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Research article

On the number of limit cycles of a class of near-Hamiltonian systems near a degenerate center

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Abstract: In this paper, we focused on studying the number of limit cycles of a class of near-Hamiltonian systems, whose unperturbed system possesses a degenerate center. Using the first order Melnikov function, we obtained the lower bound of the maximum number of limit cycles in Poincaré bifurcation under certain conditions. In addition, we obtained the number of small-amplitude limit cycles that bifurcate from the degenerate center. We also provided two examples as applications.

Keywords: Bifurcation; Melnikov function; limit cycles; degenerate center; near-Hamiltonian system

1. Introduction and main results

The latter part of the famous Hilbert's 16th problem is about the relationship between the number of limit cycles and the degree of polynomials in a planar polynomial system, as well as the distribution of these limit cycles. Over the past 100 years, many mathematicians have devoted themselves to the study of this problem and obtained a lot of achievements, but this problem has not been solved (see for example [1–4]). Therefore, the problem of the number of limit cycles has always been one of the important topics in the study of ordinary differential equations, which has posed many challenging issues. In 1977, Arnold put forward the weakened Hilbert's 16th problem, which studies the maximum number of isolated zeros for the first order Melnikov function, namely Abel integral. There have been many results in the study of the number of zeros of Melnikov function (see [5–10] and so on).

Consider the analytic differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y). \tag{1.1}$$

Assume that system (1.1) has a center. Without loss of generality, we can assume that it is at the origin of the coordinates. Then, after a linear change of variables and a rescaling of the time variable, system (1.1) can be transformed into one of the following three forms

$$\dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y), \tag{1.2}$$

$$\dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y), \tag{1.3}$$

$$\dot{x} = F_1(x, y), \qquad \dot{y} = F_2(x, y), \tag{1.4}$$

where $F_1(x, y)$ and $F_2(x, y)$ are real analytic functions without constants and linear terms. We say that the center of system (1.1) is linear type, nilpotent, or degenerate if system (1.1) can be written into Eqs (1.2)–(1.4), respectively.

There are many researchers studying how many limit cycles can be generated near a linear center, and only a few researchers studying this problem near a nilpotent or degenerate center (see for instance [11–16] and references therein). Corbera and Valls [17, 18] characterized global nilpotent center of polynomial differential systems of degree three. Llibre and Valls [19] characterized the Kukles polynomial differential systems having an invariant algebraic curve. Mujica [20] studied the following Kukles system of arbitrary odd degree

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y(x^2 + y^2 - 1)\left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{1.5}$$

where $0 < |\varepsilon| \ll 1$ and $b_{2i,2j} \in \mathbb{R}$, $i, j = 0, 1, 2, \ldots, n$, and proved the coexistence of large-amplitude limit cycles and algebraic limit cycles using the first order Melnikov function.

Liu et al. [15] studied the following two classes of near-Hamiltonian systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x^3 + \varepsilon y(x^4 + y^2 - 1)\left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{1.6}$$

and

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(n+1)x^{2n+1} + \varepsilon y(x^{2n+2} + y^2 - 1)\left(b_{0,0} + \sum_{i+j=1}^{n} b_{2i,2j} x^{2i} y^{2j}\right), \end{cases} \tag{1.7}$$

where $0 < |\varepsilon| \ll 1$ and $b_{2i,2j} \in \mathbb{R}$, $i, j = 0, 1, 2, \ldots, n$. Using the first order Melnikov function, they gained the number of small- and large-amplitude limit cycles and demonstrated the coexistence of these limit cycles with algebraic limit cycles for these two systems, respectively. Inspired by the above, in this paper, we mainly study the number of limit cycles of a near-Hamiltonian system whose unperturbed system has a center at the origin. Specifically, we consider the following near-Hamiltonian system of arbitrary odd degree

$$\begin{cases} \dot{x} = my^{2m-1}, \\ \dot{y} = -nx^{2n-1} + \varepsilon y(x^{2n} + y^{2m} - 1) \sum_{i+j=0}^{k} a_{2i,2j} x^{2i} y^{2j}, \end{cases} \tag{1.8}$$

where $0 < |\varepsilon| \ll 1$, m, n are positive integers, and $a_{2i,2j}$, $i, j = 0, 1, 2, \ldots, n$ are real numbers. It is evident that the origin of the unperturbed system $(1.8)|_{\varepsilon=0}$ is a linear center as $m = n = 1$, a nilpotent center as $m = 1$, $n \geq 2$, and a degenerate center as $m, n \geq 2$. Systems (1.5)–(1.7) are special cases of system (1.8) by taking (i) $m = 1$, $n = 1$, $k = n$; (ii) $m = 1$, $n = 2$, and $k = n$; and (iii) $m = 1$, $n = n + 1$, and $k = n$, respectively.

For $\varepsilon = 0$, system (1.8) has a first integral of the form

$$H(x, y) = \frac{1}{2}y^{2m} + \frac{1}{2}x^{2n},$$

and it possesses a family of periodic orbits with clockwise orientation as follows

$$L_h : \frac{1}{2}x^{2n} + \frac{1}{2}y^{2m} = h, \quad h \in (0, +\infty),$$

where L_h tends the origin as h approaches to 0. Let $G = \cup_{(0,+\infty)} L_h$. Then, set G is a period annulus. The problem of the number of limit cycles generated by a period annulus is called Poincaré bifurcation. By the Poincaré-Pontryagin Theorem, the total number of the zeros of the first order Melnikov function controls the number of limit cycles bifurcating from a period annulus if the first order Melnikov function is not zero identically.

In this paper, we mainly investigate the number of limit cycles of Eq (1.8) in Poincaré bifurcation and Hopf bifurcation near the center. Further, we also show that $x^{2m+2} + y^2 = 1$ is an algebraic limit cycle of Eq (1.8) under certain conditions. We state our major results below.

Theorem 1.1. *Consider system (1.8).*

(i) For $km < n$ (resp. $kn < m$), Eq (1.8) has at most $kn + 1$ (resp. $km + 1$) limit cycles in Poincaré bifurcation, counting multiplicities, if the first order Melnikov function is not zero identically. Moreover, there exist coefficients $a_{2i,2j}$, $i, j = 0, 1, 2, \ldots, n$, such that it has $\frac{k(k+3)}{2}$ limit cycles near the origin for $0 < |\varepsilon| \ll 1$.

(ii) For $m = n$, Eq (1.8) has at most $k + 1$ limit cycles in Poincaré bifurcation, counting multiplicities, if the first order Melnikov function is not zero identically. Moreover, there exist coefficients $a_{2i,2j}$, $i, j = 0, 1, 2, \ldots, n$, such that it has k limit cycles near the origin for $0 < |\varepsilon| \ll 1$.

The following results are clearly evident.

Theorem 1.2. *If $\check{M}(\frac{1}{2}) \neq 0$, then $x^{2n} + y^{2m} = 1$ is an algebraic limit cycle of Eq (1.8), where $\check{M}(h)$ can be found in Eq (2.12).*

We remark that the conclusions in Theorems 1.1 and 1.2 improve some of the conclusions in Theorem 5.2 in [20] and Theorems 3 and 4 in [15].

The rest of this paper is organized as follows. In Section 2, the expansion of the first order Melnikov function of Eq (1.8) and the number of its zeros are studied. Further, we use the obtained results to prove Theorems 1.1 and 1.2. In Section 3, we present two examples. One for the case of $m = 2$, $n = 5$, and $k = 2$, and the other for the case of $m = 2$, $n = 3$, and $k = 3$ in system (1.8).

2. Proofs of Theorems 1.1 and 1.2

In this section, we provide proofs for Theorems 1.1 and 1.2. Before that, we present some useful preliminaries. The following lemma can be found in Theorem 2.2 or 3.3 in [21].

Lemma 2.1. *([21]) Consider system (1.8). If the first order Melnikov function has at most l zeros in $h \in (0, +\infty)$, multiplicity taken into account, then for any compact set $D \subset G(= \bigcup_{(0,+\infty)} L_h)$ there is $\varepsilon_0 > 0$ such that Eq (1.8) has at most l limit cycles in D for $|\varepsilon| < \varepsilon_0$, multiplicity taken into account. In this case, we can say that the period annulus G generates at most l limit cycles.*

Next, we give an expression of the first order Melnikov function of Eq (1.8) below.

Lemma 2.2. *For $h \in (0, +\infty)$, the first order Melnikov function $M(h)$ of Eq (1.8) has the following form*

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \sum_{i+j=0}^{k} b_{i,j} h^{\frac{i}{n}+\frac{j}{m}}, \tag{2.1}$$

where

$$b_{i,j} = a_{2i,2j} 2^{\frac{i}{n}+\frac{j}{m}} B\left(\frac{2i+1}{2n}, \frac{2j+1}{2m} + 1\right) \tag{2.2}$$

with $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ being the Beta function.

Further, as $km < n$,

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \sum_{r=0}^{kn} b_r h^{\frac{r}{mn}}, \qquad h \in (0, +\infty), \tag{2.3}$$

where

$$b_r = \begin{cases} a_{2(\frac{r}{m}-\frac{n}{m}[\frac{r}{n}]),2[\frac{r}{n}]} 2^{\frac{r}{mn}} B\left(\frac{1}{mn}\left(r - n\left[\frac{r}{n}\right]\right) + \frac{1}{2n}, \frac{1}{m}\left[\frac{r}{n}\right] + \frac{1}{2m} + 1\right), \\ \qquad\qquad r = mi + nj,\ i+j = 0, 1, \dots, k, \\ 0, \qquad\qquad \text{others}, \end{cases} \tag{2.4}$$

with $[\cdot]$ being the integer part function, as $kn < m$,

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \sum_{r=0}^{km} b_r h^{\frac{r}{mn}}, \qquad h \in (0, +\infty), \tag{2.5}$$

where

$$b_r = \begin{cases} a_{2[\frac{r}{m}],2(\frac{r}{n}-\frac{m}{n}[\frac{r}{m}])} 2^{\frac{r}{mn}} B\left(\frac{1}{n}\left[\frac{r}{m}\right] + \frac{1}{2n}, \frac{1}{mn}\left(r - m\left[\frac{r}{m}\right]\right) + \frac{1}{2m} + 1\right), \\ \qquad\qquad r = mi + nj,\ i+j = 0, 1, \dots, k, \\ 0, \qquad\qquad \text{others}, \end{cases} \tag{2.6}$$

and as $m = n$,

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{n}} \sum_{r=0}^{k} b_r h^{\frac{r}{n}}, \qquad h \in (0, +\infty), \tag{2.7}$$

Table 1. The values of $r = mi + nj$ for $i + j = 0, 1, \ldots, k$.

r \ i / j	0	1	2	$\ldots$	$k-2$	$k-1$	k
0	0	m	$2m$	$\ldots$	$(k-2)m$	$(k-1)m$	km
1	n	$m+n$	$2m+n$	$\ldots$	$(k-2)m+n$	$(k-1)m+n$	/
2	$2n$	$m+2n$	$2m+2n$	$\ldots$	$(k-2)m+2n$	/	/
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$(k-1)n$	$m+(k-1)n$	/	$\ldots$	/	/	/
k	kn	/	/	$\ldots$	/	/	/

where

$$b_r = \sum_{j=0}^{r} a_{2(r-j),2j} 2^{\frac{r}{n}} B\left(\frac{2r-2j+1}{2n}, \frac{2j+1}{2n}+1\right) \tag{2.8}$$

for $r = 0, 1, \ldots, k$.

Proof. It is clear that the first order Melnikov function $M(h)$ of Eq (1.8) can be expressed as

$$M(h) = \oint_{L_h} y(x^{2n} + y^{2m} - 1) \sum_{i+j=0}^{k} a_{2i,2j} x^{2i} y^{2j} dx$$

for $h \in (0, +\infty)$. Due to symmetry and noting that $y^2 = (2h - x^{2n})^{\frac{1}{m}}$ along the curve L_h, we obtain

$$\begin{aligned} M(h) &= 4\int_0^{(2h)^{\frac{1}{2n}}} (2h - x^{2n})^{\frac{1}{2m}} (2h-1) \sum_{i+j=0}^{k} a_{2i,2j} x^{2i} (2h - x^{2n})^{\frac{j}{m}} dx \\ &= 4(2h-1)(2h)^{\frac{1}{2m}} \int_0^{(2h)^{\frac{1}{2n}}} \left(1 - \frac{x^{2n}}{2h}\right)^{\frac{1}{2m}} \sum_{i+j=0}^{k} a_{2i,2j} x^{2i} (2h)^{\frac{j}{m}} \left(1 - \frac{x^{2n}}{2h}\right)^{\frac{j}{m}} dx. \end{aligned}$$

Let $x = (2hv)^{\frac{1}{2n}}$. Based on direct calculations, we have

$$\begin{aligned} M(h) &= \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \int_0^1 v^{\frac{1}{2n}-1} (1-v)^{\frac{1}{2m}} \sum_{i+j=0}^{k} a_{2i,2j} (2h)^{\frac{i}{n}+\frac{j}{m}} v^{\frac{i}{n}} (1-v)^{\frac{j}{m}} dv \\ &= \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \sum_{i+j=0}^{k} a_{2i,2j} (2h)^{\frac{i}{n}+\frac{j}{m}} B\left(\frac{i}{n} + \frac{1}{2n}, \frac{j}{m} + \frac{1}{2m} + 1\right) \end{aligned} \tag{2.9}$$

for $h \in (0, +\infty)$. Then we get Eq (2.1) with $b_{i,j}$ given in Eq (2.2).

For $i + j = 0, 1, \ldots, k$, the values of $r = mi + nj$ are shown in Table 1. From Table 1, it can be seen that as $km < n$, for $j = \left[\frac{r}{n}\right]$, $i = \frac{r}{m} - \frac{n}{m}\left[\frac{r}{n}\right]$, we obtain Eq (2.3) with b_r given in Eq (2.4), and as $kn < m$, for $i = \left[\frac{r}{m}\right]$, $j = \frac{r}{n} - \frac{m}{n}\left[\frac{r}{m}\right]$, we obtain Eq (2.5) with b_r given in Eq (2.6).

As $m = n$, $M(h)$ in Eq (2.9) can be rewritten as

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{n}} \sum_{i+j=0}^{k} a_{2i,2j}(2h)^{\frac{i+j}{n}} B\left(\frac{2i+1}{2n}, \frac{2j+1}{2n}+1\right)$$

for $h \in (0, +\infty)$. Then we obtain Eq (2.7) with b_r given in Eq (2.8) easily. This completes the proof.

Regarding the number of zeros of $M(h)$, we have the following results.

Lemma 2.3. *(i) As $km < n$ (resp. $kn < m$), for $h \in (0, +\infty)$ the first order Melnikov function $M(h)$ of system (1.8) has at most $kn + 1$ (resp. $km + 1$) nonzero zeros if it is not zero identically, multiplicity taken into account. Furthermore, $M(h)$ can have $\frac{k(k+3)}{2}$ simple positive zeros for $0 < h \ll 1$.*

(ii) As $m = n$, for $h \in (0, +\infty)$ the first order Melnikov function $M(h)$ of system (1.8) has at most $k + 1$ nonzero zeros if it is not zero identically, and multiplicity is taken into account. Furthermore, $M(h)$ can have k simple positive zeros for $0 < h \ll 1$.

Proof. As $km < n$, according to Lemma 2.2, the first order Melnikov function $M(h)$ can be rewritten as

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}} \bar{M}(h) \tag{2.10}$$

for $h \in (0, +\infty)$, where $\bar{M}(h) = \sum_{r=0}^{kn} b_r h^{\frac{r}{mn}}$ with b_r given in Eq (2.4). It is obvious that $\bar{M}(h)$ is a polynomial in $h^{\frac{1}{mn}}$.

Let $u = h^{\frac{1}{mn}}$. Then $\bar{M}(h)$ becomes

$$\bar{M}(h) = \sum_{r=0}^{kn} b_r u^r \equiv \tilde{M}(u)$$

for $u \in (0, +\infty)$. Note that $\tilde{M}$ is a kn-degree polynomial in u. Therefore, it has at most kn zeros for $u \in (0, +\infty)$ if $\tilde{M}(u) \not\equiv 0$, multiplicity taken into account. Note that $h = \frac{1}{2}$ is a zero of $M(h)$ in Eq (2.10). Then, $M(h)$ has at most $kn + 1$ nonzero zeros for $h \in (0, +\infty)$ if it is not zero identically, and multiplicity is taken into account.

Next, we prove that $M(h)$ can have $\frac{k(k+3)}{2}$ simple positive zeros as $km < n$ for $0 < h \ll 1$.

Based on Eq (2.4), it is easy to verify that

$$\begin{aligned}
&\det \frac{\partial(b_0, b_m, \ldots, b_{km}, b_n, b_{m+n}, \ldots, b_{(k-1)m+n}, b_{2n}, b_{m+2n}, \ldots, b_{(k-2)m+2n}, b_{3n}, \ldots, b_{(k-1)n}, b_{m+(k-1)n}, b_{kn})}{\partial(a_{0,0}, a_{2,0}, \ldots, a_{k,0}, a_{0,2}, a_{2,2}, \ldots, a_{2(k-1),2}, a_{0,4}, a_{2,4}, \ldots, a_{2(k-2),4}, a_{0,6}, \ldots, a_{0,2(k-1)}, a_{2,2(k-1)}, a_{0,2k})} \\
&= B\left(\frac{1}{2n}, \frac{1}{2m}+1\right) 2^{\frac{1}{n}} B\left(\frac{3}{2n}, \frac{1}{2m}+1\right) \ldots 2^{\frac{k}{n}} B\left(\frac{2k+1}{2n}, \frac{1}{2m}+1\right) 2^{\frac{1}{m}} B\left(\frac{1}{2n}, \frac{3}{2m}+1\right) \\
&2^{\frac{1}{n}+\frac{1}{m}} B\left(\frac{3}{2n}, \frac{3}{2m}+1\right) \ldots 2^{\frac{k-1}{n}+\frac{1}{m}} B\left(\frac{2k-1}{2n}, \frac{3}{2m}+1\right) 2^{\frac{2}{m}} B\left(\frac{1}{2n}, \frac{5}{2m}+1\right) 2^{\frac{1}{n}+\frac{2}{m}} \\
&B\left(\frac{3}{2n}, \frac{5}{2m}+1\right) \ldots 2^{\frac{k-2}{n}+\frac{2}{m}} B\left(\frac{2k-3}{2n}, \frac{5}{2m}+1\right) 2^{\frac{3}{m}} B\left(\frac{1}{2n}, \frac{7}{2m}+1\right) \ldots 2^{\frac{k-1}{m}} \\
&B\left(\frac{1}{2n}, \frac{2k-1}{2m}+1\right) 2^{\frac{1}{n}+\frac{k-1}{m}} B\left(\frac{3}{2n}, \frac{2k-1}{2m}+1\right) 2^{\frac{k}{m}} B\left(\frac{1}{2n}, \frac{2k+1}{2m}+1\right).
\end{aligned}$$

Note that the Beta function is positive. Then, the above determinant is positive. Applying the implicit function theorem, these $\frac{k(k+3)}{2}+1$ coefficients b_0, b_m, …, b_{km}, b_n, b_{m+n}, …, $b_{(k-1)m+n}$, b_{2n}, b_{m+2n}, …, $b_{(k-2)m+2n}$, b_{3n}, …, $b_{(k-1)n}$, $b_{m+(k-1)n}$, and b_{kn} can be taken as free parameters. Then, we can choose appropriate values $b_{kn}, b_{m+(k-1)n}, b_{(k-1)n}, \ldots, b_{3n}, b_{(k-2)m+2n}, \ldots, b_{m+2n}, b_{2n}, b_{(k-1)m+n}, \ldots, b_{m+n}, b_n, \ldots, b_m$, and b_0 one by one so that each of the two adjacent parameters has a different sign and

$$0 < |b_0| \ll |b_m| \ll \ldots \ll |b_{km}| \ll |b_n| \ll |b_{m+n}| \ldots \ll |b_{(k-1)m+n}| \ll |b_{2n}|$$
$$\ll |b_{m+2n}| \ldots \ll |b_{(k-2)m+2n}| \ll |b_{3n}| \ldots \ll |b_{(k-1)n}| \ll |b_{m+(k-1)n}| \ll |b_{kn}| \ll 1.$$

According to Descartes' rule of signs (see [22]), we know that $\tilde{M}(u)$ can have $\frac{k(k+3)}{2}$ positive simple zeros u_1, u_2, …, $u_{\frac{k(k+3)}{2}}$ with $0 < u_{\frac{k(k+3)}{2}} < \ldots < u_2 < u_1 \ll 1$. Consequently, $h_j = \frac{1}{2}u_j^{m+1}$, $j = 1, 2, \ldots, \frac{k(k+3)}{2}$ are positive simple zeros of $M(h)$ in h for $0 < h \ll 1$.

By a similar process, we can prove the case $kn < m$.

Now, we prove the case of $m = n$ by using a similar method. In this case, the first order Melnikov function $M(h)$ in Eq (2.7) can be rewritten as

$$M(h) = \frac{2}{n}(2h-1)(2h)^{\frac{1}{n}}\bar{\bar{M}}(h) \tag{2.11}$$

for $h \in (0, +\infty)$, where $\bar{\bar{M}}(h) = \sum_{r=0}^{k} b_r h^{\frac{r}{n}}$ with b_r given in Eq (2.8). Obviously, $\bar{\bar{M}}(h)$ is a polynomial in $h^{\frac{1}{n}}$.

Let $u = h^{\frac{1}{n}}$. Then $\bar{\bar{M}}(h)$ becomes

$$\bar{\bar{M}}(h) = \sum_{r=0}^{k} b_r u^r \equiv \hat{M}(u)$$

for $u \in (0, +\infty)$. It can be seen that $\hat{M}$ is a polynomial of degree k in u. Then, it has at most k zeros for $u \in (0, +\infty)$ if $\hat{M}(u) \not\equiv 0$, and multiplicity is taken into account. It is clear that $h = \frac{1}{2}$ is a zero of $M(h)$ in Eq (2.11). Then, $M(h)$ has at most $k+1$ nonzero zeros for $h \in (0, +\infty)$ if it is not zero identically, and multiplicity is taken into account.

By direct calculation, and noting that the $B(x, y)$ function is positive, we have

$$\begin{aligned}
&\det \frac{\partial(b_0, b_1, \ldots, b_k)}{\partial(a_{0,0}, a_{2,0}, \ldots, a_{2k,0})} \\
&= B\left(\frac{1}{2n}, \frac{1}{2n}+1\right) 2^{\frac{1}{n}} B\left(\frac{3}{2n}, \frac{1}{2n}+1\right) 2^{\frac{2}{n}} B\left(\frac{5}{2n}, \frac{1}{2n}+1\right) \ldots 2^{\frac{k}{n}} B\left(\frac{2k+1}{2n}, \frac{1}{2n}+1\right) \\
&> 0.
\end{aligned}$$

Thus, combined with implicit function theorem, b_0, b_1, …, b_{k-1}, and b_k can be taken as free parameters. We can change b_k, b_{k-1}, …, b_1 and b_0 in turn such that each of the two adjacent parameters has a different sign and

$$0 < |b_0| \ll |b_1| \ll \ldots \ll |b_k| \ll 1$$

so that $\hat{M}$ can generate k positive simple zeros u_1, u_2, …, u_k. This means that $\bar{\bar{M}}(h)$ and $M(h)$ in Eq (2.11) has k positive simple zeros $h_j = u_j^n$ in h for $0 < h \ll 1$. This finishes the proof.

Next, we utilize Lemmas 2.1 and 2.3 to prove Theorem 1.1.

Proof of Theorem 1.1. Combined with Lemma 2.1 and Lemma 2.3, we get the conclusion that system (1.8) has at most $kn+1$ (resp. $km+1$) as $km<n$ (resp. $kn<m$) and $k+1$ as $m=n$ limit cycles in Poincaré bifurcation, and multiplicity is taken into account, if the first order Melnikov function is not zero identically. Applying the implicit function theorem, we obtain that Eq (1.8) can have $\frac{k(k+3)}{2}$ as $km<n$ (resp. $kn<m$) and k as $m=n$ limit cycles near the origin for $0<|\varepsilon|\ll 1$. Thus, we have completed the proof.

Proof of Theorem 1.2. For system (1.8), through direct computation, it can be obtained that the flow along the algebraic curve $C(x,y)=x^{2n}+y^{2m}-1$ satisfies

$$\dot{C}=\frac{\partial C}{\partial x}\dot{x}+\frac{\partial C}{\partial y}\dot{y}=2m\varepsilon y^{2m}C(x,y)\sum_{i+j=0}^{k}a_{2i,2j}x^{2i}y^{2j}.$$

Thus, $C(x,y)=0$ is an invariant algebraic curve of system (1.8). From Eq (2.1), we know that $M(h)$ can be written as

$$M(h)=\frac{2}{n}(2h-1)(2h)^{\frac{1}{2m}+\frac{1}{2n}}\check{M}(h)$$

for $h\in(0,\infty)$, where

$$\check{M}(h)=\sum_{i+j=0}^{k}b_{i,j}h^{\frac{i}{n}+\frac{j}{m}} \tag{2.12}$$

with $b_{i,j}$ being given in Eq (2.2). It can be concluded that if $\check{M}(\frac{1}{2})\neq 0$, then $h=\frac{1}{2}$ is a simple zero of the first order Melnikov function $M(h)$ of Eq (1.8), it corresponds to the curve $x^{2n}+y^{2m}-1=0$. Therefore, $x^{2n}+y^{2m}-1=0$ is an algebraic limit cycle of Eq (1.8) if $\check{M}(\frac{1}{2})\neq 0$. This finishes the proof.

3. Examples

In this section, we give two examples. Example 3.1 is an application of Theorems 1.1 and 1.2. From Table 1, we can see that for $km\geq n$ or $kn\geq m$, some values of $r=mi+nj$ $(i+j=0,1,\ldots,k)$ are duplicated, and their size relationship is not obvious. However, given the definite values of m, n and k in system (1.8), we can still use the method in the proof of Lemma 2.3 to calculate the number of zeros of the first order Melnikov function, and thus obtain the number of limit cycles of the system. Example 3.2 is a concrete example.

Example 3.1. *In system (1.8), we take $m=2$, $n=5$ and $k=2$ to obtain*

$$\begin{cases}\dot{x}=2y^3,\\ \dot{y}=-5x^9+\varepsilon y\left(x^{10}+y^4-1\right)\displaystyle\sum_{i+j=0}^{2}a_{2i,2j}x^{2i}y^{2j},\end{cases} \tag{3.1}$$

where $0<|\varepsilon|\ll 1$ and $a_{2i,2j}$, $i,j=0,1,2$ are real numbers.

For system (3.1), we have the following results.

Theorem 3.1. *System (3.1) has at most* 11 *limit cycles in Poincaré bifurcation, counting multiplicities, if the first order Melnikov function is not zero identically. Moreover, there exist coefficients* $a_{2i,2j}$*,* $i, j = 0, 1, 2$*, such that it has 5 limit cycles near the origin for* $0 < |\varepsilon| \ll 1$*. In addition,* $x^{10} + y^4 = 1$ *is an algebraic limit cycle of Eq (3.1) if* $\sum_{r=0}^{10} b_r 2^{-\frac{r}{10}} \neq 0$*, where* b_r *can be found in Eq (3.3).*

Proof. From Lemma 2.2 we can write the first order Melnikov function of Eq (3.1) as

$$M(h) = \frac{2}{5}(2h-1)(2h)^{\frac{7}{20}}\bar{M}(h) \tag{3.2}$$

for $0 < h \ll 1$, where $\bar{M}(h) = \sum_{r=0}^{10} b_r h^{\frac{r}{10}}$, and

$$\begin{aligned} &b_0 = B\left(\frac{1}{10}, \frac{5}{4}\right)a_{0,0}, \qquad b_1 = 0,\\ &b_2 = 2^{\frac{1}{5}}B\left(\frac{3}{10}, \frac{5}{4}\right)a_{2,0}, \quad b_3 = 0,\\ &b_4 = 2^{\frac{2}{5}}B\left(\frac{1}{2}, \frac{5}{4}\right)a_{4,0}, \quad b_5 = 2^{\frac{1}{2}}B\left(\frac{1}{10}, \frac{7}{4}\right)a_{0,2},\\ &b_6 = 0, \qquad b_7 = 2^{\frac{7}{10}}B\left(\frac{3}{10}, \frac{7}{4}\right)a_{2,2},\\ &b_8 = 0, \qquad b_9 = 0,\\ &b_{10} = 2B\left(\frac{1}{10}, \frac{9}{4}\right)a_{0,4}. \end{aligned} \tag{3.3}$$

Let $u = h^{\frac{1}{10}}$. Then $\bar{M}(h)$ becomes

$$\bar{M}(h) = \sum_{r=0}^{10} b_r u^r \equiv \tilde{M}(u) \tag{3.4}$$

for $u \in (0, +\infty)$, which is a polynomial in u of degree 10. Hence, $\tilde{M}(u)$ has at most 10 nonzero zeros if $\tilde{M}(u)$ is not equal to 0 identically, multiplicity taken into account. Note that $h = \frac{1}{2}$ is a zero of $M(h)$. This results in $M(h)$ having at most 11 nonzero zeros if $M(h) \not\equiv 0$, and multiplicity is taken into account. Therefore, from Theorem 2.2 of [21] Eq (3.1) has at most 11 limit cycles if $M(h)$ is not equal to zero identically, and multiplicity is taken into account.

By direct calculation and with the help of the Maple software, we can obtain

$$\begin{aligned} &\det \frac{\partial(b_0, b_2, b_4, b_5, b_7, b_{10})}{\partial(a_{0,0}, a_{2,0}, a_{4,0}, a_{0,2}, a_{2,2}, a_{0,4})}\\ &= B\left(\frac{1}{10}, \frac{5}{4}\right)2^{\frac{1}{5}}B\left(\frac{3}{10}, \frac{5}{4}\right)2^{\frac{2}{5}}B\left(\frac{1}{2}, \frac{5}{4}\right)2^{\frac{1}{2}}B\left(\frac{1}{10}, \frac{7}{4}\right)2^{\frac{7}{10}}B\left(\frac{3}{10}, \frac{7}{4}\right)2B\left(\frac{1}{10}, \frac{9}{4}\right)\\ &= 2^{\frac{14}{5}}\frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{27}{20}\right)}\frac{\Gamma\left(\frac{3}{10}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{31}{20}\right)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}\frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{37}{20}\right)}\frac{\Gamma\left(\frac{3}{10}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{41}{20}\right)}\frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{47}{20}\right)}\\ &= \frac{25000000\, 2^{\frac{4}{5}}\pi^{\frac{13}{2}}\Gamma\left(\frac{19}{20}\right)\left[\Gamma\left(\frac{13}{20}\right)\right]^2\csc^2\left(\frac{3\pi}{10}\right)\csc^3\left(\frac{\pi}{10}\right)}{1731807\left[\Gamma\left(\frac{9}{10}\right)\Gamma\left(\frac{3}{4}\right)\right]^3\left[\Gamma\left(\frac{7}{10}\right)\right]^2\Gamma\left(\frac{11}{20}\right)\Gamma\left(\frac{17}{20}\right)\csc\left(\frac{\pi}{20}\right)\csc^2\left(\frac{7\pi}{20}\right)} \end{aligned}$$

$$\approx 80076.375036641078617.$$

This means that b_0, b_2, b_4, b_5, b_7, and b_{10} can be used as free parameters by using the implicit function theorem. Therefore, by sequentially selecting appropriate values b_{10}, b_7, b_5, b_4, b_2, and b_0, such that

$$0 < -b_0 \ll b_2 \ll -b_4 \ll b_5 \ll -b_7 \ll b_{10} \ll 1$$

or

$$0 < b_0 \ll -b_2 \ll b_4 \ll -b_5 \ll b_7 \ll -b_{10} \ll 1,$$

the function $\tilde{M}$ has 5 positive simple zeros $u_1, u_2, \ldots, u_5$ with $0 < u_5 < u_4 < \ldots < u_1 \ll 1$. Thus, it can be seen that M has 5 positive simple zeros $h_j = u_j^{10}$, $j = 1, 2, \ldots, 5$. This implies that Eq (3.1) can have 5 limit cycles near the origin for $0 < |\varepsilon| \ll 1$. It is evident that $x^{10} + y^4 = 1$ is an algebraic limit cycle of Eq (3.1) if $\bar{M}(\frac{1}{2}) \neq 0$.

In the following, we choose parameters b_0, b_2, b_4, b_5, b_7, and b_{10} for a numerical example to illustrate that system (3.1) has 5 limit cycles near the origin.

Let $b_0 = -9.37114105578347 \times 10^{-27}$, $b_2 = 2.647554473907 \times 10^{-20}$, $b_4 = -2.6939959303882 \times 10^{-14}$, $b_5 = 1.14345572121471 \times 10^{-11}$, $b_7 = -3.76752659117270 \times 10^{-7}$, $b_{10} = 1$. Then $\tilde{M}(u)$ in Eq (3.4) becomes

$$\begin{aligned}\tilde{M}(u) \;=\; & -9.37114105578347 \times 10^{-27} + 2.647554473907 \times 10^{-20}u^2 \\ & -2.6939959303882 \times 10^{-14}u^4 + 1.14345572121471 \times 10^{-11}u^5 \\ & -3.76752659117270 \times 10^{-7}u^7 + u^{10}.\end{aligned}$$

By means of the Maple software, we can solve

$$\begin{aligned}& u_1 = 0.000779999999999999, \quad u_2 = 0.00120000000000016, \\ & u_3 = 0.00249999999999921, \quad u_4 = 0.00430000000000014, \\ & u_5 = 0.00509999999999995, \quad u_6 = -0.00547317419061131, \\ & u_7 = -0.00354052973153023 + 0.00736830481643918\,\mathrm{I}, \\ & u_8 = -0.000662883173163847 + 0.000244677080123729\,\mathrm{I}, \\ & u_9 = -0.000662883173163847 - 0.000244677080123729\,\mathrm{I}, \\ & u_{10} = -0.00354052973153023 - 0.00736830481643918\,\mathrm{I}.\end{aligned}$$

Hence, $\tilde{M}(u)$ has 5 positive simple zeros. Note that $h_j = u_j^{10}$, $j = 1, 2, \ldots, 5$. Then we obtain that

$$\begin{aligned}& h_1 = 8.335775831 \times 10^{-32}, \quad h_2 = 6.191736422 \times 10^{-30}, \\ & h_3 = 9.536743164 \times 10^{-27}, \quad h_4 = 2.161148231 \times 10^{-24}, \\ & h_5 = 1.190424238 \times 10^{-23}\end{aligned}$$

are five zeros of the function $\bar{M}(h)$, and also of the function $M(h)$ in Eq (3.2). This implies that Eq (3.1) has 5 limit cycles near the origin for $0 < |\varepsilon| \ll 1$.

Example 3.2. *In system (1.8), we take $m = 2$, $n = 3$ and $k = 3$ to obtain*

$$\begin{cases} \dot{x} = y^3, \\ \dot{y} = -3x^5 + \varepsilon y(x^6 + y^4 - 1) \sum_{i+j=0}^{3} a_{2i,2j} x^{2i} y^{2j}, \end{cases} \tag{3.5}$$

where $0 < |\varepsilon| \ll 1$ and $a_{2i,2j}$, $i, j = 0, 1, 2, 3$ are bounded.

Proceeding as in the proof of Theorem 1.1, for system (3.5) we get the following results.

Theorem 3.2. *System (3.5) has at most 10 limit cycles in Poincaré bifurcation, counting multiplicities, if the first order Melnikov function is not zero identically. Moreover, there exist coefficients $a_{2i,2j}$, $i, j = 0, 1, 2$, such that it has 8 limit cycles near the origin for $0 < |\varepsilon| \ll 1$. In addition, $x^{10} + y^4 = 1$ is an algebraic limit cycle of Eq (3.1) if $\sum_{r=0}^{9} b_r 2^{-\frac{r}{6}} \neq 0$, where b_r can be found in Eq (3.7).*

Proof. According to Lemma 2.2, the first order Melnikov function of Eq (3.5) can be expressed as

$$M(h) = \frac{2}{3}(2h-1)(2h)^{\frac{5}{12}} \bar{M}(h) \tag{3.6}$$

for $h \in (0, +\infty)$, where $\bar{M}(h) = \sum_{r=0}^{9} b_r h^{\frac{r}{6}}$, and

$$\begin{aligned}
&b_0 = B\left(\frac{1}{6}, \frac{5}{4}\right) a_{0,0}, && b_1 = 0, \\
&b_2 = 2^{\frac{1}{3}} B\left(\frac{1}{2}, \frac{5}{4}\right) a_{2,0}, && b_3 = 2^{\frac{1}{2}} B\left(\frac{1}{6}, \frac{7}{4}\right) a_{0,2}, \\
&b_4 = 2^{\frac{2}{3}} B\left(\frac{5}{6}, \frac{5}{4}\right) a_{4,0}, && b_5 = 2^{\frac{5}{6}} B(\frac{3}{6}, \frac{7}{4}) a_{2,2}, \\
&b_6 = 2B\left(\frac{7}{6}, \frac{5}{4}\right) a_{6,0} + 2B\left(\frac{1}{6}, \frac{9}{4}\right) a_{0,4}, \\
&b_7 = 2^{\frac{7}{6}} B\left(\frac{5}{6}, \frac{7}{4}\right) a_{4,2}, && b_8 = 2^{\frac{4}{3}} B\left(\frac{1}{2}, \frac{9}{4}\right) a_{2,4}, \\
&b_9 = 2^{\frac{3}{2}} B\left(\frac{1}{6}, \frac{11}{4}\right) a_{0,6}.
\end{aligned} \tag{3.7}$$

Let $u = h^{\frac{1}{6}}$. Then $\bar{M}(h)$ can be written as

$$\bar{M}(h) = \sum_{r=0}^{9} b_r u^r \equiv \tilde{M}(u) \tag{3.8}$$

for $u \in (0, +\infty)$, which is a polynomial in u of degree 9. Hence, $\tilde{M}(u)$ has at most 9 nonzero zeros if $\tilde{M}(u)$ is not equal to 0 identically, and multiplicity is taken into account. Note that $h = \frac{1}{2}$ is a zero of $M(h)$. This deduces that $M(h)$ has at most 10 nonzero zeros if $M(h) \not\equiv 0$, and multiplicity is taken into account. Therefore, Eq (3.5) has at most 10 limit cycles if $M(h)$ is not equal to 0 identically, and multiplicity is taken into account.

By direct calculations, we have

$$\begin{aligned}
&\det \frac{\partial(b_0, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)}{\partial(a_{0,0}, a_{2,0}, a_{0,2}, a_{4,0}, a_{2,2}, a_{0,4}, a_{4,2}, a_{2,4}, a_{0,6})}\\
&= B\left(\frac{1}{6}, \frac{5}{4}\right) 2^{\frac{1}{3}} B\left(\frac{1}{2}, \frac{5}{4}\right) 2^{\frac{1}{2}} B\left(\frac{1}{6}, \frac{7}{4}\right) 2^{\frac{2}{3}} B\left(\frac{5}{6}, \frac{5}{4}\right) 2^{\frac{5}{6}} B\left(\frac{1}{2}, \frac{7}{4}\right) 2B\left(\frac{1}{6}, \frac{9}{4}\right)\\
&\quad 2^{\frac{7}{6}} B\left(\frac{5}{6}, \frac{7}{4}\right) 2^{\frac{4}{3}} B\left(\frac{1}{2}, \frac{9}{4}\right) 2^{\frac{3}{2}} B\left(\frac{1}{6}, \frac{11}{4}\right)\\
&= 2^{\frac{22}{3}} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{17}{12}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{23}{12}\right)} \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{25}{12}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{9}{4}\right)} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{29}{12}\right)} \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{31}{12}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{11}{4}\right)}\\
&\quad \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{11}{4}\right)}{\Gamma\left(\frac{35}{12}\right)}\\
&= \frac{69657034752\, 2^{\frac{1}{3}} \pi^{\frac{13}{2}} \Gamma\left(\frac{7}{12}\right) \sin\left(\frac{5\pi}{12}\right)}{409003595 \Gamma\left(\frac{11}{12}\right) \Gamma\left[\left(\frac{5}{6}\right)\right]^2 \Gamma\left(\frac{3}{4}\right)^2}\\
&\approx 267334.50047032523326.
\end{aligned}$$

This implies that b_0, b_2, b_3, ..., b_8, and b_9 can be used as free parameters. Therefore, we can choose appropriate values b_9, b_8, ..., b_2, and b_0 one by one such that

$$0 < -b_0 \ll b_2 \ll -b_3 \ll b_4 \ll -b_5 \ll b_6 \ll -b_7 \ll b_8 \ll -b_9 \ll 1,$$

or

$$0 < b_0 \ll -b_2 \ll b_3 \ll -b_4 \ll b_5 \ll -b_6 \ll b_7 \ll -b_8 \ll b_9 \ll 1,$$

so that the function $\tilde{M}$ can produce 8 positive simple zeros $u_1, u_2, \ldots, u_8$ with $0 < u_8 < \ldots < u_1 \ll 1$. This leads to that M can have 8 positive simple zeros $h_j = u_j^6$, $j = 1, 2, \ldots, 8$.

Thus, we can deduce that Eq (3.5) possesses 8 limit cycles near the origin for $0 < |\varepsilon| \ll 1$. Apparently, $x^6 + y^4 = 1$ is an algebraic limit cycle associated with system (3.5) if $\bar{M}(\frac{1}{2}) \neq 0$.

Next, we let $b_0 = 1.79287565383906 \times 10^{-26}$, $b_2 = -3.86583673559000 \times 10^{-19}$, $b_3 = 1.22569182231640 \times 10^{-15}$, $b_4 = -1.71070886184070 \times 10^{-12}$, $b_5 = 1.29602736057760 \times 10^{-9}$, $b_6 = -5.64766531197460 \times 10^{-7}$, $b_7 = 0.000140687312789683$, $b_8 = -0.0185432716616353$, $b_9 = 1$. In this case, the function $\tilde{M}$ in Eq (3.8) can be written as

$$\begin{aligned}
\tilde{M}(u) \;=\; & 1.79287565383906 \times 10^{-26} - 3.86583673559000 \times 10^{-19} u^2\\
& + 1.22569182231640 \times 10^{-15} u^3 - 1.71070886184070 \times 10^{-12} u^4\\
& + 1.29602736057760 \times 10^{-9} u^5 - 5.64766531197460 \times 10^{-7} u^6\\
& + 0.000140687312789683 u^7 - 0.0185432716616353 u^8 + u^9.
\end{aligned}$$

Using the Maple software, we can obtain the zeros of the function $\tilde{M}(u)$ as follows

$$u_1 = 0.000539999999999838, \qquad u_2 = 0.000669999999975980,$$

$$u_3 = 0.00120000000087256, \quad u_4 = 0.00149999999949338,$$
$$u_5 = 0.00249999999983281, \quad u_6 = 0.00319999999987951,$$
$$u_7 = 0.00430000000000642, \quad u_8 = 0.00480000000000168,$$
$$u_9 = -0.000166728338426876.$$

It can be seen from the above that $\tilde{M}(u)$ has 8 positive simple zeros. Notice that $h_j = u_j^6$, $j = 1, 2, \ldots, 8$. Thus, we get the zeros of the function $\bar{M}(h)$:

$$h_1 = 2.479491130 \times 10^{-20}, \quad h_2 = 9.045838217 \times 10^{-20},$$
$$h_3 = 2.985984015 \times 10^{-18}, \quad h_4 = 1.139062495 \times 10^{-17},$$
$$h_5 = 2.441406250 \times 10^{-16}, \quad h_6 = 1.073741824 \times 10^{-15},$$
$$h_7 = 6.321363049 \times 10^{-15}, \quad h_8 = 1.223059046 \times 10^{-14}.$$

These are also the zeros of the function $M(h)$ in Eq (3.6). Then, the system (3.5) has 5 limit cycles near the origin for $0 < |\varepsilon| \ll 1$.

Use of AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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