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Research article

Stability and Hopf bifurcation analysis of Caputo time-fractional delayed reaction-diffusion models for sterile insect technology

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Abstract: Based on the theory of fractional differential equations, this paper studied a single-species contraception model with feedback control terms. By comprehensively applying the Mittag-Leffler function construction method, Laplace transform, and stability criteria for fractional-order systems, the existence, uniqueness, boundedness, and local stability of the model's equilibrium points were analyzed, and the key influencing factors of the system's steady-state behavior were identified. Furthermore, the mechanisms of Hopf bifurcation and Turing bifurcation were explored, the sufficient conditions for the existence of these two types of bifurcations were derived, and the theoretical framework for the system's dynamic behavior was improved. A numerical example was designed to verify the validity of the theoretical results, and the simulation results confirmed the correctness of the stability conclusions and bifurcation conditions. In the conclusion section, the numerical simulations were reviewed and correlated with the theory; future research directions were proposed in light of the limitations of the current study, providing references for subsequent explorations.

Keywords: pest control; fractional-order model; reaction-diffusion; Hopf bifurcation; turing instability

1. Introduction

Wildlife pest control stands as a paramount global challenge, exacerbated by the escalating resistance of pests to chemical pesticides and the unintended ecological damage caused by conventional control methods [1, 2]. While strategies like the sterile insect technique (SIT) , which involves releasing sterile individuals to suppress pest populations , offer promise for sustainable management, their efficacy is often undermined by environmental disturbances, such as climate variability and invasive species, which introduce complex temporal and spatial dynamics into population interactions [3, 4]. Mathematical modeling has emerged as a critical tool to decipher these dynamics, yet existing frameworks remain limited in capturing the memory-dependent behaviors and spatial heterogeneity inherent to real-world ecosystems.

Early contributions to SIT modeling primarily relied on integer-order differential equations to describe population dynamics. For example, Li et al. [5] formulated a feedback-controlled the system of ordinary differential equations to analyze the stability of fertile and sterile subpopulations, laying groundwork for delay-sensitive control strategies. However, these models overlook the memory effect and the influence of historical population states on current dynamics, a key feature of ecological systems governed by processes like age-structured reproduction or environmental legacy. In response, fractional-order calculus has gained traction for its ability to characterize such hereditary behaviors: Li and Maimaiti [6] recently applied Caputo fractional derivatives to model water-plant interactions, demonstrating enhanced accuracy in predicting stability under time-varying conditions. Concurrently, studies on spatial dynamics, such as Jiang et al. [7] and Ma et al. [8], have highlighted the role of diffusion in inducing Turing patterns and Hopf bifurcations in reaction-diffusion systems, yet these analyses remain detached from the specific mechanisms of SIT, such as sterile insect release rates and feedback control loops.

Despite these advancements, three critical gaps persist in the literature. First, most SIT models employ integer-order derivatives, failing to account for the nonlocal dependencies encoded in fractional calculus. This omission limits their ability to describe how past environmental changes or control interventions shape current population trajectories [7, 9]. Second, the synergistic effects of spatial diffusion and time delays—both pivotal in determining whether pest populations stabilize, oscillate, or form heterogeneous patterns—remain understudied in SIT contexts. While Anguelov et al. [2] and Kapranas et al. [10] emphasized the need for spatially explicit SIT models, the interaction between dispersal rates and delay-induced bifurcations has not been systematically quantified. Third, although Hopf bifurcations in delayed systems have been explored [10, 11], the conditions for Turing instability a driver of spatially heterogeneous pest distributions remain unaddressed in SIT frameworks. This gap hinders the development of targeted strategies for regions with patchy pest infestations.

To address these limitations, we propose a novel fractional-order delayed reaction-diffusion model that integrates SIT with feedback control, leveraging Caputo derivatives to capture memory effects and spatial diffusion terms to model population dispersal. Unlike conventional integer-order models [12–17], our framework rigorously analyzes the interplay between fractional-order dynamics, time delays, and spatial heterogeneity. Specifically, we derive conditions for local asymptotic stability, characterize Hopf bifurcation induced by time delays, and identify thresholds for Turing instability driven by differential diffusion coefficients. These analyses reveal how fractional-order derivatives expand the stability region of pest-control systems, how diffusion reduces the critical delay for Hopf bifurcation, and how specific diffusion ratios trigger spatially heterogeneous patterns—insights absent in prior SIT modeling. Moreover, by merging fractional calculus, reaction-diffusion theory, and SIT-specific control mechanisms, this study bridges a critical gap between theoretical ecology and pest management, offering a robust framework to optimize sterile insect release strategies in dynamically changing and spatially heterogeneous environments.

The remainder of this paper is structured as follows: Section 2 introduces mathematical preliminaries, including Caputo fractional derivatives and the Mittag-Leffler function, and formulates the SIT model with feedback control and spatial diffusion. Section 3 establishes the nonnegativity, boundedness, and uniqueness of solutions for the non-diffusive fractional-delay system, providing a foundation for subsequent stability analysis. In Section 4, we incorporate spatial dynamics, deriving stability criteria, Hopf bifurcation conditions, and Turing instability thresholds by linearizing the system and analyzing eigenvalue distributions. Numerical simulations in Section 5 validate the theoretical results, illustrating how fractional order, delay, and diffusion parameters influence system behavior. Finally, Section 6 discusses the ecological and practical implications of our findings, outlining future directions for integrating complex interactions like predation and environmental stochasticity into the model.

2. The model and preliminaries

In 2012, Li et al. [5] established a novel single-species population dynamics model integrating contraceptive control and feedback mechanisms, governed by the system

$$\begin{cases} x_1'(t) = x_1(t)\Big(b(t) - g(t)\big(x_1(t) + x_2(t)\big) - \mu(t) - c_1(t)u(t - \delta_1(t))\Big), \\ x_2'(t) = \mu(t)x_1(t) - \Big(d(t) + g(t)\big(x_1(t) + x_2(t)\big) + c_2(t)u(t - \delta_2(t))\Big)x_2(t), \\ u'(t) = -h(t)u(t) + f_1(t)x_1(t - \tau_1(t)) + f_2(t)x_2(t - \tau_2(t)), \end{cases} \tag{2.1}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of fertile and sterile subpopulations respectively, $u(t)$ represents the feedback control variable and $u(t)$ corresponds to the release rate of sterile insects, $b(t)$ represents the intrinsic growth rate, $g(t)$ represents the density-dependent regulation coefficient, $\mu(t)$ is the transition rate from fertile to sterile states, and $d(t)$ the mortality rate of sterile individuals. All system parameters $b(t), g(t), d(t), \mu(t), c_i(t), f_i(t), \tau_i(t), \delta_i(t)$ $(i = 1, 2)$ are nonnegative, bounded continuous functions.

In this work, we generalize this framework through two fundamental modifications: First by replacing the integer-order derivative with Caputo fractional derivatives, and second by incorporating spatial diffusion effects. The fractional-order extension of system (2.1) becomes

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1(t) = x_1(t)\Big(b - g\big(x_1(t) + x_2(t)\big) - \mu - c_1u(t)\Big), \\ {}_{t_0}^{C}D_t^{\alpha} x_2(t) = \mu x_1(t) - \Big(d + g\big(x_1(t) + x_2(t)\big) + c_2u(t)\Big)x_2(t), \\ {}_{t_0}^{C}D_t^{\alpha} u(t) = -hu(t) + f_1x_1(t - \tau) + f_2x_2(t - \tau), \end{cases} \tag{2.2}$$

where the Caputo fractional derivative operator is defined as

$${}_{t_0}^{C}D_t^{\alpha} f(t) := \frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^{t} \frac{f'(s)}{(t - s)^{\alpha}} ds.$$

To incorporate spatial dynamics, we extend system (2.2) to a reaction-diffusion framework under the following assumptions:

$$\begin{cases} \dfrac{\partial^{\alpha} x_1}{\partial t^{\alpha}} = x_1\left(b - g(x_1 + x_2) - \mu - c_1u\right) + \delta_1 \dfrac{\partial^2 x_1}{\partial x^2}, \\ \dfrac{\partial^{\alpha} x_2}{\partial t^{\alpha}} = \mu x_1 - \left(d + g(x_1 + x_2) + c_2u\right) x_2 + \delta_2 \dfrac{\partial^2 x_2}{\partial x^2}, \\ \dfrac{\partial^{\alpha} u}{\partial t^{\alpha}} = -hu + f_1x_1^{\tau} + f_2x_2^{\tau} + \delta_3 \dfrac{\partial^2 u}{\partial x^2}, \end{cases} \tag{2.3}$$

defined on the spatial domain $x \in (0, L_x)$ with $t > 0$, where $x_i^\tau := x_i(t-\tau, x)$. The system is complemented by positive initial conditions

$$\begin{cases} x_1(t,x) = \phi_1(t,x) \\ x_2(t,x) = \phi_2(t,x) \qquad (t,x) \in [-\tau, 0] \times \Omega, \\ u(t,x) = \phi_3(t,x) \end{cases} \tag{2.4}$$

and homogeneous Neumann boundary conditions

$$\frac{\partial x_1}{\partial \eta} = \frac{\partial x_2}{\partial \eta} = \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+. \tag{2.5}$$

The fractional time derivative operator in system (2.3) follows the Caputo definition:

$$\frac{\partial^\alpha f}{\partial t^\alpha}(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\partial f(s,x)}{\partial s} \frac{ds}{(t-s)^\alpha}.$$

The boundary conditions Eq (2.5) ensure mass conservation within the spatial domain Ω.

The following are some of the definitions, lemmas, and basic assumptions used in this paper.

Definition 1. ([18]). The Mittag-Leffler function is defined as

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)},$$

where $\alpha > 0, \gamma > 0, z \in \mathbb{C}$.In particular, for $\gamma = 1$,

$$E_{\alpha,1} = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where $\alpha > 0, z \in \mathbb{C}$.

Lemma 1. ([19]). Let $0 < \alpha < 1$, and suppose $\psi(t)$ is a continuously differentiable function on $[a, b]$ with its Caputo fractional derivative ${}_{t_0}^{C}D_t^\alpha \psi(t)$ continuous on $[a, b]$. If ${}_{t_0}^{C}D_t^\alpha \psi(t) \geq 0$ (or ${}_{t_0}^{C}D_t^\alpha \psi(t) \leq 0$) for all $t \in (a, b)$, then $\psi(t)$ is nondecreasing (or nonincreasing) on $[a, b]$.

Lemma 2. ([19]). For a function $f(t)$ that is n-times continuously differentiable on $[t_0, \infty)$ with values in $\mathbb{R}$, the Laplace transform of its Caputo fractional derivative of order $\alpha > 0$ is given by:

$$\mathscr{L}\left\{{}_{t_0}^{C}D_t^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(t_0),$$

where $\mathscr{L}\{f(t)\} = F(s)$.

Lemma 3. ([19]).For any positive constants α and γ, and a matrix $K \in \mathbb{C}^{n\times n}$, the Laplace transform of the function $t^{\gamma-1}E_{\alpha,\gamma}(Kt^\alpha)$ is expressed as:

$$\mathscr{L}\left\{t^{\gamma-1}E_{\alpha,\gamma}(Kt^\alpha)\right\} = \frac{s^{\alpha-\gamma}}{s^\alpha - K},$$

with $\mathrm{Re}(s) > \|K\|^{1/\alpha}$, where $\mathrm{Re}(s)$ represents the real part of the complex number s, and $E_{\alpha,\gamma}$ is the Mittag-Leffler function.

Lemma 4. ([20, 21]). Consider the following fractional-order delay differential equation

$$\begin{aligned} {}^{C}D_t^{\alpha}\mathbf{u}(t) &= B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau) \\ &\quad + \mathcal{G}(t, \mathbf{u}(t), \mathbf{u}(t-\tau)), \quad t \in [0, N], \\ \mathbf{u}(t) &= \Phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{2.6}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$ with lower limit zero. The state vector $\mathbf{u}(t) : [-\tau, N] \to \mathbb{R}^m$ evolves in an m-dimensional Euclidean space, where $N \in \mathbb{R}^+ \cup \{+\infty\}$ represents either a finite time horizon or an infinite time domain. The system matrices $B_0, B_1 \in \mathbb{R}^{m\times m}$ are constant, $\tau > 0$ specifies the constant delay, and the nonlinearity $\mathcal{G} \in C([0, N], \mathbb{R}^m)$ satisfies specified regularity conditions. The initial function $\Phi(t)$ belongs to the Banach space $C([-\tau, 0], \mathbb{R}^m)$ equipped with the sup-norm

$$\|\Phi\|_C := \sup_{\theta\in[-\tau,0]} \|\Phi(\theta)\|.$$

Let $|\cdot|_p$ denote an arbitrary Euclidean norm (with $p = 1, 2, \infty$) on $\mathbb{R}^m$, and $\|\cdot\|_p$ represent the corresponding induced matrix norm. We impose the following fundamental assumption:

(A1) The nonlinear function $\mathcal{G}$ satisfies the Lipschitz condition: there exists a positive constant $L > 0$ such that for all $t \in [0, N]$ and any pair of states $\mathbf{u}, \overline{\mathbf{u}} \in C([-\tau, N], \mathbb{R}^m)$,

$$\begin{aligned} &\|\mathcal{G}(t, \mathbf{u}(t), \mathbf{u}(t-\tau)) - \mathcal{G}(t, \overline{\mathbf{u}}(t), \overline{\mathbf{u}}(t-\tau))\|_p \\ &\leq L\left(\|\mathbf{u}(t) - \overline{\mathbf{u}}(t)\|_p + \|\mathbf{u}(t-\mu) - \overline{\mathbf{u}}(t-\tau)\|_p\right). \end{aligned}$$

Under assumption (A1), the fractional-order delay system (1) possesses a unique continuous solution on the interval $[0, N]$.

Consider the fractional-order delay differential equation [20]

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}\mathbf{u}(t) &= \mathcal{F}(\mathbf{u}(t), \mathbf{u}(t-\mu)), \quad t \geq t_0, \\ \mathbf{u}(t) &= \Phi(t), \quad t \in [t_0 - \mu, t_0], \end{aligned} \tag{2.7}$$

where $\mu > 0$ denotes the delay parameter, $\alpha \in (0, 1)$ is the fractional order, and $\mathbf{u} \in \mathbb{R}^n$ represents the state vector. The operator ${}_{t_0}^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative with initial time t_0.

The associated linearized system at an equilibrium point $\mathbf{u}^*$ is given by

$${}_{t_0}^{C}D_t^{\alpha}\widetilde{\mathbf{u}}(t) = A\widetilde{\mathbf{u}}(t) + B\widetilde{\mathbf{u}}(t-\mu), \tag{2.8}$$

where the Jacobian matrices are defined as

$$A = \left.\frac{\partial\mathcal{F}}{\partial\mathbf{u}(t)}\right|_{\mathbf{u}^*}, \quad B = \left.\frac{\partial\mathcal{F}}{\partial\mathbf{u}(t-\mu)}\right|_{\mathbf{u}^*}.$$

The characteristic matrix associated with equilibrium $\mathbf{u}^*$ is formulated as

$$\Delta_{\mathbf{u}^*}(s) = s^{\alpha}I - \left(A + Be^{-s\mu}\right), \tag{2.9}$$

whose determinant yields the characteristic equation

$$\det\left(\Delta_{\mathbf{u}^*}(s)\right) = \det\left[s^{\alpha} I - (A + Be^{-s\mu})\right] = 0. \tag{2.10}$$

Stability criterion 1. If all characteristic roots s_i $(i = 1, 2, \ldots, n)$ of (10) satisfy $Re(s_i) < 0$, then system (2.2) is locally asymptotically stable at equilibrium $\mathbf{u}^*$.

For the non-delayed case ($\mu = 0$), the original system reduces to

$${}^{C}_{t_0}D^{\alpha}_{t}\mathbf{u}(t) = \mathcal{F}(\mathbf{u}(t)), \tag{2.11}$$

with corresponding linearization at $\mathbf{u}^*$:

$${}^{C}_{t_0}D^{\alpha}_{t}\widetilde{\mathbf{u}}(t) = M\widetilde{\mathbf{u}}(t), \quad M = A + B. \tag{2.12}$$

Stability criterion 2. Eq (3.2) achieves local asymptotic stability at $\mathbf{u}^*$ if all eigenvalues s_i of M satisfy

$$|\arg(s_i)| > \frac{\alpha\pi}{2}, \quad i = 1, 2, \ldots, n.$$

3. Investigation into a non-diffusive fractional-order dynamical system with time delays

3.1. Nonnegativity and boundedness

Theorem 1. All solutions of system (2.2) with initial conditions in $\mathbb{R}^3_+$ are uniformly bounded and maintain nonnegativity.

Proof. Consider initial conditions $\mathfrak{X}(t_0) = (\phi_1(t_0), \phi_2(t_0), \phi_3(t_0))$ with $t_0 \in [-\tau, 0]$ belonging to the positive cone $\mathcal{K}_+ := \{(x_1, x_2, u) \in C([-\tau, 0], \mathbb{R}^3) \mid x_i \geq 0\}$. We first establish nonnegativity of solutions.

Suppose, by contradiction, that there exists a first crossing time $\xi > t_0$ and $\xi^+ > \xi$ such that:

$$\begin{cases} x_1(t) > 0, & \text{for } t_0 \leq t < \xi, \\ x_1(\xi) = 0, \\ x_1(\xi^+) < 0. \end{cases} \tag{3.1}$$

From system (2.2), we observe:

$${}^{C}_{t_0}D^{\alpha}_{t}x_1(t)\Big|_{x_1(\xi)=0} = 0.$$

By Lemma 1, it follows that $x_1(\xi^+) = 0$, which contradicts the assumption that $x_1(\xi^+) < 0$. Thus, $x_1(t) \geq 0$ for all $t \geq t_0$.

Now, suppose there exists a first crossing time $\xi > t_0$ and $\xi^+ > \xi$ such that:

$$\begin{cases} x_2(t) > 0, & \text{for } t_0 \leq t < \xi, \\ x_2(\xi) = 0, \\ x_2(\xi^+) < 0. \end{cases}$$

The dynamic of $x_2(t)$ is:

$${}^{C}_{t_0}D^{\alpha}_{t}x_2(t) = \mu x_1(t) - (d + g(x_1(t) + x_2(t)) + c_2 u(t))\, x_2(t).$$

At $t = \xi$, with $x_2(\xi) = 0$, we obtain:

$$ {}_{t_0}^{C}D_t^{\alpha} x_2(t)\Big|_{t=\xi} = \mu x_1(\xi). $$

Since we have already established that $x_1(t) \geq 0$ for all $t \geq t_0$, it follows that $\mu x_1(\xi) \geq 0$. Therefore:

$$ {}_{t_0}^{C}D_t^{\alpha} x_2(t)\Big|_{t=\xi} \geq 0. $$

By Lemma 1, if ${}_{t_0}^{C}D_t^{\alpha}\psi(t) \geq 0$, then $\psi(t)$ is nondecreasing. Thus, $x_2(t)$ cannot decrease to negative values immediately after ξ, contradicting $x_2(\xi^+) < 0$. Hence, $x_2(t) \geq 0$ for all $t \geq t_0$.

Similarly, it can be proven that $u(t)$ is positive.

Next, we demonstrate the boundedness of the solution. Define the Lyapunov-type function:

$$ G(t) = x_1(t) + x_2(t) + u(t). $$

The fractional derivative yields:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} G(t) &\leq -g(x_1 + x_2)^2 + bx_1 - dx_2 - hu + f_1 x_1^{\tau} + f_2 x_2^{\tau} \\ &\leq -g\|x\|^2 + (b + f_1)\|x\| - (d - f_2)\|x\| - h\|u\|, \end{aligned} $$

where $x_i^{\tau} := x_i(t - \tau)$. Introducing a damping coefficient $\beta > 0$ yields:

$$ {}_{t_0}^{C}D_t^{\alpha} G(t) + \beta G(t) \leq \frac{(b + f_1 + \beta)^2 + (f_2 + \beta - d)^2}{4g} =: A \quad (\beta < h). \tag{3.2} $$

Applying Laplace transforms to Eq (3.2) and by Lemma 2 , we have

$$ s^{\alpha}\mathcal{L}\{G(t)\} - s^{\alpha-1}G(0) + \beta\mathcal{L}\{G(t)\} \leq \frac{A}{s}. $$

By Lemma 3, the inverse transform gives

$$ G(t) \leq G(0)E_{\alpha,1}(-\beta t^{\alpha}) + At^{\alpha}E_{\alpha,\alpha+1}(-\beta t^{\alpha}), $$

where $E_{\alpha,\gamma}(z)$ denotes the Mittag-Leffler function. Using the identity:

$$ t^{\alpha}E_{\alpha,\alpha+1}(-\beta t^{\alpha}) = -\beta^{-1}(E_{\alpha,1}(-\beta t^{\alpha}) - 1), $$

we obtain:

$$ G(t) \leq \left(G(0) - \frac{A}{\beta}\right)E_{\alpha,1}(-\beta t^{\alpha}) + \frac{A}{\beta}, $$

$$ \Omega_+ = \left\{(x_1, x_2, u) \in \mathcal{K}_+ \,\middle|\, \|x\|_1 + \|u\| \leq \frac{A}{\beta} + \epsilon, \epsilon > 0\right\}. $$

3.2. Existence and uniqueness of the solutions

Having established the boundedness of solutions, we now proceed to prove the uniqueness of solutions. Lemma 4 demonstrates that the system satisfies a Lipschitz condition, which guarantees unique solutions for all positive initial conditions in $\mathcal{K}_+$.

Theorem 2. System (2.2) with any positive $(x_1(0), x_2(0), u(0)) \in \mathbb{R}^3_+$ admits a unique solution that remains bounded in the region

$$\Omega = \left\{ (x_1(t), x_2(t), u(t)) \in \mathbb{R}^3_+ \middle| \begin{array}{l} 0 \le x_1(t) \le \mathfrak{A}_1, \\ 0 \le x_2(t) \le \mathfrak{A}_2, \\ 0 \le u(t) \le \mathfrak{A}_3 \end{array} \right\},$$

where $\mathfrak{A}_i$ $(i = 1, 2, 3)$ are positive constants determined by the system parameters.

Proof. Consider the mapping $\mathfrak{F}(X) = (x_1(t), x_2(t), u(t))$ defined by system (2.2). For any two states $X = (x_1, x_2, u)$ and $\overline{X} = (\overline{x}_1, \overline{x}_2, \overline{u})$ in Ω, we have

$$\begin{aligned}
&\|\mathfrak{F}(X) - \mathfrak{F}(\overline{X})\| \\
&\quad \le \|x_1 (b - g(x_1 + x_2) - \mu - c_1 u + \mu x_1 - (d + g(x_1 + x_2) + c_2 u) x_2 - hu + f_1 x_+ + f_2 x_2\| \\
&\qquad - \|\overline{x}_1 (b - g(\overline{x}_1 + \overline{x}_2) - \mu - c_1 \overline{u} + \mu \overline{x}_1 - (d + g(\overline{x}_1 + \overline{x}_2) + c_2 \overline{u}) \overline{x}_v - h\overline{u} + f_1 \overline{x}_v + f_2 \overline{x}_2\| \\
&\quad \le (b + g(cx_1 + x_2) - (\overline{x}_1 + \overline{x}_2)) + f_1 + \mu + c_1(u - \overline{u}) \|x_1 - \overline{x}_1\| \\
&\qquad + (d + g(x_1 + x_2) - (\overline{x}_1 + \overline{x}_2)) + f_2 + c_2(u - \overline{u}) \|x_2 - \overline{x}_2\| \\
&\qquad + (h + c_1) + c_2 (x_2 - \overline{x}_2) \|u - \overline{u}\| \\
&\quad \le (b + 4\mathfrak{A}g + f_1 + \mu + c_1\mathfrak{A})\|x_1 - \overline{x}_1\| \\
&\qquad + (d + 4\mathfrak{A}g + f_2 + c_2\mathfrak{A})\|x_2 - \overline{x}_2\| \\
&\qquad + (h + c_1 + c_2\mathfrak{A})\|u - \overline{u}\| \\
&\quad \le L\|X - \overline{X}\|,
\end{aligned}$$

where $\mathfrak{A} = \max\{x_1, x_2, u\}$ and

$$L = \max\{b + 4\mathfrak{A}g + f_1 + \mu + c_1\mathfrak{A}, d + 4\mathfrak{A}g + f_2 + c_2\mathfrak{A}, h + c_1 + c_2\mathfrak{A}\}.$$

This establishes that $\mathfrak{F}$ is Lipschitz continuous in Ω. By Lemma 4 , system (2.2) consequently has a unique solution in Ω.

3.3. Stability and bifurcation analysis

The fractional-order system (2.2) admits four distinct equilibrium points. Among these, the trivial equilibrium $E_0 = (0, 0, 0)$ and the axial equilibrium $E_1 = \left(0, \frac{dh}{c_2 f_2 + hg}, \frac{f_2 d}{c_2 f_2 + hg}\right)$ are always present, irrespective of parameter values. The boundary equilibrium $E_2 = \left(\frac{b-\mu}{g}, 0, 0\right)$ emerges when the growth rate exceeds the transition rate $(b > \mu)$.

The coexisting equilibrium $E^* = (x_1^*, x_2^*, u^*)$ exists under the following parametric conditions

(H1). $m^2 + n > h\mu(b - \mu)$,
(H2). $h(b - \mu) > (c_1 f_2 + hg)\sqrt{m^2 + n - bh\mu + \mu^2 h}$,
(H3). $g(f_2 - f_1)\sqrt{m^2 + n - bh\mu + \mu^2 h} > f_1(b - \mu)$,

with equilibrium components given by:

$$\begin{cases} x_1^* = \dfrac{-(c_1 f_2 + hg)x_2^* + h(b-\mu)}{c_1 f_1 + hg}, \\ x_2^* = \sqrt{m^2 + n - bh\mu + \mu^2 h}, \\ u^* = \dfrac{(f_2 - f_1)g x_2^* + f_1(b-\mu)}{c_1 f_1 + hg}, \end{cases}$$

where the composite parameters are:

$$m = g(c_1(f_1 - f_2) + c_2(f_2 - f_1)),$$
$$n = b(c_2 f_1 + hg) + \mu(c_1 f_2 - c_2 f_1) + d(c_1 f_1 + hg).$$

In order to analyze the stability of the nonlinear system (2.2) around an equilibrium point $E = (\bar{x}_1, \bar{x}_2, \bar{u})$, we introduce the transformation:

$$\tilde{x}_1 = x_1 - \bar{x}_1, \quad \tilde{x}_2 = x_2 - \bar{x}_2, \quad \tilde{u} = u - \bar{u},$$

which yields the following linearized system:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}\tilde{x}_1(t) &= (b - 2g\bar{x}_1 - g\bar{x}_2 - \mu - c_1\bar{u})\{\tilde{x}_1(t)\} + (-g\bar{x}_1)\{\tilde{x}_2(t)\} + (-c_1\bar{x}_1)\{\tilde{u}(t)\}, \\ {}_{t_0}^{C}D_t^{\alpha}\tilde{x}_2(t) &= (\mu - g\bar{x}_2)\{\tilde{x}_1(t)\} + (-d - g\bar{x}_1 - 2g\bar{x}_2 - c_2\bar{u})\{\tilde{x}_2(t)\} + (-c_2\bar{x}_2)\{\tilde{u}(t)\}, \\ {}_{t_0}^{C}D_t^{\alpha}\tilde{u}(t) &= (-h)\{\tilde{u}(t)\} + f_1\tilde{x}_1(t-\tau) + f_2\tilde{x}_2(t-\tau). \end{aligned} \tag{3.3}$$

The Laplace transform is applied to both sides of system (3.3), leading to

$$\begin{aligned} s^{\alpha}\mathscr{L}\{\tilde{x}_1(t)\} - s^{\alpha-1}\phi_1(0) =& (b - 2g\bar{x}_1 - g\bar{x}_2 - \mu - c_1\bar{u})\mathscr{L}\{\tilde{x}_1(t)\} \\ &+ (-g\bar{x}_1)\mathscr{L}\{\tilde{x}_2(t)\} \\ &+ (-c_1\bar{x}_1)\mathscr{L}\{\tilde{u}(t)\}, \\ s^{\alpha}\mathscr{L}\{\tilde{x}_2(t)\} - s^{\alpha-1}\phi_2(0) =& (\mu - g\bar{x}_2)\mathscr{L}\{\tilde{x}_1(t)\} \\ &+ (-d - g\bar{x}_1 - 2g\bar{x}_2 - c_2\bar{u})\mathscr{L}\{\tilde{x}_2(t)\} \\ &+ (-c_2\bar{x}_2)\mathscr{L}\{\tilde{u}(t)\}, \\ s^{\alpha}\mathscr{L}\{\tilde{u}(t)\} - s^{\alpha-1}\phi_3(0) =& (-h)\mathscr{L}\{\tilde{u}(t)\} \\ &+ f_1 e^{-s\tau}(\mathscr{L}\{\tilde{x}_1(t)\} + \int_{-\tau}^{0} e^{-s\tau}\phi_1(t)dt) \\ &+ f_2 e^{-s\tau}(\mathscr{L}\{\tilde{x}_2(t)\} + \int_{-\tau}^{0} e^{-s\tau}\phi_2(t)dt). \end{aligned} \tag{3.4}$$

We can reformulate the above system as

$$\Im_{E(s)}\begin{pmatrix} \mathscr{L}\{x_1(t)\} \\ \mathscr{L}\{x_2(t)\} \\ \mathscr{L}\{u(t)\} \end{pmatrix}$$

$$= \begin{pmatrix} s^{\alpha-1}\phi_1(0) \\ s^{\alpha-1}\phi_2(0) \\ s^{\alpha-1}\phi_3(0) + f_1 e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\phi_1(t)dt) + f_2 e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\phi_2(t)dt) \end{pmatrix},$$

where

$$\Im_{E(s)} = \begin{pmatrix} s^{\alpha} - \Im_{\bar{E}_{11}} & g\bar{x}_1 & c_1\bar{x}_1 \\ g\bar{x}_2 - \mu & s^{\alpha} - \Im_{\bar{E}_{22}} & c_2\bar{x}_2 \\ -f_1 e^{-s\tau} & -f_2 e^{-s\tau} & s^{\alpha} + h \end{pmatrix},$$

where

$$\Im_{\bar{E}_{11}} = b - 2g\bar{x}_1 - g\bar{x}_2 - \mu - c_1\bar{u},$$
$$\Im_{\bar{E}_{22}} = -d - g\bar{x}_1 - 2g\bar{x}_2 - c_2\bar{u}.$$

As the characteristic matrix of the linear system (3.3), $\Im_{E(s)}$ has eigenvalues that are decisive for the local stability of system (2.2). The conditions for local asymptotic stability are therefore evaluated in accordance with Stability criterion 1.

3.3.1. Stability of trivial equilibrium point

The characteristic matrix at $E_0(0,0,0)$ is

$$\Im_{E_0(s)} = \begin{pmatrix} s^{\alpha} - b + \mu & 0 & 0 \\ -\mu & s^{\alpha} + d & 0 \\ -f_1 e^{-s\tau} & -f_1 e^{-s\tau} & s^{\alpha} + h \end{pmatrix},$$

and the characteristic equation is

$$\det(\Im_{E_0(s)}) = (s^{\alpha} - b + \mu)(s^{\alpha} + d)(s^{\alpha} + h) = 0. \tag{3.5}$$

The characteristic equation (3.5) possesses two distinct negative real roots, namely , $s^{\alpha} = -d$ and $s^{\alpha} = -h$, where $0 < \alpha < 1$. Consequently, the trivial equilibrium point $E_0(0,0,0)$ is locally asymptotically stable under the condition $b < \mu$.

Remarkably, the stability criteria for the equilibrium point E_0 are independent of the time delay parameter $\tau > 0$. This implies that the asymptotic stability properties of the trivial equilibrium remain invariant, regardless of whether the fractional-order system incorporates time delays ($\tau > 0$) or reduces to the non-delayed case ($\tau = 0$).

3.3.2. Local stability and Hopf bifurcation dynamics of the coexistence equilibrium point

At the coexisting equilibrium $E^*\left(x_1^*, x_2^*, u^*\right)$, the characteristic matrix is given by

$$\Im_{E^*(s)} = \begin{pmatrix} s^{\alpha} - \Im_{E^*_{11}} & g{x_1}^* & c_1{x_1}^* \\ g{x_2}^* - \mu & s^{\alpha} - \Im_{E^*_{22}} & c_2{x_2}^* \\ -f_1 e^{-s\tau} & -f_2 e^{-s\tau} & s^{\alpha} + h \end{pmatrix},$$

where $\Im_{E^*_{11}} = b - 2gx_1^* - gx_2^* - \mu - c_1u^*, \Im_{E^*_{22}} = -d - gx_1^* - 2gx_2^* - c_2u^*$, and the characteristic equation is

$$\det(\Im_{E^*(s)}) = s^{3\alpha} + P_1 s^{2\alpha} + P_2 s^{\alpha} + P_3 + P_4 e^{-s\tau} = 0, \tag{3.6}$$

where

$$\begin{aligned}
P_1 &= h - P_{11} - P_{22},\\
P_2 &= P_{11}P_{22} + P_{12}P_{21} - hP_{11} - hP_{22},\\
P_3 &= hP_{11}P_{22} + hP_{12}P_{21},\\
P_4 &= f_1\left(P_{12}P_{23} - P_{13}s^{\alpha} + P_{13}P_{22}\right) + f_2\left(P_{23}s^{\alpha} - P_{11}P_{23} - P_{13}P_{21}\right),\\
P_{11} &= b - 2gx_1^* - gx_2^* - \mu - c_1u^*,\\
P_{12} &= -gx_1^*,\\
P_{13} &= -c_1x_1^*,\\
P_{21} &= \mu - gx_2^*,\\
P_{22} &= -d - gx_1^* - 2gx_2^* - c_2u^*,\\
P_{23} &= -c_2x_2^*.
\end{aligned}$$

If the (H4) are satisfied, there are two cases here:

(H4). $f_2P_{23} - f_1P_{13} = 0$.

Case 1. When $\tau = 0$ (fractional-order system)

Then , characteristic Eq (3.6) becomes

$$s^{3\alpha} + P_1s^{2\alpha} + P_2s^{2\alpha} + P_3 + P_4 = 0. \tag{3.7}$$

Based on the Routh-Hurwitz stability criterion, all roots of Eq (3.7) possess negative real parts , i.e., the roots λ_i satisfy the condition $|\arg(\lambda_i)| > \frac{\pi}{2} > \frac{\pi\alpha}{2}$—provided that $P_1 > 0$, $P_2 > 0$, $P_3 + P_4 > 0$, and $P_1P_2 > P_3 + P_4$. Consequently, the following theorem on the stability of the coexisting equilibrium $E^*\left(x_1^*, x_2^*, u^*\right)$ is established for the case when $\tau = 0$.

Theorem 3. For the case without time delay ($\tau = 0$), the coexisting equilibrium $E^*\left(x_1^*, x_2^*, u^*\right)$ of system (2.2) is locally asymptotically stable provided that $P_1 > 0$, $P_2 > 0$, $P_3+P_4 > 0$, $P_1P_2 > P_3+P_4$, and Hypotheses (H1)–(H4) hold.

Case 2. When $\tau > 0$ (fractional-order delayed system)

Substituting $s = \mathrm{i}\omega = \omega\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)(\omega > 0)$ as a root of Eq (3.7) into the equation leads to

$$(\mathrm{i}\omega)^{3\alpha} + P_1(\mathrm{i}\omega)^{2\beta} + P_2(\mathrm{i}\omega)^{\beta} + P_3 + P_4e^{-\mathrm{i}\omega\tau} = 0. \tag{3.8}$$

Splitting this equation into its real and imaginary parts gives

$$\begin{cases} \omega^{3\alpha}\cos\frac{3}{2}\alpha\pi + P_1\omega^{2\alpha}\cos\alpha\pi + P_2\omega^{\alpha}\cos\frac{1}{2}\alpha\pi + P_3 = -P_4\cos\omega\tau,\\ \omega^{3\alpha}\sin\frac{3}{2}\alpha\pi + P_1\omega^{2\alpha}\sin\alpha\pi + P_2\omega^{\alpha}\sin\frac{1}{2}\alpha\pi + P_3 = P_4\sin\omega\tau, \end{cases} \tag{3.9}$$

and it follows that

$$\begin{cases} \cos\omega\tau = -\frac{\omega^{3\alpha}\cos\frac{3}{2}\alpha\pi + P_1\omega^{2\alpha}\cos\alpha\pi + P_2\omega^{\alpha}\cos\frac{1}{2}\alpha\pi + P_3}{P_4} = \mathfrak{G}_1(\omega),\\ \sin\omega\tau = \frac{\omega^{3\alpha}\sin\frac{3}{2}\alpha\pi + P_1\omega^{2\alpha}\sin\alpha\pi + P_2\omega^{\alpha}\sin\frac{1}{2}\alpha\pi + P_3}{P_4} = \mathfrak{G}_2(\omega). \end{cases} \tag{3.10}$$

By squaring and adding two equations in Eq (3.10), we get

$$\mathfrak{G}_1^2(\omega) + \mathfrak{G}_2^2(\omega) = 1. \tag{3.11}$$

Let us refer to Eq (3.6), then

$$\begin{aligned} &\omega^{6\alpha} + \omega^{5\alpha}\left(2P_1 \cos\frac{\alpha\pi}{2}\right) + \omega^{4\alpha}\left(2P_2 \cos\alpha\pi + 1\right) \\ &\quad + \omega^{3\alpha}\left(P_1P_2 \cos\frac{\alpha\pi}{2} + P_3 \cos\frac{3\alpha\pi}{2}\right) + \omega^{2\alpha}\left(P_1P_3 \cos\alpha\pi + 1\right) \\ &\quad + \omega^{\alpha}\left(P_2P_3 \cos\frac{\alpha\pi}{2}\right) + \left(P_3^2 - P_4^2\right) \\ &=0. \end{aligned} \tag{3.12}$$

Since $\cos\frac{\alpha\pi}{2} > 0$ and $\omega^\alpha > 0$, the existence of at least one purely imaginary root for Eq (3.6) is equivalent to the existence of a positive solution for Eq (3.12). Applying the change of variable $z = \omega^\alpha$, Eq (3.12) is transformed into

$$\begin{aligned} &z^6 + z^5\left(2P_1 \cos\frac{\alpha\pi}{2}\right) + z^4\left(2P_2 \cos\alpha\pi + 1\right) \\ &\quad + z^3\left(P_1P_2 \cos\frac{\alpha\pi}{2} + P_3 \cos\frac{3\alpha\pi}{2}\right) + z^2\left(P_1P_3 \cos\alpha\pi + 1\right) \\ &\quad + z\left(P_2P_3 \cos\frac{\alpha\pi}{2}\right) + \left(P_3^2 - P_4^2\right) \\ &=0. \end{aligned} \tag{3.13}$$

Consequently, the existence of at least one positive root for Eq (3.12) holds if , and only if , a positive root exists for Eq (3.13).

We define

$$\begin{aligned} \mathfrak{h}(z) = {} & z^6 + z^5\left(2P_1 \cos\frac{\alpha\pi}{2}\right) + z^4\left(2P_2 \cos\alpha\pi + 1\right) \\ & + z^3\left(P_1P_2 \cos\frac{\alpha\pi}{2} + P_3 \cos\frac{3\alpha\pi}{2}\right) + z^2\left(P_1P_3 \cos\alpha\pi + 1\right) \\ & + z\left(P_2P_3 \cos\frac{\alpha\pi}{2}\right) + \left(P_3^2 - P_4^2\right). \end{aligned} \tag{3.14}$$

Under the conditions $P_1 > 0$, $2P_2\cos(\alpha\pi)+1 > 0$, $P_1P_2\cos\frac{\alpha\pi}{2}+P_3\cos\frac{3\alpha\pi}{2} > 0$, $P_2P_3\cos(\alpha\pi)+1 > 0$, $P_2P_3\cos\frac{\alpha\pi}{2} > 0$, and $P_3 < P_4$, it follows that $\left(P_2^2 - P_3^2\right) < 0$. Given $h(0) = \left(P_2^2 - P_3^2\right) < 0$ and $\lim_{z\to\infty}\mathfrak{h}(z) \to +\infty$, the intermediate value theorem guarantees the existence of $z_0 > 0$ such that $\mathfrak{h}(z_0) = 0$. This establishes the existence of a positive root for the characteristic Eq (3.12). As the delay parameter τ crosses critical values, the stability of the coexisting equilibrium changes, and system (2.2) undergoes a Hopf bifurcation at the equilibrium point.

From Eq (3.10), $\cos\omega\tau = \mathfrak{G}_1(\omega)$, we obtain

$$\tau^{(n)} = \frac{1}{\omega}\left(\arccos\mathfrak{G}_1(\omega) + 2n\pi\right), \quad n = 0, 1, 2, \ldots \tag{3.15}$$

Under the assumption that Eq (3.11) admits at least one positive real root, the bifurcation point is defined as

$$\tau_0 = \min\left\{\tau^{(n)}\right\}, \quad n = 0, 1, 2, \ldots \tag{3.16}$$

To establish the conditions for Hopf bifurcation, the following hypotheses are required:
(H5). $P_1 > 0,\ 2P_2\cos\alpha\pi+1 > 0,\ P_1P_2\cos\frac{\alpha\pi}{2}+P_3\cos\frac{3\alpha\pi}{2} > 0,\ P_2P_3\cos\alpha\pi+1 > 0,\ P_2P_3\cos\frac{\alpha\pi}{2} > 0$, and $P_3 < P_4$.
(H6). $\frac{\chi_1\chi_2+\gamma_1\gamma_2}{\chi_1^2+\gamma_2^2} \neq 0$.

Let $s(\tau) = \varsigma(\tau) + i\omega(\tau)$ denote a root of Eq (3.6) satisfying $\varsigma(\tau^{(n)}) = 0$ and $\omega(\tau^{(j)}) = \omega_0$. Differentiating both sides of Eq (3.6) with respect to τ, we obtain

$$3\alpha s^{3\alpha-1}\frac{ds}{d\tau} + 2P_1\alpha s^{2\alpha-1}\frac{ds}{d\tau} + P_2\alpha s^{\alpha-1}\frac{ds}{d\tau} + P_4e^{-s\tau}\left(-s-\mu\frac{ds}{d\tau}\right) = 0,$$

which implies

$$\begin{aligned}\frac{ds}{d\tau} &= \frac{P_4 s e^{-s\tau}}{3\alpha s^{3\alpha-1} + 2P_1\alpha s^{2\alpha-1} + P_2\alpha s^{\alpha-1} - \tau P_4 e^{-s\tau}}\\ &= \frac{P_4(i\omega)e^{-i\omega\tau}}{3\alpha(i\omega)^{3\alpha-1} + 2P_1\alpha(i\omega)^{2\alpha-1} + P_2\alpha(i\omega)^{\alpha-1} - \tau P_4e^{-i\omega\tau}}.\end{aligned} \tag{3.17}$$

A direct calculation shows that

$$\mathrm{Re}\left(\frac{ds}{d\tau}\right)_{\omega=\omega_0,\tau=\tau_0} = \frac{\chi_1\chi_2+\gamma_1\gamma_2}{\chi_1^2+\gamma_2^2} \neq 0, \tag{3.18}$$

where

$$\begin{aligned}\chi_1 &= \mathrm{Re}\left\{P_4 i\omega_0 e^{-i\omega_0\tau_0}\right\} = P_4\omega_0\sin\omega_0\tau_0,\\ \gamma_1 &= P_4\omega_0\cos\omega_0\tau_0,\\ \chi_2 &= \mathrm{Re}\left\{3\alpha(i\omega_0)^{3\alpha-1} + 2P_1\alpha(i\omega_0)^{2\alpha-1} + P_2\alpha(i\omega_0)^{\alpha-1} - P_4\tau_0e^{-i\omega_0\tau_0}\right\}\\ &= 3\alpha\omega_0^{3\alpha-1}\sin\frac{3\alpha\pi}{2} + 2P_1\alpha\omega_0^{2\alpha-1}\sin\alpha\pi + P_2\alpha\omega_0^{\alpha-1}\sin\frac{\alpha\pi}{2} - P_4\tau_0\cos\omega_0\tau_0,\\ \gamma_2 &= -3\alpha\omega_0^{3\alpha-1}\cos\frac{3\alpha\pi}{2} - 2P_1\alpha\omega_0^{2\alpha-1}\cos\alpha\pi - P_2\alpha\omega_0^{\alpha-1}\cos\frac{\alpha\pi}{2} + P_4\tau_0\sin\omega_0\tau_0.\end{aligned} \tag{3.19}$$

The foregoing analysis leads to the following conclusion:
Theorem 4. Under the satisfaction of assumptions (H1)–(H6), the coexisting equilibrium $E^*\left(x_1^*, x_2^*, u^*\right)$ of the fractional-order delayed system (2.2) exhibits local asymptotic stability over the interval $[0, \tau_0)$ and loses stability for $\tau > \tau_0$. Moreover, a Hopf bifurcation occurs at this coexisting equilibrium when the time delay τ reaches the critical threshold τ_0.

The analytical framework employed for investigating the equilibrium points E_1 and E_2 follows the identical methodology as detailed above. To maintain conciseness, detailed derivations for these cases are omitted herein, and readers are advised to conduct the corresponding computations independently to reinforce their understanding of the proposed approach.

4. Dynamics of a diffusive fractional-order system with time delays

4.1. The existence and uniqueness of a postive solution

In this section, we will explore the boundedness and uniqueness of solutions to the system (2.3) when $\tau = 0$.

Theorem 5. System (2.3) has a unique positive global solution satisfying $0 \leq x_1 \leq \overline{x_1}$, $0 \leq x_2 \leq \overline{x_2}$, $0 \leq u \leq \overline{u}$.

Proof. Reconvert the system into an abstract integral equation and define a nonlinear term:

$$\begin{aligned}
\mathfrak{f}_1(x_1, x_2, u) &= x_1(b - g(x_1 + x_2) - \mu - c_1 u),\\
\mathfrak{f}_2(x_1, x_2, u) &= \mu x_1 - (d + g(x_1 + x_2) + c_2 u)x_2,\\
\mathfrak{f}_3(x_1, x_2, u) &= -h\mu + f_1 x_1 + f_2 x_2.
\end{aligned}$$

The elliptic operator $\mathcal{L} = \delta_i \frac{\partial^2}{\partial x^2}$ corresponding to the diffusion term generates an analytic semigroup $\{G_i\,(t^\alpha s)\}_{t\geq 0}$ $(i = 1, 2, 3)$ satisfying:

$$\|G_i(t^\alpha s)\|_{L^2\to H^1} \leq Ct^{-\alpha/2}e^{\omega t}, \quad t > 0.$$

Here, C denotes a constant associated with the operator, the spatial dimension L_X, and the boundary conditions; ω represents the exponential growth constant of the semi-group, which is determined by the spectral boundary of the elliptic operator (i.e., the supremum of the real parts of its spectrum); and $t^{-\alpha/2}$ signifies the time-decay factor.

First, we prove the local uniqueness of the solution to system (2.3). The Lipschitz continuity of the nonlinear term $\mathfrak{f}$ has been provided in Section 3.2. The mild solution [22, 23] of the system (2.3) is

$$\begin{aligned}
x_1(t) &= \phi_1(0, x)\int_0^{+\infty} \varrho_\alpha(s)G_1\,(t^\alpha s)\,ds + \int_0^t S_\alpha^{(1)}(t-s)\mathfrak{f}_1(x_1(s), x_2(s), u(s))ds,\\
x_1(t) &= \phi_2(0, x)\int_0^{+\infty} \varrho_\alpha(s)G_2\,(t^\alpha s)\,ds + \int_0^t S_\alpha^{(1)}(t-s)\mathfrak{f}_2(x_1(s), x_2(s), u(s))ds,\\
u(t) &= \phi_3(0, x)\int_0^{+\infty} \varrho_\alpha(s)G_3\,(t^\alpha s)\,ds + \int_0^t S_\alpha^{(1)}(t-s)\mathfrak{f}_3(x_1(s), x_2(s), u(s))ds,
\end{aligned}$$

where

$$\begin{aligned}
S_\alpha^{(i)}(t) &= \alpha\int_0^{+\infty} st^{\alpha-1}\varrho_\alpha(s)G_i\,(t^\alpha s)\,ds,\\
\varrho_\alpha(s) &= \int_{\mho} e^{s\Theta}E_{\alpha,1}(-\Theta)d\Theta,
\end{aligned}$$

with $i = 1, 2$, let the functions represent probability density functions on $[0, +\infty)$, where $\mho$ denotes a contour starting and ending at minus infinity, and $E_{\alpha,1}(-\Theta)$ is the Mittag-Leffler function. Define the Banach space $\mathcal{X} = C\left([0, T]; L^2(0, L_x)\right)^3$ equipped with the norm

$$\|(x_1, x_2, u)\| = \sum_{i=1}^{3} \sup_{t\in[0,T]} \|x_i(t)\|_{L^2}\,,$$

and the mapping $\Phi : \mathcal{X} \to \mathcal{X}$. One can readily establish the existence of a maximal time $T_{\max} > 0$ such that

$$\|\Phi(x) - \Phi(\bar{x})\| \leq CT_{\max}^\alpha\|x - \bar{x}\|,$$

for some constant $C > 0$. When $T_{\max}$ is sufficiently small to satisfy $CT_{\max}^\alpha < 1$, the Banach fixed-point theorem ensures that Φ is a contraction mapping. Consequently, system (2.3) admits a unique positive solution on the interval $(0, T_{\max})$.

Employing the method of upper and lower solutions, we now prove that the solution remains positive and global in time t.

The functionals $\mathfrak{f}_i$ ($i = 1, 2, 3$) are mixed quasimonotone with respect to x_1, x_2, and u, respectively. Define the upper solutions as

$$\begin{aligned}\overline{x_1} &= M_1 := \max\left\{\frac{b-\mu}{g}, \|\phi_1(0,x)\|_\infty\right\},\\ \overline{x_2} &= M_2 := \max\left\{\frac{-d+\sqrt{d^2+4g\mu M_1}}{2g}, \|\phi_2(0,x)\|_\infty\right\},\\ \overline{u} &= M_3 := \max\left\{\frac{f_1M_1+f_2M_2}{h}, \|\phi_3(0,x)\|_\infty\right\},\end{aligned}$$

and the lower solutions as $\underline{x_1} = 0$, $\underline{x_2} = 0$, $\underline{u} = 0$. Evidently, the pairs $(\overline{x_1}, \overline{x_2}, \overline{u})$ and $(\underline{x_1}, \underline{x_2}, \underline{u})$ form a set of upper and lower solutions for system (2.3). By the theory of upper-lower solutions, we readily derive the bounds

$$0 \le x_1 \le M_1, \quad 0 \le x_2 \le M_2, \quad 0 \le u \le M_3,$$

for all $t \ge 0$, establishing the positivity and global existence of the solution.

By means of energy estimates and Gronwall's inequality, we deduce that the solution does not exhibit finite-time blowup for all $t \ge 0$. Combining the preceding analysis, the proof of Theorem 5 is thereby completed.

4.2. Stability and Hopf bifurcation analysis of the homogeneous steady state

To facilitate the analysis of system (2.3), we define the real-valued Sobolev space

$$X := \left\{(x_1, x_2, u) \in H^2(0, L_x)^3 \;\middle|\; \begin{array}{l}\frac{\partial x_i}{\partial x}(0,t) = \frac{\partial x_i}{\partial x}(L_x,t) = 0,\\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L_x,t) = 0, \quad i = 1,2\end{array}\right\}.$$

To address the inability of the real-valued Sobolev space X to directly accommodate complex eigenfunctions, we define its complexification space $X_{\mathbb{C}}$ and the associated inner product. The complexified space is constructed as

$$X_{\mathbb{C}} := X \oplus \mathrm{i}X = \{(x_1 + \mathrm{i}y_1,\ x_2 + \mathrm{i}y_2,\ u + \mathrm{i}v) \mid x_1, x_2, y_1, y_2, u, v \in X\},$$

endowed with the inner product

$$\langle W_1, W_2\rangle = \int_0^{L_x} \left(\overline{x_1^1}x_1^2 + \overline{x_2^1}x_2^2 + \overline{u^1}u^2\right) dx,$$

where $W_1 = \left(x_1^1, x_2^1, u^1\right)^T$ and $W_2 = \left(x_1^2, x_2^2, u^2\right)^T$ are elements of $X_{\mathbb{C}}$, and $\overline{(\cdot)}$ denotes complex conjugation.

For each component $i = 1, 2, 3$, consider the eigenvalue problem governed by

$$\begin{cases}-\dfrac{\partial^2\phi_i(x)}{\partial x^2} = \lambda_i\phi_i(x), \quad x \in (0, L_x),\\ \dfrac{\partial\phi_i}{\partial x}(0) = \dfrac{\partial\phi_i}{\partial x}(L_x) = 0,\end{cases}$$

under Neumann boundary conditions. This problem admits eigenvalues $\lambda_i = k^2$, with corresponding eigenfunctions $\phi_i(x) = \cos(kx)$, where the wave number k takes the form $k = n\pi/L_x$ for $n = 0, 1, 2, \ldots$ This spectral decomposition is fundamental for analyzing the linearized stability and dynamic behavior of system (2.3) within the complexified Sobolev space framework.

We derive the Jacobian matrix of system (2.3) evaluated at the homogeneous steady state E^*

$$\Im^* = \begin{pmatrix} s^\alpha - \Im_{E^*_{11}} - \delta_1 k^2 & g\bar{x}_1 & c_1\bar{x}_1 \\ g\bar{x}_2 - \mu & s^\alpha - \Im_{E^*_{22}} - \delta_2 k^2 & c_2\bar{x}_2 \\ -f_1 e^{-s\tau} & -f_2 e^{-s\tau} & s^\alpha + h - \delta_3 k^2 \end{pmatrix}. \tag{4.1}$$

The eigenvalues of the Jacobian matrix of system (2.3) at the positive homogeneous steady state E^* satisfy the following eigenequation:

$$s^{3\alpha} + Q_1 s^{2\alpha} + Q_2 s^\alpha + Q_3 + Q_4 e^{-s\mu} = 0, \quad k \in \mathbb{N}_0, \tag{4.2}$$

where

$$\begin{aligned} Q_1 =& - P_{11} - P_{22} - \delta_1 k^2 - \delta_2 k^2 - \delta_3 k^2 + h, \\ Q_2 =& P_{11}P_{22} - P_{11}h - P_{22}h + P_{11}\delta_2 k^2 + P_{22}\delta_1 k^2 + P_{11}\delta_3 k^2 + P_{22}\delta_3 k^2 \\ & - h\delta_1 k^2 - h\delta_2 k^2 + \delta_1\delta_2 k^4 + \delta_1\delta_3 k^4 + \delta_2\delta_3 k^4, \\ Q_3 =& P_{11}P_{22}h - P_{11}P_{22}\delta_3 k^2 + P_{11}h\delta_2 k^2 - P_{22}h\delta_1 k^2 + h\delta_1\delta_2 k^4 \\ & - P_{11}\delta_2\delta_3 k^4 - P_{22}\delta_1\delta_3 k^4 - \delta_1\delta_2\delta_3 k^6, \\ Q_4 =& f_1 \left(P_{12}P_{23} - P_{13}s^\alpha + P_{13}P_{22} + P_{13}\delta_2 k^2\right) \\ & + f_2 \left(P_{23}s^\alpha - P_{11}P_{23} - P_{13}P_{21} - P_{23}\delta_1 k^2\right). \end{aligned}$$

Clearly, $s = 0$ is not a root of Eq (4.2). Specifically, the characteristic Eq (4.2) for $k = 0$ was analyzed in Section 3.3.2. For $k \geq 1$, however, the subsequent analysis addresses two distinct cases as follows:

Case 1. In the case of $\tau = 0$ (the fractional-order system with diffusion), the characteristic Eq (4.2) reduces to

$$s^{3\alpha} + Q_1 s^{2\alpha} + Q_2 s^\alpha + Q_3 + Q_4 = 0, \quad k \geq 1. \tag{4.3}$$

For $k \geq 1$, all the roots of Eq (4.3) have negative real parts if , and only if , $Q_1 > 0$, $Q_2 > 0$, $Q_3 + Q_4 > 0$, and $Q_1 Q_2 > Q_3 + Q_4$. Therefore, the following theorem can be established.

Theorem 6. In the delay-free case, the local asymptotic stability of the homogeneous steady state E^* for the reaction-diffusion system (2.3) is guaranteed provided that the following conditions hold: $Q_1 > 0$, $Q_2 > 0$, $Q_3 + Q_4 > 0$, and $Q_1 Q_2 > Q_3 + Q_4$.

Case 2. In the case of $\tau > 0$ (the fractional-order delayed system with diffusion)

Given that $s = \mathrm{i}\omega = \omega\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$, where $\omega > 0$ and $k \geq 1$, is a solution to Eq (4.2), it satisfies

$$(\mathrm{i}\omega)^{3\alpha} + Q_1(\mathrm{i}\omega)^{2\beta} + Q_2(\mathrm{i}\omega)^\beta + Q_3 + Q_4 e^{-\mathrm{i}\omega\tau} = 0. \tag{4.4}$$

Equating the real and imaginary parts separately gives

$$\begin{cases} \omega^{3\alpha}\cos\frac{3}{2}\alpha\pi + Q_1\omega^{2\alpha}\cos\alpha\pi + Q_2\omega^\alpha\cos\frac{1}{2}\alpha\pi + Q_3 = -Q_4\cos\omega\tau, \\ \omega^{3\alpha}\sin\frac{3}{2}\alpha\pi + Q_1\omega^{2\alpha}\sin\alpha\pi + Q_2\omega^\alpha\sin\frac{1}{2}\alpha\pi + Q_3 = Q_4\sin\omega\tau, \end{cases} \tag{4.5}$$

it follows that

$$\begin{cases} \cos\omega\tau = -\frac{\omega^{3\alpha}\cos\frac{3}{2}\alpha\pi+Q_1\omega^{2\alpha}\cos\alpha\pi+Q_2\omega^{\alpha}\cos\frac{1}{2}\alpha\pi+Q_3}{Q_4} = H1(\omega), \\ \sin\omega\tau = \frac{\omega^{3\alpha}\sin\frac{3}{2}\alpha\pi+Q_1\omega^{2\alpha}\sin\alpha\pi+Q_2\omega^{\alpha}\sin\frac{1}{2}\alpha\pi+Q_3}{Q_4} = H_2(\omega). \end{cases} \tag{4.6}$$

By squaring and adding two equations in Eq (4.6), we get

$$H_1^2(\omega) + H_2^2(\omega) = 1. \tag{4.7}$$

Let us refer to Eq (4.6), then

$$\begin{aligned} &\omega^{6\alpha} + \omega^{5\alpha}\left(2Q_1\cos\frac{\alpha\pi}{2}\right) + \omega^{4\alpha}\left(2Q_2\cos\alpha\pi + 1\right) \\ &\quad + \omega^{3\alpha}\left(Q_1Q_2\cos\frac{\alpha\pi}{2} + Q_3\cos\frac{3\alpha\pi}{2}\right) + \omega^{2\alpha}\left(Q_1Q_3\cos\alpha\pi + 1\right) \\ &\quad + \omega^{\alpha}\left(Q_2Q_3\cos\frac{\alpha\pi}{2}\right) + \left(Q_3^2 - Q_4^2\right) \\ &=0, \quad k \geq 1. \end{aligned} \tag{4.8}$$

Under the conditions $\cos\frac{\alpha\pi}{2} > 0$ and $\omega^{\alpha} > 0$, the existence of a purely imaginary root for Eq (4.3) is equivalent to the existence of a positive root for Eq (4.8). Letting $\omega^{\alpha} = \varpi$, Eq (4.8) can be rewritten as

$$\begin{aligned} &\varpi^6 + \varpi^5\left(2Q_1\cos\frac{\alpha\pi}{2}\right) + \varpi^4\left(2Q_2\cos\alpha\pi + 1\right) \\ &\quad + \varpi^3\left(Q_1Q_2\cos\frac{\alpha\pi}{2} + Q_3\cos\frac{3\alpha\pi}{2}\right) + \varpi^2\left(Q_1Q_3\cos\alpha\pi + 1\right) \\ &\quad + \varpi\left(Q_2Q_3\cos\frac{\alpha\pi}{2}\right) + \left(Q_3^2 - Q_4^2\right) \\ &=0. \end{aligned} \tag{4.9}$$

Consequently, a necessary and sufficient condition for Eq (4.8) to have a positive solution is that Eq (4.9) also has a positive solution. To verify this criterion, we introduce the following assumptions for a fixed integer $k_0 \in \{1, 2, \ldots\}$:

(H7). $Q_1^{k_0} > 0$, $2Q_2^{k_0}\cos\alpha\pi + 1 > 0$, $Q_1^{k_0}Q_2^{k_0}\cos\frac{\alpha\pi}{2} + Q_3^{k_0}\cos\frac{3\alpha\pi}{2} > 0$, $Q_2^{k_0}Q_3^{k_0}\cos\alpha\pi + 1 > 0$, $Q_2^{k_0}Q_3^{k_0}\cos\frac{\alpha\pi}{2} > 0$, $Q_3^{k_0} < Q_4^{k_0}$.

(H8). $\frac{\Pi_1^{k_0}\Pi_2^{k_0}+\Theta_1^{k_0}\Theta_2^{k_0}}{(\Pi_1^{k_0})^2+(\Theta_2^{k_0})^2} \neq 0$.

Let us denote

$$\begin{aligned} h(\varpi) = \varpi^6 &+ \varpi^5\left(2Q_1\cos\frac{\alpha\pi}{2}\right) + \varpi^4\left(2Q_2\cos\alpha\pi + 1\right) \\ &+ \varpi^3\left(Q_1Q_2\cos\frac{\alpha\pi}{2} + Q_3\cos\frac{3\alpha\pi}{2}\right) + \varpi^2\left(Q_1Q_3\cos\alpha\pi + 1\right) \\ &+ \varpi\left(Q_2Q_3\cos\frac{\alpha\pi}{2}\right) + \left(Q_3^2 - Q_4^2\right). \end{aligned}$$

Under the conditions $Q_1 > 0, 2Q_2\cos(\alpha\pi)+1 > 0, Q_1Q_2\cos\frac{\alpha\pi}{2}+Q_3\cos\frac{3\alpha\pi}{2} > 0, Q_2Q_3\cos(\alpha\pi)+1 > 0$, $Q_2Q_3\cos\frac{\alpha\pi}{2} > 0$, and $Q_3 < Q_4$, it follows that $Q_2^2 - Q_3^2 < 0$. Define the function $h(\varpi)$ such that

$h(0) = Q_2^2 - Q_3^2 < 0$ and observe that $\lim_{\varpi\to\infty} h(\varpi) = +\infty$. By the intermediate value theorem, there exists a $\varpi_0 > 0$ satisfying $h(\varpi_0) = 0$, which confirms that the characteristic Eq (4.8) possesses a positive root. As the delay parameter τ crosses critical values, the stability of the coexisting equilibrium flips, inducing a Hopf bifurcation in system (2.3). This bifurcation marks the transition from stable to oscillatory dynamics as the real parts of the characteristic roots cross the imaginary axis, driven by the time-delay effect.

Granted that Eq (4.7) with $k = k_0$ admits a positive real root $\omega_+^{k_0} > 0$, and recalling from Eq (4.6) that $\cos\omega\tau = H_1(\omega)$, we can solve for the corresponding τ.

$$\tau_j^{k_0} = \frac{1}{\omega_+^{k_0}}\left(\arccos H_1\left(\omega_+^{k_0}\right) + 2j\pi\right), \quad j = 0, 1, 2, \ldots. \tag{4.10}$$

Let the bifurcation point be defined as

$$\tau_*^{k_0} = \min\left\{\tau_j^{k_0}\right\}, \quad j = 0, 1, 2, \ldots. \tag{4.11}$$

Let $s(\tau) = \vartheta(\tau) + i\omega(\tau)$ denote a root of the characteristic Eq (4.2) satisfying the conditions $\vartheta\left(\tau_j^{k_0}\right) = 0$ and $\omega\left(\tau_j^{k_0}\right) = \omega_+^{k_0}$ at the critical delay parameter $\tau = \tau_j^{k_0}$, where $\tau_j^{k_0}$ corresponds to the j-th critical delay associated with the wave number k_0. By differentiating both sides of Eq (4.2) with respect to τ, we obtain

$$3\alpha s^{3\alpha-1}\frac{ds}{d\tau} + 2Q_1\alpha s^{2\alpha-1}\frac{ds}{d\tau} + Q_2\alpha s^{\alpha-1}\frac{ds}{d\tau} + Q_4 e^{-s\tau}\left(-s - \mu\frac{ds}{d\tau}\right) = 0,$$

which implies,

$$\begin{aligned}\frac{ds}{d\tau} &= \frac{Q_4 s e^{-s\tau}}{3\alpha s^{3\alpha-1} + 2_1\alpha s^{2\alpha-1} + Q_2\alpha s^{\alpha-1} - \tau Q_4 e^{-s\tau}} \\ &= \frac{Q_4(\mathrm{i}\omega)e^{-\mathrm{i}\omega\tau}}{3\alpha(\mathrm{i}\omega)^{3\alpha-1} + 2Q_1\alpha(\mathrm{i}\omega)^{2\alpha-1} + Q_2\alpha(\mathrm{i}\omega)^{\alpha-1} - \tau Q_4 e^{-\mathrm{i}\omega\tau}}.\end{aligned} \tag{4.12}$$

A direct calculation shows that

$$\mathrm{Re}\left(\frac{ds}{d\tau}\right)_{\omega=\omega_+^{k_0},\tau=\tau_*^{k_0}} = \frac{\Pi_1^{k_0}\Pi_2^{k_0} + \Theta_1^{k_0}\Theta_2^{k_0}}{(\Pi_1^{k_0})^2 + (\Theta_2^{k_0})^2}, \tag{4.13}$$

where

$$\begin{aligned}\Pi_1^{k_0} =& Q_4^{k_0}\omega_+^{k_0}\sin\omega_+^{k_0}\tau_*^{k_0}, \\ \Pi_2^{k_0} =& 3\alpha(\omega_+^{k_0})^{3\alpha-1}\sin\frac{3\alpha\pi}{2} + 2Q_1\alpha(\omega_+^{k_0})^{2\alpha-1}\sin\alpha\pi + Q_2\alpha(\omega_+^{k_0})^{\alpha-1}\sin\frac{\alpha\pi}{2} \\ &- Q_4\tau_*^{k_0}\cos\omega_+^{k_0}\tau_*^{k_0}, \\ \Theta_1^{k_0} =& Q_4^{k_0}\omega_+^{k_0}\cos\omega_+^{k_0}\tau_*^{k_0}, \\ \Theta_2^{k_0} =& -3\alpha(\omega_+^{k_0})^{3\alpha-1}\cos\frac{3\alpha\pi}{2} - 2Q_1\alpha(\omega_+^{k_0})^{2\alpha-1}\cos\alpha\pi - Q_2\alpha(\omega_+^{k_0})^{\alpha-1}\cos\frac{\alpha\pi}{2} \\ &+ Q_4\tau_*^{k_0}\sin\omega_+^{k_0}\tau_*^{k_0},\end{aligned}$$

$$
\begin{aligned}
Q_1^{k_0} =& - P_{11} - P_{22} - \delta_1 k_0^2 - \delta_2 k_0^2 - \delta_3 k_0^2 + h,\\
Q_2^{k_0} =& P_{11}P_{22} - P_{11}h - P_{22}h + P_{11}\delta_2 k_0^2 + P_{22}\delta_1 k_0^2 + P_{11}\delta_3 k_0^2 + P_{22}\delta_3 k_0^2\\
& - h\delta_1 k_0^2 - h\delta_2 k_0^2 + \delta_1\delta_2 k_0^4 + \delta_1\delta_3 k_0^4 + \delta_2\delta_3 k_0^4,\\
Q_4^{k_0} =& f_1\left(P_{12}P_{23} - P_{13}s^{\alpha} + P_{13}P_{22} + P_{13}\delta_2 k_0^2\right)\\
& + f_2\left(P_{23}s^{\alpha} - P_{11}P_{23} - P_{13}P_{21} - P_{23}\delta_1 k_0^2\right).
\end{aligned}
$$

Building on the preceding analysis, we establish the following result.

Theorem 7. Under the hypotheses (H1)–(H4), (H7), and (H8), the homogeneous steady state $E^*\left(x_1^*, x_2^*, u^*\right)$ of the time-fractional reaction-diffusion system (2.3) is locally asymptotically stable for time delays $\tau \in \left[0, \tau_*^{k_0}\right)$ and loses stability when $\tau > \tau_*^{k_0}$. Moreover, the system undergoes a Hopf bifurcation at the homogeneous steady state E^* exactly at the critical delay $\tau = \tau_*^{k_0}$.

4.3. Turing bifurcation analysis of the homogeneous steady state

In this subsection, we investigate the Turing instability of system (2.3) in the absence of time delay (i.e., when $\tau = 0$).

Theorem 8. Under the conditions that $P_1 > 0$, $P_2 > 0$, $P_3 + P_4 > 0$, $P_1P_2 > P_3 + P_4$, Hypotheses (H1)–(H3) are satisfied, and the eigenvalue λ satisfies the equation

$$
2\varsigma^3 - 9\delta_1\delta_2\delta_3\varsigma\iota - 2\left(\varsigma^2 - 3\delta_1\delta_2\delta_3\iota\right)^{\frac{3}{2}} + 27\left(\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + P_4\right)(\delta_1\delta_2\delta_3)^2 = 0,
$$

then system (2.3) experiences a Turing bifurcation at the time delay $\tau = 0$.

Proof. Under this scenario, the characteristic equation reduces to the following form:

$$
\lambda^3 + R_1(k^2)\lambda^2 + R_2(k^2)\lambda + R_3(k^2) = 0, \tag{4.14}
$$

where

$$
\begin{aligned}
R_1(k^2) =& (\delta_1 + \delta_2 + \delta_3)k^2 + P_1,\\
R_2(k^2) =& (\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3)k^4\\
& - (\delta_1(-gx_1^* - h) + \delta_2(b - 2gx_1^* - gx_2^* - c_1u^* - \mu - h)\\
& + \delta_3(b - 3gx_1^* - 3gx_2^* - c_1u^* - c_2u^* - \mu - d))k^2 + P_2,\\
R_3(k^2) =& (\delta_1 + \delta_2 + \delta_3)k^6\\
& - (\delta_2\delta_3(b - 2gx_1^* - gx_2^* - c_1u^* - \mu) + \delta_1\delta_3(-d - gx_1^* - 2gx_2^* - c_2u^*)\\
& + \delta_1\delta_2(-h))k^4,\\
& + ((-d - gx_1^* - 2gx_2^* - c_2u^*)(-h)\delta_1\\
& + (b - 2gx_1^* - gx_2^* - c_1u^* - \mu)(-h)\delta_2\\
& + (-d - gx_1^* - 2gx_2^* - c_2u^*)(b - 2gx_1^* - gx_2^* - c_1u^* - \mu)\delta_3\\
& - (-gx_1^*)(\mu - gx_2^*)\delta_3 - f_1(-c_1x_1^*)\delta_2 - f_2(-c_2x_2^*))k^2\\
& + P_3 + P_4.
\end{aligned}
$$

The stability conditions of system (2.2) in the absence of the diffusion term and with $\tau = 0$ have been previously explored in Case 1 of Section 3.3.2. Turing bifurcation takes place when the real part of at least one eigenvalue of the characteristic equation becomes positive. At the Turing bifurcation point, we have $\mathrm{Re}(\lambda(K_T^2)) = 0$ and $\mathrm{Im}(\lambda(K_T^2)) = 0$.

Consider the case when $R_3(k^2) = 0$. The eigenequation simplifies to

$$\lambda\left(\lambda^2 + R_1(k^2)\lambda + R_2(k^2)\right) = 0, \tag{4.15}$$

where $\lambda = 0$ is a root, placing the system in a critical condition. When $R_3(k^2)$ transitions from positive to negative, at least one eigenvalue of the characteristic equation shifts from having a negative real part to a positive real part, signaling the occurrence of Turing instability. The equation $R_3(k^2) = 0$ is a cubic in k^2; by Cardano's formula, its smallest positive real solution is

$$K_T^2 = \frac{-\varsigma + \sqrt{\varsigma^2 - 3\delta_1\delta_2\delta_3\iota}}{3\delta_1\delta_2\delta_3}, \tag{4.16}$$

where

$$\begin{aligned}
\varsigma =&\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3 + \delta_1\delta_2 h \\
&- \delta_2\delta_3(b - 2gx_1^* - gx_2^* - c_1u^* - \mu) - \delta_1\delta_3(-d - gx_1^* - 2gx_2^* - c_2u^*), \\
\iota =&\delta_1 + \delta_2 + \delta_3 - (\delta_1(-gx_1^* - h) + \delta_2(b - 2gx_1^* - gx_2^* - c_1u^* - \mu - h) \\
&+ \delta_3(b - 3gx_1^* - 3gx_2^* - c_1u^* - c_2u^* - \mu - d)) \\
&+ ((-d - gx_1^* - 2gx_2^* - c_2u^*)(-h)\delta_1 \\
&+ (b - 2gx_1^* - gx_2^* - c_1u^* - \mu)(-h)\delta_2 \\
&+ (-d - gx_1^* - 2gx_2^* - c_2u^*)(b - 2gx_1^* - gx_2^* - c_1u^* - \mu)\delta_3 \\
&- (-gx_1^*)(\mu - gx_2^*)\delta_3 - f_1(-c_1x_1^*)\delta_2 - f_2(-c_2x_2^*)).
\end{aligned}$$

Under the condition that the eigenvalue λ satisfies

$$\begin{aligned}
&2\varsigma^3 - 9\delta_1\delta_2\delta_3\varsigma\iota - 2\left(\varsigma^2 - 3\delta_1\delta_2\delta_3\iota\right)^{\frac{3}{2}} \\
&+ 27\left(\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + P_4\right)(\delta_1\delta_2\delta_3)^2 = 0,
\end{aligned} \tag{4.17}$$

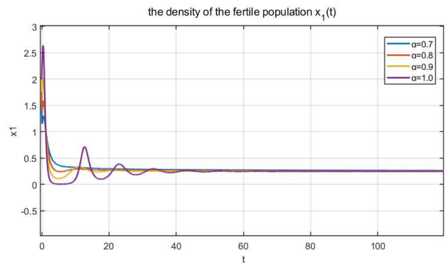
System (2.3) experiences a Turing bifurcation.

5. Numerical simulations

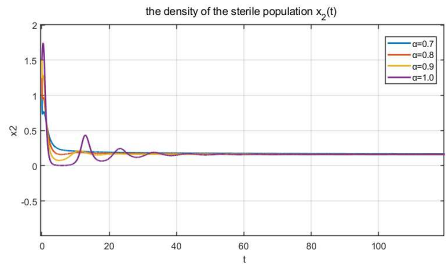
In this section, we present numerical simulation results for system (2.2) in Examples 1–3 and system (2.3) in Examples 4 and 5, respectively, to validate the theoretical findings. Consider the following parameter values: $b = 3$, $g = 0.05$, $\mu = 1$, $d = 0.05$, $c_1 = 0.8$, $c_2 = 0.6$, $h = 0.2$, $f_1 = 1.2$, $f_2 = 1.2$.

Example 1. In this example, Figures 1 and 2 illustrate the dynamic behaviors of $x_1(t)$, $x_2(t)$, and $u(t)$ at $\tau = 0$ and $\tau = 1$, respectively, for four distinct values of the fractional order α. One can observe that the convergence rate is influenced by different fractional orders: as α decreases gradually from 1, the oscillatory behavior of the solutions diminishes progressively, leading to stabilization within a shorter time frame.

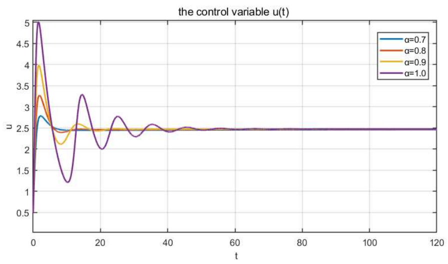
Remark 1. In Figure 2, it is observed that when $\alpha = 1$, the solution of system (2.2) exhibits unstable periodic oscillatory behavior. As the fractional order α decreases from 1, the solutions of the system tend to stabilize gradually: the smaller the value of α, the faster the solutions converge to stability. This further confirms that fractional orders can expand the stability region of the system.



(a)

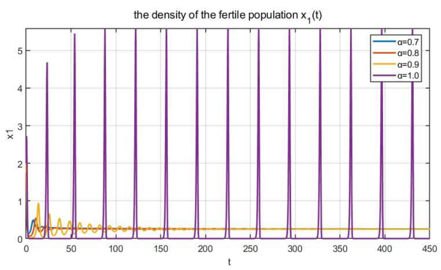


(b)

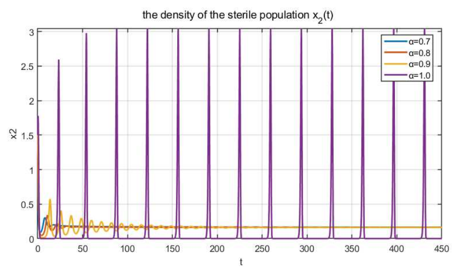


(c)

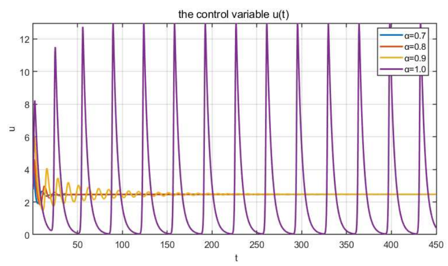
Figure 1. The time series plot of the positive equilibrium point E^* of system (2.2) when $\tau = 0$.



(a)

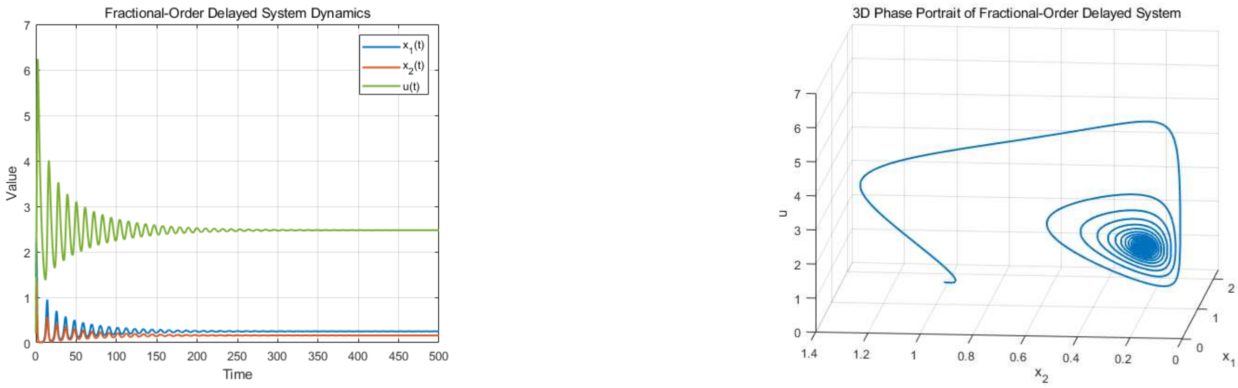


(b)



(c)

Figure 2. The time series plot of the positive equilibrium point E^* of system (2.2) when $\tau = 1$.



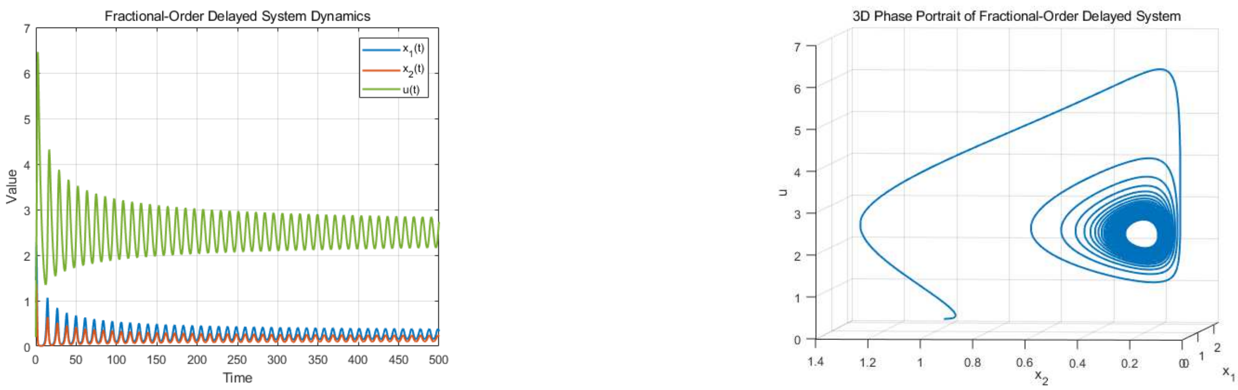
(a) (b)

Figure 3. The time series plot and phase portrait of the positive equilibrium point E^* in system (2.2) when $\tau = 0.7 < \tau_0$.

Example 2. In this part, we set $\alpha = 0.95$. Using Eqs (3.15) and (3.16), we obtain $\tau_0 = 0.724$. This indicates that a Hopf bifurcation occurs in system (2.2) when the delay parameter reaches $\tau = \tau_0$. For $\tau < \tau_0$, the system is locally asymptotically stable at the coexisting equilibrium E^*, with numerical results shown in Figure 3. Conversely, when $\tau > \tau_0$, E^* loses stability, as demonstrated by the simulations in Figure 4. The consistency between Figures 3 and 4 with the theoretical predictions validates the accuracy of Theorem 4.

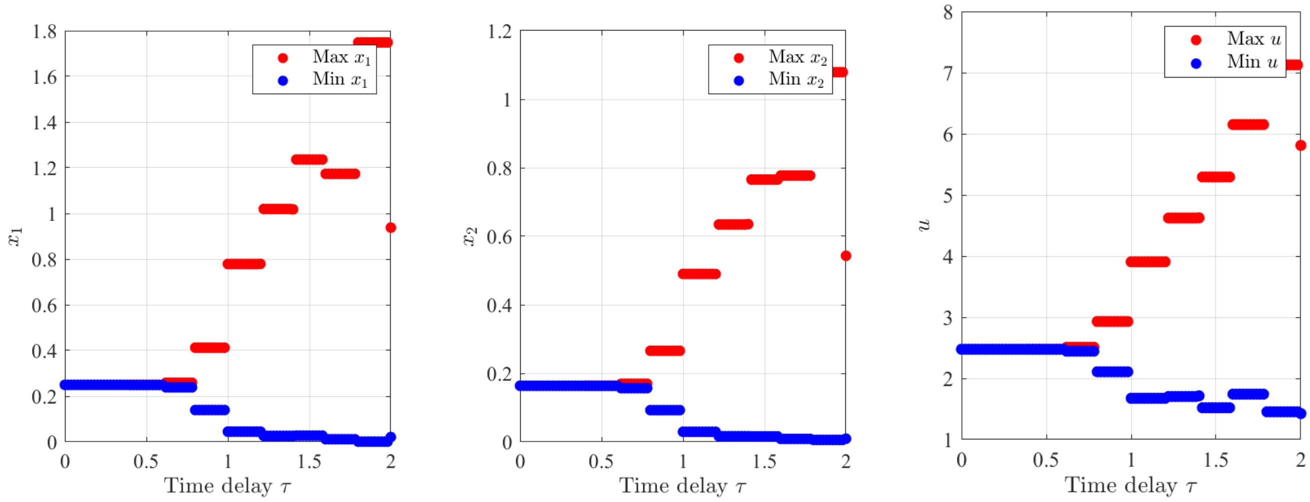
Example 3. In this section, we set $\alpha = 0.97$, 0.95, and 0.91, with corresponding numerical simulation results presented in Figures 5–7. It can be clearly observed that the critical delay τ_0 increases as the fractional order α decreases.

Remark 2. In Figure 2, it is observed that the coexistence equilibrium E^* remains stable when $\alpha = 0.9$ and $\tau = 1$. In contrast, for $\alpha = 0.95$, the equilibrium E^* loses stability when τ exceeds the critical delay $\tau_0 = 0.724$, as illustrated in Figure 4 (where $\alpha = 0.95$ and $\tau = 0.8$, demonstrating instability of E^*). This behavior demonstrates that the critical delay τ_0 decreases as the fractional order α increases.



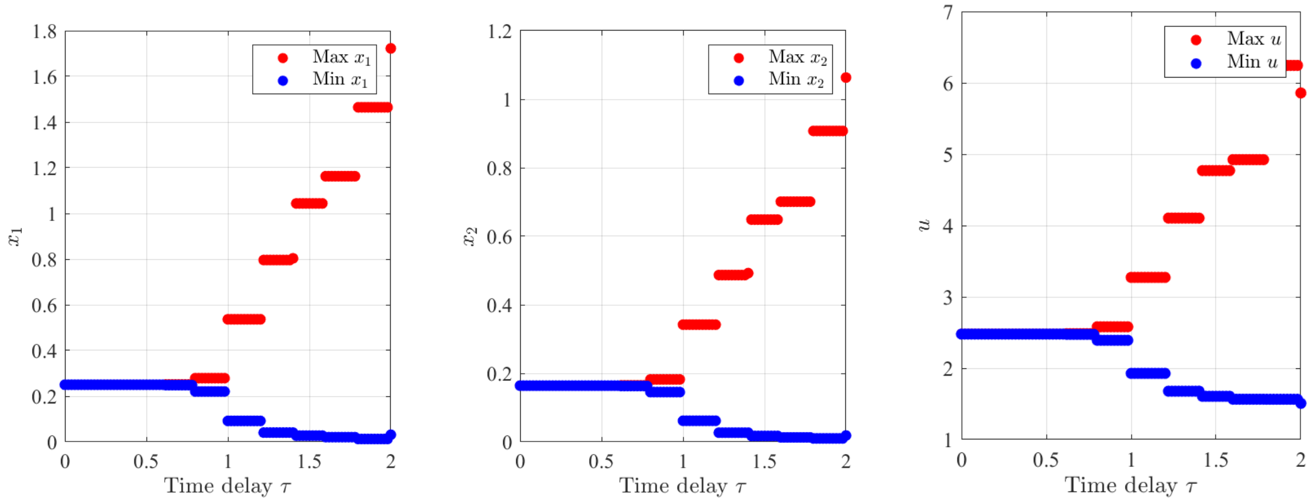
(a) (b)

Figure 4. The time series plot and phase portrait of the positive equilibrium point E^* in system (2.2) when $\tau = 0.8 > \tau_0$.



(a) **(b)** **(c)**

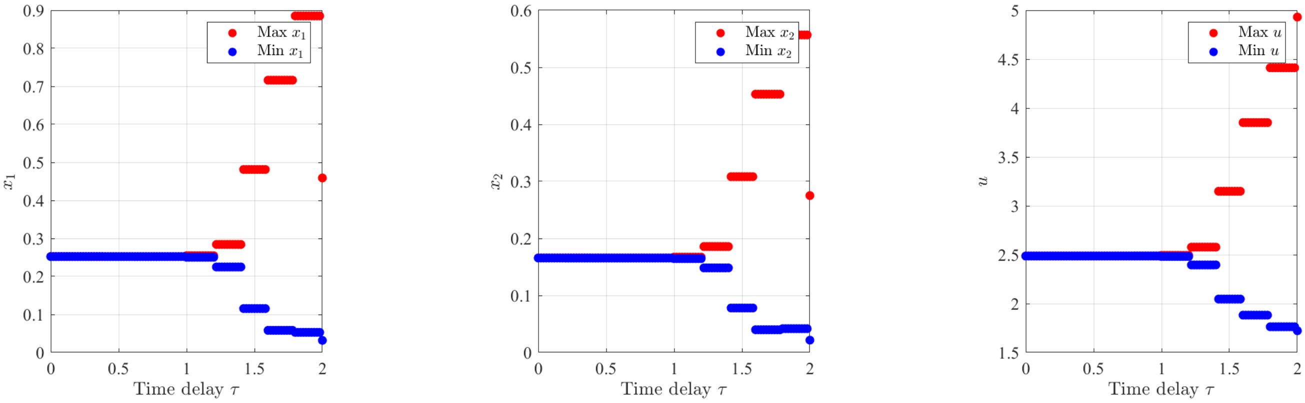
Figure 5. Hopf bifurcation phase diagrams of system (2.2) when $\alpha = 9.7$.



(a) **(b)** **(c)**

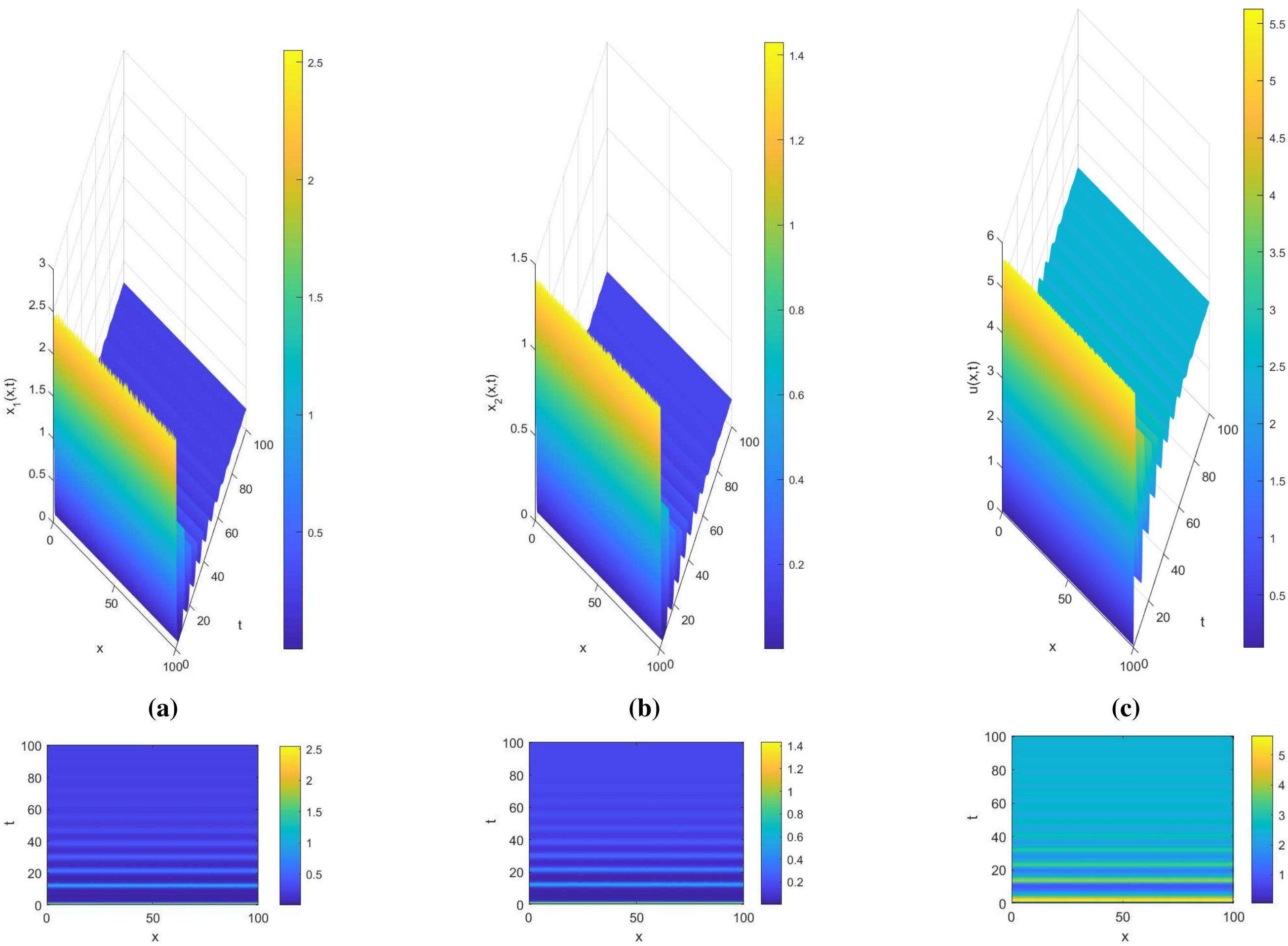
Figure 6. Hopf bifurcation phase diagrams of system (2.2) when $\alpha = 9.5$.

Example 4. In this section, we perform numerical simulations of system (2.3) to investigate the effects of time delays on the reaction-diffusion model. Grounded in the biological context of wild animal pest control—the focus of the present study—we consider the migration capabilities of insect and small mammal populations. The diffusion coefficient for the fertile population x_1 is set as $\delta_1 = 0.1$, while the infertile population exhibits slightly weaker migration ability, with $\delta_2 = 0.06$. The diffusion coefficient of the control term δ_3, which is hypothesized to be 2–5 times that of δ_1, is set to $\delta_3 = 0.2$ to avoid excessively dominating the numerical results. All other parameters are consistent with those in Example 1.



(a) **(b)** **(c)**

Figure 7. Hopf bifurcation phase diagrams of system (2.2) when $\alpha = 9.1$.



(a) **(b)** **(c)**

(d) **(e)** **(f)**

Figure 8. Space-time plots of the positive equilibrium point E^* of system (2.3) when $\tau = 0.19 < \tau_*^{k_0}$.

For $\tau < \tau_*^{k_0}$, system (2.3) is locally asymptotically stable at the positive homogeneous steady state E^*, as demonstrated by the numerical results in Figure 8. Conversely, when $\tau > \tau_*^{k_0}$, E^* loses stability, as shown in Figure 9. The consistency between these simulations and the theoretical predictions validates the findings of Theorem 7.

Remark 3. In Figures 8 and 9, it can be readily observed that the critical value $\tau_*^{k_0}$ for the Hopf bifurcation with the diffusion term is significantly smaller than the critical value τ_0 of the system in the absence of diffusion. Moreover, the inclusion of the diffusion term profoundly influences the dynamic behavior of the system.

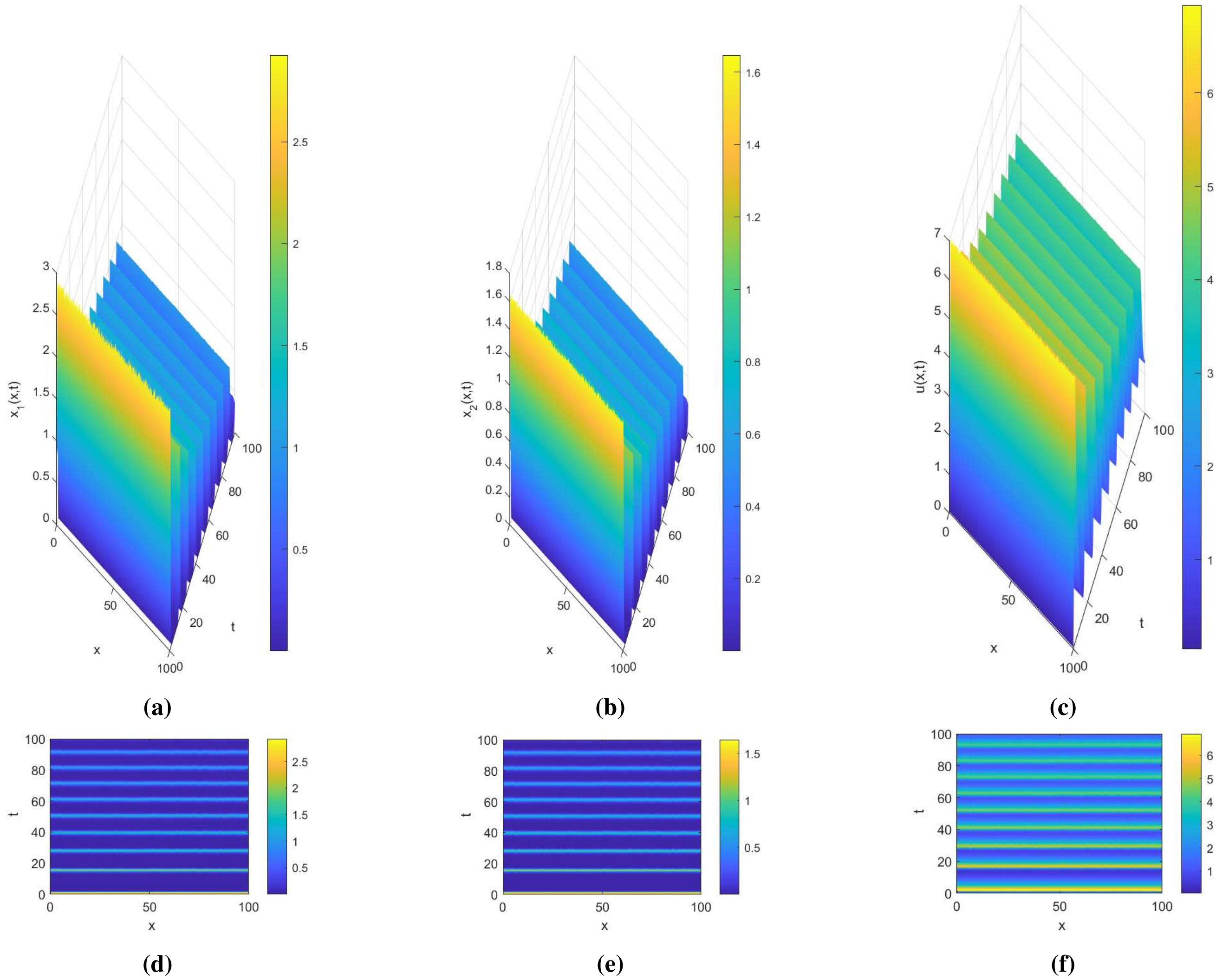


Figure 9. Space-time plots of the positive equilibrium point E^* of system (2.3) when $\tau = 0.42 > \tau_*^{k_0}$.

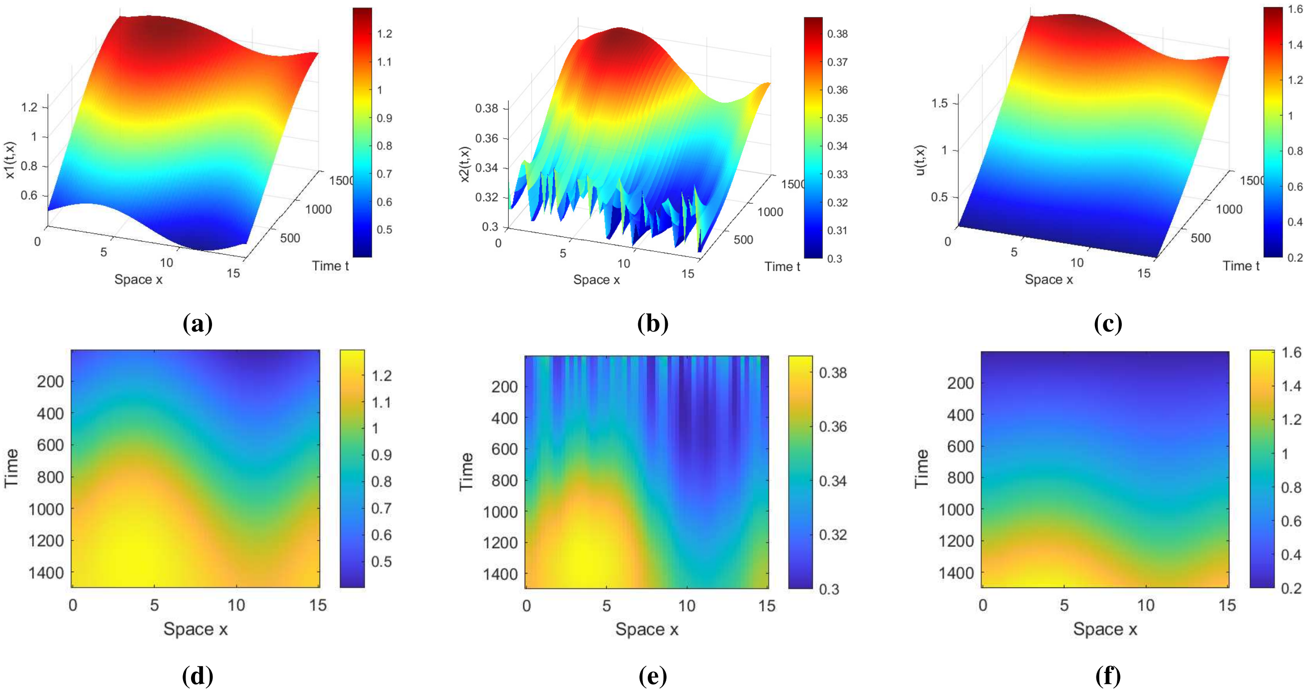


Figure 10. When $\alpha = 0.95$, $\tau = 0$, $b = 1.2$, $g = 0.1$, $\mu = 0.2$, $d = 0.4$, $c_1 = 0.5$, $c_2 = 0.3$, $h = 0.2$, $f_1 = 0.6$, $f_2 = 0.4$, $\delta_1 = 0.1$, $\delta_2 = 0.06$, $\delta_3 = 0.2$, space-time plot of the positive equilibrium point E^* of system (2.3).

Example 5. In this example, we select distinct parameter values to better demonstrate the Turing bifurcation diagram of system (2.3). Setting $\alpha = 0.95$, $\tau = 0$, $b = 1.2$, $g = 0.1$, $\mu = 0.2$, $d = 0.4$, $c_1 = 0.5$, $c_2 = 0.3$, $h = 0.2$, $f_1 = 0.6$, $f_2 = 0.4$, $\delta_1 = 0.1$, $\delta_2 = 0.06$, and $\delta_3 = 0.2$, the system exhibits Turing instability, as shown in Figures 9 and 10.

6. Conclusions

This study establishes a novel framework for understanding pest population dynamics by integrating fractional-order calculus, spatial diffusion, and time-delayed feedback control within SIT systems. The introduction of fractional-order derivatives through Caputo operators significantly broadens the stability region of pest-control models, enabling a more nuanced characterization of memory-dependent ecological interactions under environmental disturbances. Spatial heterogeneity, captured via diffusion terms, amplifies the system's sensitivity to time delays, reducing the critical threshold for Hopf bifurcation and highlighting the necessity of incorporating spatial dynamics into management protocols. Furthermore, the identification of Turing instability conditions under specific parameter combinations reveals a previously unexplored mechanism for spatially heterogeneous pattern formation, offering theoretical foundations for regionalized pest suppression strategies.

Theoretical advancements extend beyond traditional integer-order models by rigorously analyzing bifurcation behaviors and parameter sensitivity. Key parameters, such as diffusion coefficients and fractional-order values, emerge as pivotal determinants of system stability, providing actionable guidelines for model calibration. Practically, these findings advocate for adaptive SIT implementation, where sterile insect release frequencies and quantities are dynamically adjusted based on real-time population monitoring and feedback control mechanisms. Spatial management measures, including targeted buffer zones in high-risk areas, are proposed to counteract instability driven by diffusion processes.

Future research should focus on enhancing model realism through the incorporation of ecological complexities such as intraspecific competition and predator-prey interactions. Empirical validation using field data, particularly regarding Turing pattern predictions, will strengthen the model's predictive power. The framework's adaptability also opens interdisciplinary avenues: In public health, it could elucidate spatiotemporal disease transmission dynamics, while in environmental management, it supports the design of sustainable strategies for ecosystem resilience. By bridging mathematical innovation with ecological applications, this work advances both theoretical ecology and practical pest management, laying the groundwork for data-driven decision-making in complex, disturbance-prone environments.

Author contributions

Conceptualization, D.L. and A.M.; methodology, D.L.; software, D.L.; validation, D.L. and A.M.; formal analysis, D.L. and A.M.; writing-original draft preparation, D.L., A.M. and Y.M.; writing-review and editing, D.L., A.M. and Y.M.; supervision, A.M. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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