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Research article

Legendre spectral-Monte Carlo method and its error analysis for nonlinear stochastic Itô–Volterra integral equation

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Abstract: Nonlinear stochastic Itô–Volterra integral equations (NSIVIEs) represent systems whose current state is influenced by random fluctuations and is dependent on previous information. These equations appear in many real-world scenarios, including engineering systems, biological processes, financial markets, heterogeneous media, complex transport phenomena, and viscoelastic materials. Strong numerical frameworks are required because analytical solutions for these equations are rarely available, particularly when nonlinearities and random fluctuations are present. To effectively solve NSIVIEs, in this study we propose a new hybrid numerical framework that combines Monte Carlo simulation and Legendre spectral collocation. By using orthogonal polynomial basis functions to approximate the solution, this method provides spectral accuracy while handling the hereditary memory component of the Volterra equation through a high-order Legendre spectral collocation method. A precise statistical treatment of the random fluctuations is made possible by simultaneously addressing the stochastic Itô noise through Monte Carlo sampling across numerous independent realizations. We perform a thorough convergence analysis and obtain explicit error bounds that measure the decrease in approximation error with increasing spectral resolution and Monte Carlo sample count. Numerical experiments show that the method can accurately reproduce complex stochastic behaviors and validate theoretical predictions.

Keywords: stochastic Itô–Volterra integral equation; spectral methods; error estimates; Monte Carlo simulations; numerical examples

1. Introduction

Many biological, engineering, and physical systems evolve in heterogeneous, nonlinear environments that display memory effects and random fluctuations. Stochastic differential or integral equations that incorporate noise-driven dynamics and hereditary kernels are frequently used to model such systems. SIVIEs are a powerful mathematical framework for describing non-Markovian stochastic systems, in which the current state is determined by the cumulative influence of random perturbations and past events. Diffusion in heterogeneous media, viscoelastic and thermodynamic systems with after-effects, random transport and reaction processes, neural and biological networks, and uncertain financial markets are just a few of the many applications for which these equations offer a natural modeling basis. The complex interplay between stochastic forcing and deterministic hereditary dynamics is captured by combining Itô stochastic integrals with Volterra-type memory kernels. However, these equations' nonlinearity and random structure make analytical solutions unfeasible, underscoring the necessity of effective and reliable numerical techniques. Solutions can be studied using classical theory if these random functions are sufficiently regular. The classical theory cannot be applied to irregular random functions, like white noise, and the theory of stochastic differential equations (SDE) emerged as a result [1]. SDEs have become an essential tool in modeling various phenomena in fields such as physics, finance, biology, and engineering. Among them, the Itô-Volterra integral equations (IVIEs) play a crucial role in capturing the dynamics of systems where both stochasticity and memory effects are significant. The nonlinearity in these equations adds complexity, making analytical solutions challenging or even impossible to obtain. As a result, numerical methods have emerged as a powerful approach to solving these SDE and IVIEs.

Stochastic integral equations, especially nonlinear stochastic Itô–Volterra integral equations, are very important for modeling complicated systems that have both memory effects and random dynamics. These equations offer a coherent mathematical structure for characterizing non-Markovian processes where the current state is contingent upon the aggregate impact of preceding events and stochastic disturbances. Fundamental theoretical investigations have delineated the analytical framework of stochastic Volterra equations, and underscored their inherent mathematical intricacy [2, 3]. In mathematical finance, these equations are commonly used to model asset prices, interest rates, and risk measures, with historical data and random fluctuations working together to shape the system's evolution. In biological systems, they manifest in population dynamics and neural network modeling, encapsulating hereditary influences and environmental noise. Also, viscoelastic materials, signal processing, and control systems are used in physics and engineering. In control systems, the system's responses depend on past inputs and random disturbances. The existence of singular kernels and irregular stochastic forcing complicates the analytical treatment of these equations, frequently making closed-form solutions unattainable [4]. Because of this, it is now necessary to create reliable numerical methods, and different approximation methods, like collocation-based schemes, have been suggested to find accurate numerical solutions for nonlinear stochastic Itô–Volterra integral equations [5]. These challenges and applications highlight the necessity of effective numerical methods for analyzing such equations in various scientific and engineering fields.

The general form of a NSIVIEs can be expressed as:

$$f(x) = f_0 + \int_0^x k_1(x,y)\, b(y, f(y))\ \mathrm{d}y + \int_0^x k_2(x,y)\, r(y, f(y))\ \mathrm{d}B(y), \qquad x \in D, \tag{1.1}$$

where $f(x)$ is the unknown function, f_0 is the initial condition, $k_1(x, y)$ and $k_2(x, y)$ are given kernel functions, and $b(y, f(y))$ and $r(y, f(y))$ are deterministic coefficient functions. The stochasticity of the model is introduced exclusively through the Itô integral with respect to the Brownian motion $B(y)$. This formulation captures the memory-dependent nature of the process through the Volterra kernel $k_1(x, y)$ and introduces stochastic effects via the Itô integral involving $k_2(x, y)$ and the Brownian motion. The numerical solution is very sensitive to the diffusion function $r(\cdot, \cdot)$ in Eq (1.1), since this function directly controls how strong the Itô stochastic forcing is. The Itô isometry states that the mean-square contribution of the stochastic term is based on the expected value of the squared product of the kernel $k_2(x, y)$ and the diffusion function $r(y, f(y))$. This shows that if you scale the diffusion function by a factor of α, the variance of the noise-driven response will go up by about α^2. Even small changes in r can make a big difference in the spread and pathwise irregularity of the numerical solution. This is especially true when there is multiplicative noise, which means that the diffusion term depends on how the system is changing.

Over the years, significant progress has been made in both the theoretical and numerical aspects of solving Eq (1.1). Theoretical advancements have addressed issues such as the existence, uniqueness, and stability of solutions, particularly for equations with singular kernels or non-Lipschitz conditions. These studies have laid a solid foundation for developing numerical techniques tailored to these complex equations, and as a result one has to use a numerical solution for the approximate solution [6,7]. Among the various numerical methods proposed, collocation techniques have emerged as a particularly effective approach for handling stochastic Itô–Volterra integral equations. By discretizing the problem and reducing it to a system of algebraic equations, collocation methods offer a practical solution that is both computationally feasible and capable of yielding high-accuracy results. These methods have been further enhanced with the use operational matrices and special function approximations, such as delta functions and triangular functions, which improve both the efficiency and the convergence of the numerical solution [8]. Stochastic differential, Volterra, and integro-differential equations have been extensively studied, with emphasis on mean-square formulations, robustness, and reliable numerical techniques that provide a strong foundation for stochastic dynamical analysis [9–12]. In addition, mean-square schemes, simulation methods, and orthogonal function bases have enabled accurate and efficient approximation of random dynamical and variational problems [13–16]. Furthermore, nonlinear Volterra–Fredholm and multidimensional integral equations have been efficiently solved with high accuracy and low computational cost using direct methods, triangular functions, spectral polynomial expansions, and more recently, collocation-based approximation techniques for nonlinear backward stochastic Volterra integral equations [17–20]. The use of fractional modeling combined with artificial intelligence techniques to study and characterize nonlinear dynamics in complex systems arising in chaos theory is explored in [21]. A spectral representation of iterated Itô and Stratonovich stochastic integrals, particularly in the context of modeling nonlinear stochastic dynamics, is used in [22]. The author presents theoretical results and numerical methods designed to improve the mean-square and strong convergence of stochastic differential equations, essential for accurate simulations in various applications. By using the spectral method, they derive matrix-based representations of these integrals, which allow for more efficient numerical solution techniques. The work also integrates Taylor–Itô and Taylor–Stratonovich expansions, highlighting their utility in approximating solutions to complex stochastic differential equations. A computational technique for solving three-dimensional mixed Volterra–Fredholm integral equations is introduced in [23], where the author utilizes the use of Lucas

polynomials to approximate the solutions to these equations, which are then transformed into a system of linear algebraic equations. The method is evaluated through error analysis and is implemented in several numerical examples, demonstrating its effectiveness compared to other numerical techniques. The results indicate that the Lucas polynomial method provides accurate solutions, with the numerical errors being minimal, especially when the exact solutions involve polynomial forms.

Recent years have seen significant progress in numerical methods for NSIVIEs. Spectral collocation methods that utilize orthogonal polynomials have demonstrated significant accuracy by transforming NSIVIEs into deterministic algebraic systems. A shifted Jacobi polynomial operational matrix method is utilized to address SIVIEs [24]. Subsequently, this approach is expanded to multi-dimensional issues employing Lucas polynomials [25]. The versatility of these methods is underscored by using polynomial collocation to stochastic differential equations driven by fractional Brownian motion [26]. Legendre–Gauss collocation method for stochastic Volterra–Fredholm equations, thereby improving spectral accuracy [27]. Second-kind Chebyshev wavelets in conjunction with a parallel computing scheme to effectively address multi-dimensional Itô–Volterra system is used in [28]. Although they require a lot of sampling for convergence, Monte Carlo simulations have been employed as a direct method of handling the stochastic components of such integrals.

Researchers have also created hybrid approaches that combine deterministic solvers with iterative or interpolation techniques to increase stability and efficiency. A combined successive approximation algorithm with bilinear spline interpolation for two-dimensional stochastic integral equations is utilized in [29]. A moving least-squares collocation scheme for stochastic Volterra equations, which is later generalized into a fully two-dimensional meshless collocation framework, presented in [30,31]. However, none of the existing works combine spectral collocation with Monte Carlo sampling, leaving a gap that the present study addresses by integrating Legendre polynomial-based spectral collocation with Monte Carlo simulation. This hybrid approach leverages the high accuracy of spectral methods and the statistical power of Monte Carlo to more effectively solve NSIVIEs. Higher-order statistical measures such as skewness and kurtosis can be employed to assess asymmetry and tail behavior in the stochastic response, which is particularly relevant when interface defects or material heterogeneities induce non-Gaussian effects [32]. Some very recent work on numerical solution involve stochastic equations can be found in [33–36].

This paper aims to enhance the existing literature through the introduction of an improved numerical method for solving nonlinear stochastic Itô–Volterra integral equations. The method incorporates the Legendre spectral collocation technique for the integral component and employs Monte Carlo simulation to model the stochastic component. Monte Carlo methods provide a classical and well-established framework for approximating expectations in stochastic models, with rigorous convergence theory and extensive applications in science and engineering [37, 38]. These sampling methods have been successfully combined with high-order spectral and pseudospectral discretization in the last few years to solve a wide range of stochastic differential and integro-differential equations, such as fractional, Volterra-type, delay, and epidemiological models [39–42]. The theoretical foundations, stability properties, and spectral convergence of such methods are well documented in the spectral methods literature [43, 44]. These results show that spectral collocation methods can get very accurate results with polynomials of low degree. This makes them especially useful for stochastic problems when used with standard Monte Carlo sampling.

The rest of the paper is structured as follows: Section 2 includes some basic definitions and properties.

Section 3 provides the description of the proposed scheme, followed by error analysis in Section 4. To confirm the theoretical results, some numerical examples are given in Section 5. Finally, Section 6 concludes the paper.

2. Some preliminaries

In this section, we review the basic definitions and properties required for the subsequent analysis [45–47].

2.1. Approximation of function

Definition 1. The weighted inner product and its corresponding norm in the weighted space $L_w^2(I)$ are defined as follows:

$$\langle u, v\rangle_w = \int_a^b u(t)v(t)w(t)\,dt, \quad \text{for all } u, v \in L_w^2(I), \tag{2.1}$$

where $w(t)$ is a weight function, and $I = [a, b]$ denotes the interval of interest. The corresponding norm is given as:

$$\|u\|_{L_w^2} = \sqrt{\langle u, u\rangle_w}. \tag{2.2}$$

Definition 2. For any function $u \in L_w^2(I)$ and any natural number $m \in \mathbb{N}$, there exists a unique polynomial $q_m^* \in \mathbb{P}_m$ (where $\mathbb{P}_m$ is the space of polynomials of degree up to m) such that:

$$\|u - q_m^*\|_{L_w^2} = \inf_{q_m \in \mathbb{P}_m} \|u - q_m\|_{L_w^2}, \tag{2.3}$$

where the best approximation polynomial $q_m^*(x)$ is given by:

$$q_m^*(x) = \sum_{k=0}^{m} \hat{u}_k \phi_k(x), \tag{2.4}$$

with coefficients $\hat{u}_k$ defined as:

$$\hat{u}_k = \frac{\langle u, \phi_k\rangle_w}{\|\phi_k\|_w^2}. \tag{2.5}$$

Here, $\{\phi_k\}_{k=0}^m$ forms an L_w^2-orthogonal basis for $\mathbb{P}_m$.

In particular, this best approximation polynomial q_m^* is denoted by $\pi_m u$, representing the L_w^2-orthogonal projection of u onto the space of polynomials of degree up to m.

2.2. Standard Brownian motion and its properties

Definition 3. A stochastic process $\{W(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Brownian motion with variance parameter σ^2 ($\sigma > 0$) if the following conditions hold:

- $\mathbb{P}(W(0) = 0) = 1$.
- For all $t_1 > t_0 \geq 0$, the increment $W(t_1) - W(t_0)$ is normally distributed, i.e., $W(t_1) - W(t_0) \sim \mathcal{N}(0, \sigma^2(t_1 - t_0))$.
- The process $W(t)$ has independent increments.

In particular, when $\sigma = 1$, the process $\{W(t), t \geq 0\}$ is referred to as standard Brownian motion (SBM).

Definition 4. Consider the process $\{W(t), t \geq 0\}$, which is a standard Brownian motion. It has the following path properties:

- **Continuity**: Almost all sample paths of the SBM are continuous.
- **Non-differentiability**: For any $\alpha > \frac{1}{2}$, almost all sample paths of the SBM satisfy

$$\limsup_{t_1 \to t_0} \frac{|W(t_1) - W(t_0)|}{|t_1 - t_0|^\alpha} = \infty,$$

  particularly when $\alpha = 1$, almost all sample paths of the SBM are nowhere differentiable.
- **Asymptotic behavior**: The SBM almost surely satisfies the law of the iterated logarithm:

$$\limsup_{t \to \infty} \frac{W(t)}{\sqrt{2t \log\log t}} = 1, \quad \text{and} \quad \liminf_{t \to \infty} \frac{W(t)}{\sqrt{2t \log\log t}} = -1.$$

2.3. Itô integral and its properties

Definition 5. The space for the Itô integral is defined as:

$$L^2_{ab}(\Omega; L^2([0,T])) = \left\{ f_s(W) \mid f_s(W) \text{ is } \mathcal{F}_s\text{-measurable and } \mathbb{E}\left[\int_0^T f_s^2 \, dt\right] < \infty \right\}.$$

Definition 6 (Properties of the Itô Integral)**.** Let $f, g \in L^2_{ab}(\Omega; L^2([0,T]))$ for $0 \leq s \leq T$. The Itô integral satisfies the following properties:

- **Linearity:** For any scalars $\mu, \nu \in \mathbb{R}$,

$$\int_0^s (\mu g(t) + \nu f(t)) \, dW(t) = \mu \int_0^s g(t) \, dW(t) + \nu \int_0^s f(t) \, dW(t).$$

- **Itô Isometry:**

$$\mathbb{E}\left[\left(\int_0^s g(t) \, dW(t)\right)^2\right] = \int_0^s \mathbb{E}[g^2(t)] \, dt.$$

- **Generalized Itô Isometry:**

$$\mathbb{E}\left[\int_0^s g(t) \, dW(t) \int_0^s f(t) \, dW(t)\right] = \int_0^s \mathbb{E}[g(t)f(t)] \, dt.$$

- $M_s = \int_0^s g(t) \, dW(t)$ is a continuous martingale process. The quadratic variation of M_s is given by $[M]_s = \int_0^s g^2(t) \, dt$, and

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} \left(\int_0^s g(t) \, dW(t)\right)^2\right] \leq 4\mathbb{E}\left[\int_0^s \mathbb{E}[g^2(t)] \, dt\right].$$

3. Spectral method with Monte Carlo simulation

In order to solve Eq (1.1) more efficiently and fully use the properties of orthogonal polynomials, we transform Eq (1.1) from $[0, x]$ to $[-1, 1]$ by using the transformation $y = \frac{x(\xi+1)}{2}$, where $\xi \in [-1, 1]$ and $dy = \frac{x}{2}d\xi$. This changes the integrals involved in Eq (1.1) as follows:

$$\int_0^x k_1(x, y)b(y, f(y))\, dy = \frac{x}{2}\int_{-1}^1 k_1\left(x, \frac{x(\xi+1)}{2}\right)b\left(\frac{x(\xi+1)}{2}, f\left(\frac{x(\xi+1)}{2}\right)\right)d\xi, \tag{3.1}$$

$$\int_0^x k_2(x, y)r(y, f(y))\, dB(y) = \frac{x}{2}\int_{-1}^1 k_2\left(x, \frac{x(\xi+1)}{2}\right)r\left(\frac{x(\xi+1)}{2}, f\left(\frac{x(\xi+1)}{2}\right)\right)dB\left(\frac{x(\xi+1)}{2}\right). \tag{3.2}$$

Let $\{\theta_k\}_{k=0}^N$ be the Gauss-Legendre points on $[-1, 1]$, and $\{w_k\}_{k=0}^N$ be the corresponding weights. Then, the integrals involve in Eq (3.1) and Eq (3.2) are approximated as follows:

$$\frac{x}{2}\int_{-1}^1 g(\xi)d\xi \approx \frac{x}{2}\sum_{k=0}^N w_k g(\theta_k). \tag{3.3}$$

We then approximate the unknown function $f(x)$ using Lagrange interpolation polynomials at the Gauss-Legendre collocation points $\{t_j\}_{j=0}^N$, where $t_j = \frac{x(\theta_j+1)}{2}$. The function $f(x)$ is expressed as:

$$f(x) \approx \sum_{j=0}^N Y_j L_j(x), \tag{3.4}$$

where $L_j(x)$ are the Lagrange polynomials, and $Y_j = f(t_j)$ are the unknown coefficients to be determined. Substituting the Lagrange interpolation approximation into the transformed equation, we obtain:

$$\begin{aligned} f(t_i) \approx f_0 + \frac{t_i}{2}\sum_{k=0}^N w_k \Bigg[& k_1\left(t_i, \frac{t_i(\theta_k+1)}{2}\right)b\left(\frac{t_i(\theta_k+1)}{2}, \sum_{j=0}^N Y_j L_j\left(\frac{t_i(\theta_k+1)}{2}\right)\right) \\ & + k_2\left(t_i, \frac{t_i(\theta_k+1)}{2}\right)r\left(\frac{t_i(\theta_k+1)}{2}, \sum_{j=0}^N Y_j L_j\left(\frac{t_i(\theta_k+1)}{2}\right)\right)\Delta B_k\Bigg], \end{aligned} \tag{3.5}$$

where $\Delta B_k = B\left(\frac{t_i(\theta_k+1)}{2}\right) - B\left(\frac{t_i(\theta_{k-1}+1)}{2}\right)$ are the increments of the Brownian motion at the quadrature points.

Since Eq (3.5) involves stochastic terms, Monte Carlo simulations are used to approximate the solution. For each Monte Carlo simulation $m = 1, 2, \ldots, M$, generate a sample path for the Brownian motion $B(y)$ at the quadrature points $\{\theta_k\}_{k=0}^N$, and calculate the corresponding increments ΔB_k^m for each simulation m. For each sample path m, we get the system of nonlinear equations

$$\begin{aligned} Y_i^m \approx f_0 + \frac{t_i}{2}\sum_{k=0}^N w_k \Bigg[& k_1\left(t_i, \frac{t_i(\theta_k+1)}{2}\right)b\left(\frac{t_i(\theta_k+1)}{2}, \sum_{j=0}^N Y_j^m L_j\left(\frac{t_i(\theta_k+1)}{2}\right)\right) \\ & + k_2\left(t_i, \frac{t_i(\theta_k+1)}{2}\right)r\left(\frac{t_i(\theta_k+1)}{2}, \sum_{j=0}^N Y_j^m L_j\left(\frac{t_i(\theta_k+1)}{2}\right)\right)\Delta B_k^m\Bigg], \end{aligned} \tag{3.6}$$

where Y_i^m represents the solution at the ith collocation point for the mth simulation. Due to the nonlinear nature of the Eq (3.6), an iterative method such as Newton's method or fixed-point iteration is used to solve for the coefficients $\{Y_j^m\}_{j=0}^N$ in each simulation. The iterative method continues until the coefficients converge within a specified tolerance. After solving the system for all M Monte Carlo simulations, compute the average of the solutions:

$$Y_j \approx \frac{1}{M}\sum_{m=1}^{M} Y_j^m, \tag{3.7}$$

to approximate the expected value of Y_j. The final approximate solution $f(x)$ over the domain D is reconstructed using the averaged coefficients $\{Y_j\}_{j=0}^N$.

4. Error analysis

In order to provide a complete theoretical error analysis for our proposed scheme, in this section we state and prove some basic lemmas concerning polynomial approximation by Legendre polynomials and provide some useful definitions, which are essential for the analysis.

Lemma 1. *[48] (**Integration error from Gauss quadrature**) Consider an $(N+1)$-point Gauss-Legendre, Gauss-Radau, or Gauss-Lobatto quadrature formula associated with the Legendre weights, used to integrate the product $y\varphi$, where $y \in H^m(I)$ with $I := (-1, 1)$ and some $m \geq 1$, and $\varphi \in \mathcal{P}_N$. Then, there exists a constant C, independent of N, such that*

$$\left| \int_I y(x)\varphi(x)\,dx - (y,\varphi)_N \right| \leq CN^{-m}|y|_{\tilde{H}_{m,N}(I)}\|\varphi\|_{L^2(I)},$$

where

$$|y|_{\tilde{H}_{m,N}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|y^{(k)}\|_{L^2(I)}^2 \right)^{1/2},$$

$$(y,\varphi)_N = \sum_{k=0}^{N} \omega_k y(x_k)\varphi(x_k).$$

Lemma 2. ***(Estimates for interpolation error)** Assume that $y \in H^m(I)$, and denote by $I_N y$ the interpolation polynomial corresponding to the $(N+1)$ Gauss-Legendre, Gauss-Radau, or Gauss-Lobatto points $\{x_k\}_{k=0}^N$. Then,*

$$\|y - I_N y\|_{L^2(I)} \leq CN^{-m}|y|_{\tilde{H}_{m,N}(I)}, \tag{4.1}$$

$$\|y - I_N y\|_{L^\infty(I)} \leq CN^{1/2-m}|y|_{\tilde{H}_{m,N}(I)}. \tag{4.2}$$

Proof. Estimate (4.1) can be found on p. 289 of [48]. The estimate

$$\|y - I_N y\|_{H^1(I)} \leq CN^{1-m}|y|_{\tilde{H}_{m,N}(I)}, \quad 1 \leq s \leq m,$$

is also available in [48]. By applying the above estimate along with the inequality,

$$\|v\|_{L^\infty(a,b)} \leq \sqrt{\frac{1}{b-a} + 2}\,\|v\|_{L^2(a,b)}^{1/2}\|v\|_{H^1(a,b)}^{1/2}, \quad \forall v \in H^1(a,b),$$

we readily arrive at Eq (4.2).

From reference [49], we have the following result concerning the Lebesgue constant for Lagrange interpolation based on the zeros of the Legendre polynomials.

Lemma 3. ***(Lebesgue constant for the Legendre series)*** *Assume that $\{\mathcal{F}_j(x)\}_{j=0}^N$ are the Lagrange interpolation polynomials with respect to the Gauss-Legendre, Gauss-Radau, or Gauss-Lobatto points $\{x_j\}$. Then*

$$\|I_N\|_\infty := \max_{x\in(-1,1)} \sum_{j=0}^{N} |\mathcal{F}_j(x)| = O(\sqrt{N}). \tag{4.3}$$

Lemma 4. ***(Gronwall inequality for integro-stochastic differential equations)*** *Let $E : [0,T] \to [0,\infty)$ be a nonnegative function satisfying*

$$E(x) \le C_1 \int_0^x E(y)\,dy + C_2 \int_0^x \mathbb{E}\left[\left(\int_0^y E(v)\,dB(v)\right)^2\right] dy, \qquad x \in [0,T], \tag{4.4}$$

where $C_1, C_2 > 0$ are constants and $B(\cdot)$ is a standard Brownian motion. Additionally assume that $E \in L^\infty(0,T)$. Then, there exists a constant $C > 0$ such that

$$E(x) \le C \int_0^x E(y)\,dy, \qquad x \in [0,T].$$

Proof. Using the Itô isometry, we have

$$\mathbb{E}\left[\left(\int_0^y E(v)\,dB(v)\right)^2\right] = \int_0^y E(v)^2\,dv.$$

Substituting this into Eq (4.4) yields

$$E(x) \le C_1 \int_0^x E(y)\,dy + C_2 \int_0^x \int_0^y E(v)^2\,dv\,dy. \tag{4.5}$$

Since $E(v)^2 \ge 0$, Tonelli's theorem allows us to rewrite the double integral as

$$\int_0^x \int_0^y E(v)^2\,dv\,dy = \int_0^x (x-v)\,E(v)^2\,dv \le x \int_0^x E(v)^2\,dv.$$

Because $E \in L^\infty(0,T)$ and $x \le T$, we further obtain

$$x \int_0^x E(v)^2\,dv \le x\|E\|_{L^\infty(0,T)} \int_0^x E(v)\,dv \le T\|E\|_{L^\infty(0,T)} \int_0^x E(v)\,dv.$$

Therefore,

$$\int_0^x \int_0^y E(v)^2\,dv\,dy \le T\|E\|_{L^\infty(0,T)} \int_0^x E(v)\,dv.$$

Substituting this estimate into Eq (4.5), we arrive at

$$E(x) \le \left(C_1 + C_2 T\|E\|_{L^\infty(0,T)}\right) \int_0^x E(y)\,dy,$$

which completes the proof.

Theorem 1. *Consider the stochastic Itô-Volterra integral Eq (1.1) and its spectral approximation method given by Eq (3.6). If F is an approximate solution and f denotes the exact solution, then*

$$\begin{aligned}\|F - f\|_{L^\infty(D)} \leq\ & CN^{1/2-m}\,(D_1 + D_2)\,\|f\|_{L^\infty(D)} \\ & + CN^{-1/2-m}\left(\|f\|_{H_{m,N}(D)} + \|k_1 f\|_{H_{m,N}(D)} + \|k_2 f\|_{H_{m,N}(D)}\right),\end{aligned} \tag{4.6}$$

where C, D_1, and D_2, are constants which do not dependent on N.

Proof. *Let $[F]_{N,s}$ and $\Big[[F]_{N,v}\Big]_{N,s}$ denote the second and third terms on the right-hand side of Eq (3.5). Then, the spectral approximation scheme gives*

$$F(x_i) = f_0 + [k_1 F]_{N,s} + [k_2 F]_{N,s} + G(x_i) - I_{i,1} - I_{i,2},$$

where

$$I_{i,1} := \frac{x_i}{2}\int_{-1}^{1} k_1\left(\frac{x_i}{2}(\theta+1)\right) b\left(\frac{x_i}{2}(\theta+1), F\left(\frac{x_i}{2}(\theta+1)\right)\right) d\theta - [k_1 F]_{N,s},$$

$$I_{i,2} := \frac{x_i}{2}\int_{-1}^{1} k_2\left(\frac{x_i}{2}(\theta+1)\right) r\left(\frac{x_i}{2}(\theta+1), F\left(\frac{x_i}{2}(\theta+1)\right)\right) d\theta - [k_2\Big[[F]_{N,v}\Big]_{N,s},$$

It follows from Eqs (1.1) and (3.1) that:

$$F(x_i) = f_0 + \int_0^{x_i} k_1(s,v)F(v)\,dv + \int_0^{x_i} k_2(s,v)F(v)\,dB(v) - I_{i,1} - I_{i,2} + G(x_i). \tag{4.7}$$

Using Lemma 1 for the approximation, we have:

$$|I_{i,1}| \leq CN^{-m}|k_1|_{\tilde{H}_{m,N}(D)}\|F\|_{L^2(D)} \leq CN^{-m}|k_1|_{\tilde{H}_{m,N}(D)}\left(\|e\|_{L^\infty(D)} + \|f\|_{L^2(D)}\right),$$

$$|I_{i,2}| \leq CN^{-m}|k_2|_{\tilde{H}_{m,N}(D)}\|F\|_{L^2(D)} \leq CN^{-m}|k_2|_{\tilde{H}_{m,N}(D)}\left(\|e\|_{L^\infty(D)} + \|f\|_{L^2(D)}\right).$$

Multiplying by $F_i(x)$ on both sides of Eq (4.7) and summing from 0 to N gives:

$$\begin{aligned}F(x) = {} & I_N\left(\int_0^x k_1(s,v)F(v)\,dv\right) + I_N\left(\int_0^x k_2(s,v)F(v)\,dB(v)\right) \\ & + F_0 + G(x) + J_1(x),\end{aligned}$$

where

$$J_1(x) = \sum_{i=0}^{N}(I_{i,1} + I_{i,2})F_i(x).$$

Let $e(x)$ denote the error between the approximate and exact solution given by $e(x) = f(x) - F_N(x)$. Then,

$$e(x) = \int_0^x k_1(x,y)b(y,f(y))\,dy + \int_0^x k_2(x,y)r(y,f(y))\,dB(y) + J_1(x) + J_2(x). \tag{4.8}$$

where $J_2(x)$ represents additional approximation errors. Applying the Gronwall inequality given in Lemma 4 yields

$$\|e\|_{L^\infty(D)} \leq C\left(\|J_1\|_{L^\infty(D)} + \|J_2\|_{L^\infty(D)}\right). \tag{4.9}$$

Next, we estimate $\|J_1\|_{L^\infty(D)}$ *and* $\|J_2\|_{L^\infty(D)}$. *First,*

$$\|J_1\|_{L^\infty(D)} \leq CN^{1/2-m} \left(\|e\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}\right)(D_1 + D_2),$$

where D_1 *and* D_2 *are constants depending on the given functions* k_1, k_2, *and the stochastic process. For* $\|J_2\|_{L^\infty(D)}$, *we have*

$$\begin{aligned}\|J_2\|_{L^\infty(D)} \leq\ & CN^{-1/2-m}\|f\|_{\tilde{H}_{m,N}(D)} + CN^{-1/2-m}\|K_1 f\|_{\tilde{H}_{m,N}(D)} \\ & + CN^{-1/2-m}\|K_2 f\|_{\tilde{H}_{m,N}(D)} + CN^{-1/2}\|e\|_{L^\infty} + CN^{-1/2}\|K_1 e(s)\|_{L^\infty},\end{aligned}$$

where we have used Lemmas 2 and 3. Combining these estimates with Eq (4.9), we obtain the desired result:

$$\|e\|_{L^\infty(D)} \leq CN^{1/2-m}(D_1 + D_2)\left(\|e\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}\right) + CN^{-1/2-m}\|f\|_{\tilde{H}_{m,N}(D)}.$$

Remark 1. The aforementioned error analysis mainly quantifies the deterministic error resulting from the Legendre spectral discretization in relation to the polynomial degree N. The Monte Carlo approximation of stochastic expectations introduces an additional statistical sampling error of order $O(M^{-1/2})$, where M denotes the number of Monte Carlo samples. Throughout the theoretical analysis, it is assumed that M is chosen sufficiently large so that the Monte Carlo sampling error is negligible compared to the spectral discretization error. Consequently, the error bounds in Theorem 1 are stated explicitly in terms of N only.

5. Numerical illustrations

In this section, we perform some numerical experiments to confirm the theoretical justifications provided in Section 3. These numerical examples further enhance the efficiency of our proposed numerical scheme.

Example 1. In our first example, we consider a nonlinear stochastic Itô–Volterra integral equation for the function $f(x)$ defined as

$$f(x) = 1 + \int_0^x \left[-\sin^2(f(y)) - \frac{1}{4}\sin^4(f(y))\right] dy + \int_0^x \sqrt{2}\cos^2(f(y))\, dW(y),$$

with the exact solution expressed as

$$f(x) = \arctan\left[\tan(1)\exp(-x) + \sqrt{2}\int_0^x \exp(y-x)\, dW(y)\right].$$

The comparison between the exact solution and the approximate solution is made for different values of collocation points. Figure 1 shows the comparison for $N = 300$ and $N = 500$ collocation points, while Figure 2 is plotted for $N = 1000$ and $N = 1200$ collocation points. In order to further enhance the efficiency of our scheme and for justification of theoretical analysis, the error between the exact and approximate solution at different nodes for $N = 500$ collocation points is given in Table 1.

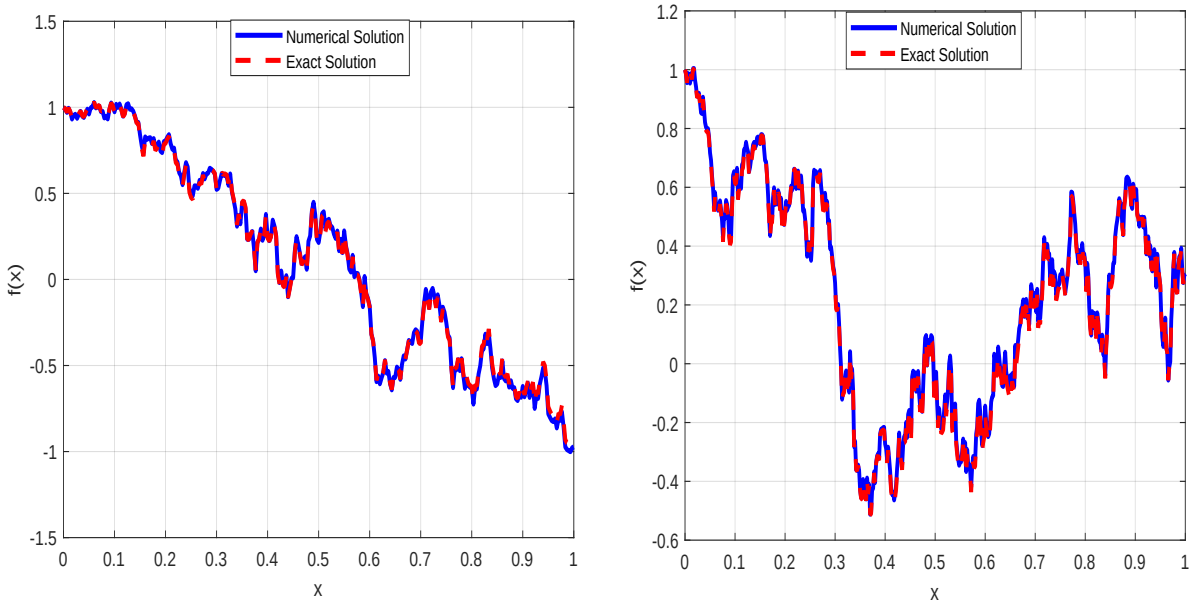

Figure 1. Example 1: Comparison of exact and approximate solutions (**left**) at $N = 300$ and (**right**) at $N = 500$ collocation points.

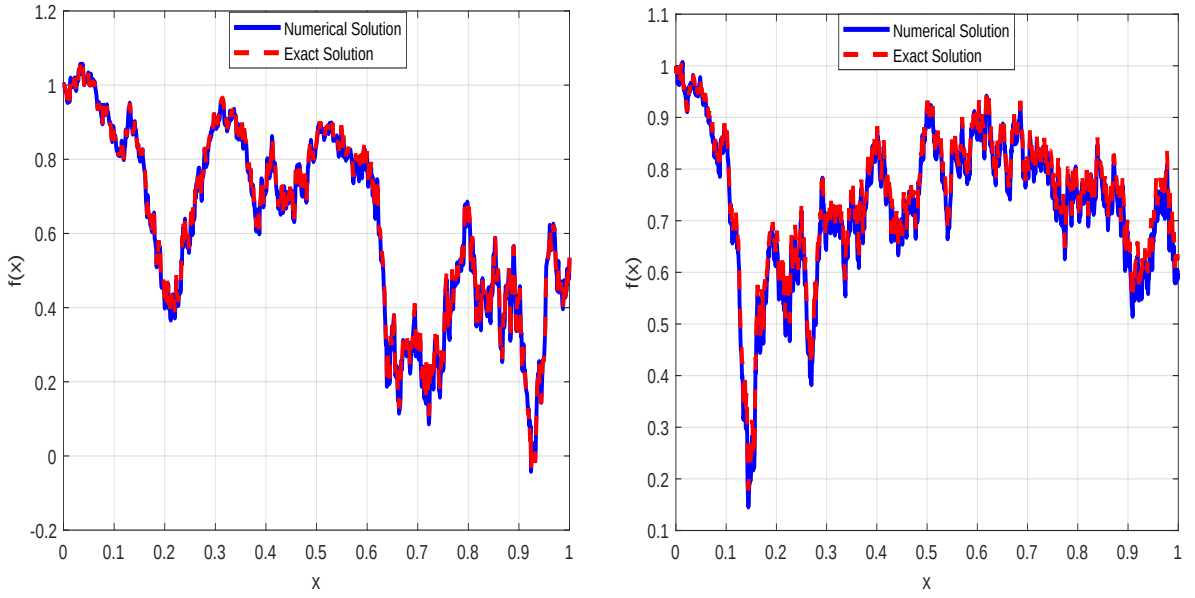

Figure 2. Example 1: Comparison of exact and approximate solutions (**left**) at $N = 1000$ and (**right**) at $N = 1200$ collocation points.

Table 1. Example 1: Error between present method and exact solution for $N = 500$ collocation points.

x_i	Exact solution	Present method	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	1.184766	1.185251	0.000485
0.2	1.155841	1.158570	0.002729
0.3	1.126984	1.126859	0.000125
0.4	1.167631	1.167471	0.000160
0.5	0.973467	0.975968	0.002501
0.6	0.978691	0.977299	0.001391
0.7	0.923261	0.917613	0.005648
0.8	0.792293	0.775928	0.016365
0.9	0.844495	0.828011	0.016485
1.0	0.705072	0.673711	0.031360

Example 2. Consider the nonlinear stochastic Itô–Volterra integral Eq [50]

$$f(x) = 0.5 + \int_0^x f(y)(1 - f(y))\,dy + \int_0^x 0.25 f(y)\,dB(y), \tag{5.1}$$

with the exact solution given by

$$f(x) = \frac{0.5 \exp(0.96875x + 0.25B(x))}{1 + 0.5 \int_0^x \exp(0.96875y + 0.25B(y))\,dy}, \tag{5.2}$$

where $f(x)$ is a known stochastic process defined on the probability space $(X, \mathcal{F}, \mathbb{P})$, and $B(x)$ is a Brownian motion process.

Figure 3 illustrates the comparison of exact and approximate solutions at $N = 300$ and $N = 500$ collocation points. Furthermore, Figure 4 presents the error metrics associated with a larger configuration of collocation points, $N = 1000$ and $N = 1200$. We also provide the error between the exact solution and the approximate method for $N = 500$ collocation points shown in Table 2. This extensive evaluation highlights not only the precision of our method, but also its comparative performance against other established methods.

Table 2. Example 2: Error between present method and exact solution for $N = 500$ collocation points.

x_i	Exact solution	Present method	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	1.184766	1.185251	0.000485
0.2	1.155841	1.158570	0.002729
0.3	1.126984	1.126859	0.000125
0.4	1.167631	1.167471	0.000160
0.5	0.973467	0.975968	0.002501
0.6	0.978691	0.977299	0.001392
0.7	0.923261	0.917613	0.005648
0.8	0.792293	0.775928	0.016365
0.9	0.844495	0.828011	0.016484
1.0	0.705072	0.673711	0.031361

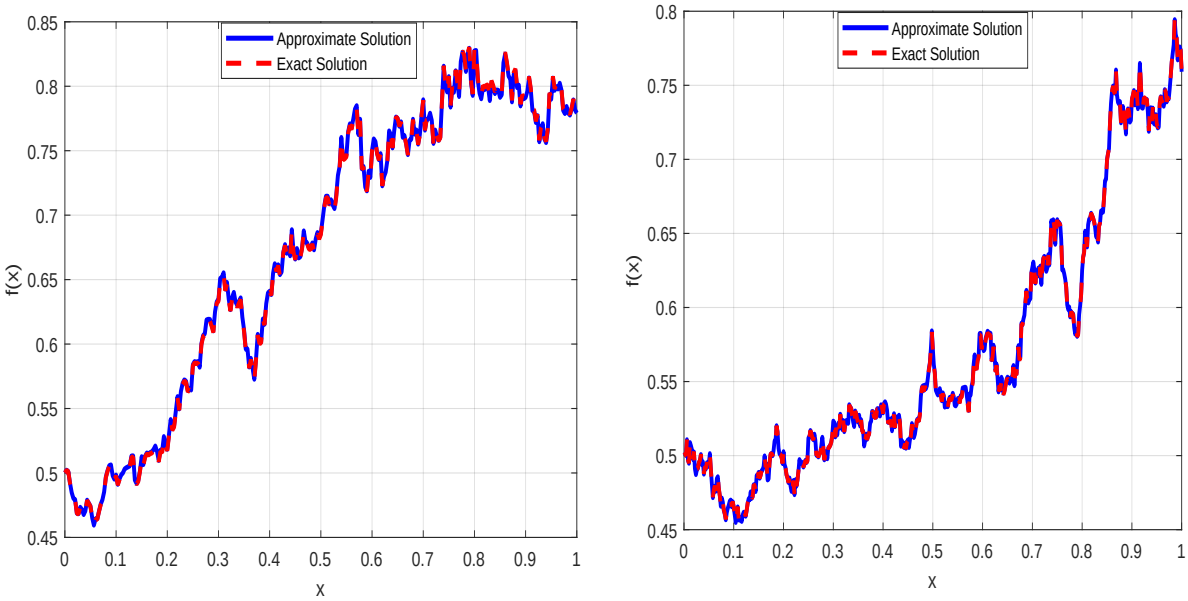

Figure 3. Example 2: Comparison of exact and approximate solutions (**left**) at $N = 300$ and (**right**) at $N = 500$ collocation points.

Example 3. Consider

$$f(x) = 1 + \int_0^x f(y)\left(\frac{1}{32} - f^2(y)\right)dy + \int_0^x 0.25 f(y)\, dB(y), \tag{5.3}$$

with the exact solution given by reference [50]:

$$f(x) = \exp\left(0.25B(x)\right)\sqrt{1 + 2\int_0^x \exp(0.5B(y))\, dy}, \tag{5.4}$$

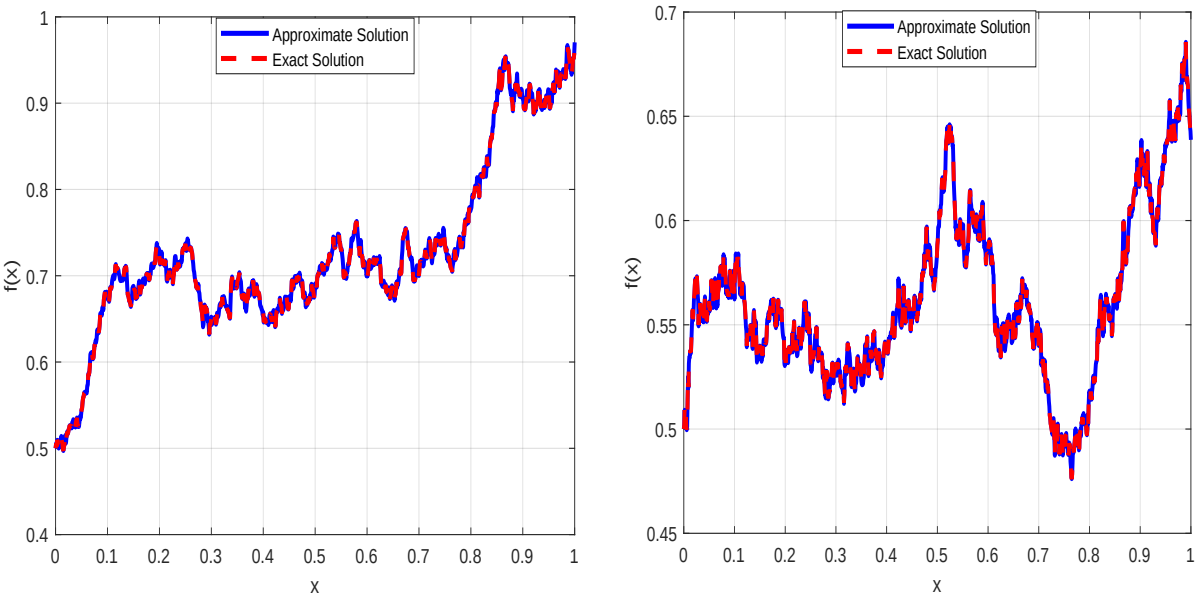

Figure 4. Example 2: Comparison of exact and approximate solutions (**left**) at $N = 1000$ and (**right**) at $N = 1200$ collocation points.

where $f(x)$ is a known stochastic process defined on the probability space $(X, \mathcal{F}, \mathbb{P})$, and $B(x)$ is a Brownian motion process. Figures 5 and 6 illustrate the comparison of the exact solution with the approximation solution for different collocation points. Table 3 provides an error analysis of the present method with an exact solution for different nodes.

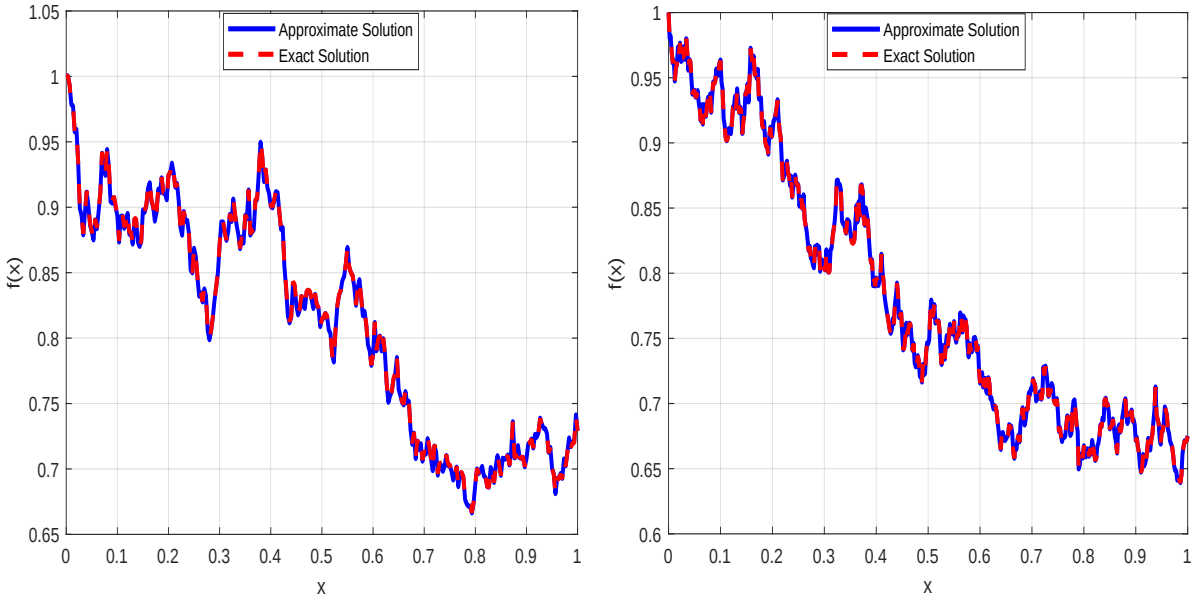

Figure 5. Example 3: Comparison of exact and approximate solutions (**left**) at $N = 300$ and (**right**) at $N = 500$ collocation points.

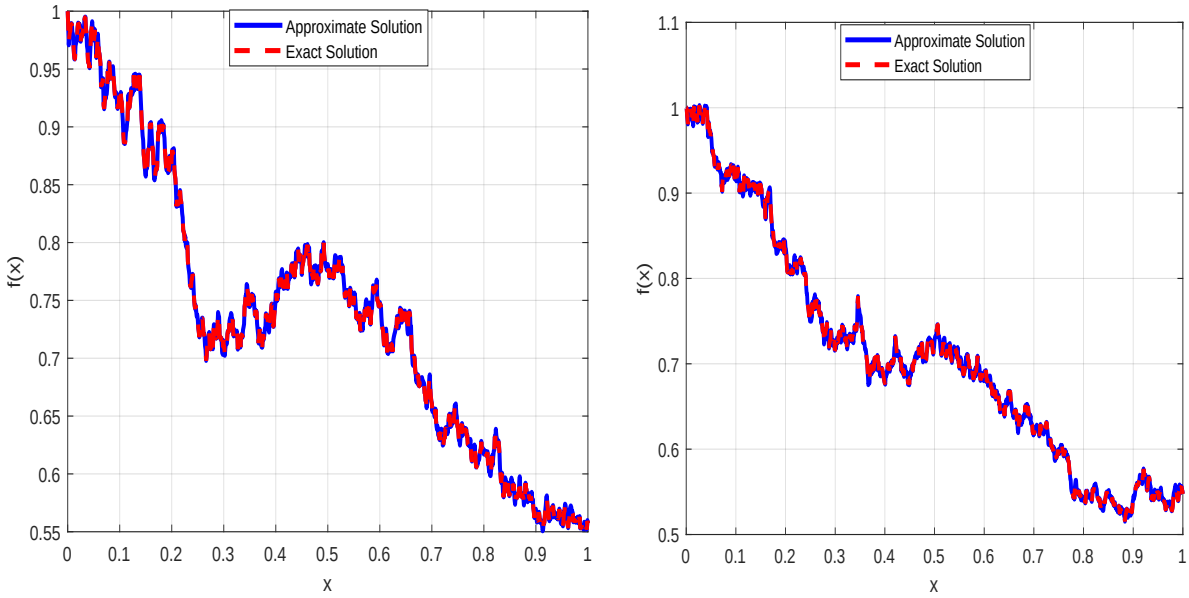

Figure 6. Example 3: Comparison of exact and approximate solutions (**left**) at $N = 1000$ and (**right**) at $N = 1200$ collocation points.

Table 3. Example 3: Error between the present method and the exact solution for $N = 500$ collocation points.

x_i	Exact solution	Present method	Absolute error
0.0	1.000000	1.000000	0.000000
0.1	0.844178	0.843029	0.001149
0.2	0.758599	0.757349	0.001250
0.3	0.788945	0.787936	0.001009
0.4	0.704230	0.703617	0.000613
0.5	0.820997	0.821043	0.000046
0.6	0.801945	0.802320	0.000375
0.7	0.756991	0.757379	0.000388
0.8	0.723725	0.723902	0.000177
0.9	0.721783	0.722562	0.000779
1.0	0.719056	0.720009	0.000953

6. Conclusions

In this work, we developed a novel numerical method for solving NSIVIEs efficiently. This method combines Monte Carlo simulations with the Legendre spectral collocation method. The proposed approach transforms complex integral equations into a more computationally feasible system of algebraic equations by considering the characteristics of orthogonal polynomials. We show through extensive error analysis that the approach achieves excellent precision, especially when it comes to capturing the complex dynamics of systems that are affected by stochastic processes and memory effects. The

numerical examples further validate the effectiveness of the proposed method, showing that our scheme performs well when compared to existing techniques, especially in terms of accuracy and efficiency. The proposed Legendre spectral-Monte Carlo method demonstrates significant accuracy and efficiency for the given NSIVIEs, however there are certain limitations; for example, the spectral accuracy of the Legendre collocation component depends on how smooth the exact solution is. If the solution is not very regular, the convergence rates may be lower. While the Monte Carlo simulation is an excellent method for estimating statistical information, its computational cost may increase for large sample sizes. Our future work is to extend this technique to integro-differential systems in heterogeneous media and to nonlinear stochastic partial differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

Author contributions

Ishtiaq Ali: Methodology, conceptualization, validation writing-original draft preparation; resources; Saeed Islam: Methodology, investigation, formal analysis, visualization, reviewing, editing. All authors have read and approved the final version of the manuscript for publication.

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