2(t, \gamma_1(t)^+),$$

$$U_{-,2}(t) = u_2(t, \gamma_2(t)^-), \quad U_{+,2}(t) = u_3(t, \gamma_2(t)^+).$$

Each shock satisfies the Rankine–Hugoniot relation as follows:

$$\dot\gamma_i(t) = \frac{f(U_{+,i}(t)) - f(U_{-,i}(t))}{U_{+,i}(t) - U_{-,i}(t)}, \qquad i = 1, 2.$$

By strict convexity of f, one has $\dot\gamma_1(t) > \dot\gamma_2(t)$, so the distance

$$d(t) := \gamma_2(t) - \gamma_1(t)$$

is strictly decreasing. There is at most one interaction time

$$T^* := \inf\{t > t_1 : \gamma_1(t) = \gamma_2(t)\}, \qquad \alpha^* := \gamma_1(T^*) = \gamma_2(T^*).$$

Let $u^-(x) := \lim_{t\uparrow T^*} u(t, x)$ be the pre-interaction profile at time T^*. Denote the left and right traces at $x = \alpha^*$ coming from the outer phases $u_1(t, \cdot)$ and $u_3(t, \cdot)$ by the following:

$$u_1^* := \lim_{x\uparrow\alpha^*} u^-(x), \qquad u_3^* := \lim_{x\downarrow\alpha^*} u^-(x).$$

Moreover, for each $t < T^*$, the solution consists of three phases, $u_1(t, \cdot)$, $u_2(t, \cdot)$, $u_3(t, \cdot)$, separated by the shocks $\gamma_1(t)$ and $\gamma_2(t)$, with

$$u_1(t, \gamma_1(t)^-) > u_2(t, \gamma_1(t)^+) \quad \text{and} \quad u_2(t, \gamma_2(t)^-) > u_3(t, \gamma_2(t)^+).$$

This ordering is preserved along characteristics. In particular, this implies the following:

$$u_1^* \;\geq\; u_3^*.$$

Case (1): $u_1^* > u_3^*$. Define the restarted data at time T^* by the following

$$U_L^*(x) := u^-(x) \quad (x < \alpha^*), \qquad U_R^*(x) := u^-(x) \quad (x > \alpha^*).$$

Then,

$$U_L^*(\alpha^{*-}) = u_1^*, \qquad U_R^*(\alpha^{*+}) = u_3^*,$$

and the inequality $u_1^* > u_3^*$ shows that (U_L^*, U_R^*) form a shock-type generalized Riemann problem at (T^*, α^*) (in the sense of Eq (4.1)). Therefore, The post-interaction evolution is given by a single outgoing Lax shock γ that connects the transported traces, exactly as in the generalized setting.

Case (2): $u_1^* = u_3^*$. In this case, u^- is continuous at $x = \alpha^*$, but the one-sided spatial derivatives of u at (T^*, α^*) may differ, so u is not C^1 at that point. We do not create a new shock; instead, we declare that a singular curve

$$\Gamma_{\text{sing}} := \{(t, \gamma_{\text{sing}}(t)) : t \geq T^*\},$$

emanates from (T^*, α^*). Along Γ_{sing}, the solution remains continuous but can be not C^1. In particular, the SH–SH interaction either produces a new shock (case (1)) or degenerates into a singular curve (case (2)); no additional fronts are generated.

SC-SC no interaction Assume that we have two singular curves issued from (t_1, a) and (t_2, b) with $a < b$. Let the corresponding inner curves be as follows:

$$\gamma_{1,+}(t) := \varphi_t^1(a, U_R^1(a)), \qquad \gamma_{2,-}(t) := \varphi_t^1(b, U_L^2(b)),$$

where $U_R^1(a)$ is the right trace of the left Riemann data at $x = a$, and $U_L^2(b)$ is the left trace of the right Riemann data at $x = b$, with

$$U_L^1(a) < U_R^1(a), \qquad U_L^2(b) < U_R^2(b).$$

The curves $\gamma_{1,+}$ and $\gamma_{2,-}$ are singular curves: the solution u is continuous across each of them, but not C^1.

By the monotonicity of the forward flow in the spatial variable (with the second component fixed), we have for all $t \geq t_1$,

$$\gamma_{1,+}(t) = \varphi_t^1(a, U_R^1(a)) \leq \varphi_t^1(b, U_R^1(a)) \leq \varphi_t^1(b, U_L^2(b)) = \gamma_{2,-}(t),$$

with strict $<$ if $a < b$. Since $\partial_{x_0}\varphi_t^1 \geq 1$, the gap between the two curves is nondecreasing and obeys the following:

$$\gamma_{2,-}(t) - \gamma_{1,+}(t) = \varphi_t^1(b, U_L^2(b)) - \varphi_t^1(a, U_R^1(a)) \;\geq\; b - a > 0.$$

Hence, the two singular curves never meet if they start from distinct points. If $a = b$, then $\gamma_{1,+} \equiv \gamma_{2,-}$ for all $t > t_1$, and the two rarefactions merge into a single fan bounded by two edges. In particular, *two distinct sing-curves do not interact* when f is strictly convex.

SC-SH interaction Now, assume that the left front is a singular curve and the right front is a shock curve. More precisely, let $\gamma_{1,+}$ be a singular curve issued from $(0, a)$, and let γ_2 be a shock curve issued from $(0, b)$, with $a < b$.

For $t < T^*$, we have the following along these two curves:

$$u_1(t, \gamma_{1,+}(t)^-) = u_2(t, \gamma_{1,+}(t)^+) =: U_{\text{sing}}(t),$$

$$u_2(t, \gamma_2(t)^-) > u_3(t, \gamma_2(t)^+).$$

Thus, $\gamma_{1,+}$ is a singular curve (no jump, only loss of C^1), while γ_2 is a shock curve between the time-dependent phases $u_2(t, \cdot)$ and $u_3(t, \cdot)$.

Let $T^* > 0$ be the first time when the shock meets the right edge of the rarefaction (the singular curve):

$$\alpha^* := \gamma_2(T^*) = \gamma_{1,+}(T^*).$$

Define the pre-interaction profile

$$u^-(x) := \lim_{t \uparrow T^*} u(t, x),$$

and denote the left/right traces at $x = \alpha^*$ coming from the outer phases $u_1(t, \cdot)$ and $u_3(t, \cdot)$ by the following:

$$u_1^* := \lim_{x \uparrow \alpha^*} u^-(x), \qquad u_3^* := \lim_{x \downarrow \alpha^*} u^-(x),$$

For each $t < T^*$, the three phases $u_1(t,\cdot)$, $u_2(t,\cdot)$, and $u_3(t,\cdot)$ occupy the regions to the left of $\gamma_{1,+}(t)$, between $\gamma_{1,+}(t)$ and $\gamma_2(t)$, and to the right of $\gamma_2(t)$, respectively. Along the two fronts, we have the following:

$$u_1(t,\gamma_{1,+}(t)^-) = u_2(t,\gamma_{1,+}(t)^+), \qquad u_2(t,\gamma_2(t)^-) > u_3(t,\gamma_2(t)^+).$$

This ordering is preserved along characteristics. In particular, this implies the following:

$$u_1^* \ \geq \ u_3^*.$$

As in the SH-SH case, we consider two possibilities,

Case (1): $u_1^* > u_3^*$. We restart at (T^*, α^*) with generalized Riemann data as follows:

$$U_L^*(x) := u^-(x) \quad (x < \alpha^*), \qquad U_R^*(x) := u^-(x) \quad (x > \alpha^*).$$

Then,

$$U_L^*(\alpha^{*-}) = u_1^*, \qquad U_R^*(\alpha^{*+}) = u_3^*,$$

and $u_1^* > u_3^*$ shows that (U_L^*, U_R^*) form a shock-type generalized Riemann problem at (T^*, α^*). For $t \geq T^*$, the solution is given by a single outgoing Lax shock γ that connects the transported traces, exactly as is in the generalized case.

Case (2): $u_1^* = u_3^*$. In this case, the solution is continuous at (T^*, α^*) but not C^1 in general. We do not create a new shock, but we continue the evolution along a new singular curve as follows

$$\Gamma_{\text{sing}} := \{(t, \gamma_{\text{sing}}(t)) : t \geq T^*\},$$

issuing from (T^*, α^*) and carrying the common value $u_1^* = u_3^*$. Along this sing-curve, the solution u is continuous but not C^1.

In summary, when f is strictly convex, interactions between shock and singular curves are resolved without creating new fronts: a Shock–Shock interaction either yields a single shock or a singular curve; two distinct singular curves never interact; and a Rarefaction–Shock interaction produces either a single shock or a singular curve. In all cases, the number of fronts does not increase, and the local configuration after the interaction is again of the generalized Riemann type.

5.3. Global solutions and the Oleinik inequality in the three canonical cases

In this subsection, we verify Oleinik's one-sided inequality in three cases: (1) continuous, nondecreasing initial data; (2) a shock Riemann problem; and (3) a rarefaction Riemann problem. These cases are sufficient because the evolution decomposes into such pieces under our simplified WFT scheme; thus, the inequality is preserved across interactions. Hence, the constructed exact solution satisfies Oleinik's bound. For clarity, we present the Riemann problems with constant left/right states; the arguments are exactly the same in the generalized setting, where the initial data consist of two piecewise C^1 nondecreasing functions.

Now, we establish the Oleinik one–sided inequality in the 3 cases mentioned above.

Case 1: Global strong solution for increasing initial data. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function. Fix $t \geq 0$ and $y_0 > x_0$. Since $u_0(y_0) \geq u_0(x_0)$,

$$\begin{aligned} X(t, y_0) - X(t, x_0) &= \varphi_t^1(y_0, u_0(y_0)) - \varphi_t^1(x_0, u_0(x_0)) \\ &\geq \varphi_t^1(y_0, u_0(x_0)) - \varphi_t^1(x_0, u_0(x_0)) \\ &= \int_{x_0}^{y_0} \partial_x \varphi_t^1(\xi, u_0(x_0))\, d\xi \ \geq\ y_0 - x_0. \end{aligned}$$

Thus, $X(t, \cdot)$ is strictly increasing and satisfies the one–sided bound $X(t, y_0) - X(t, x_0) \geq y_0 - x_0$ for all $y_0 > x_0$. Taking $y_0 = 0$ and letting $x_0 \to +\infty$ (resp. $x_0 \to -\infty$) yields the following

$$X(t, x_0) \geq X(t, 0) + x_0 \to +\infty \quad \text{and} \quad X(t, x_0) \leq X(t, 0) + x_0 \to -\infty;$$

thus,

$$\lim_{x_0 \to +\infty} X(t, x_0) = +\infty, \qquad \lim_{x_0 \to -\infty} X(t, x_0) = -\infty.$$

Since $X(t, \cdot)$ is a bijection, let $X^{-1}(t, \cdot)$ denote its inverse and define the following:

$$u(t, x) := \varphi_t^2(X^{-1}(t, x),\ u_0(X^{-1}(t, x))).$$

Then, $u \in C([0, \infty) \times \mathbb{R})$ by the continuity of φ_t, u_0, and X^{-1}. Additionally, for each x_0, we have the following:

$$(X(t, x_0),\ u(t, X(t, x_0))) = \varphi_t(x_0, u_0(x_0)),$$

thus, along the characteristics, X and u satisfy the ODEs. Because $X(t, \cdot)$ is bijective, characteristics do not intersect; hence, u is globally well defined and strong solution. Thus, the Oleinik inequality (2.9) is verified in this case using Section 2.2.

Case 2: Riemann shock $u_L > u_R$. On each side of the interface, we have the smooth profiles obtained by evolving the constant *initial* states u_L and u_R via the flow: $\varphi_t(\cdot, u_L)$ and $\varphi_t(\cdot, u_R)$ (they are not spatially constant for $t > 0$). These side profiles are global because the flow is global. The shock curve $\gamma(t)$ satisfies Rankine–Hugoniot with traces taken from the two side profiles; hence, exists for all $t \geq 0$.

Oleinik. Fix $t > 0$ and $x < y$.

- If $x < \gamma(t) < y$, then split into three terms

$$\begin{aligned} \mathcal{R} &:= H(t, y, u(t, y)) - H(t, \gamma_+(t), U_+(t)), \\ \mathcal{J} &:= H(t, \gamma_+(t), U_+(t)) - H(t, \gamma_-(t), U_-(t)), \\ \mathcal{L} &:= H(t, \gamma_-(t), U_-(t)) - H(t, x, u(t, x)), \end{aligned}$$

  so that

$$H(t, y, u(t, y)) - H(t, x, u(t, x)) = \mathcal{R} + \mathcal{J} + \mathcal{L}.$$

  On each smooth side, the equality (2.11) gives the following:

$$\mathcal{R} \leq y - \gamma(t), \qquad \mathcal{L} \leq \gamma(t) - x.$$

  At the shock point, $v \mapsto H(t, \gamma(t), v)$ is increasing and $U_-(t) > U_+(t)$ (Lax); hence,

$$\mathcal{J} \leq 0.$$

  Summing up yields $H(t, y, u(t, y)) - H(t, x, u(t, x)) \leq y - x$.

Case 3: Riemann rarefaction $u_L < u_R$. On each side, the solution is the profile obtained by evolving the initial states u_L and u_R via the flow $(\varphi_t^1, \varphi_t^2)$, which is global. Define the fan edges as follows:

$$\gamma_-(t) := \varphi_t^1(0, u_L), \qquad \gamma_+(t) := \varphi_t^1(0, u_R).$$

By forward–flow monotonicity in the second variable, $\gamma_-(t) < \gamma_+(t)$ for all $t > 0$, so the fan is *global* (the solution is smooth away from $\gamma_\pm$).

Oleinik. Fix $t > 0$ and $x < y$.

- If x, y lie *inside the fan*, then the two points trace back to the same footpoint $x_0 = 0$, and by definition, we have the following:

$$H(t, y, u(t, y)) - H(t, x, u(t, x)) = y - x.$$

- If x, y lie on the same smooth side *outside* the fan, then the Oleinik inequality is automatically verified.
- *Across the whole fan,* if $x < \gamma_-(t)$ and $y > \gamma_+(t)$, then it is split at the two edges as follows:

$$\begin{aligned} H(t, y, u(t, y)) - H(t, x, u(t, x)) &= \Big[H(t, y, u(t, y)) - H(t, \gamma_+(t), u(t, \gamma_+(t)))\Big] \\ &\quad + \Big[H(t, \gamma_+(t), u(t, \gamma_+(t))) - H(t, \gamma_-(t), u(t, \gamma_-(t)))\Big] \\ &\quad + \Big[H(t, \gamma_-(t), u(t, \gamma_-(t))) - H(t, x, u(t, x))\Big]. \end{aligned}$$

The first and third brackets compare points within smooth regions; hence, $\Delta x = \Delta x_0 + \Delta H$,

$$\begin{aligned} H(t, y, u(t, y)) - H(t, \gamma_+(t), u(t, \gamma_+(t))) &\le y - \gamma_+(t), \\ H(t, \gamma_-(t), u(t, \gamma_-(t))) - H(t, x, u(t, x)) &\le \gamma_-(t) - x. \end{aligned}$$

For the middle bracket, both points lie *inside the fan*, so $\Delta x_0 = 0$; thus,

$$H(t, \gamma_+(t), u(t, \gamma_+(t))) - H(t, \gamma_-(t), u(t, \gamma_-(t))) = \gamma_+(t) - \gamma_-(t).$$

Summing the three contributions yields the following

$$H(t, y, u(t, y)) - H(t, x, u(t, x)) \le (y - \gamma_+) + (\gamma_+ - \gamma_-) + (\gamma_- - x) = y - x.$$

Finally, in all the 3 cases, the Oleinik inequality (3.1) is verified.

Completion of the proof of Theorem 3.1. Approximate the initial datum by a piecewise-constant, monotone mesh u_0^Δ. For each mesh, solve exactly the finite family of Riemann problems and all ensuing wave interactions. Because f is strictly convex, no new waves are created at the interactions and the number of fronts does not increase; hence, the front-tracked solution u^Δ exists globally for all $t > 0$. By the results above, Oleinik's one-sided inequality holds for each elementary piece (continuous monotone data, shock, and rarefaction). Moreover, by the *additivity of H*, the one-sided bound is preserved under concatenation of pieces and across interactions; therefore, $u^\Delta(t, \cdot)$ satisfies Oleinik's inequality for every $t > 0$. Finally, letting $\Delta \to 0$ and using Kružkov's L^1-stability theorem for scalar balance laws (see Kružkov [20]), the inequality passes to the limit, thus yielding Oleinik's estimate for Eq (1.1).

5.4. Direct consequences of the Oleinik inequality

Now, we turn to the proofs of the regularizing effects, which directly follow from the Oleinik inequality. first, we prove Corollary 3.2.

Proof. Fix $t > 0$ and set

$$v(x) := H(t, x, u(t, x)).$$

We begin by noting that the generalized Oleinik inequality implies the following:

$$(\Delta H)^+ \leq (\Delta x)^+, \tag{5.1}$$

where $x^+ := \max(x, 0)$.

We recall that the space BV^+ is defined by the following:

$$BV^+ := \{w \,:\, \mathrm{TV}^+(w) < +\infty\},$$

where the *positive total variation* is

$$\mathrm{TV}^+(w) = \sup_{\{x_i\}\in\mathcal{P}} \sum_{i=2}^{n} (w(x_i) - w(x_{i-1}))^+, \tag{5.2}$$

and

$$\mathcal{P} = \Big\{\{x_1, \ldots, x_n\} \,:\, x_1 < \cdots < x_n,\ 2 \leq n \in \mathbb{N}\Big\}$$

is the set of all finite subdivisions of $\mathbb{R}$.

Since $u(t, \cdot)$ satisfies a maximum principle, it remains uniformly bounded in $L^\infty(\mathbb{R})$. Consequently, the composition $v(\cdot) = H(t, \cdot, u(t, \cdot))$ remains uniformly bounded in $L^\infty(\mathbb{R})$ for every fixed $t > 0$.

Additionally, we recall the identity

$$L^\infty \cap BV^+ = BV, \tag{5.3}$$

together with the estimate

$$\mathrm{TV}(w) \leq 2\|w\|_{L^\infty} + 2\,\mathrm{TV}^+(w). \tag{5.4}$$

Combining Eq (5.1) with the bound on v in L^∞, and using Eqs (5.3) and (5.4), we conclude that

$$H(t, \cdot, u(t, \cdot)) = v \in BV_{\mathrm{loc}}(\mathbb{R}).$$

Furthermore, the derivative of H with respect to u is given by the following:

$$\frac{\partial H}{\partial u}(t, x, u) = \int_0^t f''(\varphi^2_{-s}(x, u))\, \partial_u \varphi^2_{-s}(x, u)\, ds.$$

Since f is uniformly convex (i.e., $f''(u) \geq \alpha > 0$ for all $u \in \mathbb{R}$), using Proposition 2.2, it follows that $\partial_u H(t, x, u) > 0$. Therefore, for each fixed (t, x), the map $u \mapsto H(t, x, u)$ is a smooth diffeomorphism. Since $H(t, \cdot, u(t, \cdot)) \in BV_{\mathrm{loc}}(\mathbb{R})$, we conclude that

$$u(t, \cdot) \in BV_{\mathrm{loc}}(\mathbb{R}).$$

Now, we turn to the proof of Corollary 3.4.

Proof. Fix $t > 0$. We already know that the function

$$x \longmapsto H(t, x, u(t, x))$$

belongs to $BV_{\mathrm{loc}}(\mathbb{R})$.

Moreover, it is straightforward to verify that if the flux function f satisfies the degeneracy condition (3.2) with exponent p, then H inherits the same degeneracy condition with the same exponent p (for each fixed (t, x)). Indeed, let $u \neq v \in K$, and assume $u > v$ without loss of generality. Since the map $u \mapsto H(t, x, u)$ is increasing for every fixed (t, x), it follows that

$$H(t, x, u) - H(t, x, v) > 0.$$

Hence,

$$|H(t, x, u) - H(t, x, v)| = H(t, x, u) - H(t, x, v) = \int_0^t \Big(f'(\varphi^2_{-s}(x, u)) - f'(\varphi^2_{-s}(x, v)) \Big) \, ds.$$

Since f' satisfies the degeneracy condition of order p, and the flow $\varphi^2_{-s}(x, \cdot)$ is Lipschitz continuous in u, we obtain the following:

$$\Big| f'(\varphi^2_{-s}(x, u)) - f'(\varphi^2_{-s}(x, v)) \Big| \geq \frac{c}{L^p} \, |u - v|^p.$$

Integrating over $s \in [0, t]$, we conclude that

$$|H(t, x, u) - H(t, x, v)| \geq \frac{ct}{L^p} \, |u - v|^p.$$

Now, we may apply Corollary A.4. Hence, we deduce that

$$u(t, \cdot) \in BV^s_{\mathrm{loc}}(\mathbb{R}) \quad \text{for every } t > 0.$$

Now, we establish the proof of Theorem 3.5.

Proof. Recall the following expression of the derivative of H with respect to v:

$$\partial_v H(t, x, v) = \int_0^t f''(\varphi^2_{-s}(x, v)) \, \partial_v \varphi^2_{-s}(x, v) \, ds.$$

Assume by contradiction that $\partial_v H(t, x, v) = 0$. The integrand is continuous and nonnegative. By Proposition 2.2, $\partial_v \varphi^2_{-s}(x, v) > 0$; hence,

$$f''(\varphi^2_{-s}(x, v)) = 0 \quad \text{for all } s \in [0, t].$$

Because f is strictly convex, f' is strictly increasing. In particular, the set $\{f'' = 0\}$ has an empty interior; hence, it contains no nontrivial connected subsets. Define the following:

$$\Gamma := \{\varphi^2_{-s}(x, v) : \; s \in [0, t]\}.$$

Since $s \mapsto \varphi^2_{-s}(x, v)$ is continuous on the connected interval $[0, t]$, its image Γ is connected, and $\Gamma \subset \{f'' = 0\}$. Thus, Γ is a singleton, and

$$\varphi^2_{-s}(x, v) = v \quad \text{for all } s \in [0, t].$$

However, the (backward) characteristic flow $\varphi_{-s} = (\varphi^1_{-s}, \varphi^2_{-s})$ satisfies the following:

$$\frac{d}{ds}\varphi^1_{-s}(x,v) = -f'(\varphi^2_{-s}(x,v)), \tag{5.5}$$

$$\frac{d}{ds}\varphi^2_{-s}(x,v) = -g(\varphi^1_{-s}(x,v)), \tag{5.6}$$

Using Eq (5.6), we get a contradiction since $g(x) \neq 0$ for all $x \in \mathbb{R}$. Therefore $\partial_v H(t,x,v) > 0$ for all u. In particular, since $v(t,x) := H(t,x,u(t,x))$ has a locally bounded variation and $H(t,x,\cdot)$ is a diffeomorphism for each $t > 0$, it follows that $u(t,\cdot) \in BV_{\mathrm{loc}}(\mathbb{R})$.

Now, we discuss the regularizing effect in the presence of a spatially varying source with $g'(x) > 0$. In this case, there exists at most one point $\tilde{x}$ such that $g(\tilde{x}) = 0$, and at most one point $\tilde{u}$ such that $f'(\tilde{u}) = 0$. Even if g vanishes at such a point, we can still recover the BV_{loc} regularity for the entropy solution under a pointwise condition on f'', as stated in the following theorem.

Theorem 5.2. *Let $u_0, g \in L^\infty(\mathbb{R})$. Assume that $g'(x) > 0$ for all x and that $f \in C^2(\mathbb{R})$ is strictly convex. Additionally, if $f''(\tilde{u}) \neq 0$, then for every $t > 0$, the unique entropy solution $u(t,\cdot)$ of Eq* (1.1) *belongs to $BV_{\mathrm{loc}}(\mathbb{R})$.*

Proof. Arguing in the same way as in the proof of Theorem 3.5, we obtain that

$$f''(\varphi^2_{-s}(x,v)) = 0 \quad \text{for all } s \in [0,t], \quad \text{and thus} \quad \varphi^2_{-s}(x,v) = v \quad \text{for all } s.$$

Now, using both relations for Eqs (5.5) and (5.6), and the fact that $\tilde{x}$ is the only point such that $g(\tilde{x}) = 0$ and $\tilde{u}$ is the only point such that $f'(\tilde{u}) = 0$, we deduce that

$$\varphi^2_{-s}(x,v) = \tilde{u} \quad \text{and} \quad \varphi^1_{-s}(x,v) = \tilde{x} \quad \text{for all } s.$$

This is a contradiction, since $f''(\tilde{u}) \neq 0$.

Remark 5.3. As we have seen, $\partial_v H(t,x,v)$ is nonnegative for all (t,x,v), and it can be equal to 0 only on pairs (x_0, v_0) that satisfy the following:

$$g(x_0) = 0, \qquad f'(v_0) = 0, \qquad f''(v_0) = 0.$$

5.5. *The case $g' \equiv 0$*

In this section, we record, without proofs, the key formulas and consequences for the following constant source case:

$$\begin{cases} \partial_t u + \partial_x(f(u)) = \lambda, \\ u(0,x) = u_0(x), \end{cases} \qquad \lambda \in \mathbb{R}. \tag{5.7}$$

The special case $g = \lambda$ greatly simplifies the formulas and is already interesting in its own right.

Set

$$H(t,v) := \int_0^t f'(v - \lambda s)\, ds, \quad \text{for } v \in \mathbb{R}, \text{ and } t > 0. \tag{5.8}$$

For f strictly convex and $u_0 \in L^\infty(\mathbb{R})$, the unique entropy solution satisfies the following:

$$\Delta H \leq \Delta x, \tag{5.9}$$

where H is given by Eq (5.8).

The Riemann problem

Shock ($u_L > u_R$). The entropy solution is as follows:

$$u(t,x) = \begin{cases} u_L(t) := \lambda t + u_L, & x < \gamma(t), \\ u_R(t) := \lambda t + u_R, & x > \gamma(t), \end{cases}$$

with the Rankine–Hugoniot speed

$$\dot{\gamma}(t) = \frac{f(\lambda t + u_R) - f(\lambda t + u_L)}{u_R - u_L}.$$

Rarefaction ($u_L < u_R$). The unique entropy solution is given by the following:

$$u(t,x) = \begin{cases} u_L(t) := \lambda t + u_L, & x < H(t, u_L(t)), \\ \Delta H(t,u) = \Delta x, & H(t, u_L(t)) \leq x \leq H(t, u_R(t)), \\ u_R(t) := \lambda t + u_R, & x > H(t, u_R(t)). \end{cases}$$

Example: Burgers with constant source term

Consider the Burgers equation with the constant source term λ.

Shock case $u_L > u_R$:

$$u(t,x) = \begin{cases} \lambda t + u_L, & x < \gamma(t), \\ \lambda t + u_R, & x > \gamma(t), \end{cases} \qquad \gamma(t) = \frac{u_L + u_R}{2} t + \frac{\lambda}{2} t^2.$$

Rarefaction case $u_L < u_R$:

$$u(t,x) = \begin{cases} \lambda t + u_L, & x \leq t\, u_L + \dfrac{\lambda t^2}{2}, \\ \dfrac{x}{t} + \dfrac{\lambda t}{2}, & t\, u_L + \dfrac{\lambda t^2}{2} \leq x \leq t\, u_R + \dfrac{\lambda t^2}{2}, \\ \lambda t + u_R, & x \geq t\, u_R + \dfrac{\lambda t^2}{2}. \end{cases}$$

Regularizing effect

In this case, the regularization effect directly follows from the general source term $g(x)$. Assume that $\lambda \in \mathbb{R} \setminus \{0\}$. Under the assumptions of Theorem 3.1, the unique entropy solution $u(t,\cdot)$ of Eq (5.7) belongs to $BV_{\mathrm{loc}}(\mathbb{R})$ for all $t > 0$. This is an illustration of Theorems 3.5 and 5.2.

6. No global smooth solution with decreasing source term

In this section, we highlight how a *decreasing* spatial source can destroy regularity, even for smooth initial data and smooth sources. We present two Burgers examples: (i) with a linear decreasing source, the smooth solution blows up in finite time (as $t \to (\pi/2)^-$), so there is *no* global L^∞ entropy solution; and (ii) with a piecewise bounded, globally Lipschitz source that is decreasing on a finite interval and constant outside, a *single* Lax shock forms in finite time (at $t = \pi/2$ when the outer bound is large enough, and later otherwise), after which the entropy solution continues globally for all times. This example parallels the one in [21], which analyzed the Burgers equation without a source term under piecewise-continuous, decreasing initial data and proved the shock formation at a finite time t^*. Thus, allowing a decreasing source term makes the problem considerably more complicated.

Example 6.1. We consider the following Burgers equation with a decreasing source term and constant initial data given by the following:

$$\partial_t u \;+\; \partial_x\Big(\tfrac{u^2}{2}\Big) \;=\; -\,x, \qquad u(0,x) \equiv 1. \tag{6.1}$$

Characteristic flow and values along characteristics

Let $X = X(t, x_0)$ be the characteristic issued from x_0, and write $u(t, X(t, x_0))$ for the solution value along it. The characteristic system is as follows:

$$\frac{d}{dt}X(t,x_0) = u(t, X(t,x_0)), \qquad \frac{d}{dt}u(t, X(t,x_0)) = -\,X(t,x_0),$$

with $X(0, x_0) = x_0$, $u(0, X(0, x_0)) = 1$. Solving gives the rotation flow as follows:

$$X(t,x_0) = \varphi_t^1(x_0,1) = x_0\cos t + \sin t, \tag{6.2a}$$

$$u(t, X(t,x_0)) = \varphi_t^2(x_0,1) = -x_0\sin t + \cos t. \tag{6.2b}$$

Method 1 — Jacobian (diffeomorphism criterion)

The Lagrangian–Eulerian Jacobian is as follows:

$$\partial_{x_0}X(t,x_0) = \cos t. \tag{6.3}$$

Hence, $X(t,\cdot)$ is a C^1 diffeomorphism for $0 \le t < \frac{\pi}{2}$, and loses invertibility at

$$t_* = \frac{\pi}{2} \quad \text{since} \quad \partial_{x_0}X(t_*, x_0) = 0.$$

Moreover, from Eq (6.2a),

$$X\Big(\tfrac{\pi}{2}, x_0\Big) = 1 \quad \text{for all } x_0;$$

thus, all characteristics focus at $x_* = 1$. Therefore, no global strong solution exists; the classical description breaks at $t = \pi/2$ by characteristic crossing (shock formation).

Method 2 — Riccati (slope blow-up)

Let $w = u_x$. Differentiating Eq (6.1) in x gives the following:

$$w_t + u\,w_x + w^2 = -1. \tag{6.4}$$

Evaluating along the characteristic $X(t, x_0)$ (so $\frac{d}{dt} = \partial_t + u\,\partial_x$), define $W(t; x_0) := w(t, X(t, x_0))$. Then, W solves the following:

$$\dot{W} = -1 - W^2, \qquad W(0) = u_0'(x_0) = 0. \tag{6.5}$$

This integrates explicitly as follows:

$$W(t) = -\tan t, \tag{6.6}$$

so

$$\lim_{t \uparrow T_*} W(t) = -\infty \quad \text{at the finite time} \quad T_* = \frac{\pi}{2}.$$

Thus, $W(t; x_0) \to -\infty$ as $t \uparrow \frac{\pi}{2}$, which proves the finite-time blow-up of the gradient and rules out a global C^1 solution.

Remark 6.2 (Blow-up for Burgers with linear source; no shock at $t = \frac{\pi}{2}$)**.** For $t < \frac{\pi}{2}$, $\cos t > 0$ and

$$x = X(t; x_0) \implies x_0 = \frac{x - \sin t}{\cos t};$$

hence, the Eulerian formula is as follows:

$$u(t, x) = -x\,\tan t + \sec t.$$

Therefore, as $t \to (\frac{\pi}{2})^-$,

$$u(t, x) \sim \frac{1 - x}{\frac{\pi}{2} - t}.$$

In particular, for any $a \in (0, 1)$,

$$\sup_{x \in [-a, a]} u(t, x) \longrightarrow +\infty \qquad \text{as } t \to \left(\tfrac{\pi}{2}\right)^-.$$

Conclusion. The solution undergoes a focusing blow-up at $t = \frac{\pi}{2}$; there are no finite left/right traces at a discontinuity, so this is *not* a Lax shock, and there is no entropy solution (in L^∞) that continues for $t > \frac{\pi}{2}$ in this case.

Example 6.3. Consider the following Burgers equation:

$$\partial_t u + \partial_x\Big(\frac{u^2}{2}\Big) = g(x), \qquad u(0, x) \equiv 1,$$

with a piecewise source term parametrized by A, a positive constant,

$$g(x) = \begin{cases} A, & x < -A, \\ -x, & |x| < A, \\ -A, & x > A, \end{cases}$$

Here, g is bounded ($\|g\|_\infty = A$) and globally Lipschitz (a.e. $g'(x) \in \{0, -1\}$); hence, by the standard theory for balance laws, there exists a unique global entropy solution. A single shock forms, with the formation time given by the following:

$$t^*(A) = \begin{cases} \dfrac{\pi}{2}, & A \geq 1, \\ \arcsin(A) + \dfrac{\sqrt{1-A^2}}{A}, & 0 < A < 1, \end{cases} \qquad \left(\text{so } t^*(A) > \frac{\pi}{2} \text{ when } 0 < A < 1\right). \tag{6.7}$$

After t^*, the solution consists of one Lax shock that persists for all later times.

Sketch of proof. To justify the above expression for the formation time t^*, we follow the characteristic curves. Writing the equation as

$$u_t + u\, u_x = g(x),$$

the characteristic system is as follows:

$$\dot{X} = U, \qquad \dot{U} = g(X), \qquad X(0) = x_0,\ U(0) = 1.$$

For the central characteristic $x_0 = 0$ and as long as $|X(t)| < A$, we have $g(X) = -X$; thus,

$$\dot{X} = U, \qquad \dot{U} = -X, \qquad X(0) = 0,\ U(0) = 1,$$

which gives $X(t) = \sin t$, $U(t) = \cos t$. Differentiating the partial differential equation (PDE) in x shows that the slope $W(t) := u_x(t, X(t))$ satisfies $\dot{W} = g'(X(t)) - W^2$ with $W(0) = 0$. In the inner region $|x| < A$, we have $g'(x) = -1$; hence, $W(t) = -\tan t$. If $A \geq 1$, then the central characteristic stays in $|x| < A$ up to $t = \frac{\pi}{2}$, and $W(t) \to -\infty$ as $t \uparrow \frac{\pi}{2}$, so $t^* = \frac{\pi}{2}$.

If $0 < A < 1$, then the central characteristic leaves the inner region at the time $t_1 = \arcsin A$, where $X(t_1) = A$ and $W(t_1) = -A/\sqrt{1-A^2}$. For $x > A$, one has $g'(x) = 0$; thus, for $t > t_1$ while $X(t) > A$, the slope solves $\dot{W} = -W^2$, which gives the following:

$$W(t) = \frac{1}{t - t_1 - \frac{\sqrt{1-A^2}}{A}}.$$

Hence, there is a blow-up at the following:

$$t^*(A) = t_1 + \frac{\sqrt{1-A^2}}{A} = \arcsin A + \frac{\sqrt{1-A^2}}{A}.$$

In both cases, namely $A \geq 1$ and $0 < A < 1$, we showed that the classical solution develops an infinite negative gradient along the central characteristic at the finite time as follows:

$$t^* = \begin{cases} \dfrac{\pi}{2}, & A \geq 1, \\ \arcsin A + \dfrac{\sqrt{1-A^2}}{A}, & 0 < A < 1. \end{cases}$$

This is precisely the time of the first gradient blow-up (i.e., the classical solution ceases to be C^1). For scalar conservation/balance laws with a strictly convex flux, this loss of regularity corresponds to the formation of a (Lax) shock.

Because g is bounded and globally Lipschitz, the entropy solution exists globally in time and is unique. The construction of the entropy solution after t^* follows the standard recipe: One replaces the multi-valued region generated by crossing characteristics with a single shock curve $\gamma(t)$, whose speed is given by the Rankine–Hugoniot condition as follows:

$$\dot{\gamma}(t) = \frac{f(u^+(t)) - f(u^-(t))}{u^+(t) - u^-(t)} = \frac{\frac{(u^+(t))^2}{2} - \frac{(u^-(t))^2}{2}}{u^+(t) - u^-(t)} = \frac{u^+(t) + u^-(t)}{2},$$

with left/right states determined by the transported characteristics.

In this example, the first and only breakdown of the flow map $x_0 \mapsto X(t; x_0)$ occurs at the central characteristic; thus, exactly one shock is created at $t = t^*$. After its formation, the solution consists of a single Lax shock that connects two smooth states, and this shock persists for all later times.

7. Conclusions and perspectives

We established a generalized Oleinik-type one-sided inequality for the class of heterogeneous scalar balance laws considered in this work. The proof is based on an eWFT construction, in which wave interactions are treated as interactions between SH and SC. On each bounded domain of dependence $\mathcal{K}_T$, the eWFT solution consists of finitely many such curves, and between any two consecutive fronts, the solution is C^1 and nondecreasing with respect to x. This piecewise smooth non-decreasing structure of approximate solutions allows us to verify the generalized Oleinik inequality on each elementary piece and to propagate it in time. The proved Oleinik inequality yields a regularizing effect for the unique entropy solution.

Several perspectives naturally follow. When the source g is decreasing, this piecewise C^1 nondecreasing structure is no longer suitable, and a new strategy is required. Since the Oleinik inequality obtained here is a property of the exact solution, it is also natural to seek numerical schemes that preserve this one-sided control, which is analogous with known results for the Lax–Friedrichs scheme in the homogeneous case, see [21]. Finally, the regularizing effect and the refined front description may be useful both in control problems for scalar laws, as in [3], and in the large-time analysis of balance laws and their coherent structures, in the spirit of [22].

Use of AI tools declaration

The authors used AI-assisted tools for English language editing and readability improvements. All scientific content, results, and conclusions are the authors' own, and the authors take full responsibility for the final manuscript.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

Rida Harb: Conceptualization, methodology, formal analysis, writing—original draft. Stephane Junca: Supervision, validation, writing—review & editing. All authors approved the final version of the manuscript.

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Appendix

A. BV^s spaces

In this section, the definition of fractional BV spaces are recalled [23–25].

Definition A.1. Let $p = \frac{1}{s} \geq 1$. The TV^s variation, also called the total p-variation of any real function v, is defined as follows:

$$\mathrm{TV}^s v = \sup_{\{x_i\} \in \mathcal{P}} \sum_{i=2}^{n} |v(x_i) - v(x_{i-1})|^p, \tag{A.1}$$

where $\mathcal{P} = \{\{x_1, \ldots, x_n\},\ x_1 < \cdots < x_n,\ 2 \leq n \in \mathbb{N}\}$ is the set of subdivisions of $\mathbb{R}$.

The space $BV^s(\mathbb{R})$ is the subset of real functions such that

$$BV^s(\mathbb{R}) = \{v : \mathrm{TV}^s(v) < \infty\}. \tag{A.2}$$

Remark A.2. For $s = 1$, we recover the classical space of functions of bounded variation as follows:

$$BV(I, \mathbb{R}) = BV^1(I).$$

This theorem characterizes the space BV^s with the holder space Lip^s and the BV space due to Michel Bruneau [23].

Theorem A.3. *(Bruneau's factorization, 1974) For any $u \in BV^s(\mathbb{R})$, there exists the following factorization by an s-Hölder function and a BV function:*

$$u \in BV^s(\mathbb{R}) \quad \Leftrightarrow \quad \exists L \in \mathrm{Lip}^s(\mathbb{R}, \mathbb{R}),\ \exists v \in BV(\mathbb{R}) \text{ such that } u = L \circ v.$$

That means that

$$BV^s(\mathbb{R}, \mathbb{R}) = \mathrm{Lip}^s(\mathbb{R}, \mathbb{R}) \circ BV(\mathbb{R}, \mathbb{R}).$$

As a direct consequence of Bruneau's factorization theorem, we obtain the following regularity result.

Corollary A.4. *(Regularity Result). Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function that satisfies the degeneracy condition* (3.2) *with the degeneracy exponent p.*

Suppose that for each fixed $t > 0$, the composition $x \mapsto h(u(t, x))$ belongs to $BV_{\mathrm{loc}}(\mathbb{R})$.

Then, it follows that

$$u(t, \cdot) \in BV^s_{\mathrm{loc}}(\mathbb{R}), \quad \text{with } s = \frac{1}{p}.$$

In other words, the function $u(t, \cdot)$ inherits a fractional bounded variation regularity of order $s = 1/p$, thanks to the non-degeneracy and monotonicity of h.