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Research article

Weighted Morrey space boundedness for Hörmander-type singular integrals

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Abstract: We investigate the boundedness properties of singular integral operators characterized by L^r-Hörmander kernel conditions (for $1 < r < \infty$) within the framework of weighted Morrey spaces. Additionally, the analysis is extended to commutators generated by these operators and functions in the Bounded Mean Oscillation (BMO) classes, establishing corresponding norm estimates under comparable geometric and weight hypotheses.

Keywords: weighted Morrey space; singular integral operator; L^r Hörmander condition; commutator

1. Introduction and main results

This paper examines the Calderón–Zygmund singular integral operator, a foundational object in harmonic analysis and partial differential equations. It is defined by integration against a kernel that exhibits a singularity. Such kernels are classically required to satisfy certain smoothness and decay properties, which underpin the boundedness and regularity-preserving behavior of the operator in function space theory. Formally, a classical Calderón–Zygmund singular integral operator T is defined via a principal value integral.

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)\,dy,$$

where *p.v.* denotes the principal value integral and the kernel K is the classical Calderón-Zygmund kernel which adheres to the following canonical Calderón-Zygmund-type hypotheses:

(a) For all ε and N with $0 < \varepsilon < N < \infty$,

$$\int_{\varepsilon<|x|<N} K(x)\,dx = 0, \tag{1.1}$$

(b) There exists a constant C such that

$$|K(x)| \leq \frac{C}{|x|^n}, \tag{1.2}$$

(c) There exists a constant C such that

$$|K(x-y) - K(x)| \leq \frac{C|y|}{|x|^{n+1}}, \quad |x| > 2|y|. \tag{1.3}$$

Let $1 < r < \infty$, $1/r + 1/r' = 1$, a function $K \in L^1_{Loc}\{\mathbb{R}^n \setminus \{0\}\}$ is called L^r- Hörmander type Calderón-Zygmund kernel if K satisfies Eqs (1.1) and (1.2)

$$\sup_{0<|y|<h} \sum_{k=1}^{\infty} \left(2^k h\right)^{\frac{n}{r'}} \left(\int_{2^k h \leq |x| \leq 2^{k+1} h} |K(x-y) - K(x)|^r dx\right)^{\frac{1}{r}} < \infty. \tag{1.4}$$

Note that if $r = 1$, Eq (1.4) agrees with the standard Hörmander condition

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx < \infty. \tag{1.5}$$

When $r = \infty$, Eq (1.4) can be understood as

$$\sup_{0<|y|<h} \sum_{k=1}^{\infty} (2^k h)^n \sup_{2^k h \leq |x| \leq 2^{k+1} h} |K(x-y) - K(x)| < \infty.$$

In this paper, we denote by H^r the class of kernels that satisfy the L^r -Hörmander condition and $H^{\infty,*}$ the class of kernels that fulfill the classical Lipschitz condition Eq (1.3). It is evident that these classes are nested. That is,

$$H^{\infty,*} \subset H^\infty \subset H^s \subset H^r \subset H^1, \quad 1 < r < s < \infty. \tag{1.6}$$

The foundational theory of Calderón-Zygmund singular integral operators with L^1-Hörmander-type kernels traces back to the work of García-Cuerva and Rubio de Francia [1]. For homogeneous kernels K, the standard condition Eq (1.5) is strengthened to an L^1-Dini requirement, as detailed in Lu, Ding, and Yang et al. [2]. Subsequent advances by Kurtz and Wheeden [3] established weighted norm inequalities on $\mathbb{R}^n$ for homogeneous singular integrals satisfying L^r-Dini conditions ($1 < r < \infty$). A pivotal refinement by Watson [4] demonstrated that the L^r-Dini smoothness hypothesis could be relaxed, thereby expanding the class of admissible operators through the substitution of the L^r-Hörmander condition Eq (1.4)—A generalization mirroring the classical replacement of Dini constraints by Hörmander-type criteria Eq (1.5) in Calderón-Zygmund theory.

Watson's methodology, inspired by techniques for Hilbert transforms along curves, leveraged Fourier transform estimates from Duoandikoetxea, Rubio, and Wang [5], Fan [6], and Nagel, Ricci, and Wainger et al. [7]. Later, Lee, Lin, Lin, and Yan et al. [8] extended Watson's Theorem 2 [4] to variable kernel settings. Their work further derived weighted estimates on Hardy spaces by imposing supplementary Dini-type regularity on K, as articulated in Theorems 4 and 5 of [8].

In the fractional case, there is fractional L^r-Hörmander condition which can be stated as

$$\sup_{0<|y|<h} \sum_{k=1}^{\infty} \left(2^k h\right)^{\frac{n-\alpha}{r'}} \left(\int_{2^k h\leq|x|\leq 2^{k+1}h} |K_\alpha(x-y) - K_\alpha(x)|^r dx\right)^{\frac{1}{r}} < \infty, \tag{1.7}$$

where $0 < \alpha < n$, $1 < r < \infty$. We will write $H^{r,\alpha}$ for the class of kernels satisfying the fractional L^r-Hörmander condition. When $r = \infty$, $H^{\infty,\alpha}$ can be understood as

$$\sup_{0<|y|<h} \sum_{k=1}^{\infty} (2^k h)^{n-\alpha} \sup_{2^k h\leq|x|\leq 2^{k+1}h} |K_\alpha(x-y) - K_\alpha(x)| < \infty.$$

The kernel K_α is deemed to satisfy the $H^{\infty,\alpha,*}$ regularity if there exist constants $c \geq 1$, $C > 0$ ensuring that

$$|K_\alpha(x-y) - K_\alpha(x)| \leq \frac{C|y|}{|x|^{n+1-\alpha}}, |x| > c|y|.$$

It is easy to see that [9]

$$H^{\infty,\alpha,*} \subset H^{\infty,\alpha} \subset H^{r,\alpha}, 0 < \alpha < n, 1 < r < \infty.$$

It is noteworthy that, by employing the Mean Value Theorem, we can demonstrate that the kernel of the fractional integral I_α, represented as $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$, belongs to the space $H^{\infty,\alpha,*}$. For further information regarding the fractional L^r -Hörmander condition, please refer to [9–11].

Morrey space was first introduced by Morrey [12] in his foundational work on characterizing the local regularity of solutions to second-order elliptic partial differential equations (PDEs). He established that essential analytical properties of PDE solutions—such as Hölder continuity and blow-up behavior—are intrinsically linked to the boundedness of singular integral operators acting on these spaces. This discovery catalyzed sustained interest in norm estimates for operators on Morrey-type function spaces, as evidenced by subsequent developments in [13–16], where researchers have systematically employed methodologies structured around the following axiomatic framework:

$$M_{p,q}(\mathbb{R}^n) = \left\{ f : \|f\|_{M_{p,q}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{p}{q}}} \int_B |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

where $f \in L^p_{loc}(\mathbb{R}^n)$ and $1 \leq p \leq q < \infty$. Here and in what follows, we denote by B any balls in $\mathbb{R}^n$, $B(x,R)$ the ball centered at $x \in \mathbb{R}^n$ with radius $R > 0$ and $\mu B(x,R) = B(x,\mu R)$ with $\mu > 0$. $M_{p,q}(\mathbb{R}^n)$ is an expansion of $L^p(\mathbb{R}^n)$ in the sense that $M_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

The validity of various operators persists when the Lebesgue measure dx is substituted by a weighted measure $w(x)dx$. A significant body of research has focused on weighted inequalities involving weights $w(x)$ within the Muckenhoupt classes. For further foundational insights, we direct readers to the seminal works [1] and [17]. The Muckenhoupt classes A_p and $A_{(p,q)}$, introduced in [18], are defined as collections of non-negative locally integrable functions w satisfying precise integral conditions that ensure the boundedness of key operators in harmonic analysis.

$$A_p : \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} \leq C, 1 < p < \infty$$

and

$$A_{(p,q)}:\sup_{B}\left(\frac{1}{|B|}\int_{B}w(x)^{q}\,dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_{B}w(x)^{-p'}\,dx\right)^{\frac{1}{p'}}\leq C,1<p,q<\infty,$$

respectively, where $1/p+1/p'=1$. Komori and Shirai pioneered the development of weighted Morrey space theory in their seminal work [19], offering a natural framework that extends the classical weighted Lebesgue spaces. Furthermore, they established fundamental boundedness results for central operators in harmonic analysis within these refined function spaces, most notably for the Hardy–Littlewood maximal operator

$$Mf(x)=\sup_{B\ni x}\frac{1}{|B|}\int_{B}|f(y)|dy,$$

the Calderón-Zygmund singular integral operator T and the fractional integral which is defined by

$$I_{\alpha}f(x)=\int_{\mathbb{R}^{n}}\frac{f(y)}{|y-x|^{n-\alpha}}dy,\quad 0<\alpha<n.$$

Let $f\in L^{p}_{loc}(\mathbb{R}^{n})$, $1\leq p<\infty$, $0<\lambda<1$ and w be functions, then the weighted Morrey space $M_{p,\lambda}(w)$ in [19] is defined by

$$M_{p,\lambda}(w)=\left\{f:\|f\|_{M_{p,\lambda}(w)}=\sup_{B}\left(\frac{1}{w(B)^{\lambda}}\int_{B}|f(x)|^{p}w(x)\,dx\right)^{\frac{1}{p}}<\infty\right\},$$

where $w(B)=\int_{B}w(x)dx$. It is obvious that if $w=1,\lambda=1-\frac{p}{q}$, then $M_{p,\lambda}(w)=M_{p,q}(\mathbb{R}^{n})$. For $w\in A_{p}(1\leq p<\infty)$, if $\lambda=0$, then $M_{p,0}(w)=L^{p}(w)$ and if $\lambda=1$, $M_{p,1}(w)=L^{\infty}(w)$.

The corresponding Morrey space related to the boundedness for I_{α} is the weighted Morrey space $M_{p,\lambda}(w_1,w_2)$ with two weights which is also introduced by Komori and Shirai in [19]. Let $1\leq p<\infty$, $0<\lambda<1$. For two weights w_1 and w_2,

$$M_{p,\lambda}(w_1,w_2)=\left\{f:\|f\|_{M_{p,\lambda}(w_1,w_2)}=\sup_{B}\left(\frac{1}{w_2(B)^{\lambda}}\int_{B}|f(x)|^{p}w_1(x)\,dx\right)^{\frac{1}{p}}<\infty\right\}.$$

If $w_1=w_2=w$, we denote by $M_{p,\lambda}(w_1,w_1)=M_{p,\lambda}(w_2,w_2)=M_{p,\lambda}(w)$.

Over the past decade, the theory of weighted Morrey spaces has undergone substantial diversification, [20–23] discuss the properties and applications of Morrey spaces, while [24–26] establish the boundedness of some singular integral operators on Morrey spaces. Building on this foundation, the present work seeks to extend the foundational results of Komori and Shirai [19] by establishing operator estimates under refined L^{r}-Hörmander-type conditions. Our principal contributions are organized as follows:

Theorem 1.1. *Let $0<\lambda<1$, $1<r<\infty$, $1/r+1/r'=1$, $r'\leq p<\infty$ and $K\in H^{r}$. Then T is bounded on $M_{p,\lambda}(w)$ with $w\in A_{\frac{p}{r'}}$.*

Denote by

$$T_{\alpha}f(x)=\int_{\mathbb{R}^{n}}K_{\alpha}(x-y)f(y)dy,$$

where K_α satisfy Eq (1.7), $0<\alpha<n$ and

$$|K_\alpha(x)| \le \frac{1}{|x|^{n-\alpha}}. \tag{1.8}$$

It is obvious that when $|K_\alpha(x)| = \frac{1}{|x|^{n-\alpha}}$, T_α agrees with the fractional integral I_α. For the fractional case T_α, we have

Theorem 1.2. *Let* $0<\lambda<\frac{p}{q}$, $0<\alpha<n$, $1<r<\infty$, $1/r+1/r'=1$, $r'\le p<n/\alpha$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. *Then* T_α *is bounded from* $M_{p,\lambda}(w^p,w^q)$ *to* $M_{q,\frac{q\lambda}{p}}(w^q)$ *with* $w^{r'}\in A_{(\frac{p}{r'},\frac{q}{r'})}$.

For any ball $B\in R^n$,$BMO(R^n)$ is defined to be the set of all locally integrable functions f on R^n such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(y)-f_B|\,dy<\infty,$$

where the supreme is taken over all balls $B\subset\mathbb{R}^n$ and $b_B=\frac{1}{|B|}\int_B b(y)dy$. An early work about $BMO(\mathbb{R}^n)$ space can attribute to John and Nirenberg [27]. Given an operator T acting on a generic function f and a function b, the commutator T_b is formally defined as

$$T_bf=[b,T]f=bT(f)-T(bf).$$

Given the strict inclusion $L^\infty(\mathbb{R}^n)\subsetneq BMO(\mathbb{R}^n)$, the commutator operator T_b exhibits weaker boundedness properties compared to its non-commutator counterpart T particularly in contexts involving singular integral behavior (see, e.g., [28] for quantitative manifestations of this phenomenon). This discrepancy has motivated sustained inquiry into whether T_b retains boundedness properties analogous to those of T? A substantial body of literature addresses commutators of operators with BMO-class functions over Lebesgue spaces, tracing its origins to the seminal work of Coifman, Rochberg, and Weiss et al. [29], who introduced these objects in their analysis of factorization theorems for generalized Hardy spaces. Subsequent investigations have bifurcated into two principal directions: (i) the use of commutators to characterize function spaces, as exemplified by [30–33]; and (ii) applications of commutator theory to regularity and solvability problems in partial differential equations, as developed in [34–36].

We will extend the boundedness of T and T_α to T_b and $T_{\alpha,b}$ on the weighted Morrey spaces, respectively.

Theorem 1.3. *Let* r,p,λ,K *and* w *be as in Theorem* 1.1 *and* $b\in BMO(\mathbb{R}^n)$. *Then* T_b *is bounded on* $M_{p,\lambda}(w)$.

Theorem 1.4. *Let* $p,r,q,\lambda,\alpha,w,K_\alpha$ *be as in Theorem* 1.2 *and* $b\in BMO(\mathbb{R}^n)$. *Then* $T_{\alpha,b}$ *is bounded from* $M_{p,\lambda}(w^p,w^q)$ *to* $M_{q,\frac{q\lambda}{p}}(w^q)$.

In the foregoing and following, the letter C stands for a positive constant which may change from line to line. For $a,b\in\mathbb{R}$, $a\lesssim b$ (resp. $a\gtrsim b$) means $a\leqslant Cb$ (resp. $a\geqslant Cb$) and $a\approx b$ equals $a\lesssim b\lesssim a$. $|\cdot|$ means the Lebesgue measure and $\omega^q(B)=\int_B \omega^q(x)\,dx$.

This paper demonstrates broad application potential in harmonic analysis and the theory of partial differential equations. Specifically, Theorems 1.1 and 1.2 establish boundedness properties in weighted Morrey spaces, while Theorems 1.3 and 1.4 extend the boundedness theory for commutators in these spaces. These results provide new tools for studying the regularity of solutions to elliptic and parabolic partial differential equations in inhomogeneous media, offering distinct advantages when addressing equations with non-smooth coefficients or complex geometric structures. In image processing and signal analysis, this theoretical framework supports the design of adaptive filters based on singular integral operators, thereby enhancing the stability of edge detection and texture analysis algorithms. The classes of weights A_p and $A_{(p,q)}$ mentioned in this paper have become central to the study of weighted norm inequalities in modern analysis and have found applications in several branches of Analysis, from Complex function theory to PDEs [1]. Future research may develop in several directions, such as extending the framework to variable exponent weighted Morrey spaces to address nonlinear scaling problems, and investigating the boundedness of higher-order commutators and multilinear operators.

2. Proofs of the main results

The proofs of Theorems 1.1 and 1.2 rely greatly on specific properties of A_p weights, which are extensively discussed in the literature on weighted boundedness for operators in harmonic analysis, including references such as [17]. To facilitate the reader's understanding, we summarize some relevant properties of the A_p weights without providing proofs, thereby ensuring that our exposition remains self-contained.

Lemma 2.1. *Let $1 \leq p < \infty$ and $w \in A_p$. Then $\omega > 0$ almost everywhere and the following statements are true*

(a) *[37, Lemma 4.1.3] There exists a constant C such that*

$$w(2B) \leq Cw(B). \tag{2.1}$$

Condition (a) *is referred to as the doubling condition.*

(b) *[37, Theorem 4.2.6] There exists a constant $C > 1$ such that*

$$w(2B) \geq Cw(B). \tag{2.2}$$

Condition (b) *is referred to as the reverse doubling condition.*

(c) *[1, IV. Lemma 2.2] For all $\mu > 1$, we have*

$$w(\mu B) \leq C\mu^{np} w(B). \tag{2.3}$$

(d) *[1, IV. Theorem 2.9] There exist two constants C and $\delta > 0$ such that for any measurable set $Q \subset B$*

$$\frac{w(Q)}{w(B)} \leq C\left(\frac{|Q|}{|B|}\right)^{\delta}. \tag{2.4}$$

If w satisfies Eq (2.4)*, we say $w \in A_\infty$.*

(e) *[1, IV. Corollary 2.13] For all $p < q < \infty$, we have*

$$A_\infty = \cup_p A_p, \quad A_p \subset A_q. \tag{2.5}$$

The next two lemmas give the boundedness of T and T_α on weighted Lebesgue spaces, which are needed in the proof of our main results.

Lemma 2.2. *[8, Theorem 1] Let* $1 < r < \infty$*,* $K \in H^r$*,* $r' \le p < \infty$ *and* $w \in A_{\frac{p}{r'}}$*. Then* T *is bounded on* $L^p(w)$*.*

Lemma 2.3. *[11, Theorem 2.3] Let* $0 < \alpha < n$*,* $r' \le p < \frac{n}{\alpha}$*,* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*. Then there exists* $C > 0$ *such that*

$$\|T_\alpha f\|_{L^q(w^q)} \le C\|f\|_{L^p(w^p)}.$$

Proof of Theorem 1.1. Let $w \in A_{p/r'}$ and $0 < \lambda < 1$. By Eq (2.5), we have $w \in A_p$. For a fixed ball $B = B(x_0, R)$. We decompose

$$f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{w(B)^\lambda}\int_B |Tf(x)|^p w(x)dx \\
&\le \frac{C}{w(B)^\lambda}\int_B |Tf_1(x)|^p w(x)dx + \frac{C}{w(B)^\lambda}\int_B |Tf_2(x)|^p w(x)dx \\
&:= I + II.
\end{aligned}$$

Using Lemma 2.2, it is easy to obtain that

$$I \le \frac{C}{w(B)^\lambda}\int_{\mathbb{R}^n} |Tf_1(x)|^p w(x)dx \le \frac{C}{w(B)^\lambda}\int_{2B} |f(x)|^p w(x)dx \le C\|f\|^p_{M_{p,\lambda}(w)}.$$

For the term II, after observing for $x \in B$ and $y \in (2B)^c$, $|x_0 - y| < C|x - y|$, we get

$$\begin{aligned}
|Tf_2(x)| &\le C\int_{|x_0-y|>2R} \frac{|f(y)|}{|x_0 - y|^n}dy \\
&= C\sum_{k=1}^{\infty}\int_{2^kR<|x_0-y|<2^{k+1}R} \frac{|f(y)|}{|x_0 - y|^n}dy \\
&\le C\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B} |f(y)|dy.
\end{aligned}$$

It follows from Hölder's inequality and the definition of A_p that

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B} |f(y)|dy &\le \sum_{k=1}^{\infty}\frac{1}{|2^kB|}\left(\int_{2^{k+1}B} |f(y)|^p w(y)\,dy\right)^{\frac{1}{p}}\left(\int_{2^{k+1}B} w(y)^{\frac{-p'}{p}}\,dy\right)^{\frac{1}{p'}} \\
&\le C\|f\|_{M_{p,\lambda}(w)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B|}{|2^kB|w(2^{k+1}B)^{\frac{(1-\lambda)}{p}}}.
\end{aligned}$$

Thus,

$$II \le C\|f\|^p_{M_{p,\lambda}(w)}\left(\sum_{k=1}^{\infty}\frac{w(B)^{\frac{(1-\lambda)}{p}}}{w(2^{k+1}B)^{\frac{(1-\lambda)}{p}}}\right)^p \le C\|f\|^p_{M_{p,\lambda}(w)},$$

we have used Eq (2.2) in the last inequality, which gives the desired result with a constant $C = C(n, p, \lambda)$.We have thus completed the proof of Theorem 1.1.

Proof of Theorem 1.2. We can use the similar arguments as in the proof of Theorem 1.1. It suffices to show that

$$\frac{1}{(w^q(B))^{\frac{q\lambda}{p}}}\int_B |T_\alpha f(x)|^q w(x)^q dx \le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}.$$

For a fixed ball $B = B(x_0, R)$, we decompose $f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2$. Since T_α is a linear operator, we get

$$\begin{aligned}
&\frac{1}{(w^q(B))^{\frac{q\lambda}{p}}}\int_B |T_\alpha f(x)|^q w(x)^q dx\\
&\le \frac{C}{(w^q(B))^{\frac{q\lambda}{p}}}\int_B (|T_\alpha f_1(x)|^q + |T_\alpha f_2(x)|^q) w^q(x) dx\\
&:= J + JJ.
\end{aligned}$$

To estimate the term J, Lemma 2.3 shows that

$$\int_B |T_\alpha f_1(x)|^q w^q(x)\,dx \le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}(w^q(B))^{\frac{q\lambda}{p}},$$

which inturn implies that

$$J \le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}.$$

For the term JJ, by Hölder's inequality and the definition of $A_{(p,q)}$

$$\begin{aligned}
JJ &\le C\sum_k\left(\int_{2^kR<|x_0-y|<2^{k+1}R}\frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy\right)^q (w^q(B))^{1-\frac{q\lambda}{p}}\\
&\le C\sum_k\left(2^{-k(n-\alpha)}\int_{2^{k+1}B}|f(y)|dy\right)^q(\omega^q(B))^{1-\frac{q\lambda}{p}}\\
&\le C\sum_k\left(2^{-k(n-\alpha)}\left(\int_{2^{k+1}B}|f(y)|^p\omega(y)^p\,dy\right)^{\frac{1}{p}}\left(\int_{2^{k+1}B}\omega(y)^{-p'}\,dy\right)^{\frac{1}{p'}}\right)^q(\omega^q(B))^{1-\frac{q\lambda}{p}}\\
&\le C\sum_k\left(2^{-k(n-\alpha)}\|f\|_{M_{p,\lambda}(w^p,w^q)}|2^{k+1}B|^{1-\frac{\alpha}{n}}\frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}-\frac{\lambda}{p}}}\right)^q(w^q(B))^{1-\frac{q\lambda}{p}}\\
&\le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}\left(\sum_{k=1}^{\infty}\frac{w^q(B)^{\frac{1}{q}-\frac{\lambda}{p}}}{w^q(2^{k+1}B)^{\frac{1}{q}-\frac{\lambda}{p}}}\right)^q\\
&\le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}.
\end{aligned}$$

We have used Eq (2.2) in the last inequality. This yields the desired result, with a constant $C = C(n, p, q, \lambda)$.

The following lemmas about $BMO(\mathbb{R}^n)$ functions will help us to prove Theorems 1.3 and 1.4.

Lemma 2.4. *Let $1 \le p < \infty$, $b \in BMO(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$, the following statements are true*

(a) [1, Theorem II.3.8] There exist constants C_1, C_2 such that for all $\alpha > 0$

$$|\{x \in B : |b(x) - b_B| > \alpha\}| \le C_1 |B| e^{-\frac{C_2 \alpha}{\|b\|_{BMO(R^n)}}}. \tag{2.6}$$

Inequality (2.6) is called the John-Nirenberg inequality, which has a weighted version:

(b) There exist constants C_1, C_2 such that for all $\alpha > 0$

$$\omega(\{x \in B : |b(x) - b_B > \alpha|\}) \le C_1 \omega(B) e^{-\frac{C_2 \alpha}{\|b\|_{BMO(R^n)}}}. \tag{2.7}$$

(c) The notation $2^\mu B$ denotes the ball concentric with the ball B and a radius 2^μ times that of B, where μ is a positive integer.

$$|b_{2^\mu B} - b_B| \le 2^n \mu \|b\|_{BMO(\mathbb{R}^n)}. \tag{2.8}$$

Lemma 2.5. [17, Proposition 7.1.2](see also [18, Theorem 5]) Let $w \in A_\infty$ and $1 < p < \infty$. Then the following statements are true.

(a) $\|b\|_{BMO(\mathbb{R}^n)} \approx \sup_B \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p dx\right)^{\frac{1}{p}}$;

(b) $\|b\|_{BMO(\mathbb{R}^n)} \approx \sup_B \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |b(x) - a| dx$;

(c) $\|b\|_{BMO(w)} \approx \sup_B \left(\frac{1}{w(B)} \int_B |b(x) - b_{b,w}|^p w(x) dx\right)^{\frac{1}{p}}$ where $BMO(w) = \{b : \|b\|_{BMO(w)} < \infty\}$ and $b_{B,w} = \frac{1}{w(B)} \int_B b(y) w(y) dy$.

Lemma 2.6. *Let $b \in BMO(\mathbb{R}^n)$, $B = B(x_0, R)$ be a generic fixed ball, $0 < \lambda < 1$, $1 < p < \infty$, $b_{B,\omega} = \frac{1}{\omega(B)} \int_B b(z) \omega(z) dz$ and $w \in A_p$. Then the inequality*

$$\left(\int_{|x_0 - y| > 2R} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy\right)^p w(B)^{1-\lambda} \le C \|f\|^p_{M_{p,\lambda}(w)} \tag{2.9}$$

holds for every $y \in (2B)^c =: \mathbb{R}^n \setminus (2B)$.

Proof. Using Hölder's inequality to the left-hand-side of Eq (2.9), we have

$$\begin{aligned}
&\left(\int_{|x_0 - y| > 2R} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy\right)^p w(B)^{1-\lambda} \\
&\le \left(\sum_{j=1}^{\infty} \int_{2^j R < |x_0 - y| < 2^{j+1} R} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy\right)^p w(B)^{1-\lambda} \\
&\le \left(\sum_{j=1}^{\infty} \frac{1}{|2^j B|} \int_{2^{j+1} B} |f(y)| |b_{B,w} - b(y)| dy\right)^p w(B)^{1-\lambda} \\
&\le C \left[\sum_{j=1}^{\infty} \frac{1}{|2^j B|} \left(\int_{2^{j+1} B} |f(y)|^p w(y) dy\right)^{\frac{1}{p}} \left(\int_{2^{j+1} B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} dy\right)^{\frac{1}{p'}}\right]^p w(B)^{1-\lambda} \\
&\le C \|f\|^p_{M_{p,\lambda}(w)} \left[\sum_{j=1}^{\infty} \frac{w\left(2^{j+1} B\right)^{\frac{\lambda}{p}}}{|2^j B|} \left(\int_{2^{j+1} B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} dy\right)^{\frac{1}{p'}}\right]^p w(B)^{1-\lambda}.
\end{aligned}$$

For the simplicity of analysis, we denote

$$A = \left(\int_{2^{j+1}B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'}\, dy\right)^{\frac{1}{p'}}.$$

By an elementary estimate, we have

$$\begin{aligned} A &\le \left(\int_{2^{j+1}B} \left(|b_{2^{j+1}B,w^{1-p'}} - b(y)| + |b_{2^{j+1}B,w^{1-p'}} - b_{B,w}|\right)^{p'} w(y)^{1-p'}\, dy\right)^{\frac{1}{p'}} \\ &\le \left(\int_{2^{j+1}B} |b_{2^{j+1}B,w^{1-p'}} - b(y)|^{p'} w(y)^{1-p'}\, dy\right)^{\frac{1}{p'}} + |b_{2^{j+1}B,w^{1-p'}} - b_{B,w}| w^{1-p'}\left(2^{j+1}B\right)^{\frac{1}{p'}} \\ &=: A_1 + A_2. \end{aligned}$$

For the term A_1, Lemma 2.5 gives

$$A_1 \le C\|b\|_{BMO(w^{1-p'})} w^{1-p'}\left(2^{j+1}B\right)^{\frac{1}{p'}} \le C w^{1-p'}\left(2^{j+1}B\right)^{\frac{1}{p'}}. \tag{2.10}$$

To deal with A_2, we first observe by Eq (2.8) that

$$\begin{aligned} &|b_{2^{j+1}B,w^{1-p'}} - b_{B,w}| \\ &\le |b_{2^{j+1}B,w^{1-p'}} - b_{2^{j+1}B}| + |b_{2^{j+1}B} - b_B| + |b_B - b_{B,w}| \\ &\le \frac{1}{w^{1-p'}(2^{j+1}B)} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| w(y)^{1-p'} dy + 2^n(j+1)\|b\|_{BMO(\mathbb{R}^n)} \\ &\qquad + \frac{1}{w(B)} \int_B |b(y) - b_B| w(y) dy \\ &:= A_{21} + A_{22} + A_{23}. \end{aligned}$$

Combining Eq (2.3) with Eq (2.6), we deduce that

$$\begin{aligned} A_{23} &= \frac{1}{w(B)} \int_0^\infty w(\{x \in B : |b(y) - b_B| > \alpha\}) d\alpha \\ &\le C \int_0^\infty e^{-\frac{C\alpha\delta}{\|b\|_{BMO(R^n)}}} d\alpha \\ &\le C. \end{aligned}$$

Similar arguments apply to the term A_{21}, we see that

$$A_{21} \le C.$$

It follows immediately that

$$A_2 \le C\left(2^n(j+1) + 2\right) w^{1-p'}\left(2^{j+1}B\right)^{\frac{1}{p'}}. \tag{2.11}$$

We conclude from Eqs (2.10) and (2.11) that

$$A \le C(j+1) w^{1-p'}\left(2^{j+1}B\right)^{\frac{1}{p'}}.$$

Hence, the proof of Eq (2.9) is concluded from Eq (2.2) and the following observation

$$\left[\sum_{j=1}^{\infty}\frac{w\left(2^{j+1}B\right)^{\frac{\lambda}{p}}}{|2^jB|}\left(\int_{2^{j+1}B}|b\left(y\right)-b_{B,w}|^{p'}w\left(y\right)^{1-p'}dy\right)^{\frac{1}{p'}}\right]^p w\left(B\right)^{1-\lambda}$$
$$\leq C\left[\sum_{j=1}^{\infty}\frac{\left(j+1\right)w\left(B\right)^{\frac{(1-\lambda)}{p}}}{w\left(2^{j+1}B\right)^{\frac{(1-\lambda)}{p}}}\right]^p = C\left(n,p,\lambda\right)=C.$$

Having disposed of these steps, we can now return to the proof of our main results as follows.

Proof of Theorem 1.3. The task is now to find a constant C such that for fixed ball $B = B(x_0, R)$, we can obtain

$$\frac{1}{w(B)^\lambda}\int_B |T_bf(x)|^p\, w(x)dx \leq C\|f\|^p_{M_{p,\lambda}(w)}. \tag{2.12}$$

We decompose $f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2$, and consider the corresponding splitting

$$\int_B |T_bf(x)|^p\, w(x)dx \leq C\left(\int_B |T_bf_1(x)|^p w(x)dx + \int_B |T_bf_2(x)|^p w(x)dx\right)$$
$$=: K + KK.$$

Since T is a linear operator, by the well-known result that the weighted L^p boundedness of T_b can be attribute to the weighted L^p boundedness of T ([2, Theorem 2.4.3]), we deduce that T_b is bounded on $L^p(w)$ with $w \in A_{\frac{p}{r'}}$. Therefore

$$K \leq C\int_{2B}|f(x)|^p w(x)dx \leq C\|f\|^p_{M_{p,\lambda}(w)}w(B)^\lambda.$$

Then a further use of Eq (1.2) derives that

$$|T_bf_2(x)|^p \leq C\left(\int_{\mathbb{R}^n}\frac{|f_2(y)||b(x)-b(y)|}{|x-y|^n}dy\right)^p$$
$$\leq C\left(\int_{|x_0-y|>2R}\frac{|f(y)|}{|x_0-y|^n}\{|b(x)-b_{B,w}|+|b_{B,w}-b(y)|\}dy\right)^p.$$

Thus, we have

$$KK \leq C\left(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy\right)^p\int_B |b(x)-b_{B,w}|^p w(x)dx$$
$$+\, C\left(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}|b(y)-b_{B,w}|dy\right)^p w(B)$$
$$:= KK_1 + KK_2.$$

Using Lemma 2.6, we have

$$KK_2 \leq C\|f\|^p_{M_{p,\lambda}(w)}w(B)^\lambda.$$

KK_1 can be estimated taking into account Eq (2.1), Lemma 2.5, Hölder's inequality, and the definition of A_p. In fact,

$$\begin{aligned}
KK_1 &= \left(\sum_{j=1}^{\infty}\int_{2^jR<|x_0-y|<2^{j+1}R}\frac{|f(y)|}{|x_0-y|^n}dy\right)^p\int_B|b(x)-b_{B,w}|^p w(x)dx\\
&\le \left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^{j+1}B}|f(y)|dy\right)^p\int_B|b(x)-b_{B,w}|^p w(x)dx\\
&\le \left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}\left(\int_{2^{j+1}B}w(y)^{-\frac{p'}{p}}dy\right)^{\frac{1}{p'}}\right)^p\int_B|b(x)-b_{B,w}|^p w(x)dx\\
&\le C\|f\|^p_{M_{p,\lambda}(w)}\left(\sum_{j=1}^{\infty}\frac{|2^{j+1}B|}{|2^jB|}w(2^{j+1}B)^{\frac{\lambda-1}{p}}\right)^p\int_B|b(x)-b_{B,w}|^p w(x)dx\\
&\le C\|f\|^p_{M_{p,\lambda}(w)}\|b\|^p_{BMO(w)}\sum_{j=1}^{\infty}\left(\frac{w(B)^{\frac{1-\lambda}{p}}}{w(2^{j+1}B)^{\frac{1-\lambda}{p}}}\right)^p w(B)^{\lambda}\\
&\le C\|f\|^p_{M_{p,\lambda}(w)}w(B)^{\lambda}.
\end{aligned}$$

We have used Eq (2.2) in the last inequality to obtain the desired result, with a constant $C = C(n,p,\lambda)$, hence

$$KK \le C\|f\|^p_{M^{p,\lambda}(w)}w(B)^{\lambda}.$$

Proof of Theorem 1.4. The proof of Theorem 1.4 is similar as that of Theorem 1.3, except using $w^{r'} \in A_{(\frac{p}{r'},\frac{q}{r'})}$, we omit its proof here.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

All authors wrote the main manuscript text and reviewed the manuscript.

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