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Research article

Computational analysis of fractional Drinfeld-Sokolov-Wilson equation associated with regularized form of Hilfer-Prabhakar derivative

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Abstract: In this research paper, we utilize an analytical technique to investigate the behavior of the Drinfeld-Sokolov-Wilson equation of arbitrary order. The implemented technique is an adequate composition of the Kharrat-Toma transform and the q-homotopy analysis approach. Here, a regularized form of the Hilfer-Prabhakar derivative of arbitrary order is used to formulate the problem. The Drinfeld-Sokolov-Wilson equation of arbitrary order is utilized to model the dispersive water waves and plays a very significant role in fluid dynamics. The results of the discussed model are presented graphically to show the efficiency and reliability of the obtained results.

Keywords: Prabhakar fractional calculus; regularized version of Hilfer-Prabhakar fractional derivative; Drinfeld-Sokolov-Wilson equation; Kharrat-Toma transform

1. Introduction

Mathematical models provide a practical way to describe and analyze various real-world problems by offering simplified representations of physical phenomena through mathematical expressions. Among these, some models are formulated using partial differential equations. In particular, partial differential equations of arbitrary order are highly effective in capturing the physical characteristics of many everyday challenges. A quantitative and qualitative study of the nonlinear (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation was done by Wang et al. [1]. Here, by utilizing the semi-inverse method the authors constructed the variational principle.

Fractional calculus, a field with origins dating back approximately four centuries, has seen a surge in interest and applications among mathematicians and researchers over the past few decades [2–5]. Various fractional-order integrals and derivatives, such as the Katugampola, Caputo, Atangana-Baleanu, Caputo-Fabrizio, Riemann-Liouville, and Hilfer-Prabhakar, have been developed by eminent mathematicians to extend traditional differential equations to fractional orders. Liang and Wang [6] employed a local fractional derivative on Vakhnenko-Parker equation for the fractal relaxation medium and obtained exact fractal wave solutions. Wang [7] derived a new fractal active low-pass filter within the local fractional derivative on the cantor set. A new $\mathfrak{I}$-order non differentiable R-C zero state response circuit was derived by Wang and Liu [8] by utilizing the local fractional derivative for the first time on the cantor set. A new exothermic reaction model of fractional order with constant heat source in porous media was proposed by Wang [9] using the He's fractional derivative and solved utilizing the Ritz technique. These arbitrary order models usually yield more accurate results than their classical counterparts because they incorporate the system's memory effects. This unique characteristic of fractional calculus has led to its application in diverse fields, including the study of viscoelastic materials, earthquake modeling, chemical process analysis, traffic flow dynamics, mathematical biology, engineering, and ecology [10–15].

Here, we are analyzing the behavior of the Drinfeld-Sokolov-Wilson (DSW) equation, which is used in dispersive water waves and fluid mechanics. The generic form of the DSW equation [16] is provided as

$$u_{\vartheta}(\xi,\vartheta)+a_1\omega(\xi,\vartheta)\omega_{\xi}(\xi,\vartheta)=0,$$

$$\omega_{\vartheta}(\xi,\vartheta)+a_2\omega_{\xi\xi\xi}(\xi,\vartheta)+a_3u(\xi,\vartheta)\omega_{\xi}(\xi,\vartheta)+a_4u_{\xi}(\xi,\vartheta)\omega(\xi,\vartheta)=0. \tag{1}$$

In Eq (1), a_1, a_2, a_3, and $a_4$are constants, and $u(\xi,\vartheta)$ and $\omega(\xi,\vartheta)$ represent the amplitude of the wave modes with respect to (w.r.t.) time ϑ and space ξ. In this paper, we have taken the particular values of these constants as $a_1=3, a_2=2, a_3=2$, and $a_4=1$.

In this work, we employ a regularized form of the Hilfer-Prabhakar (HP) derivative of non-integer order to model the problem. The HP derivative serves as a generalized framework that encompasses the Prabhakar, Hilfer, Caputo, Caputo-Fabrizio, and Riemann-Liouville derivatives for specific values of its parameters. Consequently, the regularized HP derivative is capable of retaining more system memory compared to other fractional derivatives, allowing it to more effectively capture and describe the physical behavior of the system. Thus, the time fractional DSW equation associated with the regularized form of the HP derivative of non-integer order is given as

$$^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}u_{\vartheta}(\xi,\vartheta)+3\omega(\xi,\vartheta)\omega_{\xi}(\xi,\vartheta)=0,$$

$$^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\omega_{\vartheta}(\xi,\vartheta)+2\omega_{\xi\xi\xi}(\xi,\vartheta)+2u(\xi,\vartheta)\omega_{\xi}(\xi,\vartheta)+u_{\xi}(\xi,\vartheta)\omega(\xi,\vartheta)=0. \tag{2}$$

Here, $^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}u_{\vartheta}(\xi,\vartheta)$ and $^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\omega_{\vartheta}(\xi,\vartheta)$ stand for the regularized form of HP derivative of order μ of u and ω, respectively, w.r.t. time ϑ. At $\mu=1$, the fractional DSW equation becomes a classical DSW equation.

In general, exact solutions for a nonlinear differential equation of integer order are not available. The same occurs for a nonlinear fractional differential equation. Therefore, scientists have developed numerous analytical as well as numerical techniques [17–19] to obtain approximate results to those equations. Researchers have also proposed and utilized several techniques for solving the DSW equation. Singh et al. [16] employed the homotopy analysis Sumudu transform method to attain the effective results of fractional DSW equation. Here, the authors utilized the Caputo derivative of arbitrary order to model the problem. The homotopy analysis technique was employed by Arora and Kumar [20] to find out the approximate series solution of the DSW equation, and the author compared the attained solutions with the exact solution. Homotopy perturbation transform method was implemented by Singh et al. [21] to acquire the result of the arbitrary order DSW equation. Jin and Lu [22] implemented the variational iteration approach to acquire the solution of the DSW equation. Gao et al. [23] employed q-homotopy analysis transform approach to find out the solution of coupled DSW equation. The homotopy perturbation transform technique and Sumudu transform decomposition method was implemented by Noor et al. [24] to acquire the result of arbitrary order coupled DSW equation. Shahen et al. [25] attained a distinct set of analytic results of the fractional DSW equation utilizing the $exp(-\phi(\xi))$-expansion technique and expressed that in terms of trigonometric, hyperbolic, and rational functions. Shahzad et al. [26] employed the ϕ^6-model expansion technique to obtain the solitary wave solution of the DSW equation. Nadeem and Alsayaad [27] suggested a new iterative procedure to attain the approximate analytical solution of the fractional DSW equation.

Here, we employ an analytical approach, namely q-homotopy analysis Kharrat-Toma transform technique (q-HAKTM), which is an adequate amalgamation of the Kharrat-Toma (KT) transform [28] and q-homotopy analysis approach (q-HAM) [29]. The implemented technique is very reliable and efficient for solving and analyzing the behavior of partial differential equations and requires less computational work. The paper is organized as: Some basic definitions are discussed in Section 2; Section 3 contains an elementary description of the implemented analytical method; In Section 4; the q-HAKTM solution of DSW equation of arbitrary order is provided; Graphical behavior of obtained solution and its discussion are given in Section 5; and finally, Section 6 provides the concluding observations of this research work.

2. Some basic definitions

Definition 1: [2] Suppose that $\hbar \in L[a_1, \ell_1]$, where $\ell_1 > a_1$, is a locally integrable and real-valued function. The Riemann-Liouville derivative of $\hbar(\vartheta)$ of non-integer order μ $(k-1 < \mu \leq k, k \in \mathbb{N})$ is defined as

$$D^{\mu}_{a_1^+}\hbar(\vartheta) = \frac{1}{\Gamma(k-\mu)}\frac{d^k}{d\vartheta^k}\int_{a_1}^{\vartheta}(\vartheta - x)^{k-1-\mu}\hbar(x)dx. \tag{3}$$

Definition 2: [2,33] Suppose that $\hbar \in L[a_1, \ell_1]$, where $\ell_1 > a_1$, is a locally integrable and real-valued function. The Caputo fractional derivative of $\hbar(\vartheta)$ of non-integer order μ $(k-1 < \mu \leq k, k \in \mathbb{N})$ is given as

$${}^{C}D^{\mu}_{a_1^+}\hbar(\vartheta) = \frac{1}{\Gamma(k-\mu)}\int_{a_1}^{\vartheta}(\vartheta - x)^{k-1-\mu}\frac{d^k}{dx^k}\hbar(x)dx. \tag{4}$$

Definition 3: [31,32] T. R. Prabhakar introduced the following function, also known as the three parameter Mittag-Leffler function, given as

$$E^{\beta}_{\rho,\mu}(\vartheta) = \sum_{m=0}^{\infty} \frac{(\beta)_m}{\Gamma(\rho m+\mu)} \frac{\vartheta^m}{m!}, \tag{5}$$

for $\beta, \rho, \mu \in \mathbb{C}$ and $Re(\rho), Re(\mu) > 0$, here $(\beta)_m$ represents for Pochhammer symbol and is provided as $(\beta)_m = \frac{\Gamma(\beta+m)}{\Gamma(m)}$ and $\mathbb{C}$ stands for the set of complex numbers.

Definition 4: [33] Suppose that $\mu \in (0,1]$, $\lambda \in [0,1]$, $\hbar \in L^1[a_1, b_1], b_1 > a_1$, and $\left(\hbar * I_{a_1^+}^{(1-\mu)(1-\lambda)}\right)(\vartheta) \in AC^1[a_1, b_1]$, the Hilfer derivative of non-integer order μ of the locally integrable function $\hbar(\vartheta)$ is defined as

$$\left(D_{a_1^+}^{\mu,\lambda} \hbar\right)(\vartheta) = \left(I_{a_1^+}^{\lambda(1-\mu)} \frac{d}{d\vartheta} I_{a_1^+}^{(1-\mu)(1-\lambda)} \hbar\right)(\vartheta). \tag{6}$$

In the above definition, λ is a parameter, I stands for the integral operator, and $AC^1[a_1, b_1]$ represents the set of absolutely continuous functions in the interval $[a_1, b_1]$.

At $\lambda = 0$ the Hilfer derivative given in Eq (6) reduces to the Riemann-Liouville fractional derivative, and at $\lambda = 1$, it reduces to Caputo fractional derivative.

Definition 5: [32,34] Consider that $\hbar \in L^1(0, b_1)$, and $0 < \vartheta < b_1 \leq \infty$, then the Prabhakar integral is provided as

$$\mathbb{E}^{\beta}_{\rho,\mu,\varsigma,0^+} \hbar(\vartheta) = \int_0^{\vartheta} (\vartheta - x)^{\mu-1} E^{\beta}_{\rho,\mu}[\varsigma(\vartheta - x)^{\rho}] \hbar(x) dx = \left(\hbar * e^{\beta}_{\rho,\mu,\varsigma}\right)(\vartheta), \tag{7}$$

where, $\rho,\ \beta,\ \mu,\ \varsigma \in \mathbb{C}$ with $Re(\rho), Re(\mu) > 0$ and * denotes the convolution of two functions. Here, $e^{\beta}_{\rho,\mu,\varsigma}(\vartheta) = \vartheta^{\mu-1} E^{\beta}_{\rho,\mu}(\varsigma\, \vartheta^{\rho})$.

Definition 6: [31,34] Suppose that a locally integrable function $\hbar \in L^1(0, b_1), 0 < b_1 < \infty$, and $\hbar * e^{-\beta}_{\rho,k-\mu,\varsigma}(\cdot) \in W^{k,1}[0, b_1]$, $k = \lceil \mu \rceil$, accordingly the Prabhakar fractional derivative is given as

$$D^{\beta}_{\rho,\mu,\varsigma,0^+} \hbar(\vartheta) = \frac{d^k}{d\vartheta^k} \mathbb{E}^{-\beta}_{\rho,k-\mu,\varsigma,0^+} \hbar(\vartheta), \tag{8}$$

where, $\rho,\ \beta,\ \mu,\ \varsigma \in \mathbb{C}$ with $Re(\rho),\ Re(\mu) > 0$, and $W^{k,1}[0, b_1]$ is a sobolev space.

Definition 7: [35] Suppose that $\mu \in (0,1]$, $\lambda \in [0,1]$, $\hbar \in L^1[0, b_1]$, $0 < \vartheta < b_1 \leq \infty$, and also consider $\left(\hbar * e^{-\beta(1-\lambda)}_{\rho,(1-\lambda)(1-\mu),\varsigma}\right)(\vartheta) \in AC^1[0, b_1]$, then the HP derivative of fractional order μ of function $\hbar(\vartheta)$ can be defined as

$$D^{\beta,\mu,\lambda}_{\rho,\varsigma,0^+} \hbar(\vartheta) = \left(\mathbb{E}^{-\beta\lambda}_{\rho,\lambda(1-\mu),\tau,0^+} \frac{d}{d\vartheta} \left(\mathbb{E}^{-\beta(1-\lambda)}_{\rho,(1-\lambda)(1-\mu),\varsigma,0^+} \hbar\right)\right)(\vartheta), \tag{9}$$

where $\beta,\ \varsigma \in \mathbb{R}$, $\rho > 0$ and $\mathbb{E}^{0}_{\rho,0,\varsigma,0^+} \hbar = \hbar$. Thus, we can say that the HP derivative of order μ becomes the Hilfer fractional derivative for $\beta = 0$.

Definition 8: [35] Consider that $0 < \vartheta < b_1 < \infty$, $h \in AC^1[0, b_1]$, also $\mu \in (0,1]$, $\lambda \in [0,1]$, $\beta,\ \varsigma \in \mathbb{R}$, and $\rho > 0$. The regularized form of the HP derivative of fractional order μ of function $\hbar(\vartheta)$ is denoted by ${}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+} \hbar(\vartheta)$ and can be defined as

$${}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+} \hbar(\vartheta) = \left(\mathbb{E}^{-\beta\lambda}_{\rho,\lambda(1-\mu),\varsigma,0^+} \mathbb{E}^{-\beta(1-\lambda)}_{\rho,(1-\lambda)(1-\mu),\tau,0^+} \frac{d}{d\vartheta} \hbar\right)(\vartheta). \tag{10}$$

Also, we know the property of the Prabhakar integral as

$$\left(\mathbb{E}^{\beta}_{\rho,\mu,\varsigma,0^+}\mathbb{E}^{\varrho}_{\rho,\lambda,\varsigma,0^+}\hbar\right)(\vartheta) = \left(\mathbb{E}^{\beta+\varrho}_{\rho,\mu+\lambda,\varsigma,0^+}\hbar\right)(\vartheta). \quad (11)$$

Using Eq (11), the aforementioned Eq (10) transformed as

$$^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\hbar(\vartheta) = \left(\mathbb{E}^{-\beta}_{\rho,1-\mu,\varsigma,0^+}\frac{d}{d\vartheta}\hbar\right)(\vartheta). \quad (12)$$

Definition 9: [28] Suppose that a real-valued function $\hbar(\vartheta)$ subject to (s.t.) $\hbar(\vartheta) > 0$ for $\vartheta \geq 0$ and $\hbar(\vartheta) = 0$ for $\vartheta < 0$. If $\hbar(\vartheta)$ is a piecewise continuous function and of exponential order then, the KT transform of $\hbar(\vartheta)$ is given as

$$B[\hbar(\vartheta)] = F(w) = w^3 \int_0^{\infty} e^{-\frac{\vartheta}{w^2}}\hbar(\vartheta)\, d\vartheta\,; w > 0, \quad (13)$$

where, w stands for the transform variable, and B represents the KT transform.

Definition 10: [28] The inverse KT transform is defined as

$$B^{-1}[F(w)](\vartheta) = \hbar(\vartheta) = B^{-1}\left[w^3 \int_0^{\infty} \hbar(\vartheta) e^{-\frac{\vartheta}{w^2}} d\vartheta\right], \vartheta > 0. \quad (14)$$

In above Eq (14), $F(w)$ denotes the KT transform of $\hbar(\vartheta)$.

Definition 11: [36] The KT transform of the regularized form of the HP fractional derivative $^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\hbar(\vartheta)$ given in Eq (12) is provided as

$$B\left({}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\hbar(\vartheta)\right)(w) = [1-\varsigma w^{2\rho}]^{\beta}[w^{-2\mu}B[\hbar(\vartheta)](w) - s^{5-2\mu}\hbar(0^+)]. \quad (15)$$

3. Elementary description of analytical technique

To demonstrate the fundamental working plan of the implemented analytical approach [37], let us suppose a non-homogeneous nonlinear fractional differential equation of order μ:

$$^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\hbar_{\vartheta}(\xi,\vartheta) + \mathcal{R}\hbar(\xi,\vartheta) + \mathcal{N}\hbar(\xi,\vartheta) = \phi(\xi,\vartheta), \quad k-1 < \mu \leq k,\ k \in \mathbb{N}, \quad (16)$$

where, $^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}$ represents the regularized form of the HP derivative of non-integer order μ, $\hbar(\xi,\vartheta)$ is a function of ξ and ϑ, $\mathcal{R}$ is a bounded linear operator of ξ and ϑ, $\mathcal{N}$ stands for the general nonlinear operator that is Lipschitz continuous, and $\phi(\xi,\vartheta)$ represents the source term.

On employing the KT transform on Eq (16), we attain the following equation:

$$B\left[{}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\hbar_{\vartheta}(\xi,\vartheta)\right] + B[\mathcal{R}\hbar(\xi,\vartheta) + \mathcal{N}\hbar(\xi,\vartheta)] = B[\phi(\xi,\vartheta)]. \quad (17)$$

Using the KT transform of the regularized form of the HP derivative of arbitrary order, we obtain the consequent equation:

$$[1-\varsigma w^{2\rho}]^{\beta}[w^{-2\mu}B[\hbar(\xi,\vartheta)](w) - w^{5-2\mu}\hbar(\xi,0^+)] + B[\mathcal{R}\hbar(\xi,\vartheta) + \mathcal{N}\hbar(\xi,\vartheta)] = B[\phi(\xi,\vartheta)]. \quad (18)$$

After simplification, Eq (18) becomes

$$B[\hbar(\xi,\vartheta)](w) - w^5\hbar(\xi,0^+) + w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}\big[B[\mathcal{R}\hbar(\xi,\vartheta)+\mathcal{N}\hbar(\xi,\vartheta)] - B[\phi(\xi,\vartheta)]\big] = 0. \tag{19}$$

According to Eq (19), the nonlinear operator can be written as

$$\mathcal{N}[\Psi(\xi,\vartheta;q)] = B[\Psi(\xi,\vartheta;q)] - w^5\Psi(\xi,0;q) + w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}\big[B[\mathcal{R}\Psi(\xi,\vartheta;q) + \mathcal{N}\Psi(\xi,\vartheta;q)] - B[\phi(\xi,\vartheta)]\big], \tag{20}$$

where $\Psi(\xi,\vartheta;q)$ is a function of $\xi,\ \vartheta$, and q, and also q is an embedding parameter s. t. $q \in \left[0,\frac{1}{n}\right]$, where $n \geq 1$ and the homotopy can be given as

$$(1-nq)B[\Psi(\xi,\vartheta;q) - \hbar_0(\xi,\vartheta)] = q\hbar H(\xi,\vartheta)\mathcal{N}[\Psi(\xi,\vartheta;q)], \tag{21}$$

where B stands for KT transform, $\Psi(\xi,\vartheta;q)$ is an unknown function, $\hbar_0(\xi,\vartheta)$ is an initial approximation of $\hbar(\xi,\vartheta)$, $H(\xi,\vartheta) \neq 0$ denotes an auxiliary function, and $\hbar \neq 0$ is an auxiliary parameter. Furthermore, we can observe that, on putting the values of the embedding parameter $q = 0$ along with $q = \frac{1}{n}$, it gives

$$\Psi(\xi,\vartheta;0) = \hbar_0(\xi,\vartheta), \qquad \Psi\left(\xi,\vartheta;\frac{1}{n}\right) = \hbar(\xi,\vartheta), \tag{22}$$

respectively. Thus, from Eq (22) we can notice that when the value of q varies from 0 to $\frac{1}{n}$, the outcome of $\Psi(\xi,\vartheta;q)$ varies from the initial guess $\hbar_0(\xi,\vartheta)$ to the solution $\hbar(\xi,\vartheta)$. The Taylor's series expansion for the $\Psi(\xi,\vartheta;q)$ can be given as

$$\Psi(\xi,\vartheta;q) = \hbar_0(\xi,\vartheta) + \sum_{m=1}^{\infty}\hbar_m(\xi,\vartheta)q^m, \tag{23}$$

where,

$$\hbar_m(\xi,\vartheta) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}\{\Psi(\xi,\vartheta;q)\}\Big|_{q=0}. \tag{24}$$

If the asymptotic parameter n, convergence control parameter $\hbar$, arbitrary function $H(\xi,\vartheta)$, and the initial approximation $\hbar_0(\xi,\vartheta)$ are chosen adequately, s.t. Eq (23) converges at $q = \frac{1}{n}$. Then we acquire the subsequent equation:

$$\hbar(\xi,\vartheta) = \hbar_0(\xi,\vartheta) + \sum_{m=1}^{\infty}\hbar_m(\xi,\vartheta)\left(\frac{1}{n}\right)^m. \tag{25}$$

The solution obtained in Eq (25) represents one of the solutions of the discussed nonlinear differential equation of non-integer order μ. The governing equation is obtained by using Eqs (25) and (21) as

$$\vec{\hbar}_m = \{\hbar_1(\xi,\vartheta), \hbar_2(\xi,\vartheta), \hbar_3(\xi,\vartheta), \ldots, \hbar_m(\xi,\vartheta)\}. \tag{26}$$

Now, differentiating Eq (21) m times w.r.t. q and then setting $q = 0$ subsequently, dividing by $m!$, yields the consequent equation:

$$B[\hbar_m(\xi,\vartheta)-\chi_m\hbar_{m-1}(\xi,\vartheta)]=\hbar H(\xi,\vartheta)\Re_m(\vec{\hbar}_{m-1}). \quad (27)$$

Next, by employing the inverse KT transform on Eq (27), we obtain the subsequent equation:

$$\hbar_m(\xi,\vartheta)=\chi_m\hbar_{m-1}(\xi,\vartheta)+\hbar B^{-1}\left[H(\xi,\vartheta)\Re_m(\vec{\hbar}_{m-1})\right], \quad (28)$$

where, χ_m is defined as

$$\chi_m=\begin{cases}0, & m\leq 1\\ n, & m>1\end{cases}, \quad (29)$$

and the value of $\Re_m(\vec{\hbar}_{m-1})$ can be written as

$$\Re_m(\vec{\hbar}_{m-1})=B[\hbar_{m-1}(\xi,\vartheta)]-\left(1-\frac{\chi_m}{n}\right)\left[w^5\hbar(\xi,0)+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B[\phi(\xi,\vartheta)]\right]+$$
$$w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B[\mathcal{R}\hbar_{m-1}+A_{m-1}]. \quad (30)$$

In the above Eq (30), A_m narrates the homotopy polynomial [38] and is provided as

$$A_m=\frac{1}{\Gamma(m)}\left[\frac{\partial^m}{\partial q^m}\mathcal{N}\Psi(\xi,\vartheta;q)\right]_{q=0}, \quad (31)$$

and

$$\Psi(\xi,\vartheta;q)=\Psi_0+q\Psi_1+q^2\Psi_2+\cdots. \quad (32)$$

Utilizing Eq (30) in Eq (28), we obtain the approximate analytical solution $\hbar(\xi,\vartheta)$ of the subsequent form:

$$\hbar(\xi,\vartheta)=\sum_{m=0}^{\infty}\hbar_m(\xi,\vartheta)\left(\frac{1}{n}\right)^m. \quad (33)$$

4. Solution of the fractional DSW equation

The fractional DSW equation associated with the regularized form of the HP derivative of fractional order is given as

$${}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}u_\vartheta(\xi,\vartheta)+3\omega(\xi,\vartheta)\omega_\xi(\xi,\vartheta)=0,$$

$${}^{C}D^{\beta,\mu}_{\rho,\varsigma,0^+}\omega_\vartheta(\xi,\vartheta)+2\omega_{\xi\xi\xi}(\xi,\vartheta)+2u(\xi,\vartheta)\omega_\xi(\xi,\vartheta)+u_\xi(\xi,\vartheta)\omega(\xi,\vartheta)=0, \quad (34)$$

with initial conditions

$$u_0(\xi,\vartheta)=u(\xi,0)=3sech^2(\xi),$$

$$\omega_0(\xi,\vartheta)=\omega(\xi,0)=2sech(\xi). \quad (35)$$

The exact solution [13] of the integer order DSW equation obtained by substituting $\mu=1$ in Eq (34) is given as

$$u(\xi,\vartheta)=\frac{3c}{2}sech^2\left(\sqrt{\frac{c}{2}}(\xi-c\vartheta)\right),$$

$$\omega(\xi,\vartheta)=\pm csech\left(\sqrt{\frac{c}{2}}(\xi-c\vartheta)\right). \tag{36}$$

Now, implementing the KT transform on Eq (34) both sides and utilizing the initial guess given by Eq (35), we acquire

$$B[u(\xi,\vartheta)]-w^5u(\xi,0)+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B\big[3\omega(\xi,\vartheta)\omega_\xi(\xi,\vartheta)\big]=0,$$

$$B[\omega(\xi,\vartheta)]-w^5\omega(\xi,0)+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B\big[2\omega_{\xi\xi\xi}(\xi,\vartheta)+2u(\xi,\vartheta)\omega_\xi(\xi,\vartheta)+ u_\xi(\xi,\vartheta)\omega(\xi,\vartheta)\big]=0. \tag{37}$$

Next, the nonlinear operator for the discussed problem is provided as

$$\mathcal{N}_1\big[\Psi_{(1)}(\xi,\vartheta;q),\Psi_{(2)}(\xi,\vartheta;q)\big]$$
$$=B\big[\Psi_{(1)}(\xi,\vartheta;q)\big]-w^5\Psi_{(1)}(\xi,0;q)$$
$$+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}\Big[B\big[3\Psi_{(2)}(\xi,\vartheta;q)\Psi_{(2)\xi}(\xi,\vartheta;q)\big]\Big],$$

$$\mathcal{N}_2\big[\Psi_{(1)}(\xi,\vartheta;q),\Psi_{(2)}(\xi,\vartheta;q)\big]=B\big[\Psi_{(2)}(\xi,\vartheta;q)\big]-w^5\Psi_{(2)}(\xi,0;q)+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}\Big[B\big[2\Psi_{(2)\xi\xi\xi}(\xi,\vartheta;q)+2\Psi_{(1)}(\xi,\vartheta;q)\Psi_{(2)\xi}(\xi,\vartheta;q)+\Psi_{(1)\xi}(\xi,\vartheta;q)\Psi_{(2)}(\xi,\vartheta;q)\big]\Big], \tag{38}$$

and the value of $\Re_{1m}(\vec{u}_{m-1},\vec{\omega}_{m-1}),\Re_{2m}(\vec{u}_{m-1},\vec{\omega}_{m-1})$ can be written as

$$\Re_{1m}(\vec{u}_{m-1},\vec{\omega}_{m-1})=B[u_{m-1}(\xi,\vartheta)]-\left(1-\frac{\chi_m}{n}\right)[w^5u(\xi,0)]+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B[3A_{m-1}],$$

$$\Re_{2m}(\vec{u}_{m-1},\vec{\omega}_{m-1})=B[\omega_{m-1}(\xi,\vartheta)]-\left(1-\frac{\chi_m}{n}\right)[w^5\omega(\xi,0)]+w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B\big[2\omega_{(m-1)\xi\xi\xi}(\xi,\vartheta)+2B_{m-1}+C_{m-1}\big]. \tag{39}$$

Now, substituting the values of $\Re_{1m}(\vec{u}_{m-1},\vec{\omega}_{m-1}),\Re_{2m}(\vec{u}_{m-1},\vec{\omega}_{m-1})$ in the iterative formula given in Eq (28), we get

$$u_m(\xi,\vartheta)=(\chi_m+\hbar)u_{m-1}(\xi,\vartheta)-\hbar\left(1-\frac{\chi_m}{n}\right)B^{-1}[w^5u(\xi,0)]$$
$$+\hbar B^{-1}\Big[w^{2\mu}[1-\varsigma w^{2\rho}]^{-\beta}B[3A_{m-1}]\Big],$$

$$\omega_m(\xi,\vartheta) = (\chi_m + \hbar)\omega_{m-1}(\xi,\vartheta) - \hbar\left(1 - \frac{\chi_m}{n}\right)B^{-1}[w^5\omega(\xi,0)] + \hbar B^{-1}\Big[w^{2\mu}[1 - \varsigma w^{2\rho}]^{-\beta}B\big[2\omega_{(m-1)\xi\xi\xi}(\xi,\vartheta) + 2B_{m-1} + C_{m-1}\big]\Big]. \quad (40)$$

Now, by substituting $m = 1$ in Eq (40) and using the initial approximation given in Eq (35), we get

$$u_1(\xi,\vartheta) = -12\hbar\tanh(\xi)\,sech^2(\xi)e^{\beta}_{\rho,\mu+1,\varsigma}(\vartheta),$$

$$\omega_1(\xi,\vartheta) = -4\hbar\,\mathrm{sech}(\xi)\tanh(\xi)\,e^{\beta}_{\rho,\mu+1,\varsigma}(\vartheta). \quad (41)$$

Thus, proceeding in the same way, one can find many components $u_m(\xi,\vartheta)$ and $\omega_m(\xi,\vartheta)$ for $m \geq 2$, and the approximate solution using q-HAKTM is obtained.

Consequently, the q-HAKTM solution is given as

$$u(\xi,\vartheta) = \lim_{M\to\infty}\sum_{m=0}^{M} u_m(\xi,\vartheta)\left(\frac{1}{n}\right)^m,$$

$$\omega(\xi,\vartheta) = \lim_{M\to\infty}\sum_{m=0}^{M} \omega_m(\xi,\vartheta)\left(\frac{1}{n}\right)^m. \quad (42)$$

5. Numerical results and discussion

Graphical representation serves as a way to depict the characteristics of the approximate solution. Thus, in this section, we analyze the behavior of the outcomes of the DSW equation of arbitrary order utilizing an analytical method, namely q-HAKTM. A numerical simulation is conducted for numerous values of fractional order μ, space variable ξ, and time variable ϑ.The outcomes of this numerical simulation are described in the form of Figures 1–16. Figures 1–4 corresponds to $u(\xi,\vartheta)$, in which Figures 1–3 depict the surface of the q-HAKTM solution $u(\xi,\vartheta)$at $\mu = 1$, $\mu = 0.90$, and $\mu = 0.80$, respectively. Figure 4 plots the exact solution of $u(\xi,\vartheta)$. Figures 5–8 are plotted for $\omega(\xi,\vartheta)$, in which Figures 5–7 depict the surface of the q-HAKTM solution $\omega(\xi,\vartheta)$ at $\mu = 1$, $\mu = 0.90$, and $\mu = 0.80$, respectively. Figure 8 plots the exact solution of $\omega(\xi,\vartheta)$. From Figures 1–8, we see the obtained solutions are very much similar to exact solution. Figures 9 and 10 exhibit the influence of arbitrary order μ w.r.t. ξ and ϑ, respectively for $u(\xi,\vartheta)$. Similarly, Figures 11 and 12 exhibit the influence of arbitrary order μ w.r.t. ξ and ϑ, respectively for $\omega(\xi,\vartheta)$. Figures 13 and 14 are drawn between the exact and approximate results of $u(\xi,\vartheta)$ and $\omega(\xi,\vartheta)$, respectively to show the accuracy of the obtained solution. Next, Figures 15 and 16 are n-curve $u(\xi,\vartheta)$ and $\omega(\xi,\vartheta)$, which show the asymptotic behavior of the obtained solution via q-HAKTM. From Figures 15 and 16 we can observe that initially the value of u and ω increase rapidly and then decrease but after some time, it becomes almost constant, which is due to the appearance of the term $\left(\frac{1}{n}\right)^m$ in the solution.

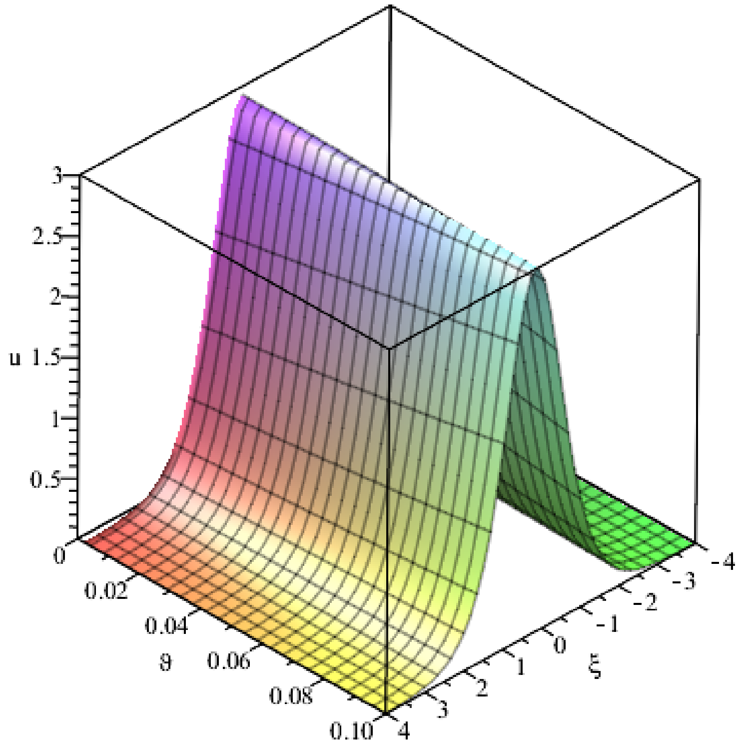

Figure 1. Nature of $u(\xi, \vartheta)$ for the q-HAKTM outcome at $\mu = 1$.

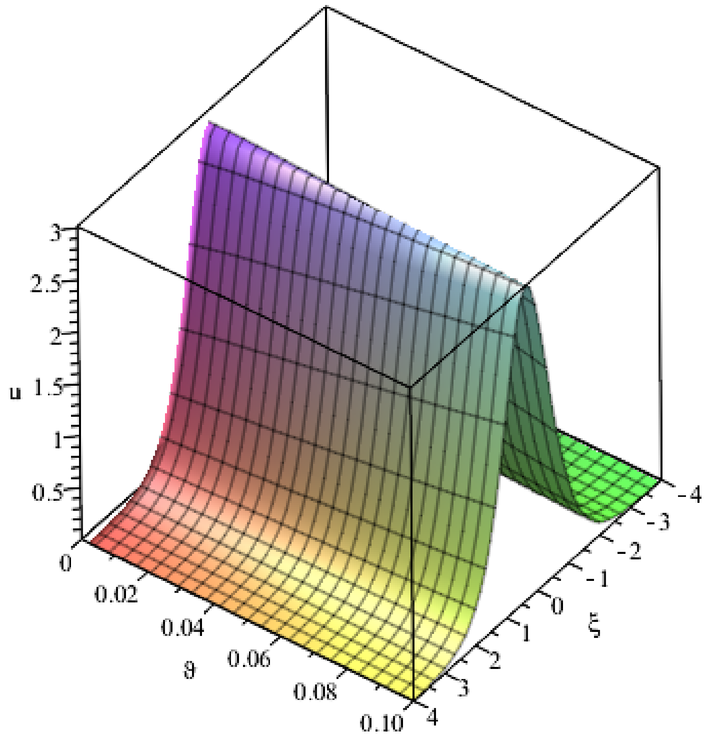

Figure 2. Nature of $u(\xi, \vartheta)$ for the q-HAKTM outcome at $\mu = 0.90$.

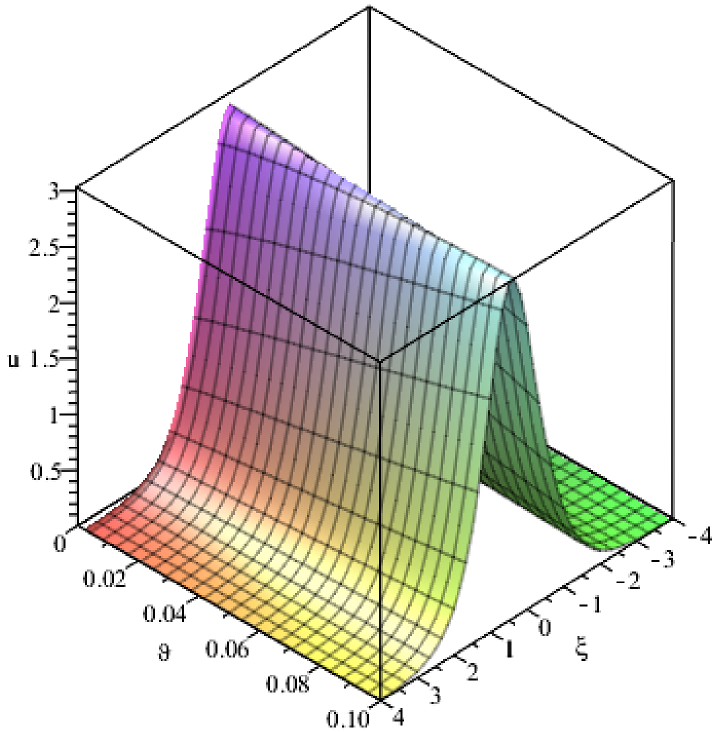

Figure 3. Nature of $u(\xi,\vartheta)$ for the q-HAKTM outcome at $\mu = 0.80$.

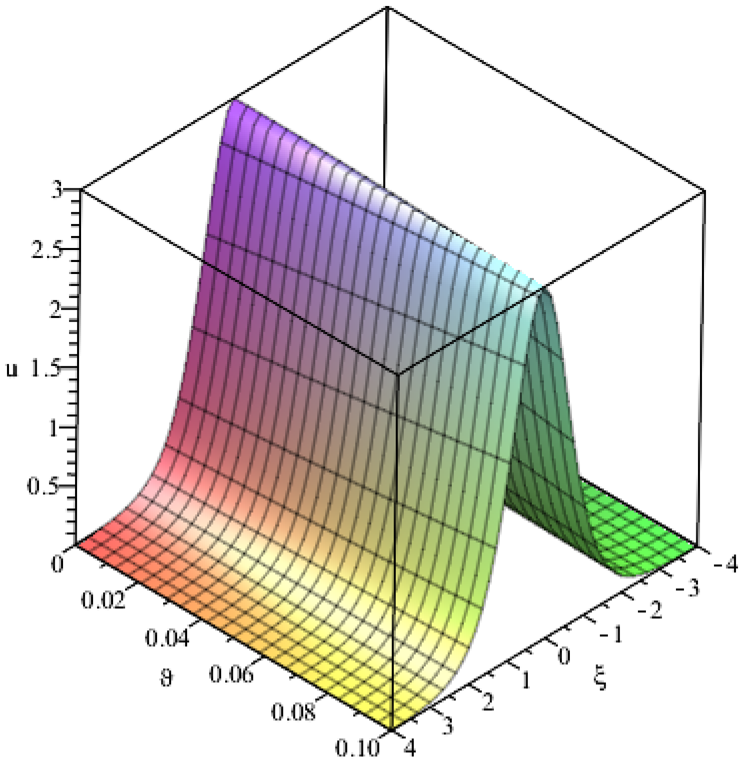

Figure 4. The surface of the exact solution $u(\xi,\vartheta)$.

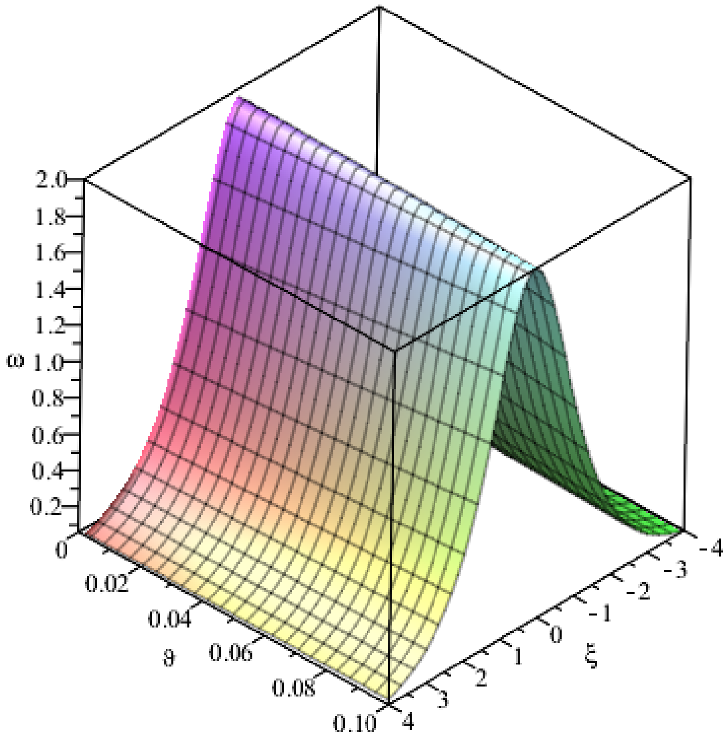

Figure 5. The surface of $\omega(\xi, \vartheta)$ for the q-HAKTM solution at $\mu = 1$.

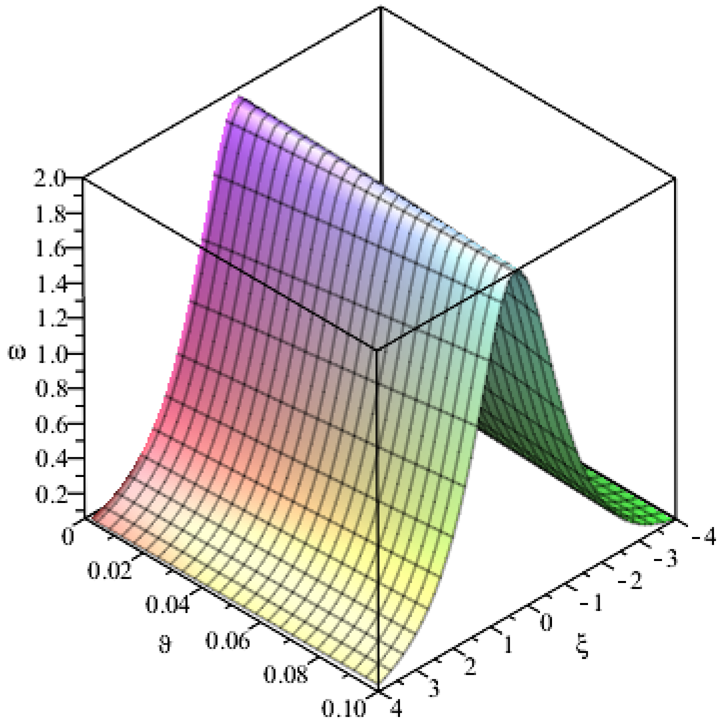

Figure 6. The surface of $\omega(\xi, \vartheta)$ for the q-HAKTM solution at $\mu = 0.90$.

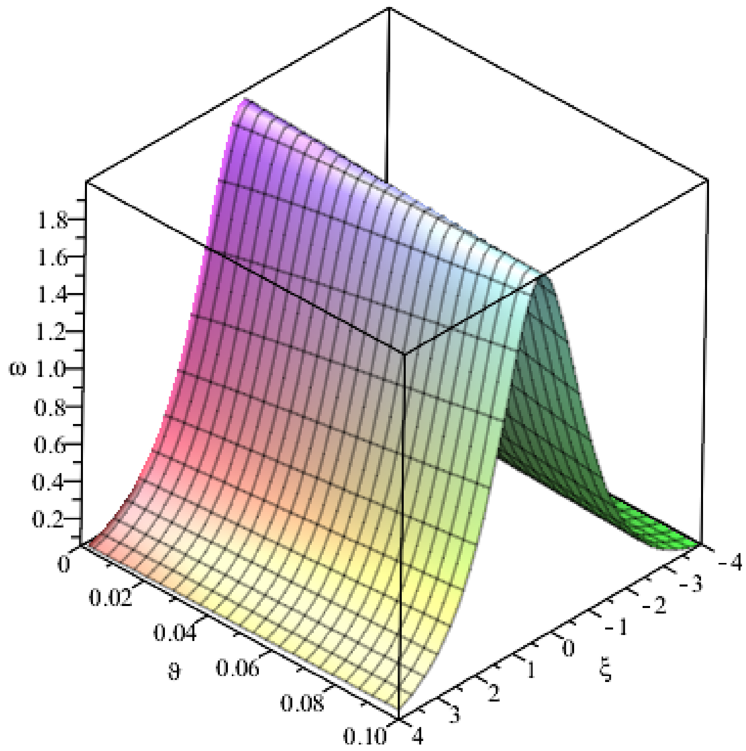

Figure 7. The surface of $\omega(\xi,\vartheta)$ for the q-HAKTM solution at $\mu = 0.80$.

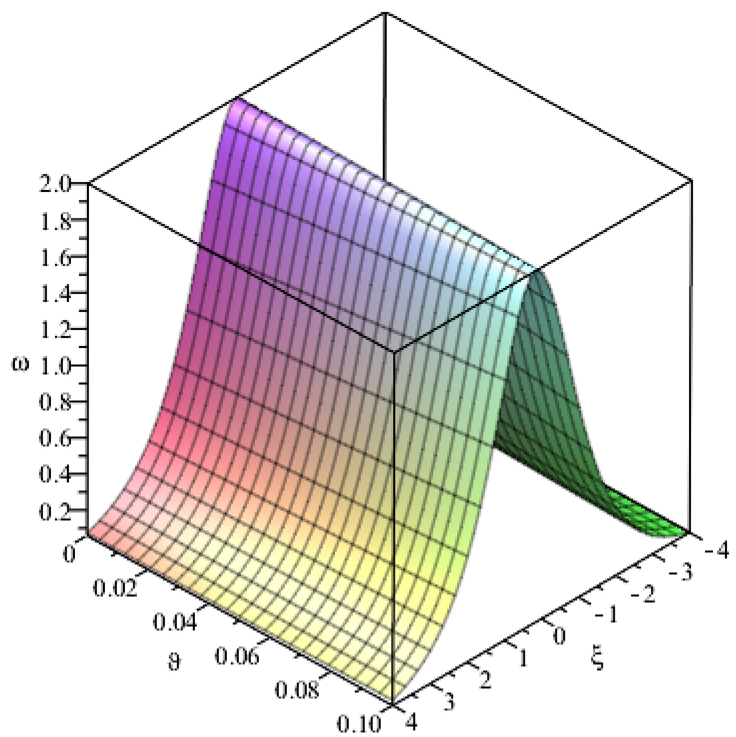

Figure 8. The surface of the exact solution $\omega(\xi,\vartheta)$.

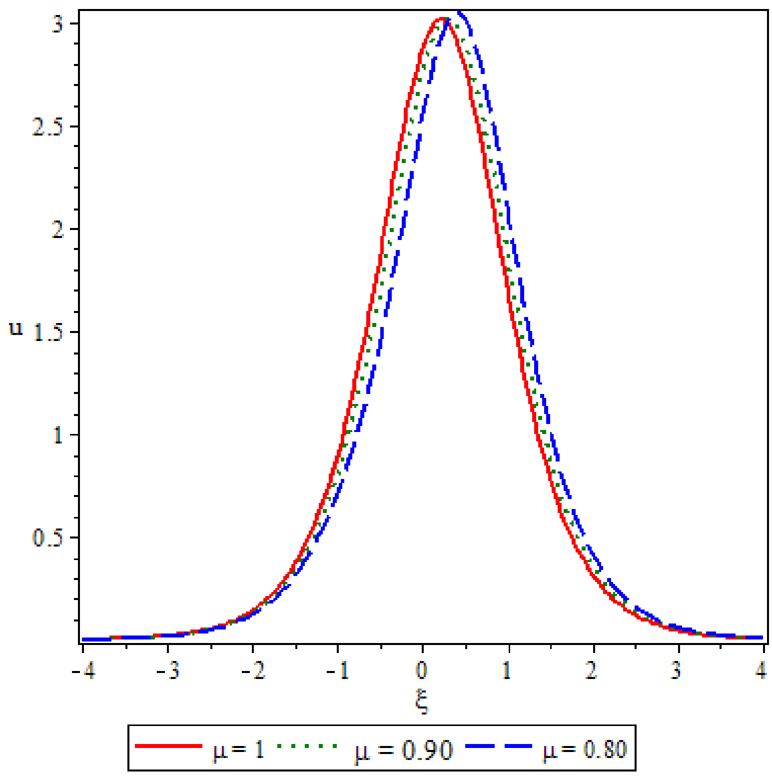

Figure 9. Characteristic of $u(\xi,\vartheta)$ w. r. t. ξ for distinct values of μ.

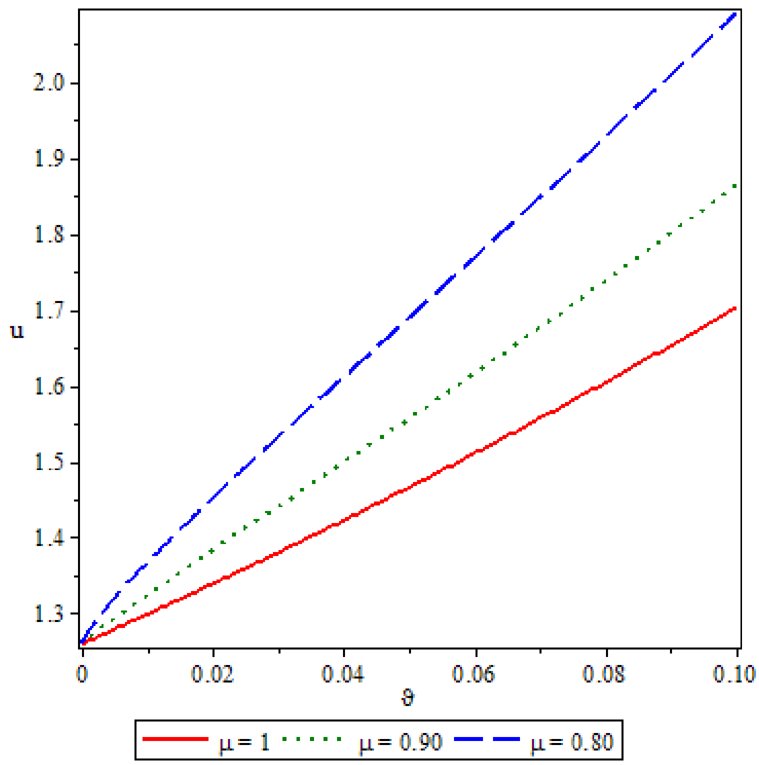

Figure 10. Characteristic of $u(\xi,\vartheta)$ w. r. t. ϑ for distinct values of μ.

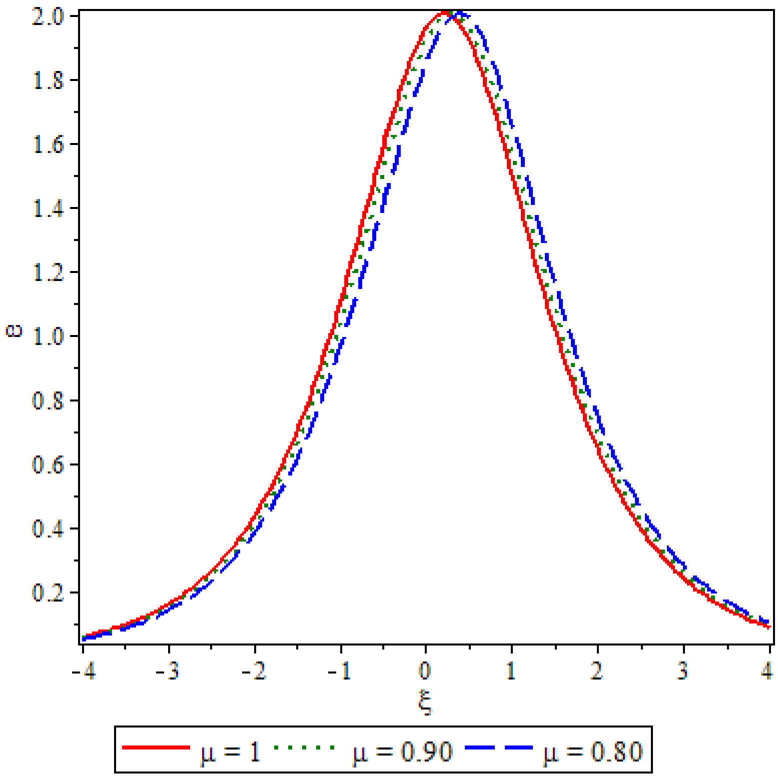

Figure 11. Characteristic of $\omega(\xi,\vartheta)$ w. r. t. ξ for distinct values of μ.

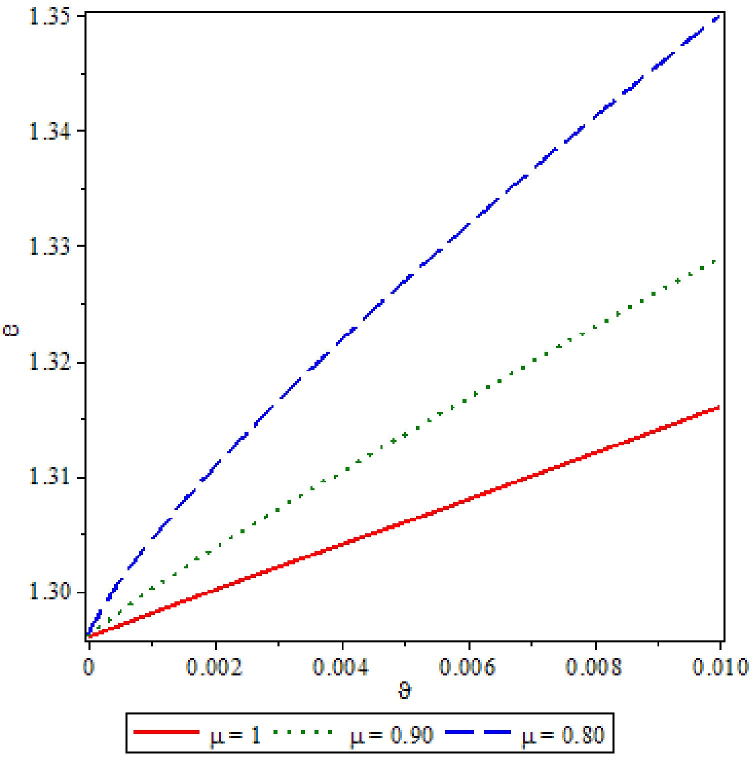

Figure 12. Characteristic of $\omega(\xi,\vartheta)$ w. r. t. ϑ for distinct values of μ.

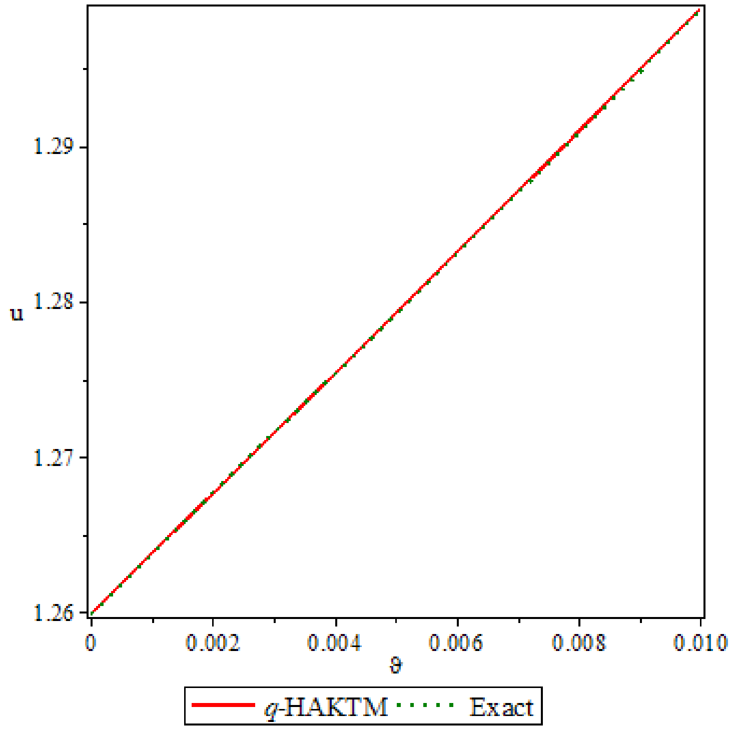

Figure 13. Comparative graph for the approximate and exact solutions of $u(\xi, \vartheta)$.

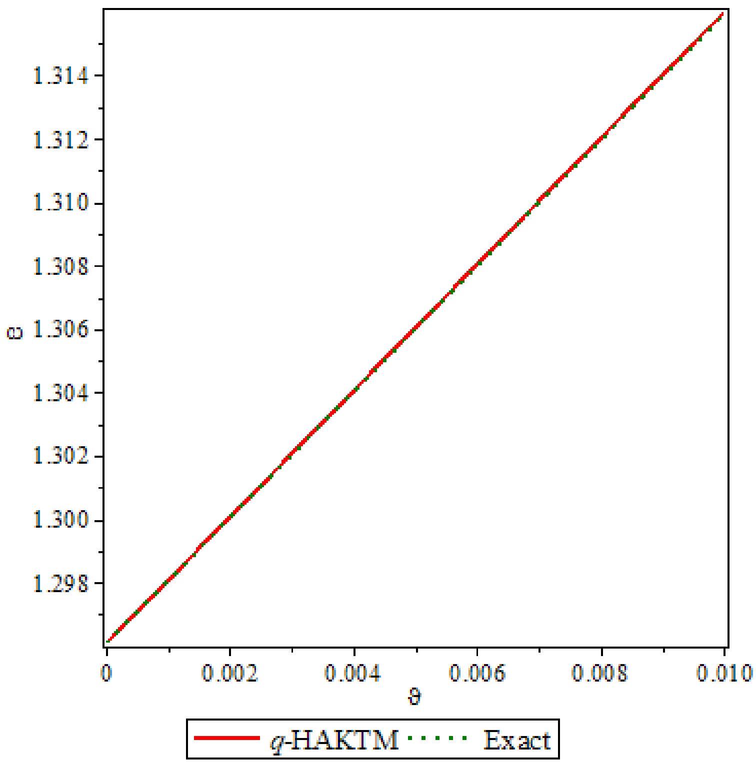

Figure 14. Comparative graph for the exact and approximate outcomes of $\omega(\xi, \vartheta)$.

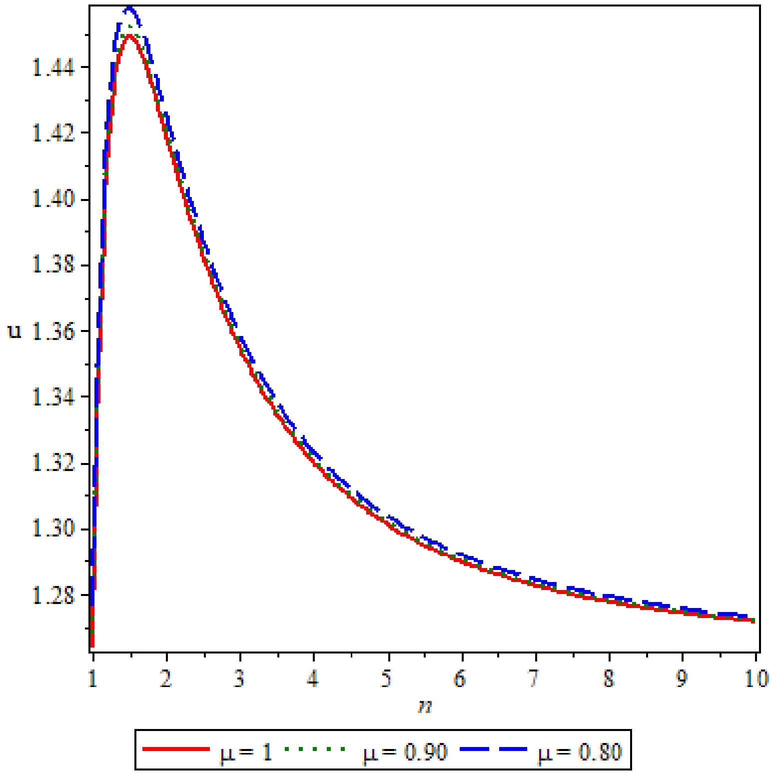

Figure 15. n-curve of $u(\xi, \vartheta)$ for distinct values of μ.

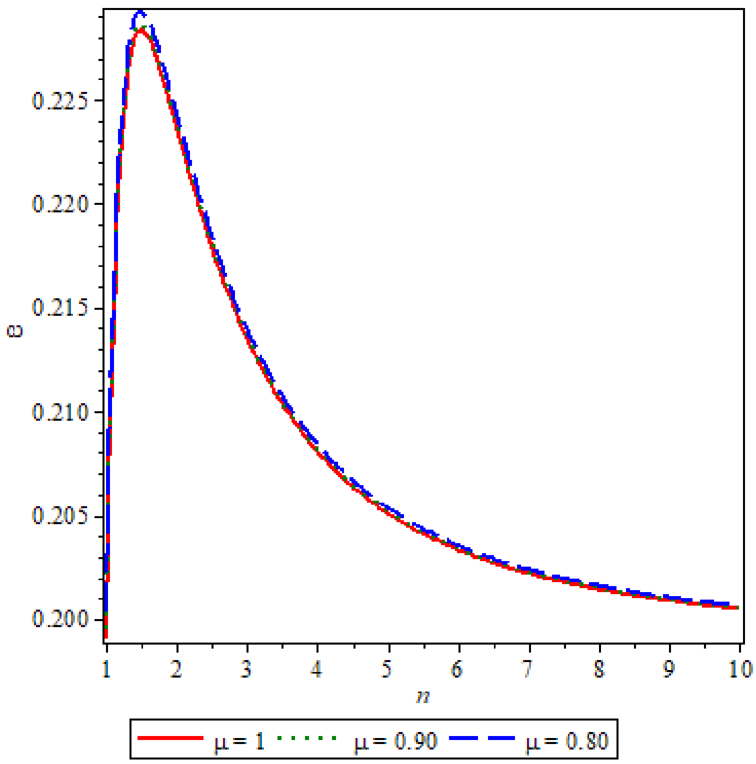

Figure 16. n-curve of $\omega(\xi, \vartheta)$ for distinct values of μ.

6. Conclusions

To describe the nature of dispersive water waves, the DSW equation is widely used. Here, we implemented an analytical method, q-HAKTM, to obtain the result of fractional DSW associated with the regularized form of the HP derivative of arbitrary order. Graphical behavior of the attained solutions is given to show the authenticity and efficiency of the results. Hence, we can say that the implemented method is very reliable, powerful, and needs less computational work to analyze the nature of arbitrary order differential equations.

Use of AI tools declaration

The authors declare we have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Juan J. Nieto is an editorial board member for Networks and Heterogeneous Media, but was not involved in the editorial review or the decision to publish this article. All authors declare no competing interests.

Author contributions

Conceptualization: Jagdev Singh, Arpita Gupta and Juan J. Nieto; methodology: Jagdev Singh, Arpita Gupta; software: Arpita Gupta and Moisés Rutkoski; validation: Jagdev Singh, Arpita Gupta. And Juan J. Nieto; formal analysis: Jagdev Singh, Arpita Gupta and Juan J. Nieto, Moisés Rutkoski; investigation: Jagdev Singh, Arpita Gupta. and Juan J. Nieto; resources: Arpita Gupta and Moisés Rutkoski; data curation: Arpita Gupta and Moisés Rutkoski; writing—original draft preparation: Jagdev Singh, Arpita Gupta and Juan J. Nieto, Moisés Rutkoski; writing—review and editing: Jagdev Singh, Arpita Gupta and Juan J. Nieto, Moisés Rutkoski; visualization: Jagdev Singh, Arpita Gupta and Juan J. Nieto, Moisés Rutkoski; supervision: Jagdev Singh and Juan J. Nieto; All authors have read and agreed to the published version of the manuscript.

Data availability statement

The original contributions presented in the study are included in the article, and further inquiries can be directed to the corresponding authors.

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