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Research article

Supercloseness of the NIPG method on Bakhvalov-type meshes for a system of singularly perturbed reaction-diffusion equations

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Abstract: In this paper, a higher order nonsymmetric interior penalty Galerkin (NIPG) method on a Bakhvalov mesh is developed for a weakly coupled system of singularly perturbed reaction-diffusion equations. At first, by selecting special penalty parameters at different mesh points, the supercloseness of the $k + \frac{1}{2}$ order of our proposed method is derived, where k is the degree of polynomials space. Then, an optimal order of uniform convergence analysis in a balanced norm is performed. Finally, some numerical experiments are given to support our theoretical findings.

Keywords: singularly perturbed; the NIPG method; Bakhvalov-type mesh; balanced norm; supercloseness

1. Introduction

In this paper, we develop a nonsymmetric interior penalty Galerkin (NIPG) method for the following weakly coupled system of singularly perturbed reaction-diffusion equations

$$\begin{cases} L\vec{u} := -\varepsilon^2 \frac{\mathrm{d}^2\vec{u}}{\mathrm{d}x^2} + \vec{A}\vec{u} = \vec{f}, \quad x \in \Omega = (0, 1), \\ \vec{u}(0) = \vec{0}, \quad \vec{u}(1) = \vec{0}, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, $\vec{u} := (u_1(x), u_2(x))^T$, $\vec{f} := (f_1(x), f_2(x))^T$, and

$$\vec{A} = \begin{pmatrix} a_{11}(x), & a_{12}(x) \\ a_{21}(x), & a_{22}(x) \end{pmatrix}.$$

Assume that $a_{ij}(x)$,$f_i(x)$, $i, j = 1, 2$ are sufficiently smooth functions and there exists a constant β

such that

$$a_{11}(x) > |a_{12}(x)|\,,\ a_{22}(x) > |a_{21}(x)|\,,\ a_{12}(x) \le 0,\ a_{21}(x) \le 0,\quad \forall x \in \Omega, \tag{1.2}$$

$$\min_{x\in\bar{\Omega}} \{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\} > \beta^2 > 0. \tag{1.3}$$

Singularly perturbed problems are characterized by the presence of a small parameter $0 < \varepsilon \ll 1$ which multiplies the highest-order derivative in the differential equation. This leads to the formation of thin boundary layers near the domain boundaries where the solution rapidly changes. Outside these layers, the solution typically varies smoothly. The multi-scale nature of these problems poses significant challenges for numerical approximation, as standard methods on uniform meshes often fail to capture the sharp transitions in boundary layers, unless an impractically large number of mesh points is used.

The study of singularly perturbed problems is motivated by their widespread occurrence in applied sciences and engineering, including the following examples: Convection-dominated flows in fluid dynamics, where ε represents the inverse of the Reynolds number; semiconductor device modeling, where the small parameter arises from the Debye length; chemical reaction-diffusion systems with fast and slow reaction rates; and problems in elasticity and plasticity with small hardening parameters. Therefore, it is important to develop robust numerical methods for such problems.

When standard numerical methods (such as finite differences or finite elements on uniform meshes) are applied to singularly perturbed problems, they often produce non-physical oscillations that pollute the numerical solution throughout the domain. This phenomenon occurs when the mesh is not fine enough to resolve the boundary layers. To obtain reliable results, the mesh size h must satisfy $h < \varepsilon$, which becomes computationally prohibitive for very small ε. This limitation necessitates the development of specialized numerical techniques that can accurately capture the multi-scale behavior of the solution without requiring excessive computational resources.

Layer-adapted meshes, such as the Bakhvalov and Shishkin meshes used in this work, provide an effective solution to this challenge. These meshes are specially designed to be fine within boundary layers and coarse in the smooth regions of the solution. This nonuniform refinement allows for an accurate resolution of the sharp transitions in boundary layers while maintaining the computational efficiency. The Bakhvalov-type mesh employed in this paper is particularly advantageous as it typically leads to optimal convergence rates without the logarithmic factors that appear in Shishkin mesh analyses.

The numerical solutions of various systems of singularly perturbed differential equations have been considered by several authors in the recent research literature; see [3,4] for surveys of these results. In particular, finite element methods were used to solve systems of reaction-diffusion equations in [4–7]; however, the analyses were carried out in the standard-weighted energy norm associated with Eq (1.1), which is too weak to recognize the features of the layers. In recent years, researchers have addressed this issue by introducing balanced norms that can better capture each layer of components. To the best of our knowledge, most of the current studies on the balanced norm are limited to the scalar reaction-diffusion equations on Shishkin meshes, such as [8–13]. At the same time, Liu et al. [14–16] analyzed the supercloseness properties of the scalar reaction-diffusion equations on the balanced norm. However, for the systems of reaction-diffusion problems, there is only [17], which proved that the classical finite element method (FEM) using quadratic C^1 splines is an order of $O(N^{-1}\ln N)$ in the balanced norm.

As is stated in [18], the standard Galerkin method may be unstable even on the layer adapted mesh. It is very necessary to study more stabilization FEMs on layer-adapted meshes for singularly perturbed problems. To the best of our knowledge, the nonsymmetric interior penalty Galerkin (NIPG) method, which was proposed in [19], is a stabilization FEM. It employs penalty parameters to solve the discontinuity of finite element functions across the element interface, and allows the construction of high-order schemes in a natural way to approximate the exact solution. Recently, in [20], the authors developed a nonsymmetric interior penalty Galerkin (NIPG) method to solve a singularly perturbed convection-diffusion equation with two small parameters and proved the supercloseness in a balanced norm of the proposed method.

In this paper, the supercloseness in a balanced norm of a NIPG method for a weakly coupled system of singularly perturbed reaction-diffusion equation is studied on a Bakhvalov-type mesh. For this purpose, we use a locally weighted L^2 projection outside the layer, and a Gauß Lobatto projection inside the layer. On that basis, we derive the supercloseness of the $k + \frac{1}{2}$ order and prove an optimal order of uniform convergence in a balanced norm. Here, "superclosenes" means that the convergence order for the error between some interpolation of the solution u and the numerical solution u^N in some norm is greater than the order for $u - u^N$ in the same norm.

The paper is organized as follows: In Section 2, we present a priori information of the solution, describe the Bakhvalov-type mesh suitable for the problem, and on the basis of that, the NIPG method for the singularly perturbed problem is introduced; in Section 3, we design a new interpolation and obtain the corresponding interpolation error estimate; in Section 4, the main conclusion about supercloseness in the balanced norm is presented; finally, numerical tests are performed to verify our proposed theoretical results.

Notations. Throughout this article, the notations C and C_i denote the generic constant and the fixed constant respectively, which are independent of ε and N.

Assumption 1. In our following analysis, we shall assume that $\varepsilon \leq CN^{-1}$.

2. Decomposition of the solution

Similar to [21], the continuous solution $\vec{u}$ of (1.1) can be divided into the smooth part $\vec{S} = (S_1, S_2)^T$ and the boundary layer part $\vec{E} = (E_1, E_2)^T$, which is given by the following:

$$\vec{u} = \vec{S} + \vec{E}. \tag{2.1}$$

Note that these two parts satisfy the following two boundary value problems, respectively,

$$L\vec{S} = \vec{f} \quad \text{on} \quad \Omega \quad \textit{and} \quad \vec{S} = \vec{A}^{-1}\vec{f} \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

$$L\vec{E} = \vec{0} \quad \text{on} \quad \Omega \quad \textit{and} \quad \vec{E} = \vec{u} - \vec{S} \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

For $k = 0, 1, \cdots, 4$ and $i = 1, 2$, one has the following:

$$\left|S_i^{(k)}(x)\right| \leq C, \quad \forall x \in \Omega, \tag{2.4}$$

$$\left|E_i^{(k)}(x)\right| \leq C\varepsilon^{-k}D_\varepsilon(x), \forall x \in \Omega, \tag{2.5}$$

where $D_\varepsilon(x) = exp(-\beta x/\varepsilon) + exp(-\beta(1-x)/\varepsilon)$.

Proof. The proof can be found in [21].

3. The discretization method

3.1. The Bakhvalov-type Mesh

In order to construct the Bakhvalov-type mesh (see [22]), we first define the following mesh generating function:

$$x = \psi(t) = \begin{cases} -\frac{\sigma\varepsilon}{\beta}\ln(1-4(1-\varepsilon)t), & t \in [0, 1/4], \\ d_1(t-1/4) + d_2(t-3/4), & t \in [1/4, 3/4], \\ 1 + \frac{\sigma\varepsilon}{\beta}\ln(1-4(1-\varepsilon)(1-t)), & t \in [3/4, 1], \end{cases} \tag{3.1}$$

where $\sigma \geq k+1$ and d_1, d_2 are used to ensure the continuity of $\psi(t)$ at $t = 1/4$ and $t = 3/4$, respectively.

Then for a given integer $N \geq 8$ divisible by 4, let $x_j = \psi(j/N)$ for $0 \leq j \leq N$. Set $\tau_N = \{I_j = [x_{j-1}, x_j], j = 1, 2, \ldots, N\}$. The length of I_j is defined by $h_j = x_j - x_{j-1}$ and a generic subinterval by I.

Remark 3.1. *The Bakhvalov-type mesh defined by Eq (3.1) can be viewed as a specific instance of a more general class of layer-adapted meshes constructed via mesh-generating functions. The core idea is to utilize a function $\psi(t)$ that is steep in the boundary layer regions ($x \in [0, \tau] \cup [1-\tau, 1]$) and relatively flat in the regular region ($x \in [\tau, 1-\tau]$), where τ is a transition point often chosen as $O(\varepsilon \ln N)$.*

The convergence analysis presented in this paper, particularly the interpolation error estimates (Lemma 4.2, Theorem 4.5) and the subsequent supercloseness analysis (Section 5), relies on key properties of the mesh, such as the estimates provided in Lemmas 3.2 and 3.3. These properties can be guaranteed for a general mesh-generating function $\psi(t)$ that satisfies conditions such as the following:

- *$\psi'(t) = O(\varepsilon)$ for t in the layer regions,*
- *$\psi'(t) = O(1)$ for t in the regular region,*
- *The transition points are located at $t = \tau$ and $t = 1-\tau$ with $\tau = O(\varepsilon \ln N)$.*

Therefore, the methodologies and results established herein can be extended to such a general framework, which leads to a unified convergence theory for the NIPG method on a broad class of layer-adapted meshes.

Lemmas 3.2 and 3.3 provide some important properties of the Bakhvalov-type mesh. The reader is referred to [23] for the detailed proof.

Lemma 3.2. *For $j = 1, \cdots, N$, the mesh steps h_j of the above Bakhvalov mesh (3.1) satisfy the following estimations:*

$$C_1\varepsilon \leq h_j \leq C_2\varepsilon, \quad 1 \leq j \leq N/4-1, \tag{3.2}$$

$$C_3\varepsilon \leq h_{N/4} \leq C_4 N^{-1}, \tag{3.3}$$

$$C_5 N^{-1} \leq h_j \leq C_6 N^{-1}, \quad N/4+1 \leq j \leq N/2, \tag{3.4}$$

$$x_{N/4-1} = 1 - x_{3N/4+1} \geq C\sigma\varepsilon \ln N, \quad x_{N/4} = 1 - x_{3N/4} \geq C\sigma\varepsilon \ln(1/\varepsilon). \tag{3.5}$$

Note, $h_j = h_{N-j+1}$ for $j = 1, \cdots, N/2$.

Lemma 3.3. *For the Bakhvalov-type mesh (3.1), we have:*

$$h_j^{\mu}\max_{x\in I_j} D_{\varepsilon}(x) \le C\varepsilon^{\mu}N^{-\mu} \text{ for } j \le N/4-1 \text{ and } 0 \le \mu \le k+1,$$
$$\max_{x\in I_j} D_{\varepsilon}(x) \le CN^{-(k+1)} \text{ for } j \ge N/4.$$

3.2. NIPG Method

For each element I_j, $j = 1, \cdots, N$ and a nonnegative s, let $H^s(\Omega, \tau_N) = \{u \in L^2(\Omega) : u|_{I_j} \in H^s(I_j), \forall j = 1, 2, \cdots, N\}$ be the broken Sobolev space of order s. The corresponding norm and semi-norm are defined by the followings:

$$\left\|\vec{u}\right\|^2_{s,\tau_N} = \sum_{j=1}^{N} \|u_1\|^2_{s,I_j} + \sum_{j=1}^{N} \|u_2\|^2_{s,I_j}, \quad |\vec{u}|^2_{s,\tau_N} = \sum_{j=1}^{N} |u_1|^2_{s,I_j} + \sum_{j=1}^{N} |u_2|^2_{s,I_j},$$

where $\|\cdot\|_{s,I_j}$ and $|\cdot|_{s,I_j}$ are the usual Sobolev norm and semi-norm in $H^s(I_j)$, respectively. It is worth noting that $\|\cdot\|_I$ and $(\cdot,\cdot)_I$ are usually used to represent the $L^2(I)$-norm and the $L^2(I)$-inner product, respectively. In the case of $I = \Omega$, the subscript will be removed from the notation. Then, on the Bakhvalov-type mesh, the finite element space can be defined by the followings:

$$V_N^k = \{u \in L^2(\Omega) : u|_{I_j} \in \mathbb{P}_k(I_j), \ \forall j = 1, 2, \cdots, N\},$$

where $\mathbb{P}_k(I_j)$ represents the space of polynomials of degree at most k on I_j.

It should be pointed out that the functions in V_N^k are completely discontinuous at the boundaries between two adjacent cells. Therefore, we define the jump and average of the trace of u at the interior node x_j by the followings:

$$[v(x_j)] = v(x_j^+) - v(x_j^-), \ \ \{v(x_j)\} = \frac{1}{2}\left(v(x_j^+) + v(x_j^-)\right) \ \ \forall j = 1, \cdots, N-1,$$

where $v(x_j^+)$ and $v(x_j^-)$ are the values of u at x_j, from the right cell and the left cell of x_j, respectively. Additionally, for the boundary nodes x_0 and x_N, we provide the definitions as follows:

$$[v(x_0)] = v(x_0^+), \ \ \{v(x_0)\} = v(x_0^+), \ \ [v(x_N)] = -v(x_N^-), \ \ \{v(x_N)\} = v(x_N^-).$$

Based on the above definitions, the NIPG finite element method of (1.1) reads as follows: Find a numerical solution $\vec{u}_N = \left(u_1^N, u_2^N\right)^T \in V_N^k$ such that

$$a(\vec{u}_N, \vec{v}_N) = a_1(\vec{u}_N, \vec{v}_N) + a_2(\vec{u}_N, \vec{v}_N) = L(\vec{v}_N), \ \forall \vec{v}_N = (v_1^N, v_2^N)^T \in V_N^k, \tag{3.6}$$

where

$$a_1(\vec{u}_N,\vec{v}_N)=\varepsilon^2\sum_{j=1}^{N}\int_{I_j}u_1^N(x)'v_1^N(x)'dx+\varepsilon^2\sum_{j=0}^{N}\left(\{u_1^N(x_j)'\}[v_1(x_j)]-\{v_1^N(x)'\}[u_1(x_j)]\right)$$
$$+\sum_{j=0}^{N}\rho(x_j)[u_1^N(x_j)][v_1^N(x_j)]+\sum_{j=1}^{N}\int_{I_j}\left(a_{11}(x)u_1^N(x)+a_{12}(x)u_2^N(x)\right)v_1^N(x)dx,$$
$$a_2(\vec{u}_N,\vec{v}_N)=\varepsilon^2\sum_{j=1}^{N}\int_{I_j}u_2^N(x)'v_2^N(x)'dx+\varepsilon^2\sum_{j=0}^{N}\left(\{u_2^N(x_j)'\}[v_2(x_j)]-\{v_2^N(x)'\}[u_2(x_j)]\right)$$
$$+\sum_{j=0}^{N}\rho(x_j)[u_2^N(x_j)][v_2^N(x_j)]+\sum_{j=1}^{N}\int_{I_j}\left(a_{21}(x)u_1^N(x)+a_{22}(x)u_2^N(x)\right)v_2^N(x)dx,$$
$$L(\vec{v}_N)=\sum_{j=1}^{N}\int_{I_j}(f_1^N(x)v_1^N(x)+f_2^N(x)v_2^N(x))dx.$$

Here, for $j=0,1,\cdots,N$, $\rho(x_j)$ represent the corresponding penalty parameters associated with the node x_j, which are defined by the followings:

$$\rho(x_j)=\begin{cases}\varepsilon N^2, & 0\le j\le N/4-2,\ 3N/4+2\le j\le N-1,\\ \varepsilon N, & j=N/4-1,3N/4+1,\\ \varepsilon, & N/4\le j\le 3N/4.\end{cases}\tag{3.7}$$

Lemma 3.4. *Let $\vec{u}$ be the solution of problem (1.1) and $\vec{u}_N$ be the corresponding NIPG finite element solution. Then, the bilinear form $a(\cdot,\cdot)$ defined in Eq (3.6) has the following Galerkin orthogonality property:*

$$a(\vec{u}-\vec{u}_N,\vec{v}_N)=0,\ \ \forall\vec{v}_N\in V_N^k.\tag{3.8}$$

Proof. The proof is similar to [24, Lemma 3.1].

For any $\vec{v}\in V_N^k$, define the following energy norm and balanced norm:

$$\begin{aligned}\left\|\vec{v}\right\|_\varepsilon^2=&\varepsilon^2\sum_{j=1}^{N}\left(\left\|v_1'\right\|_{I_j}^2+\left\|v_2'\right\|_{I_j}^2\right)+\beta^2\sum_{j=1}^{N}\left(\|v_1\|_{I_j}^2+\|v_2\|_{I_j}^2\right)\\&+\sum_{j=0}^{N}\rho(x_j)[v_1(x_j)]^2+\sum_{j=0}^{N}\rho(x_j)[v_2(x_j)]^2,\end{aligned}\tag{3.9}$$

$$\begin{aligned}\left\|\vec{v}\right\|_b^2=&\varepsilon\sum_{j=1}^{N}\left(\left\|v_1'\right\|_{I_j}^2+\left\|v_2'\right\|_{I_j}^2\right)+\beta^2\sum_{j=1}^{N}\left(\|v_1\|_{I_j}^2+\|v_2\|_{I_j}^2\right)\\&+\sum_{j=0}^{N}\rho(x_j)[v_1(x_j)]^2+\sum_{j=0}^{N}\rho(x_j)[v_2(x_j)]^2.\end{aligned}\tag{3.10}$$

Then, by simple calculation, we obtain the following coercivity

$$\left\|\vec{v}_N\right\|_\varepsilon^2\le a(\vec{v}_N,\vec{v}_N),\ \forall\vec{v}_N\in V_N^k.\tag{3.11}$$

Based on the Lax-Milgram lemma, our proposed NIPG scheme (3.6) has a unique solution $\vec{u}_N$.

4. Interpolation error analysis

4.1. Interpolation

In this section, we first introduce a new interpolation operator Π as follows:

$$(\Pi\vec{u})|_{I_j} = \begin{cases} (L_k\vec{u})|_{I_j}, & 1 \le j \le N/4 - 1 \text{ and } 3N/4 + 2 \le j \le N, \\ (P\vec{u})|_{I_j}, & N/4 \le j \le 3N/4 + 1, \end{cases} \tag{4.1}$$

where $(L_k\vec{u})|_{I_j}$ is the k-degree Gauß Lobatto interpolation on I_j, and $(P\vec{u})|_{I_j}$ is the locally weighted L^2 projection on I_j.

On the one hand, we assume that $x_{j-1} = t_0 \le t_1 \le \cdots \le t_k = x_j$ are the Gauß Lobatto points, where $t_1, t_2, \cdots, t_{k-1}$ are zeros of the derivative of the k-degree Legendre polynomial on I_j. For $u_m \in C(\bar{\Omega})$ ($m = 1, 2$) and $j = 1, 2, \cdots, N$, let $(L_k u_m)|_{I_j}$ be the k-degree Lagrange interpolation at the Gauß Lobatto points $\{t_s\}_{s=0}^{k}$. Then, for $u_m(x) \in H^{k+2}(I_j)$, we have the following:

$$\left|(u_m' - (L_k u_m)', v')_{I_j}\right| \le Ch_j^{k+1} |u_m|_{k+2,I_j} |v|_{1,I_j}, \quad \forall v \in \mathbb{P}_k(I_j). \tag{4.2}$$

On the other hand, define the locally weighted L^2 projection $Pu_m \in P^k(I_j)$ as follows (see [25]):

$$(\sum_{m=1}^{2} a_{mm}(Pu_m - u_m), v)_{I_j} = 0, \quad \forall v \in \mathbb{P}_k(I_j), \quad j = 1, 2, \ldots, N. \tag{4.3}$$

The well-posedness of the coupled weighted L^2 projection P defined in Eq (4.3) follows from the standard finite element theory. For each element I_j, Eq (4.3) defines a linear system for the coefficients of Pu_1 and Pu_2 in the polynomial basis.

The uniqueness of the solution can be established as follows: if $u_1 = u_2 = 0$, then for any $v \in \mathbb{P}_k(I_j)$, we have

$$\int_{I_j} [a_{11}(x)Pu_1(x) + a_{22}(x)Pu_2(x)]\, v(x)dx = 0.$$

In particular, taking $v = a_{11}Pu_1 + a_{22}Pu_2$ (which belongs to $\mathbb{P}_k(I_j)$ since a_{11} and a_{22} are smooth and Pu_1, Pu_2 are polynomials), we obtain the following:

$$\int_{I_j} [a_{11}(x)Pu_1(x) + a_{22}(x)Pu_2(x)]^2\, dx = 0.$$

This implies $a_{11}(x)Pu_1(x) + a_{22}(x)Pu_2(x) = 0$ on I_j. Given that $a_{11}(x) > 0$ and $a_{22}(x) > 0$ by assumption (1.2), standard arguments in the polynomial approximation theory ensure that $Pu_1 = Pu_2 = 0$. Therefore, the homogeneous system only has the trivial solution, thus guaranteeing uniqueness. Since we are dealing with a finite-dimensional linear system, existence follows from the uniqueness.

Remark 4.1. *According to the above definition of interpolation (4.1), we obtain the following:*

$$(u_m - L_k u_m)(x_{N/4-1}^-) = 0 \;\; \textit{and} \;\; (u_m - L_k u_m)(x_{3N/4+1}^+) = 0, \quad m = 1, 2.$$

4.2. Interpolation error

According to the interpolation theories in Sobolev spaces [26, Theorem 3.1.4], the following holds:

$$\|v - v^I\|_{W^{l,q}(I_j)} \leq Ch_i^{k+1-l+1/p-1/q}|v|_{W^{k+1,p}(I_j)}, \quad \forall v \in W^{k+1,q}(I_j), \tag{4.4}$$

where $p, q \geq 1$, and $l = 0, 1$. Furthermore, for all $v \in H^{k+1}(I_j)$ and $j = 1, 2, \cdots, N$, by using the projection results [27, Lemma 5.1], one has the following:

$$\|Pv\|_{I_j} \leq C\,\|v\|_{I_j}, \tag{4.5}$$

$$\|Pv\|_{L^\infty(I_j)} \leq C\,\|v\|_{L^\infty(I_j)}, \tag{4.6}$$

$$\|v - Pv\|_{L^r(I_j)} \leq Ch_j^{k+1}\left\|v^{(k+1)}\right\|_{L^r(I_j)}, \quad r = 2, \infty. \tag{4.7}$$

Lemma 4.2. *Let $\Pi\vec{S}$ and $\Pi\vec{E}$ be the interpolations of $\vec{S}$ and $\vec{E}$, respectively. Then, for $m = 1, 2$, we have*

$$\|u_m - \Pi u_m\|_{L^\infty[0,1]} \leq CN^{-(k+1)}, \tag{4.8}$$

$$\sum_{j=1}^{N/4-1}\|u_m - L_k u_m\|_{I_j}^2 + \sum_{j=3N/4+2}^{N}\|u_m - L_k u_m\|_{I_j}^2 \leq C\varepsilon N^{-(2k+1)}, \tag{4.9}$$

$$\sum_{j=N/4}^{3N/4+1}\|u_m - Pu_m\|_{I_j}^2 \leq CN^{-2(k+1)}, \tag{4.10}$$

$$\sum_{j=1}^{N/4-1}\left\|(u_m - L_k u_m)^{(l)}\right\|_{I_j}^2 \leq C\varepsilon^{1-2l}N^{-2(k+1-l)}, \quad l = 1, 2, \tag{4.11}$$

$$\sum_{j=N/4}^{3N/4+1}\left\|(u_m - Pu_m)^{(l)}\right\|_{I_j}^2 \leq C\varepsilon^{1-2l}N^{-(2k+1)}, \quad l = 1, 2, \tag{4.12}$$

$$\sum_{j=3N/4+2}^{N}\left\|(u_m - L_k u_m)^{(l)}\right\|_{I_j}^2 \leq C\varepsilon^{1-2l}N^{-2(k+1-l)}, \quad l = 1, 2. \tag{4.13}$$

Proof. First, we begin to prove Eqs (4.8)–(4.10). For $j \leq N/4 - 1$, it follows from Eqs (2.2) and (4.4) that

$$\begin{aligned}\|E_m - L_k E_m\|_{L^\infty(I_j)} &\leq Ch_j^{k+1}\left\|E_m^{(k+1)}\right\|_{L^\infty(I_j)}\\ &\leq C\varepsilon^{-(k+1)}h_j^{k+1}\max_{x\in I_j}D_\varepsilon(x)\\ &\leq CN^{-(k+1)},\end{aligned} \tag{4.14}$$

where we used Lemma 3.3 with $\mu = k + 1$. Similarly, for $N/4 \leq j \leq 3N/4 + 1$, by using Eq (4.8) and Lemma 3.3, the following holds:

$$\begin{aligned}\|E_m - PE_m\|_{L^\infty(I_j)} &\leq \|E_m\|_{L^\infty(I_j)} + \|PE_m\|_{L^\infty(I_j)}\\ &\leq C\,\|E_m\|_{L^\infty(I_j)}\\ &\leq C\max_{x\in I_j}D_\varepsilon(x)\\ &\leq CN^{-(k+1)}.\end{aligned} \tag{4.15}$$

For $3N/4+2 \le j \le N$, one has the following:

$$\|E_m - L_k E_m\|_{L^\infty(I_j)} \le CN^{-(k+1)}. \tag{4.16}$$

Collecting Eqs (4.14)–(4.16), it is easy to obtain the following:

$$\|E_m - \Pi E_m\|_{L^\infty[0,1]} \le CN^{-(k+1)}.$$

Meanwhile, by using Eqs (4.4), (4.6) and Lemma 3.2, the following holds:

$$\|S_m - \Pi S_m\|_{L^\infty(\Omega)} \le CN^{-(k+1)}.$$

Therefore, the desirable result (4.8) can be followed from Eq (2.1). Furthermore, we obtain Eqs (4.9) and (4.10).

Next, we prove Eqs (4.11) and (4.13):

From Eqs (4.4) and (4.7) and Lemma 3.2, for $l = 1, 2$, we have the following:

$$\begin{aligned}\sum_{j=1}^{N} \left\|(S_m - \Pi S_m)^{(l)}\right\|_{I_j}^2 &\le C \sum_{j=1}^{N} h_j^{2(k+1-l)} \left\|S_m^{(k+1)}\right\|_{I_j}^2 \\ &\le C \sum_{j=1}^{N} h_j^{2(k+3/2-l)} \left\|S_m^{(k+1)}\right\|_{L^\infty(I_j)}^2 \\ &\le CN^{-2(k+1-l)}.\end{aligned} \tag{4.17}$$

Furthermore, in light of Eq (3.2) and Lemma 3.3 with $\mu = k+3/2-l,\ l = 1, 2$, we find the following:

$$\begin{aligned}\sum_{j=1}^{N/4-1} \left\|(E_m - L_K E_m)^{(l)}\right\|_{I_j}^2 &\le C \sum_{j=1}^{N/4-1} h_j^{2(k+1-l)} \left\|E_m^{(k+1)}\right\|_{I_j}^2 \\ &\le C \sum_{j=1}^{N/4-1} h_j^{2(k+3/2-l)} \left\|E_m^{(k+1)}\right\|_{L^\infty(I_j)}^2 \\ &\le C \sum_{j=1}^{N/4-1} h_j^{2(k+3/2-l)} \varepsilon^{-2(k+1)} \max_{x\in I_j} D_\varepsilon^2(x) \\ &\le C \sum_{j=1}^{N/4-1} \varepsilon^{-2(k+1)} \left(h_j^{(k+3/2-l)} \max_{x\in I_j} D_\varepsilon(x)\right)^2 \\ &\le C \sum_{j=1}^{N/4-1} \varepsilon^{-2(k+1)} \varepsilon^{2(k+3/2-l)} N^{-2(k+3/2-l)} \\ &\le C\varepsilon^{1-2l} N^{-2(k+1-l)}.\end{aligned} \tag{4.18}$$

Combining Eqs (4.17), (4.18), and (2.1), the desirable result Eq (4.11) can be followed. Similarly, we can prove Eq (4.13).

Finally, we begin to show Eq (4.12), From a triangle inequality, Eqs (2.5) and (4.15), Lemma 3.2, an inverse inequality, and $\varepsilon \le CN^{-1}$, one has the following:

$$\begin{aligned}\sum_{j=N/4}^{3N/4+1} \|(E_m - PE_m)'\|_{I_j}^2 &\le C \sum_{j=N/4}^{3N/4+1} \left\|E_m'\right\|_{I_j}^2 + C \|(PE_m)'\|_{I_{N/4}}^2 \\ &\quad + C \sum_{j=N/4+1}^{3N/4+1} \|(PE_m)'\|_{I_j}^2 \\ &\le C\varepsilon^{-2} \int_{x_{N/4-1}}^{x_{3N/4+1}} D_\varepsilon^2(x)dx + Ch_{N/4}^{-2} \|PE_m\|_{I_{N/4}}^2 \\ &\quad + C \sum_{j=N/4+1}^{3N/4+1} h_j^{-1} \|E_m\|_{L^\infty(I_j)}^2 \\ &\le C\left(\varepsilon^{-1}N^{-2\sigma} + \varepsilon^{-1}N^{-2\sigma} + N^2\varepsilon^{2\sigma}\right) \\ &\le C\varepsilon^{-1}N^{-(2k+1)}.\end{aligned} \tag{4.19}$$

In a same way, we have the following:

$$\sum_{j=N/4}^{3N/4+1} \|(E_m - PE_m)''\|_{I_j}^2 \le C\varepsilon^{-3}N^{-(2k+1)}. \tag{4.20}$$

Combining Eqs (4.19), (4.20), and (2.1), we complete the proof.

Before deriving some error estimations for the interpolation on the element boundaries, we introduce the following multiplicative trace inequality.

Lemma 4.3. *[28, Lemma 4] Let $w \in H^1(I_j)$, $j = 1, \cdots, N$. Then,*

$$|w(x_s)|^2 \le 2(h_j^{-1} \|w\|_{I_j}^2 + \|w\|_{I_j} \|w'\|_{I_j}), \; s \in \{j-1, j\}.$$

Lemma 4.4. *Let $\eta_m(x) = (u_m - \Pi u_m)(x)$, $m = 1, 2$ and $\rho(x_j)$ is given by Eq (3.7). Then, on the Bakhvalov-type mesh (3.1), one has the following:*

$$\left(\sum_{j=0}^{N} \frac{\varepsilon^4}{\rho(x_j)} \{\eta_m'(x_j)^2\}\right)^{\frac{1}{2}} \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})}. \tag{4.21}$$

Proof. It follows from Lemma 4.3 that

$$\begin{aligned}\{\eta_m'(x_j)\}^2 &= \frac{1}{4}(\eta'(x_j^-) + \eta_m'(x_j^+))^2 \le \frac{1}{2}(\eta_m'(x_j^-)^2 + \eta_m'(x_j^+)^2) \\ &\le h_j^{-1} \left\|\eta_m'\right\|_{I_j}^2 + \left\|\eta_m'\right\|_{I_j} \left\|\eta_m''\right\|_{I_j} + h_{j+1}^{-1} \left\|\eta_m'\right\|_{I_{j+1}}^2 + \left\|\eta_m'\right\|_{I_{j+1}} \left\|\eta_m''\right\|_{I_{j+1}}.\end{aligned}$$

Then, by simple calculation, we obtain the following:

$$\begin{aligned}\sum_{j=1}^{N}\frac{\varepsilon^4}{\rho(x_j)}\{\eta_m'(x_j)^2\} &\le C\sum_{j=1}^{N}\frac{\varepsilon^4}{\rho(x_j)}\left(h_j^{-1}\left\|\eta_m'\right\|_{I_j}^2+\left\|\eta_m'\right\|_{I_j}\left\|\eta_m''\right\|_{I_j}\right)\\ &\quad + C\sum_{j=1}^{N-1}\frac{\varepsilon^4}{\rho(x_j)}\left(h_{j+1}^{-1}\left\|\eta_m'\right\|_{I_{j+1}}^2+\left\|\eta_m'\right\|_{I_{j+1}}\left\|\eta_m''\right\|_{I_{j+1}}\right)\\ &\le C\varepsilon N^{-(2k+1)},\end{aligned} \tag{4.22}$$

where Lemmas 3.2 and 4.2, and Eq (3.7) were used. This completes the proof of this conclusion.

Theorem 4.5. *Let $\vec{\eta}=\vec{u}-\Pi\vec{u}$. Then under the Bakhvalov-type mesh (3.1), one has the following:*

$$\left\|\vec{\eta}\right\|_b \le CN^{-k}.$$

Proof. It follows from the definition of the balanced norm (3.10) that

$$\begin{aligned}\left\|\vec{\eta}\right\|_b^2 &= \varepsilon\sum_{j=1}^{N}(\left\|\eta_1'\right\|_{I_j}^2+\left\|\eta_2'\right\|_{I_j}^2)+\beta^2\sum_{j=1}^{N}(\|\eta_1\|_{I_j}^2+\|\eta_2\|_{I_j}^2)\\ &\quad+\sum_{j=0}^{N}\rho(x_j)([\eta_1(x_j)]^2+[\eta_2(x_j)]^2)\\ &=: \Lambda_1+\Lambda_2+\Lambda_3.\end{aligned} \tag{4.23}$$

Obviously, from Lemma 4.2, it is easy to get

$$\Lambda_1+\Lambda_2 \le C(\varepsilon\varepsilon^{-1}N^{-2k}+N^{-2(k+1)}) \le CN^{-2k}. \tag{4.24}$$

According to Eq (4.1) and Remark 1, the following holds:

$$[(u_m-L_k u_m)(x_j)]=0,\ \text{for}\ \ m=1,2\ \text{and}\ \ 0\le j\le N/4-2\ \mathit{or}\ 3N/4+2\le j\le N.$$

Furthermore, by using $\rho(x_j)$ defined in Eq (3.7), one has the following:

$$\begin{aligned}\Lambda_3 &= \rho(x_{N/4-1})[(u_m-\Pi u_m)(x_{N/4-1})]^2+\sum_{j=N/4}^{3N/4}\rho(x_j)[(u_m-\Pi u_m)(x_j)]^2\\ &\quad+\rho(x_{3N/4+1})[(u_m-\Pi u_m)(x_{3N/4+1})]^2\\ &\le C\rho(x_{N/4-1})\|u_m-Pu_m\|_{L^\infty(I_{N/4})}^2+C\sum_{j=N/4}^{3N/4}\rho(x_j)\|u_m-Pu_m\|_{L^\infty(I_j\cup I_{j+1})}^2\\ &\quad+C\rho(x_{3N/4+1})\|u_m-Pu_m\|_{L^\infty(I_{3N/4+1})}^2\\ &\le C\varepsilon N^{-(2k+1)},\ \ \text{for}\ \ m=1,2.\end{aligned} \tag{4.25}$$

Substituting Eqs (4.24) and (4.25) into Eq (4.23), we obtain the following:

$$\left\|\vec{\eta}\right\|_b \le CN^{-k},$$

which completes the proof.

5. Supercloseness

Let $\vec{\xi} = \Pi\vec{u} - \vec{u}_N$, and recall that $\vec{\eta} = \Pi\vec{u} - \vec{u}$. Then, it follows from the Galerkin orthogonality and Eq (3.11) that

$$\begin{aligned}\left\|\vec{\xi}\right\|_\varepsilon^2 &= a(\Pi\vec{u} - \vec{u} + \vec{u} - \vec{u}_N, \vec{\xi}) = a(\vec{\eta}, \vec{\xi}) \\ &= \sum_{m=1}^{2}\sum_{k=1}^{5} T_k(\eta_m, \xi_m) =: \sum_{k=1}^{5} T_k(\vec{\eta}, \vec{\xi}),\end{aligned} \tag{5.1}$$

where

$$T_1(\eta_m, \xi_m) = \varepsilon^2 \sum_{j=1}^{N} \int_{I_j} \eta_m' \xi_m' dx,\ \ T_2(\eta_m, \xi_m) = \varepsilon^2 \sum_{j=0}^{N} \{\eta_m'(x_j)\}[\xi_m(x_j)],$$

$$T_3(\eta_m, \xi_m) = -\varepsilon^2 \sum_{j=0}^{N} \{\xi_m'(x_j)\}[\eta_m(x_j)],\quad T_4(\eta_m, \xi_m) = \sum_{j=0}^{N} \rho(x_j)[\eta_m(x_j)][\xi_m(x_j)],$$

$$T_5(\eta_m, \xi_m) = \sum_{j=1}^{N} \int_{I_j} (a_{m1}\eta_1\xi_m + a_{m2}\eta_2\xi_m)\, dx$$

and $T_k(\vec{\eta}, \vec{\xi}) = T_k(\eta_1, \xi_2) + T_k(\eta_2, \xi_2)$, $k = 1, 2, 3, 4, 5$.

In order to derive the bound of $\left\|\vec{\xi}\right\|_\varepsilon^2$, we only need to estimate $T_k(\eta, \xi)$, $k = 1, 2, 3, 4, 5$.

First, we rewrite $T_1(\eta_m, \xi_m)$ into the following form:

$$\begin{aligned}T_1(\eta_m, \xi_m) =& \varepsilon^2 \left(\sum_{j=1}^{N/4-1} \int_{I_j} \eta_m' \xi_m' dx + \sum_{j=N/4}^{3N/4+1} \int_{I_j} \eta_m' \xi_m' dx \right. \\ &\left. + \sum_{j=3N/4+2}^{N} \int_{I_j} \eta_m' \xi_m' dx \right).\end{aligned} \tag{5.2}$$

For the first term of Eq (5.2), using the Hölder inequalities, Eqs (4.2) and (4.17), Lemma 3.3, and $\varepsilon \le N^{-1}$, we have the following:

$$\begin{aligned}&\left| \varepsilon^2 \sum_{j=1}^{N/4-1} \int_{I_j} \eta_m' \xi_m' dx \right| \\ \le& \left| \varepsilon^2 \sum_{j=1}^{N/4-1} \int_{I_j} (S_m - L_k S_m)' \xi_m' dx \right| + \left| \varepsilon^2 \sum_{j=1}^{N/4-1} \int_{I_j} (E_m - L_k E_m)' \xi_m' dx \right| \\ \le& C \left(\sum_{j=1}^{N/4-1} \varepsilon^2 \left\|(S_m - L_k S_m)'\right\|_{I_j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{N/4-1} \varepsilon^2 \left\|\xi_m'\right\|_{I_j}^2 \right)^{\frac{1}{2}} \\ &+ C\varepsilon^2 \sum_{j=1}^{N/4-1} h_j^{k+1} \left\|E_m^{(k+2)}\right\|_{I_j} \left\|\xi_m'\right\|_{I_j}\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&\le C\left(\sum_{j=1}^{N/4-1} \varepsilon^2 \|(S_m - L_k S_m)'\|_{I_j}^2\right)^{\frac{1}{2}} \left\|\vec{\xi}\right\|_{\varepsilon} \\
&\quad + C\left(\varepsilon^2 \sum_{j=1}^{N/4-1} h_j h_j^{2(k+1)} \left\|E_0^{(k+2)}\right\|_{L^\infty(I_j)}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N/4-1} \varepsilon^2 \left\|\xi_m'\right\|_{I_j}^2\right)^{\frac{1}{2}} \\
&\le C\left(\varepsilon^2 N^{-2k}\right)^{\frac{1}{2}} \left\|\vec{\xi}\right\|_{\varepsilon} + C\left(\sum_{j=1}^{N/4-1} h_j h_j^{2(k+1)} \varepsilon^{-2(k+1)} D_\varepsilon^2(x)\right)^{\frac{1}{2}} \left\|\vec{\xi}\right\|_{\varepsilon} \\
&\le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}.
\end{aligned}
$$

Similarly, from the Hölder inequalities and Eq (4.12), we obtain

$$
\begin{aligned}
|\varepsilon^2 \sum_{j=N/4}^{3N/4+1} \int_{I_j} \eta_m' \xi_m' dx| &\le C\left(\sum_{j=N/4}^{3N/4+1} \varepsilon^2 \left\|\eta_m'\right\|_{I_j}^2\right)^{\frac{1}{2}} \left(\sum_{j=N/4}^{3N/4+1} \varepsilon \left\|\xi_m'\right\|_{I_j}^2\right)^{\frac{1}{2}} \\
&\le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon},
\end{aligned} \tag{5.4}
$$

and

$$
|\varepsilon^2 \sum_{j=3N/4+2}^{N} \int_{I_j} \eta_m' \xi_m' dx| \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.5}
$$

Substituting Eqs (5.3)–(5.5) into Eq (5.2) and using the triangle inequality, we obtain the following:

$$
|T_1(\vec{\eta}, \vec{\xi})| \le \sum_{m=1}^{2} |T_1(\eta_m, \xi_m)| \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.6}
$$

From the Hölder inequalities and Lemma 4.4, we have the following:

$$
\begin{aligned}
|\varepsilon^2 \sum_{j=0}^{N} \{\eta_m'(x_j)\}[\xi_m(x_j)]| &\le \left(\sum_{j=0}^{N} \frac{\varepsilon^4}{\rho_j} \{\eta_m'(x_j)\}^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{N} \rho_j [\xi_m(x_j)]\}^2\right)^{\frac{1}{2}} \\
&\le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}.
\end{aligned}
$$

Using the triangle inequality, we obtain the following:

$$
|T_2(\vec{\eta}, \vec{\xi})| \le \sum_{m=1}^{2} |T_2(\eta_m, \xi_m)| \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.7}
$$

Next, according to Remark 1 and the properties of the Gauß Lobatto interpolation, we have the following:

$$
\begin{aligned}
&\left|-\varepsilon^2\sum_{j=0}^{N}\{\xi_m'(x_j)\}[\eta_m(x_j)]\right| = \left|-\varepsilon^2\sum_{j=N/4-1}^{3N/4+1}\{\xi_m'(x_j)\}[\eta_m(x_j)]\right| \\
\le{}& \left|\varepsilon^2[\eta_m(x_{N/4-1})]\{\xi_m'(x_{N/4-1})\}\right| + \left|\varepsilon^2\sum_{j=N/4}^{3N/4}[\eta_m(x_j)]\{\xi_m'(x_j)\}\right| \\
&+ \left|\varepsilon^2[\eta_m(x_{3N/4+1})]\{\xi_m'(x_{3N/4+1})\}\right| \\
\le{}& C\varepsilon^2\,\|\eta_m\|_{L^\infty(I_{N/4})}\left\|\xi_m'\right\|_{L^\infty(I_{N/4-1}\cup I_{N/4})} + C\varepsilon^2\sum_{j=N/4}^{3N/4}\|\eta_m\|_{L^\infty(I_j)}\left\|\xi_m'\right\|_{L^\infty(I_j\cup I_{j+1})} \\
&+ C\varepsilon^2\,\|\eta_m\|_{L^\infty(I_{3N/4+1})}\left\|\xi_m'\right\|_{L^\infty(I_{3N/4+1}\cup I_{3N/4+2})} \\
\le{}& C\varepsilon^2\,\|\eta_m\|_{L^\infty(I_{N/4})}\,\varepsilon^{-\frac{1}{2}}\left\|\xi_m'\right\|_{I_{N/4-1}\cup I_{N/4}} + C\varepsilon^2\sum_{j=N/4}^{3N/4}\|\eta_m\|_{L^\infty(I_j)}\,h_j^{-\frac{1}{2}}\;\left\|\xi_m'\right\|_{I_j\cup I_{j+1}} \\
&+ C\varepsilon^2\,\|\eta_m\|_{L^\infty(I_{3N/4+1})}\,\varepsilon^{-\frac{1}{2}}\left\|\xi_m'\right\|_{I_{3N/4+1}\cup I_{3N/4+2}} \\
\le{}& C\varepsilon^{\frac{1}{2}}N^{-(k+1)}\varepsilon\left\|\xi_m'\right\|_{I_{N/4-1}\cup I_{N/4}} + C\varepsilon^{\frac{1}{2}}N^{-(k+1)}\varepsilon\left\|\xi_m'\right\|_{I_{3N/4+1}\cup I_{3N/4+2}} \\
&+ C\varepsilon^{\frac{1}{2}}\left(\sum_{j=N/4}^{3N/4}\|\eta_m\|^2_{L^\infty(I_j)}\right)^{\frac{1}{2}}\left(\sum_{j=N/4}^{3N/4}\varepsilon^2\left\|\xi_m'\right\|^2_{I_j\cup I_{j+1}}\right)^{\frac{1}{2}} \\
\le{}& C\varepsilon^{\frac{1}{2}}N^{-(k+\frac{1}{2})}\left\|\vec{\xi}\right\|_{\varepsilon},
\end{aligned}
$$

where the inverse inequality, Lemma 3.2, and Eq (4.8) were used. Using the triangle inequality, we can obtain the following:

$$
|T_3(\vec{\eta},\vec{\xi})| \le \sum_{m=1}^{2}|T_3(\eta_m,\xi_m)| \le C\varepsilon^{\frac{1}{2}}N^{-(k+\frac{1}{2})}\left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.8}
$$

Similarly, the Hölder inequalities and Eqs (3.7) and (4.8) yield the following:

$$
\begin{aligned}
&\left|\sum_{j=0}^{N}\rho(x_j)[\eta_m(x_j)][\xi_m(x_j)]\right| = \left|\sum_{j=N/4-1}^{3N/4+1}\rho(x_j)[\eta_m(x_j)][\xi_m(x_j)]\right| \\
\le{}& C\left(\sum_{j=N/4-1}^{3N/4+1}\rho(x_j)[\eta_m(x_j)]^2\right)^{\frac{1}{2}}\left(\sum_{j=N/4-1}^{3N/4+1}\rho(x_j)[\xi_m(x_j)]^2\right)^{\frac{1}{2}} \\
\le{}& C\Bigg(\rho(x_{N/4-1})\,\|\eta_m\|^2_{L^\infty(I_{N/4})} + \sum_{j=N/4}^{3N/4}\rho(x_j)\,\|\eta_m\|^2_{L^\infty(I_j\cup I_{j+1})} \\
&+\rho(x_{3N/4+1})\,\|\eta_m\|^2_{L^\infty(I_{3N/4+1})}\Bigg)^{\frac{1}{2}}\left\|\vec{\xi}\right\|_{\varepsilon} \\
\le{}& C\varepsilon^{\frac{1}{2}}N^{-(k+\frac{1}{2})}\left\|\vec{\xi}\right\|_{\varepsilon}.
\end{aligned}
$$

Using the triangle inequality, we can obtain the following:

$$|T_4(\vec{\eta},\vec{\xi})| \le \sum_{m=1}^{2} |T_4(\eta_m,\xi_m)| \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.9}$$

In addition, recalling the definition of the locally weighted L^2 projection (4.3), we have the following:

$$\begin{aligned}
|T_5(\eta_m,\xi_m)| \le & \left|\sum_{j=1}^{N/4-1} \int_{I_j} (a_{m1}\eta_1\xi_m + a_{m2}\eta_2\xi_m)\, dx\right| \\
& + \left|\sum_{j=3N/4+2}^{N} \int_{I_j} (a_{m1}\eta_1\xi_m + a_{m2}\eta_2\xi_m)\, dx\right| \\
\le & C\left(\sum_{j=1}^{N/4-1} (\|\eta_1\|_{I_j}^2 + \|\eta_2\|_{I_j}^2)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N/4-1} \|\xi_m\|_{I_j}^2\right)^{\frac{1}{2}} \\
& + C\left(\sum_{j=3N/4+2}^{N} (\|\eta_1\|_{I_j}^2 + \|\eta_2\|_{I_j}^2)\right)^{\frac{1}{2}} \left(\sum_{j=3N/4+2}^{N} \|\xi_m\|_{I_j}^2\right)^{\frac{1}{2}} \\
\le & C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon},
\end{aligned}$$

where the Hölder inequalities and Eqs (4.9) and (4.10) were used. Using the triangle inequality yields the following:

$$|T_5(\vec{\eta},\vec{\xi})| \le \sum_{m=1}^{2} |T_5(\eta_m,\xi_m)| \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon}. \tag{5.10}$$

Finally, according to Eqs (5.6)–(5.10), one has the following:

$$\left\|\vec{\xi}\right\|_{\varepsilon}^2 \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})} \left\|\vec{\xi}\right\|_{\varepsilon},$$

which implies the following estimate holds true:

$$\left\|\Pi\vec{u} - \vec{u}_N\right\|_{\varepsilon} \le C\varepsilon^{\frac{1}{2}} N^{-(k+\frac{1}{2})}. \tag{5.11}$$

Theorem 5.1. *On the Bakhvalov-type mesh (3.1), one has the following:*

$$\left\|\Pi\vec{u} - \vec{u}_N\right\|_{b} \le CN^{-(k+\frac{1}{2})}.$$

Proof. Combined with the definition of $\|\cdot\|_b$ and

$$\left\|(\Pi\vec{u} - \vec{u}_N)'\right\|_{[0,1]} \le \varepsilon^{-1} \left\|\Pi\vec{u} - \vec{u}_N\right\|_{\varepsilon},$$

we can obtain the following:

$$\begin{aligned}
\left\|\Pi\vec{u} - \vec{u}_N\right\|_{b} &= (\varepsilon\|(\Pi\vec{u} - \vec{u}_N)'^2\|_{[0,1]} + \beta^2\|(\Pi\vec{u} - \vec{u}_N)^2\|_{[0,1]} + \sum_{j=0}^{N} \rho(x_j)[\Pi\vec{u} - \vec{u}_N]^2)^{1/2} \\
&\le (\varepsilon * C\varepsilon^{-1} N^{-(2k+1)} + C\varepsilon N^{-(2k+1)} + C\varepsilon N^{-(2k+1)})^{1/2} \\
&\le CN^{-(k+1/2)}.
\end{aligned}$$

Theorem 5.2. *Let $\vec{u}$ and $\vec{u^N}$ be the solutions of Eq (1.1) and Eq (3.6), respectively. On Bakhvalov-type mesh (3.1), one has the following:*

$$\left\|\vec{u}-\vec{u}_N\right\|_b \leq CN^{-k}.$$

Proof. From the triangle inequality and Theorems 4.5 and Theorems 5.1, we have the following:

$$\left\|\vec{u}-\vec{u}_N\right\|_b \leq \left\|\vec{u}-\Pi\vec{u}\right\|_b + \left\|\Pi\vec{u}-\vec{u}_N\right\|_b \leq CN^{-k}.$$

6. Numerical experiments and discussion

In this section, we verify our presented theoretical results by considering the following test problem:

$$\begin{cases} -\varepsilon^2 u_1''(x) + 2u_1(x) - u_2(x) = f_1(x), \ x \in \Omega = (0,1), \\ -\varepsilon^2 u_2''(x) - u_1(x) + 2u_2(x) = f_2(x), \ x \in \Omega = (0,1), \\ u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0, \end{cases} \tag{6.1}$$

where $f_1(x)$, $f_2(x)$ are chosen such that

$$u_1(x) = 1 + e^{-1/\varepsilon} - e^{-x/\varepsilon} - e^{-(1-x)/\varepsilon},$$
$$u_2(x) = 2\left(1 - \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1+e^{-1/\varepsilon}}\right).$$

For all the computations below, we set $\sigma = k+1$.

For a fixed ε, μ, and N, the balanced-norm errors E_N^B and supercloseness errors E_N^S are given by

$$E_N^B = \left\|\vec{u}-\vec{u}_N\right\|_b, E_N^S = \left\|\Pi\vec{u}-\vec{u}_N\right\|_b,$$

respectively, where $\vec{u}$ is the exact solution and $\vec{u}_N$ is the NIPG finite element solution of Eq (3.6). Then, the corresponding rates of convergence are defined by the following:

$$r^B = \frac{\ln E_N^B - \ln E_{2N}^B}{\ln 2}, \quad r^S = \frac{\ln E_N^S - \ln E_{2N}^S}{\ln 2}.$$

For $\varepsilon = 10^{-i} (i = 3, \cdots, 7)$, $N = 2^j (j = 4, \cdots, 9)$, and $k = 1, 2, 3$, we use our presented NIPG method on a Bakhvalov-type mesh to solve this test problem. The results of the errors and the corresponding convergence orders are displayed in Tables 1–3. We can observe that under the Bakhvalov-type mesh, the errors E_N^B display k-order uniform convergence, which is in full agreement with Theorem 5.2. Furthermore, it is found that the supercloseness errors E_N^S can almost reach $k+1$. This is a half order higher than the convergence rate given in Theorem 5.1. Additionally, Figure 1 can demonstrate that the numerical results satisfy Theorems 5.1 and 5.2.

Table 1. Errors and rates in the case of $k = 1$.

ε		N 16	32	64	128	256	512
10^{-3}	E_N^B	4.579e-01	2.283e-01	1.141e-02	5.701e-02	2.850e-02	1.425e-02
		1.00	1.00	1.00	1.00	1.00	-
	E_N^S	8.741e-02	2.001e-02	4.727e-03	1.117e-03	2.644e-04	6.388e-05
		2.12	2.08	2.08	2.08	2.07	-
10^{-4}	E_N^B	4.564e-01	2.282e-01	1.141e-02	5.701e-02	2.853e-02	1.426e-02
		1.00	1.00	1.00	1.00	1.00	-
	E_N^S	9.788e-02	2.271e-02	5.487e-03	1.320e-03	3.169e-04	7.544e-05
		2.11	2.05	2.05	2.06	2.07	-
10^{-5}	E_N^B	4.556e-01	2.280e-01	1.141e-02	5.705e-02	2.853e-02	1.426e-02
		1.00	1.00	1.00	1.00	1.00	-
	E_N^S	1.045e-01	2.442e-02	5.946e-03	1.457e-03	3.561e-04	8.661e-05
		2.10	2.04	2.03	2.03	2.04	-
10^{-6}	E_N^B	4.556e-01	2.280e-01	1.141e-02	5.705e-02	2.853e-02	1.426e-02
		1.00	1.00	1.00	1.00	1.00	-
	E_N^S	1.088e-01	2.553e-02	6.252e-03	1.543e-03	3.808e-04	9.371e-05
		2.09	2.03	2.02	2.02	2.02	-
10^{-7}	E_N^B	4.559e-01	2.280e-01	1.141e-01	5.705e-02	2.853e-02	1.426e-02
		1.00	1.00	1.00	1.00	1.00	-
	E_N^S	1.118e-01	2.629e-02	6.457e-03	1.600e-03	3.968e-04	9.826e-05
		2.09	2.03	2.01	2.01	2.01	-

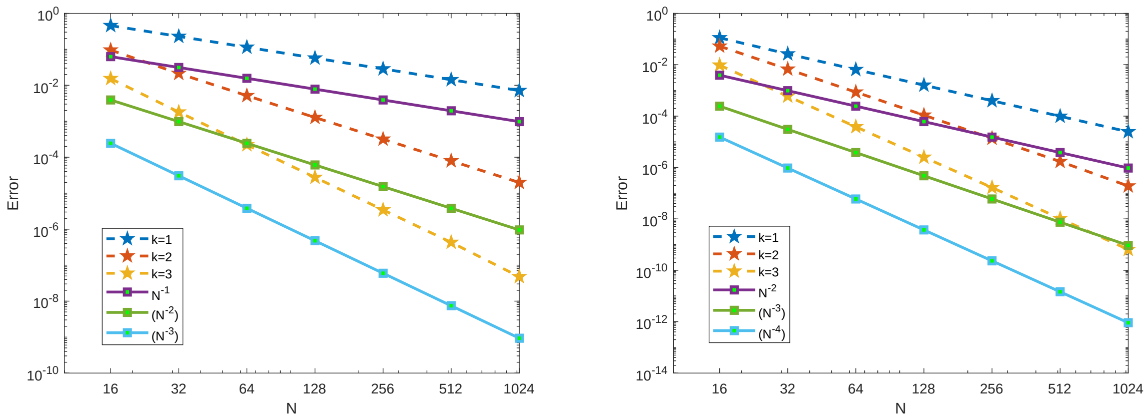

Figure 1. The left and right figures represent E_N^B and E_N^S in the case of $\varepsilon = 10^{-8}$, respectively.

Table 2. Errors and rates in the case of $k = 2$.

ε		N					
		16	32	64	128	256	512
10^{-3}	E_N^B	9.673e-02	2.140e-02	5.143e-03	1.271e-03	3.169e-04	7.915e-05
		2.17	2.06	2.02	2.00	2.00	-
	E_N^S	5.162e-02	6.626e-03	8.512e-04	1.095e-04	1.414e-05	1.865e-06
		2.96	2.96	2.96	2.95	2.93	-
10^{-4}	E_N^B	9.602e-02	2.136e-02	5.147e-03	1.273e-03	3.174e-04	7.929e-05
		2.17	2.05	2.01	2.00	2.00	-
	E_N^S	5.196e-02	6.642e-03	8.504e-04	1.077e-04	1.357e-05	1.709e-06
		2.97	2.97	2.98	2.99	2.99	-
10^{-5}	E_N^B	9.553e-02	2.132e-02	5.144e-03	1.273e-03	3.174e-04	7.930e-05
		2.16	2.05	2.01	2.00	2.00	-
	E_N^S	5.223e-02	6.672e-03	8.537e-04	1.080e-04	1.357e-05	1.702e-06
		2.97	2.97	2.98	2.99	3.00	-
10^{-6}	E_N^B	9.532e-02	2.130e-02	5.142e-03	1.273e-03	3.174e-04	7.930e-05
		2.16	2.05	2.01	2.00	2.00	-
	E_N^S	5.243e-02	6.694e-03	8.565e-04	1.083e-04	1.362e-05	1.707e-06
		2.97	2.97	2.98	2.99	3.00	-
10^{-7}	E_N^B	9.527e-02	2.129e-02	5.142e-03	1.273e-03	3.174e-04	7.930e-05
		2.16	2.05	2.01	2.00	2.00	-
	E_N^S	5.256e-02	6.709e-03	8.584e-04	1.086e-04	1.366e-05	1.713e-06
		2.97	2.97	2.98	2.99	3.00	-

Table 3. Errors and rates in the case of $k = 3$.

ε		N					
		16	32	64	128	256	512
10^{-3}	E_N^B	1.480e-02	1.780e-03	2.201e-04	2.745e-05	3.429e-06	4.286e-07
		3.06	3.02	3.00	3.00	3.00	-
	E_N^S	6.221e-03	3.920e-04	2.475e-05	1.570e-06	9.989e-08	6.591e-09
		3.99	3.99	3.98	3.97	3.92	-
10^{-4}	E_N^B	1.498e-02	1.791e-03	2.210e-04	2.753e-05	3.439e-06	4.298e-07
		3.06	3.02	3.00	3.00	3.00	-
	E_N^S	7.580e-03	4.730e-04	2.944e-05	1.813e-06	1.121e-07	7.062e-09
		4.00	4.00	4.02	4.02	3.99	-
10^{-5}	E_N^B	1.509e-02	1.795e-03	2.212e-04	2.755e-05	3.440e-06	4.299e-07
		3.07	3.02	3.00	3.00	3.00	-
	E_N^S	8.575e-03	5.350e-04	3.438e-05	2.150e-06	1.322e-07	8.252e-09
		4.00	3.96	4.02	4.02	4.00	-
10^{-6}	E_N^B	1.520e-02	1.798e-03	2.213e-04	2.755e-05	3.440e-06	4.299e-07
		3.08	3.02	3.00	3.00	3.00	-
	E_N^S	9.185e-03	5.713e-04	3.692e-05	2.397e-06	1.510e-07	9.423e-09
		4.00	3.95	3.95	3.99	4.00	-
10^{-7}	E_N^B	1.532e-02	1.802e-03	2.214e-04	2.755e-05	3.440e-06	4.299e-07
		3.09	3.02	3.00	3.00	3.00	-
	E_N^S	9.630e-03	5.950e-04	3.830e-05	2.505e-06	1.659e-07	1.032e-08
		4.02	3.96	3.93	3.92	4.00	-

7. Conclusions

In this paper, we analyzed the supercloseness and uniform convergence properties of the NIPG method on Bakhvalov-type meshes for weakly coupled systems of singularly perturbed reaction-diffusion equations. We established a supercloseness result of the order $k + \frac{1}{2}$ and proved the optimal uniform convergence of the order k in the balanced norm. The theoretical findings were supported by numerical experiments that confirmed the predicted convergence rates.

Author contributions

Xiaobing Bao and Lei Xu conceived the study, designed and performed the experiments, and wrote the manuscript. Yong Zhang analyzed the data and provided critical revisions to the manuscript. All authors reviewed and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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