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Research article

Dynamics of the discrete-time man-environment-man epidemic model with a free boundary

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Abstract: In this paper, we investigate the discrete-time man-environment-man epidemic model with a free boundary, which can be viewed as a time-discrete version of the free boundary model studied by Ahn, Beak, and Lin. First, applying the properties of the principal eigenvalue of the corresponding eigenvalue problem, we obtain the global dynamics of the corresponding fixed boundary problem. Then, we solve the problem step by step and establish the well-posedness of the solution. Moreover, we provide some sufficient conditions for the diseases spreading and vanishing by using the modified comparison principle. Finally, we give the long-time behavior of the solution by making use of the above results about the corresponding fixed boundary problem.

Keywords: discrete-time; epidemic model; free boundary; spreading-vanishing dichotomy

1. Introduction

Fecally–orally transmitted diseases have long caused great harm to human health, such as cholera. During the spreading of cholera, healthy humans will be infected after drinking water containing bacteria, and infected humans in return contribute to the spread of the disease. To describe the cholera epidemic which spread in the European/Mediterranean region in 1973, Capasso and Paveri-Fontana [1] proposed the following ordinary differential equation:

$$\begin{cases} u' = -a_{11}u + a_{12}v, & t > 0, \\ v' = -a_{22}v + G(u), & t > 0, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ denote the concentration of the bacteria and infective human population at time t, respectively. The positive constants a_{11} and a_{22} denote the removal rate of the bacteria and the infective human population, respectively, the positive constant a_{12} represents the growth rate of the bacteria contributed by the infective human, and $G(u)$ represents the infection rate of humans. Under the assumption that the total susceptible human population is constant during the evolution of the epidemic, they showed that there exists a threshold parameter determining whether the epidemic will spread or not.

However, model (1.1) ignores spatial diffusion. Supposing that the diffusion rate of the human population is smaller than that of bacteria and can be ignored, Capasso and Maddalena [2] proposed the following model:

$$\begin{cases} u_t(t,x) = d\Delta u(t,x) - a_{11}u(t,x) + a_{12}v(t,x), & (t,x) \in (0,+\infty)\times\Omega, \\ v_t(t,x) = -a_{22}v(t,x) + G(u(t,x)), & (t,x) \in (0,+\infty)\times\Omega, \\ \frac{\partial u}{\partial \eta} + \alpha u = 0, & (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

where $u(x,t)$ and $v(x,t)$ represent the spatial densities of the bacteria and the infective humans at time $t \geq 0$ and a point x in the habitat $\Omega \subset \mathbb{R}^n$. The positive constant d denotes the dispersal rate of the bacteria and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative. They obtained a threshold parameter $\widehat{R}_0 > 0$ such that the diseases will spread for $\widehat{R}_0 > 1$ and the diseases will tend to extinction for $\widehat{R}_0 < 1$. From the point of view of epidemic waves, the existence of Fisher-type monotone traveling waves and the minimal wave speed of problem (1.2) were obtained by [3].

The above results show that the disease will always spread when $\widehat{R}_0 > 1$ no matter what the initial infective region and the initial size of the infective human population are. Obviously, this does not match the facts well. To overcome this shortcoming, Ahn et al. [4] introduced the free boundary conditions into Eq (1.2) and proposed the following problem:

$$\begin{cases} u_t(t,x) = du_{xx}(t,x) - a_{11}u(t,x) + a_{12}v(t,x), & t>0,\ g(t)<x<h(t), \\ v_t(t,x) = -a_{22}v(t,x) + G(u(t,x)), & t>0,\ g(t)<x<h(t), \\ u(t,x) = v(t,x) = 0, & t \geq 0,\ x = g(t) \text{ or } x = h(t), \\ g(0) = -h_0,\ g'(t) = -\mu u_x(t,g(t)), & t>0, \\ h(0) = h_0,\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & -h_0 \leq x \leq h_0. \end{cases} \tag{1.3}$$

They found that when $\widehat{R}_0 > 1$ (here $\widehat{R}_0$ was given in [2]), there exists a spatial-temporal risk index $R_0^F(t)$ such that if $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, the bacteria must spread, while if $R_0^F(0) < 1 < \widehat{R}_0$, the spreading or vanishing of the bacteria depends on the initial number of bacteria, the length of the initial habitat, the diffusion rate, and other factors. This result seems more realistic than that in [2]. Following the work of [4], many free boundary problems related to Eq (1.3) have been studied. For example, Chen et al. [5] considered the effect of time delay on the spreading of diseases. Wang and Du [6] further extended the results in [4] to the case of that the diffusion rate of human population is not zero. Later, Chang and Du [7] considered the nonlocal version of [6], and obtained a rather complete description of the long-time dynamics of the model. Recently, Li et al. [8] studied the nonlocal version of Eq (1.3) and further assumed that the infective rate of humans is a nonlocal term due to the nonlocal diffusion of the infectious agents.

We note that a derivation of the free boundary conditions in Eq (1.3) can be found in [9]. To consider the spreading of invasive species, Du and Lin [10] introduced these free boundary conditions into a logistic type local diffusion model, and established a spreading-vanishing dichotomy. Subsequently, many researchers have studied the free boundary problems; we refer to the epidemic models [11–15] and other models [16–20] and the references therein for some of the recent works in this direction.

Note that all the above models are continuous-time models. However, for some non-overlapping generations like some annual plants or insects, they first have a growth phase during which spatial dispersal is negligible, and then they have a dispersal phase during which no growth occurs [21]. Further, statistical data for scientific research is collected in discrete time, especially during an epidemic [22–24]. Hence, it is necessary to study discrete models. According to [25], the general form of discrete-time single-species models is:

$$N_{n+1}(x) = \mathcal{F}(N_n(x)),$$

where $N_n(x)$ represents the density of population at time n and location x, and $\mathcal{F}$ is a specified operator which models the growth, interaction, and migration of the species. Many related discrete-time models have been studied, such as Naik et al. [26] considered the complex dynamics of a discrete-time seasonally-forced SIR epidemic model. Recently, Li et al. [27] considered the following temporally discrete diffusion equation:

$$\begin{cases} u(n+1,x) = D\Delta u(n+1,x) + (1-d)u(n,x) + \eta b(u(n,x)), & n \geq 0,\ x \in (0,h_{n+1}), \\ u_x(n,0) = 0,\ u(n,x) = 0, & n \geq 0,\ x \geq h_n, \\ h_{n+1} = h_n - \mu u_x(n,h_n), & n \geq 0, \\ u(0,x) = u_0(x), & x \in [0,h_0], \end{cases} \tag{1.4}$$

where $u(n,x)$ denotes the population density at time $n \in \mathbb{N}$ and location $x \in [0,h_n]$. They first proved the well-posedness and boundedness of the global solution, then obtained a spreading-vanishing dichotomy, and finally gave some sufficient conditions for spreading or vanishing. For more temporally discrete reaction-diffusion models, we can refer to [28–30] and references therein.

Since the spatial densities of the bacteria and the infective humans are normally observed at discrete points in time, it is more reasonable to use temporal discrete models to describe the spreading of the diseases. Motivated by [4] and [27], we propose the following discrete-time version of Eq (1.3):

$$\begin{cases} u(n+1,x) = du_{xx}(n+1,x) + (1-a_{11})u(n,x) + a_{12}v(n,x), & n \geq 0,\ x \in (0,h_{n+1}), \\ v(n+1,x) = (1-a_{22})v(n,x) + G(u(n,x)), & n \geq 0,\ x \in (0,h_{n+1}), \\ u_x(n,0) = u(n,x) = 0,\ v_x(n,0) = v(n,x) = 0, & n \geq 0,\ x \geq h_n, \\ h_{n+1} = h_n - \mu u_x(n,h_n), & n \geq 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in [0,h_0], \end{cases} \tag{1.5}$$

where the parameters a_{11} and a_{22} satisfy

(A): $0 < a_{11}, a_{22} < 1$ and $a_{12}G'(0) \neq (1-a_{11})(1-a_{22})$,

and the function $G(u)$ satisfies:

(G1): $G \in C^1([0,\infty)),\ G(0) = 0,\ G'(u) > 0,\ \forall u \geq 0;$

(G2): $\frac{G(u)}{u}$ is decreasing and $\lim\limits_{u\to+\infty} \frac{G(u)}{u} < \frac{a_{11}a_{22}}{a_{12}}$.

The initial functions $u_0(x)$ and $v_0(x)$ belong to

$$\chi(h_0) := \{\phi \in C^1([0, h_0)) \mid \phi(x) > 0 \text{ for } x \in [0, h_0)\}.$$

Throughout this paper, we always assume that the conditions **(A)**, **(G1)**, and **(G2)** hold. While the dynamics of the continuous model (1.3) were obtained in [4], much less is known for the time-discrete model (1.5). Following the approach developed in [27] where a time-discrete single-species model with a free boundary was treated, we extend [27] to the case of a system with free boundary Eq (1.5) and obtain some initial results. Since discretization of time makes the eigenvalue problem of the systems more challenging, we introduce new methods to address these challenges. We believe that the methods of this work can provide insights into the investigation of discrete time system with free boundary, and the conclusions in this paper can provide the theoretical basis for controlling disease.

This paper is organized as follows. In Section 2, we discuss the global dynamics of the solution to the corresponding fixed bounded boundary by the related eigenvalue problems. In Section 3, we first prove the well-posedness of the solution to Eq (1.5), then establish the criteria for spreading and vanishing, and finally we give the long-time behavior of the solution.

2. Global dynamical behavior of the fixed boundary problem

To investigate the dynamical properties of Eq (1.5), we first consider the following temporally discrete initial boundary value problem in this section:

$$\begin{cases} u(n+1,x) = du_{xx}(n+1,x) + (1-a_{11})u(n,x) + a_{12}v(n,x), & n \geq 0,\ x \in (0,L), \\ v(n+1,x) = (1-a_{22})v(n,x) + G(u(n,x)), & n \geq 0,\ x \in (0,L), \\ u_x(n,0) = u(n,L) = 0,\ v_x(n,0) = v(n,L) = 0, & n \geq 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in [0,L], \end{cases} \tag{2.1}$$

where $u(n,x)$ and $v(n,x)$ represent the spatial densities of bacteria and the infective human, respectively, at time $n \in \mathbb{N}$ and location $x \in [0,L]$ with finite domain length $L > 0$. Assume that the initial functions u_0 and v_0 are nonnegative and belong to $C^1([0,L])$.

For convenience, we define

$$F(u(n,\cdot), v(n,\cdot)) = (1-a_{11})u(n,\cdot) + a_{12}v(n,\cdot).$$

For nonnegative integers a and b, if $a < b$, we set the discrete time point intervals

$$\mathbb{N}(a,b) = \{a+1, a+2, \ldots, b-1\},\ \mathbb{N}[a,b] = \{a, a+1, \ldots, b\}.$$

Similarly, we can define $\mathbb{N}(a,b]$ and $\mathbb{N}[a,b)$.

2.1. Preliminaries

We first give the following three lemmas about the corresponding ODEs, which are proved in [27].

Lemma 2.1. *Suppose that $f \in C([0,L])$. Then, the problem*

$$\begin{cases} -dU''(x) + U(x) = f(x),\ x \in (0,L), \\ U'(0) = U(L) = 0, \end{cases} \tag{2.2}$$

admits a unique classical solution $U(x) = \int_0^L K(x,y)f(y)dy$*, where*

$$K(x,y) = \begin{cases} \dfrac{\sinh \frac{L-x}{\sqrt{d}} \cosh \frac{y}{\sqrt{d}}}{\sqrt{d}\cosh \frac{L}{\sqrt{d}}}, & 0 \le y \le x, \\ \dfrac{\cosh \frac{x}{\sqrt{d}} \sinh \frac{L-y}{\sqrt{d}}}{\sqrt{d}\cosh \frac{L}{\sqrt{d}}}, & x \le y \le L, \end{cases} \tag{2.3}$$

$\sinh x = \frac{e^x - e^{-x}}{2}$ *and* $\cosh x = \frac{e^x + e^{-x}}{2}$.

Lemma 2.2. *Let* $K(x,y)$ *be given as in Eq* (2.3)*. Then, it satisfies:*

(i) *for any* $(x,y) \in [0,L] \times [0,L]$, $K(x,y)$ *is continuously differentiable, except for* $x = y$, $K(x,y) \ge 0$*, and* $K(x,y) = K(y,x)$*;*

(ii) $\max_{(x,y)\in[0,L]\times[0,L]} K(x,y) = K(0,0) = \frac{1}{\sqrt{d}} \tanh \frac{L}{\sqrt{d}}$*, where* $\tanh x = \frac{\sinh x}{\cosh x}$*;*

(iii) *for any* $x \in [0,L]$, $\int_0^L K(x,y)dy = 1 - \frac{\cosh \frac{x}{\sqrt{d}}}{\cosh \frac{L}{\sqrt{d}}}$ *and is decreasing in x.*

Lemma 2.3. *Let* $U(x)$ *be given in Lemma 2.1. Then,*

$$||U||_{C^2([0,L])} \le C||f||_{C([0,L])}, \tag{2.4}$$

where $C > 0$ is a constant independent of f and L.

Applying Lemmas 2.1 and 2.2, we can obtain the well-posedness and boundedness of the global solution to Eq (2.1) by using an iterative method.

Proposition 2.4. *For any given* $u_0, v_0 \in C^1([0,L])$ *and* $u_0(x), v_0(x) \ge 0,\ x \in [0,L]$*, problem* (2.1) *admits a unique nonnegative bounded classical solution* $(u(n,x), v(n,x))$ *for* $n \in \mathbb{N}(0,+\infty)$.

Proof. For any given nonnegative $u_0, v_0 \in C^1([0,L])$, we take $G_0 := \max_{x\in[0,L]} G(u_0(x))$. Then, we directly calculate $v(1,x) \ge 0$ and

$$||v(1,\cdot)||_{C([0,L])} \le (1-a_{22})||v_0||_{C([0,L])} + ||G(u_0)||_{C([0,L])} = (1-a_{22})||v_0||_{C([0,L])} + G_0.$$

By Eq (2.1), we have

$$\begin{cases} u(1,x) - du_{xx}(1,x) = F(u_0(x), v_0(x)),\ x \in (0,L), \\ u_x(1,0) = 0,\ u(1,L) = 0. \end{cases} \tag{2.5}$$

Thanks to $F(u_0(x), v_0(x)) \in C([0,L])$, it follows from Lemma 2.1 that Eq (2.5) has a unique classical solution $u(1,x) = \int_0^L K(x,y)F(u_0(y), v_0(y))dy$, where $K(x,y)$ is given in Eq (2.3) and $u(1,x)$ is clearly nonnegative. Define $\theta = 1 - \frac{1}{\cosh \frac{L}{\sqrt{d}}} \in (0,1)$. Using Lemma 2.2(iii), we have

$$\begin{aligned} ||u(1,\cdot)||_{C([0,L])} &\le \theta||F(u_0(\cdot), v_0(\cdot))||_{C([0,L])} \le \theta(1-a_{11})||u_0||_{C([0,L])} + \theta a_{12}||v_0||_{C([0,L])} \\ &\le (1-a_{11})||u_0||_{C([0,L])} + a_{12}||v_0||_{C([0,L])}. \end{aligned}$$

For above $(u(1,x), v(1,x))$, we set $G_1 := \max_{x\in[0,L]} G(u(1,x))$. Then, we have $v(2,x) \geq 0$ and

$$\begin{aligned}\|v(2,\cdot)\|_{C([0,L])} &\leq (1-a_{22})^2\|v_0\|_{C([0,L])} + (1-a_{22})\|G(u_0)\|_{C([0,L])} + \|G(u(1,x))\|_{C([0,L])} \\ &\leq (1-a_{22})^2\|v_0\|_{C([0,L])} + (1-a_{22})G_0 + G_1.\end{aligned}$$

By Eq (2.1), we have

$$\begin{cases} u(2,x) - du_{xx}(2,x) = F(u(1,x), v(1,x)), \ x \in (0,L), \\ u_x(2,0) = 0, \ u(2,L) = 0. \end{cases} \tag{2.6}$$

Similarly, Eq (2.6) has an unique nonnegative classical solution $u(2,x)$ and

$$\begin{aligned}&\|u(2,\cdot)\|_{C([0,L])} \leq \theta\|F(u(1,\cdot), v(1,\cdot))\|_{C([0,L])} \\ &\leq \theta^2(1-a_{11})^2\|u_0\|_{C([0,L])} + (\theta^2 a_{12}(1-a_{11}) + \theta a_{12}(1-a_{22}))\|v_0\|_{C([0,L])} + \theta a_{12}G_0 \\ &\leq (1-a_{11})^2\|u_0\|_{C([0,L])} + (a_{12}(1-a_{11}) + a_{12}(1-a_{22}))\|v_0\|_{C([0,L])} + a_{12}G_0.\end{aligned}$$

Let $\widetilde{G} = \max\{G_0, G_1, G_2, \ldots, G_n\}$, where $G_i = \max_{x\in[0,L]} G(u(i,x))$ for $i \in \mathbb{N}[0,+\infty)$. Thus, $\widetilde{G}$ is only dependent on u_0, v_0. Repeating the above process and noting that $0 < 1-a_{11},\ 1-a_{22} < 1$, we can easily find that the nonnegative classical solution $(u(n,x), v(n,x))$ exists uniquely for $n \in \mathbb{N}[3,+\infty)$, and it satisfies that

$$\begin{aligned}\|u(n,\cdot)\|_{C([0,L])} &\leq (1-a_{11})^n\|u_0\|_{C([0,L])} + a_{12}N\sum_{i=0}^{n-1}(1-a_{11})^i\sum_{j=0}^{n-i-1}(1-a_{22})^j \\ &\leq (1-a_{11})\|u_0\|_{C([0,L])} + \frac{a_{12}N}{a_{11}a_{22}},\end{aligned}$$

and

$$\begin{aligned}\|v(n,\cdot)\|_{C([0,L])} &\leq (1-a_{22})^n\|v_0\|_{C([0,L])} + \widetilde{G}\sum_{i=0}^{n-1}(1-a_{22})^i \\ &\leq (1-a_{22})N + \frac{N}{a_{22}},\end{aligned}$$

where $N = \max\left\{\|v_0\|_{C([0,L])}, \widetilde{G}\right\}$. We complete the proof.

Next, we will give the following comparison principle, which can be proved by the similar arguments in [29, Lemma 3.3].

Lemma 2.5. *Suppose that* $T \in \mathbb{N}(0,+\infty), \overline{u} = (\overline{u}(0,\cdot), \overline{u}(1,\cdot), \ldots, \overline{u}(T,\cdot)), \overline{v} = (\overline{v}(0,\cdot), \overline{v}(1,\cdot), \ldots, \overline{v}(T,\cdot)) \in \mathbb{X}_T = C^1([0,L]) \times (C^2([0,L]))^T$ *and satisfies*

$$\begin{cases} \overline{u}(n+1,x) - d\overline{u}_{xx}(n+1,x) \geq F(\overline{u}(n,x), \overline{v}(n,x)), & (n,x) \in \mathbb{N}[0,T-1] \times (0,L), \\ \overline{v}(n+1,x) \geq (1-a_{22})\overline{v}(n,x) + G(\overline{u}(n,x)), & (n,x) \in \mathbb{N}[0,T-1] \times (0,L), \\ \overline{u}_x(n,0) \leq 0, \ \overline{u}(n,L) \geq 0, \ \overline{v}_x(n,0) \leq 0, \ \overline{v}(n,L) \geq 0, & n \in \mathbb{N}[0,T], \\ \overline{u}(0,x) \geq u(0,x), \ \overline{v}(0,x) \geq v(0,x), & x \in [0,L]. \end{cases} \tag{2.7}$$

Then, the solution (u,v) *of Eq* (2.1) *satisfies* $\overline{u}(n,x) \geq u(n,x),\ \overline{v}(n,x) \geq v(n,x)$ *for* $(n,x) \in \mathbb{N}(0,T] \times (0,L)$.

Proof. Let $w = \overline{u} - u,\ z = \overline{v} - v$. It follows from Eqs (2.1) and (2.7) that we have

$$\begin{cases} w(n+1,x) - dw_{xx}(n+1,x) - F(w(n,x), z(n,x)) \geq 0, & (n,x) \in \mathbb{N}[0,T-1] \times (0,L), \\ z(n+1,x) - (1-a_{22})z(n,x) - G'(\xi(n,x))w(n,x) \geq 0, & (n,x) \in \mathbb{N}[0,T-1] \times (0,L), \\ w_x(n,0) \leq 0,\ w(n,L) \geq 0,\ z_x(n,0) \leq 0,\ z(n,L) \geq 0, & n \in \mathbb{N}[0,T], \\ w(0,x) \geq 0,\ z(0,x) \geq 0, & x \in [0,L], \end{cases} \tag{2.8}$$

where ξ is between $\overline{u}$ and u. Taking $n = 0$ in the first two equations of Eq (2.8) and letting $n = 1$ in the third equation of Eq (2.8) yield

$$\begin{cases} w(1,x) - dw_{xx}(1,x) - F(w(0,x), z(0,x)) \geq 0, & x \in (0,L), \\ z(1,x) - (1-a_{22})z(0,x) - G'(\xi(0,x))w(0,x) \geq 0, & x \in (0,L), \\ w_x(1,0) \leq 0,\ w(1,L) \geq 0,\ z_x(1,0) \leq 0,\ z(1,L) \geq 0, & \\ w(0,x) \geq 0,\ z(0,x) \geq 0, & x \in [0,L]. \end{cases} \tag{2.9}$$

By $0 < 1 - a_{22} < 1$ and $G'(u) > 0$ for any $u \geq 0$, we have that $z(1,x) \geq 0$ for $x \in (0,L)$. Applying the maximum principle, we immediately obtain $w(1,x) \geq 0$ for $x \in (0,L)$. Repeating the above process, we can prove $w(n,x) \geq 0,\ z(n,x) \geq 0$ for $(n,x) \in \mathbb{N}(0,T] \times (0,L)$. We finish this proof.

Remark 2.6. The pair $(\overline{u}, \overline{v})$ is usually called an upper solution of Eq (2.1), while a lower solution $(\underline{u}, \underline{v})$ can be similarly defined by reversing all the inequalities.

2.2. Eigenvalue problems

For any given constant $L > 0$, we consider the following eigenvalue problem:

$$\begin{cases} \lambda\phi = d\lambda\phi'' + (1-a_{11})\phi - \frac{a_{12}G'(0)}{1-a_{22}}\phi,\ x \in (0,L), \\ \phi'(0) = \phi(L) = 0. \end{cases} \tag{2.10}$$

A simple calculation yields that the principal eigenvalue is

$$\widetilde{\lambda}_1 = \frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{[1 + d\pi^2/(4L^2)](1-a_{22})}$$

and its corresponding eigenfunction $\widetilde{\phi} \gg 0$ in $[0,L)$.

Linearizing Eq (2.1) at $(0,0)$, we have

$$\begin{cases} u(n+1,x) = du_{xx}(n+1,x) + F(u(n,x), v(n,x)), & n \geq 0,\ x \in (0,L), \\ v(n+1,x) = (1-a_{22})v(n,x) + G'(0)u(n,x), & n \geq 0,\ x \in (0,L), \\ u_x(n,0) = u(n,L) = 0,\ v_x(n,0) = v(n,L) = 0, & n \geq 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in [0,L]. \end{cases} \tag{2.11}$$

Letting $u(n,x) = \lambda^n\phi(x)$ and $v(n,x) = \lambda^n\psi(x)$, we have

$$\begin{cases} \lambda\phi = d\lambda\phi'' + (1-a_{11})\phi + a_{12}\psi, & x \in (0,L), \\ \lambda\psi = G'(0)\phi + (1-a_{22})\psi, & x \in (0,L), \\ \phi'(0) = \phi(L) = 0,\ \psi'(0) = \psi(L) = 0. & \end{cases} \tag{2.12}$$

For the convenience, we define an operator

$$\mathcal{L} := \begin{pmatrix} d\lambda \frac{d^2}{dx^2} + 1 - a_{11} & a_{12} \\ G'(0) & 1 - a_{22} \end{pmatrix}$$

with $D(\mathcal{L}) = \{(\phi, \psi) \in H^2(0, L) \times L^2(0, L) : \phi_x(0) = \phi(L) = \psi_x(0) = \psi(L) = 0\}$. Then, problem (2.12) can be rewritten as

$$\mathcal{L}(\phi, \psi)^T = \lambda(\phi, \psi)^T.$$

In the following, we give some important properties of the principal eigenvalue of Eq (2.12).

Proposition 2.7. *Assume that* $(1 - a_{11})(1 - a_{22}) \neq a_{12}G'(0)$. *Let* $\sigma(\mathcal{L})$ *be the spectral set of* $\mathcal{L}$ *and* $\mathcal{S}(\mathcal{L}) := \sup\{Re\lambda : \lambda \in \sigma(\mathcal{L})\}$ *be the spectral bound of* $\mathcal{L}$*. Then, the following statements hold:*

(i) $\mathcal{S}(\mathcal{L})$ *is the principal eigenvalue of Eq* (2.12)*;*

(ii) *if*

$$L > \frac{\pi}{2} \sqrt{\frac{d}{-a_{11} + a_{12}G'(0)/a_{22}}} =: L^*, \tag{2.13}$$

then $\lambda_1 > 1$.

Proof. The proof is similar to [31, Theorem 3.1]. For $\lambda > 1 - a_{22}$, we define the following linear operator

$$\mathcal{L}_\lambda := d\lambda \frac{d^2}{dx^2} + 1 - a_{11} + \frac{a_{12}G'(0)}{\lambda - 1 + a_{22}}$$

on $C([0, L], \mathbb{R})$, and set

$$\begin{aligned} Q(\lambda) = &\left(-1 + a_{11} + \frac{a_{12}G'(0)}{1 - a_{22}}\right)\lambda^2 + \left[(1 - a_{11})(1 - a_{22}) - a_{12}G'(0) + \widetilde{\lambda}_1(1 - a_{11})\right]\lambda \\ &+ \widetilde{\lambda}_1 \left[(1 - a_{11})(a_{22} - 1) + a_{12}G'(0)\right]. \end{aligned}$$

(i) **Cases 1**: $(1 - a_{11})(1 - a_{22}) > a_{12}G'(0)$. In this case, we have $\widetilde{\lambda}_1 \in (0, 1)$ and

$$Q(0) = \widetilde{\lambda}_1 \left[(1 - a_{11})(a_{22} - 1) + a_{12}G'(0)\right] < 0, \; Q(1 - a_{22}) = a_{12}G'(0)\widetilde{\lambda}_1 > 0.$$

It follows that equation $Q(\lambda) = 0$ has two distinct real roots. We denote the largest root by λ_1. Then, $\lambda_1 > 1 - a_{22}$.

Cases 2: $(1 - a_{11})(1 - a_{22}) < a_{12}G'(0)$. In this case, we have $\widetilde{\lambda}_1 < 0$ and

$$Q(0) = \widetilde{\lambda}_1 \left[(1 - a_{11})(a_{22} - 1) + a_{12}G'(0)\right] < 0, \; Q(1 - a_{22}) = a_{12}G'(0)\widetilde{\lambda}_1 < 0.$$

Thus, the equation $Q(\lambda) = 0$ has two distinct real roots. We denote the largest root by λ_1, and then $\lambda_1 > 1 - a_{22}$.

Direct calculation yields $\mathcal{L}_{\lambda_1}\widetilde{\phi} = \lambda_1\widetilde{\phi}$. Then, it follows from [32, Theorem 2.3(i)] that $\mathcal{S}(\mathcal{L}) = \lambda_1$ is the principal eigenvalue of Eq (2.12).

(ii) According to $Q(\lambda_1) = 0$, we can derive that

$$\widetilde{\lambda_1} = \lambda_1 - \frac{a_{12}G'(0)\lambda_1^2}{(1-a_{22})[(1-a_{11})\lambda_1 + a_{12}G'(0) - (1-a_{11})(1-a_{22})]} =: f(\lambda_1), \tag{2.14}$$

where $\lambda_1 > 1 - a_{22}$.

Cases 1: $(1-a_{11})(1-a_{22}) > a_{12}G'(0)$. In this case, we will prove that $f(\lambda_1)$ is strictly increasing in λ_1. It is easy to calculate that

$$f'(\lambda_1) = \frac{[(1-a_{11})(1-a_{22}) - a_{12}G'(0)]g(\lambda_1)}{(1-a_{22})[(1-a_{11})\lambda_1 + a_{12}G'(0) - (1-a_{11})(1-a_{22})]^2},$$

where

$$\begin{aligned} g(\lambda_1) = (1-a_{11})\lambda_1^2 - 2[(1-a_{11})(1-a_{22}) - a_{12}G'(0)]\lambda_1 \\ + (1-a_{22})[(1-a_{11})(1-a_{22}) - a_{12}G'(0)]. \end{aligned}$$

It is easy to deduce that $g(\lambda_1) > 0$ for $\lambda_1 > 1 - a_{22}$. Then, we have that $f(\lambda_1)$ is strictly increasing in λ_1 for $\lambda_1 > 1 - a_{22}$. For $L > L^*$,

$$\begin{aligned} & f(\lambda_1) - f(1) = \widetilde{\lambda_1} - f(1) \\ =& \frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{[1 + d\pi^2/(4L^2)](1-a_{22})} - 1 + \frac{a_{12}G'(0)}{(1-a_{22})[1 - a_{11} + a_{12}G'(0) - (1-a_{11})(1-a_{22})]} \\ =& \frac{1}{1-a_{22}}\left[\frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{[1 + d\pi^2/(4L^2)]} - (1-a_{22}) + \frac{a_{12}G'(0)}{1 - a_{11} + a_{12}G'(0) - (1-a_{11})(1-a_{22})}\right] \\ =& \frac{1}{1-a_{22}}\left[\frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{[1 + d\pi^2/(4L^2)]} - \frac{a_{22}[(1-a_{11})(1-a_{22}) - a_{12}G'(0)]}{a_{22}(1-a_{11}) + a_{12}G'(0)}\right] \\ =& \frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{1-a_{22}}\left[\frac{1}{1 + d\pi^2/(4L^2)} - \frac{a_{22}}{a_{22}(1-a_{11}) + a_{12}G'(0)}\right] \\ =& \frac{(1-a_{11})(1-a_{22}) - a_{12}G'(0)}{1-a_{22}}\left[\frac{1}{1 + d\pi^2/(4L^2)} - \frac{1}{1 - a_{11} + \frac{a_{12}}{a_{22}}G'(0)}\right] > 0. \end{aligned}$$

Hence, $\lambda_1 > 1$.

Cases 2: $(1-a_{11})(1-a_{22}) < a_{12}G'(0)$. In this case, we can prove that $g(\lambda_1) > 0$ for $\lambda_1 > 1 - a_{22}$. Then, we have that $f(\lambda_1)$ is strictly decreasing in λ_1 for $\lambda_1 > 1 - a_{22}$. For $L > L^*$,

$$f(\lambda_1) - f(1) < 0.$$

Hence, $\lambda_1 > 1$.

2.3. *Global dynamics analysis*

At first, we consider the following equation:

$$\begin{cases} u_{n+1} = (1-a_{11})u_n + a_{12}v_n, & n \geq 0, \\ v_{n+1} = (1-a_{22})v_n + G(u_n), & n \geq 0 \end{cases} \tag{2.15}$$

with $u_0, v_0 \geq 0$. Then, we have its dynamics as follows.

Theorem 2.8. *If $a_{12}G'(0) < a_{11}a_{22}$, then the unique equilibrium $(0,0)$ of Eq (2.15) is globally asymptotically stable. If $a_{12}G'(0) > a_{11}a_{22}$, then $(0,0)$ is unstable, and the unique positive equilibrium (u^*, v^*) of Eq (2.15) is stable.*

Proof. It is easy to find that $(0,0)$ is always an equilibrium of Eq (2.15), while Eq (2.15) possesses a unique positive equilibrium (u^*, v^*) if and only if $a_{12}G'(0) \geq a_{11}a_{22}$. Next, we will analyze the stability of them.

First, for the equilibrium $(0,0)$, we calculate the Jacobian matrix J_1 as follows:

$$J_1 = \begin{pmatrix} 1-a_{11} & a_{12} \\ G'(0) & 1-a_{22} \end{pmatrix}.$$

According to the Jury conditions [21], $(0,0)$ is asymptotically stable if and only if $|\lambda_{J_1}| < 1$, which is equivalent to the following conditions:

$$1 - \mathrm{tr}J_1 + \det J_1 > 0,\ 1 + \mathrm{tr}J_1 + \det J_1 > 0,\ 1 - \det J_1 > 0,$$

namely, $a_{12}G'(0) < a_{11}a_{22}$.

From what we have discussed above, if $a_{12}G'(0) > a_{11}a_{22}$, $(0,0)$ is unstable. For the unique positive equilibrium (u^*, v^*), the Jacobian matrix J_2 is given by

$$J_2 = \begin{pmatrix} 1-a_{11} & a_{12} \\ G'(u^*) & 1-a_{22} \end{pmatrix}.$$

The assumption **(G2)** implies that $G'(u) \leq \frac{G(u)}{u}$ for any $u > 0$. Combined with $\frac{G(u^*)}{u^*} = \frac{a_{11}a_{22}}{a_{12}}$, we can obtain that $a_{12}G'(u^*) \leq a_{11}a_{22}$, which implies that $|\lambda_{J_2}| \leq 1$. Thus, (u^*, v^*) is stable.

Theorem 2.9. *If $a_{12}G'(0)/a_{22} < a_{11} + d\pi^2/(4L^2)$, then the unique equilibrium $(0,0)$ of Eq (2.1) is globally attractive. If $a_{12}G'(0)/a_{22} > a_{11} + d\pi^2/(4L^2)$, then $(0,0)$ is unstable and the unique positive equilibrium $(u^*(x), v^*(x))$ of Eq (2.1) is globally attractive.*

Proof. First, we consider the steady-state equation of Eq (2.1):

$$\begin{cases} -du''(x) = -a_{11}u(x) + a_{12}v(x), & x \in (0,L), \\ 0 = -a_{22}v(x) + G(u(x)), & x \in (0,L), \\ u'(0) = u(L) = 0,\ v'(0) = v(L) = 0. \end{cases} \tag{2.16}$$

Case 1: $a_{12}G'(0)/a_{22} < a_{11}$. It is obvious that Eq (2.16) has a solution $(0,0)$. We first prove $(0,0)$ is globally attractive. The solution $(u(n,x), v(n,x))$ of Eq (2.1) satisfies $u(n,x) \leq u_n,\ v(n,x) \leq v_n$ for $n > 0,\ x \in [0,L]$ by applying Lemma 2.5, where (u_n, v_n) is the solution of the problem

$$\begin{cases} u_{n+1} = (1-a_{11})u_n + a_{12}v_n, & n \geq 0, \\ v_{n+1} = (1-a_{22})v_n + G(u_n), & n \geq 0, \\ u_0 = \|u_0\|_\infty,\ v_0 = \|v_0\|_\infty. \end{cases} \tag{2.17}$$

From Theorem 2.8, we immediately obtain that $\lim_{n\to+\infty} u(n,x) \leq \lim_{n\to+\infty} u_n = 0,\ \lim_{n\to+\infty} v(n,x) \leq \lim_{n\to+\infty} v_n = 0$. Therefore, $(0,0)$ is globally attractive when $a_{12}G'(0) < a_{11}a_{22}$.

Case 2: $a_{11} \le a_{12}G'(0)/a_{22} < a_{11} + d\pi^2/(4L^2)$. It is well-known that the eigenvalue problem

$$\begin{cases} -\phi''(x) = \alpha\phi(x), & x \in (0, L), \\ \phi'(0) = \phi(L) = 0 \end{cases} \tag{2.18}$$

has a principal eigenvalue $\alpha_1 = (\frac{\pi}{2L})^2$ and its corresponding positive eigenfunction $\phi(x)$. We can rewrite Eq (2.16) into

$$\begin{cases} -du''(x) = -a_{11}u(x) + \frac{a_{12}}{a_{22}}G(u(x)), & x \in (0, L), \\ u'(0) = u(L) = 0. \end{cases} \tag{2.19}$$

Next, we apply the similar arguments in [33, Theorem 5.1] to prove that Eq (2.16) has no positive solution when $\frac{a_{12}G'(0)}{a_{22}} < a_{11} + d(\frac{\pi}{2L})^2$. On the contrary, we assume that Eq (2.16) has a positive solution $(u^*(x), v^*(x))$ with $v^*(x) = \frac{G(u^*(x))}{a_{22}}$, namely, Eq (2.19) has a positive solution $u^*(x)$. Multiplying Eq (2.19) by $\phi(x)$, integrating over $(0, L)$ and applying the divergence theorem yields

$$\int_0^L -d\phi'' u^* dx = \int_0^L \phi \left[-a_{11}u^* + \frac{a_{12}}{a_{22}}G(u^*)\right] dx.$$

Due to the monotonicity of the $\frac{G(u)}{u}$, then it follows that

$$d(\frac{\pi}{2L})^2 \int_0^L \phi u^* dx = \int_0^L \phi \left[-a_{11}u^* + \frac{a_{12}}{a_{22}}G(u^*)\right] dx \le \left(-a_{11} + \frac{a_{12}G'(0)}{a_{22}}\right) \int_0^L \phi u^* dx,$$

and thus $a_{12}G'(0)/a_{22} \ge a_{11} + d\pi^2/(4L^2)$, which is a contradiction to $a_{12}G'(0)/a_{22} < a_{11} + d\pi^2/(4L^2)$. This implies that Eq (2.16) has only one solution $(0, 0)$. According to [34, Theorem 2.2.1], we can prove that the unique equilibrium $(0, 0)$ of the monotone problem (2.1) is globally attractive.

Case 3: $a_{12}G'(0)/a_{22} > a_{11} + d\pi^2/(4L^2)$. First, we examine the stability of $(0, 0)$. Let λ_1 be the principal eigenvalue of Eq (2.12). By Proposition 2.7, it is easy to deduce that $a_{12}G'(0)/a_{22} > a_{11} + d\pi^2/(4L^2)$ is equivalent to $\lambda_1 > 1$. Thus, $(0, 0)$ is unstable.

Next, we prove the existence, uniqueness, and global stability of the positive solution $(u^*(x), v^*(x))$ when $a_{12}G'(0)/a_{22} > a_{11} + d\pi^2/(4L^2)$. Since $G \in C^1([0, +\infty))$, we can find sufficiently small $\epsilon > 0$ such that $a_{12}G'(\epsilon\phi(x))/a_{22} \ge a_{11} + d\pi^2/(4L^2)$, where $\phi(x)$ is the principal eigenfunction of Eq (2.18). We define

$$\underline{u}(x) = \epsilon\phi(x) \text{ for } x \in [0, L].$$

Direct calculations yield

$$\begin{aligned}
& -d\underline{u}''(x) + a_{11}\underline{u}(x) - \frac{a_{12}}{a_{22}}G(\underline{u}(x)) \\
= & -d\epsilon\phi''(x) + a_{11}\epsilon\phi(x) - \frac{a_{12}}{a_{22}}G(\epsilon\phi(x)) \\
\le & \, d(\frac{\pi}{2L})^2\epsilon\phi(x) + a_{11}\epsilon\phi(x) - \frac{a_{12}}{a_{22}}\epsilon\phi(x)G'(\epsilon\phi(x)) \\
= & \, \epsilon\phi(x)\left[d(\frac{\pi}{2L})^2 + a_{11} - \frac{a_{12}}{a_{22}}G'(\epsilon\phi(x))\right] \le 0,
\end{aligned}$$

where we use the result $(\frac{G(u)}{u})' = \frac{G'(u)u-G(u)}{u^2} \le 0$ by the fact that the condition $G(u)/u$ is decreasing in u. Thus, $\underline{u}(x)$ is a lower solution to Eq (2.19). On the other hand, it follows from $\lim\limits_{u\to+\infty} G(u)/u < \frac{a_{11}a_{22}}{a_{12}}$ that we can find a sufficiently large constant M such that $G(M) < \frac{a_{11}a_{22}}{a_{12}}M$. Now, we prove that

$$\overline{u}(x) \equiv M$$

is an upper solution to Eq (2.19). In fact,

$$\begin{aligned} & -d\overline{u}''(x) + a_{11}\overline{u}(x) - \frac{a_{12}}{a_{22}}G(\overline{u}(x)) \\ = {} & a_{11}M - \frac{a_{12}}{a_{22}}G(M) > a_{11}M - a_{11}M = 0. \end{aligned}$$

Thus, we find a pair of upper and lower solutions of Eq (2.19). According to [33], we can conclude the Eq (2.19) has a positive solution $u^*(x)$.

To prove the positive solution is unique, we suppose u_1 and u_2 are two different positive solutions of Eq (2.19) and u_* is the minimal solution in $[\underline{u},\ \overline{u}]$. Thus, $u_* \le u_i,\ i = 1, 2$. Now we multiply u_i by the equation satisfied by u_*, integrate by parts over $[0, L]$, and obtain

$$\frac{a_{12}}{da_{22}} \int_0^L u_* u_i \left[\frac{G(u_*)}{u_*} - \frac{G(u_i)}{u_i} \right] = 0. \tag{2.20}$$

Since $u_* \le u_i$ and $\frac{G(u)}{u}$ is decreasing, Eq (2.20) holds if and only if $u_* = u_i$. This means that Eq (2.16) has a unique positive solution $(u^*(x), v^*(x))$.

Moreover, it follows from [29, 35] that the unique positive equilibrium $(u^*(x), v^*(x))$ is globally attractive. The proof is complete.

3. Long-time behavior of the free boundary problem

In this section, we consider the corresponding free boundary problem (1.5). All the assumptions which are provided in problem (2.1) still hold.

3.1. The global solution

Similar to Proposition 2.4, we first give the global solvability of Eq (1.5) by using an iterative method.

Theorem 3.1. *For any given $h_0 > 0, u_0$, and $v_0 \in C^1([0, h_0])$ with $u_0(x), v_0(x) > 0,\ x \in [0, h_0)$, problem* (1.5) *admits a unique solution $(u, v; h_n)$ for $n \in \mathbb{N}(0, +\infty)$ that satisfies:*

(i) *there exists a constant $K_1 > 0$ such that $0 < u(n, x), v(n, x) \le K_1$ for $x \in [0, h_n)$ and $u_x(n, h_n) < 0$;*

(ii) *there exists a constant $K_2 > 0$ such that $0 < h_{n+1} - h_n \le K_2$. In particular, $h_1 \ge h_0$.*

Proof. For any given h_0, u_0, and v_0, we know that $h_1 = h_0 - \mu u_x(0, h_0)$ by the Stefan condition in Eq (1.5). Combined with $\mu > 0$ and $u_0'(h_0) \le 0$, we have $h_1 \ge h_0$. By Eq (1.5), we have

$$\begin{cases} u(1, x) = du_{xx}(1, x) + F(u_0(x), v_0(x)), & x \in (0, h_1), \\ v(1, x) = (1 - a_{22})v_0(x) + G(u_0(x)), & x \in (0, h_1), \\ u_x(1, 0) = u(1, h_1) = 0,\ v_x(1, 0) = v(1, h_1) = 0. \end{cases} \tag{3.1}$$

It follows from Lemma 2.1 that problem (3.1) has a unique classical solution

$$(u(1,x), v(1,x)) = \left(\int_0^{h_1} K(x,y;h_1)F(u_0(y), v_0(y))dy, (1-a_{22})v_0(x) + G(u_0(x))\right),$$

where $K(x,y;h_1)$ is defined as Eq (2.3) with L replaced by h_1. Since $\int_0^{h_1} K(x,y;h_1)dy \le 1$, we have

$$\|u(1,\cdot)\|_{C([0,h_1])} \le \|F(u_0(\cdot), v_0(\cdot))\|_{C([0,h_0])} \le (1-a_{11})\|u_0\|_{C([0,h_0])} + a_{12}\|v_0\|_{C([0,h_0])}$$

and

$$\|v(1,\cdot)\|_{C([0,h_1])} \le (1-a_{22})\|v_0\|_{C([0,h_0])} + \widehat{G},$$

where $\widehat{G} = \max\{G_0, G_1, G_2, \ldots, G_n\}$ with $G_i = \max_{x\in[0,h_i]} G(u(i,x))$ for $i \in \mathbb{N}[0,+\infty)$. Moreover, we can easily get $u(1,x) > 0$ for $x \in [0,h_1)$ and $u_x(1,h_1) < 0$ by the strong maximum principle and the Hopf lemma.

Furthermore, according to $h_2 = h_1 - \mu u_x(1,h_1)$, we can obtain that $h_2 > h_1$ immediately. From Eq (1.5), we obtain that

$$\begin{cases} u(2,x) = du_{xx}(2,x) + F(u(1,x), v(1,x)), & x \in (0,h_2), \\ v(2,x) = (1-a_{22})v(1,x) + G(u(1,x)), & x \in (0,h_2), \\ u_x(2,0) = u(2,h_2) = 0,\ v_x(2,0) = v(2,h_2) = 0. \end{cases} \tag{3.2}$$

Then, Eq (3.2) exists a unique classical solution

$$(u(2,x), v(2,x)) = \left(\int_0^{h_2} K(x,y;h_2)F(u(1,y), v(1,y))dy, (1-a_{22})v(1,x) + G(u(1,x))\right),$$

where $K(x,y;h_2)$ is defined as Eq (2.3) with L replaced by h_2. Since $\int_0^{h_2} K(x,y;h_2)dy \le 1$, we have

$$\begin{aligned} \|u(2,\cdot)\|_{C([0,h_2])} \le\ & \|F(u(1,\cdot), v(1,\cdot))\|_{C([0,h_1])} \\ \le\ & (1-a_{11})^2\|u_0\|_{C([0,h_0])} + (a_{12}(1-a_{11}) + a_{12}(1-a_{22}))\|v_0\|_{C([0,h_0])} + a_{12}\widehat{G} \end{aligned}$$

and

$$\|v(2,\cdot)\|_{C([0,h_2])} \le (1-a_{22})^2\|v_0\|_{C([0,h_0])} + (1-a_{22})\widehat{G} + \widehat{G}.$$

Furthermore, we can deduce that $u(2,x) > 0$ for $x \in (0,h_2)$ and $u_x(2,h_2) < 0$.

Repeating the above process, we can have that problem (1.5) has a unique solution $(u,v;h_n)$ for $n \in \mathbb{N}(0,+\infty)$, and it satisfies

$$\begin{aligned} \|u(n,\cdot)\|_{C([0,h_n])} &\le (1-a_{11})^n\|u_0\|_{C([0,h_0])} + a_{12}\widehat{N}\sum_{i=0}^{n-1}(1-a_{11})^i\sum_{j=0}^{n-i-1}(1-a_{22})^j \\ &\le (1-a_{11})\|u_0\|_{C([0,h_0])} + \frac{a_{12}\widehat{N}}{a_{11}a_{22}}, \end{aligned}$$

and

$$\begin{aligned}\|v(n,\cdot)\|_{C([0,h_n])} &\le (1-a_{22})^n\|v_0\|_{C([0,h_0])} + \widehat{G}\sum_{i=0}^{n-1}(1-a_{22})^i \\ &\le (1-a_{22})\widehat{N} + \frac{\widehat{N}}{a_{22}},\end{aligned}$$

where $\widehat{N} = \max\left\{\|v_0\|_{C([0,h_0])}, \widehat{G}\right\}$. Hence, there exists a positive constant K_1 such that

$$0 < u(n,x),\ v(n,x) \le K_1,\ n \in \mathbb{N}(0,+\infty),\ x \in [0,h_n).$$

Moreover, $h_1 \ge h_0$, $u_x(n,h_n) < 0$ and $h_{n+1} - h_n > 0$ for $n \in \mathbb{N}(0,+\infty)$.

Finally, it remains to show that there exists a positive constant K_2 such that $h_{n+1} - h_n \le K_2$ for $n \in \mathbb{N}(0,+\infty)$. Define

$$\Omega = \left\{(n,x) : n \in \mathbb{N}[0,+\infty),\ x \in (h_{n+1} - M^{-1}, h_{n+1})\right\}.$$

Let

$$w(n,x) = K_1\left[2M(h_n - x) - M^2(h_n - x)^2\right].$$

We will choose some suitable M such that $w(t,x) \ge u(t,x)$ for $(n,x) \in \Omega$. Direct calculation yields that, for $(n,x) \in \Omega$,

$$\begin{aligned}&w(n+1,x) - w(n,x) - dw_{xx}(n+1,x) + a_{11}u(n,x) - a_{12}v(n,x) \\ &= K_1M(h_{n+1} - h_n)\left[2 - M(h_{n+1} + h_n - 2x)\right] + 2dK_1M^2 + a_{11}u(n,x) - a_{12}v(n,x) \\ &\ge K_1M(h_{n+1} - h_n)\left([1 - M(h_{n+1} - x)] + [1 - M(h_n - x)]\right) + 2dK_1M^2 - a_{12}K_1 \ge 0,\end{aligned}$$

if

$$M^2 \ge \frac{a_{12}}{2d}.$$

On the other hand,

$$w(n, h_n - M^{-1}) = K_1 \ge u(n, h_n - M^{-1}),\ w(n,h_n) = u(n,h_n) = 0.$$

If we choose

$$M = \max\left\{\sqrt{\frac{a_{12}}{2d}}, \frac{4\|u_0\|_{C^1([0,h_0])}}{3K_1}\right\},$$

we can use the similar arguments in the proof of [10, Lemma 2.2] to obtain that

$$w(0,x) \ge u(0,x) \text{ for } x \in [h_0 - M^{-1}, h_0].$$

Applying the maximum principle to $w(n+1,h_{n+1}) - u(n+1,h_{n+1})$ over Ω, we can deduce that

$$w(n,x) \ge u(n,x),\ (n,x) \in \mathbb{N}(0,+\infty) \times (h_n - M^{-1}, h_n).$$

It follows that $u_x(n,h_n) \ge w_x(n,h_n) = -2MK_1$, and then

$$h_{n+1} - h_n = -\mu u_x(n,h_n) \le 2\mu MK_1 := K_2,\ n \in (0,+\infty).$$

The proof is complete.

For convenience, we write the free boundary problem (1.5) in the following form:

$$\begin{cases} u(n+1,x) = \int_0^{h_{n+1}} K(x,y;h_{n+1})F(u(n,y),v(n,y))dy, & n \geq 0,\ x \in [0,h_{n+1}], \\ v(n+1,x) = (1-a_{22})v(n,x) + G(u(n,x)), & n \geq 0,\ x \in [0,h_{n+1}], \\ h_{n+1} = h_n - \mu u_x(n,h_n), & n \geq 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in [0,h_0], \end{cases}$$

where

$$K(x,y;h_{n+1}) = \begin{cases} \dfrac{\sinh \frac{h_{n+1}-x}{\sqrt{d}} \cosh \frac{y}{\sqrt{d}}}{\sqrt{d}\cosh \frac{h_{n+1}}{\sqrt{d}}}, & 0 \leq y \leq x, \\ \dfrac{\cosh \frac{x}{\sqrt{d}} \sinh \frac{h_{n+1}-y}{\sqrt{d}}}{\sqrt{d}\cosh \frac{h_{n+1}}{\sqrt{d}}}, & x \leq y \leq h_{n+1}. \end{cases}$$

Let

$$\gamma = (1-a_{11}) \max\{\|u_0\|_\infty, K_1\} + a_{12} \max\{\|v_0\|_\infty, K_1\}.$$

In the following, we will prove the comparison principle.

Lemma 3.2. *Assume that $T \in \mathbb{N}(0,+\infty)$, $\overline{h}_n \in \mathbb{R}^+$ for any $n \in \mathbb{N}[0,T]$, $\overline{u} = (\overline{u}(0,\cdot), \overline{u}(1,\cdot), \cdots, \overline{u}(T,\cdot))$ and $\overline{v} = (\overline{v}(0,\cdot), \overline{v}(1,\cdot), \cdots, \overline{v}(T,\cdot)) \in \mathbb{X}_T^* = C^1([0,\overline{h}_0]) \times C^2([0,\overline{h}_1]) \times \cdots \times C^2([0,\overline{h}_T])$ satisfy*

$$\begin{cases} \overline{u}(n+1,x) - d\overline{u}_{xx}(n+1,x) \geq F(\overline{u}(n,x),\overline{v}(n,x)), & (n,x) \in \mathbb{N}[0,T-1] \times (0,\overline{h}_{n+1}), \\ \overline{v}(n+1,x) \geq (1-a_{22})\overline{v}(n,x) + G(\overline{u}(n,x)), & (n,x) \in \mathbb{N}[0,T-1] \times (0,\overline{h}_{n+1}), \\ \overline{u}_x(n,0) \leq 0,\ \overline{v}_x(n,0) \leq 0,\ \overline{u}(n,x) = \overline{v}(n,x) = 0, & n \in \mathbb{N}[0,T],\ x \geq \overline{h}_n, \\ \overline{h}_{n+1} \geq \overline{h}_n - \mu\overline{u}_x(n,\overline{h}_n),\ \overline{h}_0 \geq h_0, & n \in \mathbb{N}[0,T-1], \\ \overline{u}(0,x) \geq u(0,x),\ \overline{v}(0,x) \geq v(0,x), & x \in [0,h_0], \\ \overline{u}_x(0,\overline{h}_0) \leq u_x(0,h_0). \end{cases} \tag{3.3}$$

If $\mu\gamma \leq d$, then the solution $(u,v;h_n)$ of Eq (1.5) satisfies

$$\overline{h}_n \geq h_n,\ n \in \mathbb{N}(0,T];\ \overline{u}(n,x) \geq u(n,x),\ \overline{v}(n,x) \geq v(n,x),\ (n,x) \in \mathbb{N}(0,T] \times (0,h_n).$$

Proof. This lemma can be proved by the similar arguments in the proof of [27, Lemma 3.2]. For the given initial data, we have

$$\overline{h}_1 \geq \overline{h}_0 - \mu\overline{u}_x(0,\overline{h}_0) \geq h_0 - \mu u_x(0,h_0) = h_1.$$

From the monotonicity of G, we can obtain that, for $x \in (0,h_1)$,

$$\overline{v}(1,x) \geq (1-a_{22})\overline{v}_0 + G(\overline{u}_0) \geq (1-a_{22})v_0 + G(u_0) = v(1,x).$$

It is easy to check that, for $x \in (0,h_1)$,

$$\begin{aligned} \overline{u}(1,x) - d\overline{u}_{xx}(1,x) &\geq (1-a_{11})\overline{u}_0 + a_{12}\overline{v}_0 \geq (1-a_{11})u_0 + a_{12}v_0 \\ &= u(1,x) - du_{xx}(1,x), \end{aligned}$$

$$\overline{u}_x(1,0) \le 0 = u_x(1,0),\ \overline{u}(1,h_1) \ge 0 = u(1,h_1).$$

Applying the maximum principle to $\overline{u} - u$, we can deduce that $\overline{u}(1,x) \ge u(1,x)$ for $x \in (0,h_1)$.

Next, we will prove $\overline{h}_2 \ge h_2$. If $\overline{h}_1 = h_1$, it is obvious that $\overline{u}_x(1,\overline{h}_1) \le u_x(1,h_1)$ by using $\overline{u}(1,x) \ge u(1,x)$ for $x \in (0,h_1)$ and $\overline{u}(1,\overline{h}_1) = u(1,h_1) = 0$, and then $\overline{h}_2 \ge h_2$. If $\overline{h}_1 > h_1$, we claim that $\overline{h}_2 \ge h_2$ still holds. It is easy to see that Eq (3.3) is equivalent to the following IDEs:

$$\begin{cases} \overline{u}(n+1,x) \ge \displaystyle\int_0^{\overline{h}_{n+1}} K(x,y;\overline{h}_{n+1})F(\overline{u}(n,y),\overline{v}(n,y))dy, & (n,x) \in \mathbb{N}[0,T-1] \times [0,\overline{h}_{n+1}], \\ \overline{v}(n+1,x) \ge (1-a_{22})\overline{v}(n,x) + G(\overline{u}(n,x)) & (n,x) \in \mathbb{N}[0,T-1] \times [0,\overline{h}_{n+1}], \\ \overline{h}_{n+1} \ge \overline{h}_n - \mu\overline{u}_x(n,\overline{h}_n), & n \in \mathbb{N}[0,T-1]. \end{cases} \tag{3.4}$$

Noticing that $u(1,x) = \int_0^{h_1} K(x,y;h_1)F(u_0(y),v_0(y))dy$ for $x \in [0,h_1]$, we have

$$u_x(1,h_1) = -\frac{1}{d\cosh\frac{h_1}{\sqrt{d}}}\int_0^{h_1}\cosh\frac{y}{\sqrt{d}}F(u_0(y),v_0(y))dy.$$

On the other hand, we can obtain from Eq (3.4) that, for $x \in [0,\overline{h}_1]$,

$$\overline{u}(1,x) \ge \int_0^{\overline{h}_1} K(x,y;\overline{h}_1)F(\overline{u}_0(y),\overline{v}_0(y))dy =: \widetilde{u}(1,x).$$

It is easy to see $\overline{u}(1,\overline{h}_1) = \widetilde{u}(1,\overline{h}_1) = 0$. By the Hopf lemma, we can deduce that

$$\overline{u}_x(1,\overline{h}_1) \le \widetilde{u}_x(1,\overline{h}_1) = -\frac{1}{d\cosh\frac{\overline{h}_1}{\sqrt{d}}}\int_0^{\overline{h}_1}\cosh\frac{y}{\sqrt{d}}F(\overline{u}_0(y),\overline{v}_0(y))dy.$$

Therefore, we have

$$\begin{aligned} &\overline{h}_2 - h_2 \\ &\ge \overline{h}_1 - h_1 + \mu\left(u_x(1,h_1) - \overline{u}_x(1,\overline{h}_1)\right) \\ &\ge \mu\left(\frac{1}{d\cosh\frac{\overline{h}_1}{\sqrt{d}}}\int_0^{\overline{h}_1}\cosh\frac{y}{\sqrt{d}}F(\overline{u}_0(y),\overline{v}_0(y))dy - \frac{1}{d\cosh\frac{h_1}{\sqrt{d}}}\int_0^{h_1}\cosh\frac{y}{\sqrt{d}}F(u_0(y),v_0(y))dy\right) \\ &\quad + \overline{h}_1 - h_1 \\ &\ge \overline{h}_1 - h_1 + \mu\left(\frac{1}{d\cosh\frac{\overline{h}_1}{\sqrt{d}}} - \frac{1}{d\cosh\frac{h_1}{\sqrt{d}}}\right)\int_0^{h_1}\cosh\frac{y}{\sqrt{d}}F(u_0(y),v_0(y))dy \\ &= \overline{h}_1 - h_1 - \mu(\overline{h}_1 - h_1)\frac{\sqrt{d}\sinh\frac{\xi}{\sqrt{d}}}{(d\cosh\frac{\xi}{\sqrt{d}})^2}\int_0^{h_1}\cosh\frac{y}{\sqrt{d}}F(u_0(y),v_0(y))dy \\ &\ge \overline{h}_1 - h_1 - \frac{\mu\gamma}{d}(\overline{h}_1 - h_1) \ge 0, \end{aligned}$$

where $\xi \in (h_1,\overline{h}_1)$. Similarly, we can have $\overline{u}(2,x) \ge u(2,x)$ and $\overline{v}(2,x) \ge v(2,x)$ for $x \in (0,h_2)$. Repeating the above argument, we can complete this proof.

3.2. The spreading-vanishing dichotomy

It follows from Theorem 3.1 that there exists a $h_\infty \in (0, +\infty]$ such that $\lim_{n\to+\infty} h_n = h_\infty$. Next, we will give an estimate of the solution of Eq (1.5).

Lemma 3.3. *Let $(u, v; h_n)$ be a solution of Eq (1.5). Then, there exists a positive constant K_3 which depends only on h_0, u_0, and v_0 such that*

$$\|u(n,\cdot)\|_{C^2([0,h_n])} \le K_3,\ n \in \mathbb{N}(0, +\infty).$$

Proof. It follows from Lemma 2.3 that the solution of Eq (1.5) satisfies

$$\|u(n+1,\cdot)\|_{C^2[0,h_{n+1}]} \le C\|F(u(n,\cdot), v(n,\cdot))\|_{C([0,h_n])},\ n \in \mathbb{N}[0, +\infty),$$

where the positive constant C is independent of F and h_n. By Theorem 3.1, we have that for any given h_0, u_0, and v_0,

$$\|u(n,\cdot)\|_{C([0,h_n])}, \|v(n,\cdot)\|_{C([0,h_n])} \le K_1.$$

Then,

$$\begin{aligned}\|F(u(n,\cdot), v(n,\cdot))\|_{C([0,h_n])} &\le (1-a_{11})\|u(n,\cdot)\|_{C([0,h_n])} + a_{12}\|v(n,\cdot)\|_{C([0,h_n])}\\ &\le (1-a_{11}+a_{12})K_1.\end{aligned}$$

Therefore, there exists a positive constant K_3 such that

$$\|u(n,\cdot)\|_{C^2([0,h_n])} \le K_3,\ n \in \mathbb{N}(0, +\infty).$$

The proof is completed.

Lemma 3.4. *Let $(u, v; h_n)$ be a solution of Eq (1.5). If $h_\infty < +\infty$, then*

$$\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0,\ \lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0.$$

Proof. The proof is motivated by the proof of [27, Lemma 3.5] and [4, Lemma 3.2].

According to Theorem 3.1 and Lemma 3.3, there exists a constant $\widehat{C} > 0$ depending only on h_0 and u_0 such that, if $h_\infty < +\infty$, then

$$\|u(n,\cdot)\|_{C^2([0,h_n])} \le \widehat{C} \text{ and } h_{n+1} - h_n \le \widehat{C} \text{ for } n \in \mathbb{N}(0, +\infty),\ \lim_{n\to+\infty}(h_{n+1} - h_n) = 0,$$

which implies that $\|u(n,\cdot)\|_{C^1([0,h_n])} \le \widehat{C}$ for $n \in \mathbb{N}(0, +\infty)$.

First, we prove $\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0$. On the contrary, we suppose that there exists a $\xi > 0$ such that

$$\liminf_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = \xi.$$

Then, there exists a monotone increasing sequence $(n_k, x_k) \in \mathbb{N}(0, +\infty) \times (0, h_{n_k})$ satisfying $n_k \to +\infty$ as $k \to +\infty$ such that

$$u(n_k, x_k) \ge \frac{\xi}{2} \text{ for any } k \in \mathbb{N}.$$

Since $0 < x_k < h_{n_k} \le h_\infty < +\infty$, there exists a convergent subsequence of $\{x_k\}$ denoted by itself. Let $x_k \to \widetilde{x}$ as $k \to +\infty$. Next, we claim that $\widetilde{x} \in (0, h_\infty)$. Otherwise, if $\widetilde{x} = h_\infty$, then $\lim\limits_{k\to+\infty}(x_k - h_{n_k}) = \widetilde{x} - h_\infty = 0$. On the other hand,

$$\frac{\xi}{2} \le u(n_k, x_k) = u(n_k, x_k) - u(n_k, h_{n_k}) = u_x(n_k, \delta_k)(x_k - h_{n_k}) \le \frac{\widehat{C}}{\mu}(h_{n_k} - x_k),$$

where $\delta_k \in (x_k, h_{n_k})$. This implies that $h_{n_k} - x_k \ge \frac{\mu\xi}{2\widehat{C}} > 0$ for any $k \in \mathbb{N}$, and then $\widetilde{x} \ne h_\infty$.

Now define

$$W_k(n, x) = u(n_k + n, x),\ Z_k(n, x) = v(n_k + n, x) \text{ for } n \in \mathbb{N}(-n_k, +\infty) \text{ and } x \in (0, h_{n_k+n}).$$

Noticing that

$$h_\infty < +\infty,\ \|u(n,\cdot)\|_{C^2([0,h_n])} \le \widehat{C},\ \|v(n,\cdot)\|_{C([0,h_n])} \le K_1,\ n \in \mathbb{N}(0, +\infty), \tag{3.5}$$

then $(W_k(n, x), Z_k(n, x))$ has a convergent subsequence denoted by itself, and there exists $(\widetilde{W}(n, x), \widetilde{Z}(n, x))$ such that

$$\lim_{k\to+\infty} W_k(n, x) = \widetilde{W}(n, x),\ \lim_{k\to+\infty} Z_k(n, x) = \widetilde{Z}(n, x) \text{ for any } n \in \mathbb{N}(-\infty, +\infty)$$

locally uniformly in $x \in [0, h_\infty]$. Next, we prove that if $\{W_k(n, x), Z_k(n, x)\}$ is a solution sequence of Eq (1.5), then $(\widetilde{W}(n, x), \widetilde{Z}(n, x))$ is a solution of Eq (1.5). The first three equations of Eq (1.5) can be rewritten into

$$\begin{cases} u(n+1, x) = \displaystyle\int_0^{h_{n+1}} K(x, y; h_{n+1}) F(u(n, y), v(n, y))dy := P[u(n, x)], & n \ge 0,\ x \in [0, h_{n+1}], \\ v(n+1, x) = (1 - a_{22})v(n, x) + G(u(n, x)) := Q[v(n, x)], & n \ge 0,\ x \in [0, h_{n+1}], \end{cases} \tag{3.6}$$

where F is monotone increasing and satisfies $F(0, 0) = 0$. It is easy to see that the operators P and Q are monotone increasing. By [21, Theorem 3.1], we know P is compact in the space of continuous functions. In the following, we consider the iterative form of Eq (3.6):

$$u(n_k + n + 1, x) = P\left[u(n_k + n, x)\right],\ v(n_k + n + 1, x) = Q\left[v(n_k + n, x)\right].$$

By Eq (3.5), the boundedness of K and F, and the uniform convergence of $F(W_k, Z_k)$, it follows from the dominated convergence theorem that we can obtain

$$\begin{aligned} \lim_{k\to+\infty} W_k(n+1, x) &= \lim_{k\to+\infty} \int_0^{h_{n_k+n+1}} K(x, y; h_{n_k+n+1}) F(W_k(n, y), Z_k(n, y))dy \\ &= \int_0^{h_\infty} K(x, y; h_\infty) \lim_{k\to+\infty} F(W_k(n, y), Z_k(n, y))dy, \end{aligned}$$

$$\lim_{k\to+\infty} Z_k(n+1, x) = (1 - a_{22}) \lim_{k\to+\infty} Z_k(n, x) + \lim_{k\to+\infty} G(W_k(n, x)),$$

and then

$$\widetilde{W}(n+1, x) = \int_0^{h_\infty} K(x, y; h_\infty) F(\widetilde{W}(n, y), \widetilde{Z}(n, y))dy,$$

$$\widetilde{Z}(n+1,x) = (1-a_{22})\widetilde{Z}(n,x) + G(\widetilde{W}(n,x)).$$

Therefore, $(\widetilde{W}(n,x), \widetilde{Z}(n,x))$ is a solution of Eq (3.6), namely, $(\widetilde{W}(n,x), \widetilde{Z}(n,x))$ satisfies

$$\begin{cases} \widetilde{W}(n+1,x) = d\widetilde{W}_{xx}(n+1,x) + (1-a_{11})\widetilde{W}(n,x) + a_{12}\widetilde{Z}(n,x), & n \in \mathbb{N}(-\infty,+\infty),\ x \in (0,h_\infty), \\ \widetilde{Z}(n+1,x) = (1-a_{22})\widetilde{Z}(n,x) + G(\widetilde{W}(n,x)), & n \in \mathbb{N}(-\infty,+\infty),\ x \in (0,h_\infty), \\ \widetilde{W}_x(n,0) = \widetilde{W}(n,x) = \widetilde{Z}_x(n,0) = \widetilde{Z}(n,x) = 0, & n \in \mathbb{N}(-\infty,+\infty),\ x \geq h_\infty, \\ \widetilde{W}(n,x) \geq 0,\ \widetilde{Z}(n,x) \geq 0, & n \in \mathbb{N}(-\infty,+\infty),\ x \in [0,h_\infty). \end{cases} \tag{3.7}$$

Since $W_k(0,x_k) = u(n_k,x_k) \geq \frac{\xi}{2}$ for $k \in \mathbb{N}$, we have $\widetilde{W}(0,\tilde{x}) \geq \frac{\xi}{2} > 0$. From Eq (3.7), we obtain

$$\begin{cases} \widetilde{W}(1,x) = d\widetilde{W}_{xx}(1,x) + (1-a_{11})\widetilde{W}(0,x) + a_{12}\widetilde{Z}(0,x), & x \in (0,h_\infty), \\ \widetilde{W}_x(1,0) = 0,\ \widetilde{W}(1,h_\infty) = 0. \end{cases}$$

Applying the maximum principle and the Hopf lemma, we have

$$\widetilde{W}(1,x) > 0 \text{ for } x \in [0,h_\infty),\ \text{and } \widetilde{W}_x(1,h_\infty) < 0.$$

Repeating the above process, one obtains that

$$\widetilde{W}(n,x) > 0,\ \widetilde{W}_x(n,h_\infty) < 0 \text{ for } n \in \mathbb{N}(0,+\infty) \text{ and } x \in [0,h_\infty).$$

On the other hand, recalling that $\lim_{n\to+\infty}(h_{n+1}-h_n) = 0$, then it follows from the Stefan condition that $u_x(n_k+n, h_{n_k+n}) \to 0$ as $n_k \to \infty$. Thanks to $\|u(n,\cdot)\|_{C^2([0,h_n])} \leq \widehat{C}$, we have that $(W_k)_x$ is continuous with respect to x. By the uniform convergence of $\{W_k\}$, we obtain

$$u_x(n_k+n, h_{n_k+n}) = (W_k)_x(n, h_{n_k+n}) \to \widetilde{W}_x(n,h_\infty),\ k \to +\infty,$$

namely, $\widetilde{W}_x(n,h_\infty) = 0$ for $n \in \mathbb{N}(0,+\infty)$, which is a contradiction. Therefore,

$$\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0.$$

Then, we prove $\lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0$. It follows from $\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0$ that, for any $\epsilon > 0$, there exists a sufficiently large $T \in \mathbb{N}(0,+\infty)$ such that

$$u(n,\cdot) < \epsilon \text{ for } n \in \mathbb{N}(T,+\infty).$$

From Eq (1.5), we obtain that

$$v(n+1,x) < (1-a_{22})v(n,x) + G(\epsilon) \text{ for } n \in \mathbb{N}(T,+\infty) \text{ and } x \in (0,h_{n+1}).$$

By the comparison principle, we have

$$v(n,x) \leq \bar{v}(n) \text{ for } n \in \mathbb{N}(T,+\infty) \text{ and } x \in [0,h_n],$$

where $\overline{v}(n)$ is the solution of the following problem:

$$\begin{cases}\overline{v}(n+1) = (1-a_{22})\overline{v}(n) + G(\epsilon), & n \in \mathbb{N}(T, +\infty),\\ \overline{v}(T) = \|v(T,\cdot)\|_\infty.\end{cases}$$

It is easy to derive that

$$\overline{v}(T+n) = (1-a_{22})^n \|v(T,\cdot)\|_\infty + \sum_{i=1}^{n} (1-a_{22})^{i-1} G(\epsilon),$$

and then

$$\lim_{n\to+\infty} \overline{v}(n) = \frac{G(\epsilon)}{a_{22}}.$$

By the arbitrariness of ϵ, we have

$$\lim_{n\to+\infty} v(n,x) \le \lim_{n\to+\infty} \overline{v}(n) = 0 \text{ for } x \in [0, h_n].$$

Then, we have $\lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0$. This proof is completed.

Lemma 3.5. *If* $a_{12}G'(0) \le a_{11}a_{22}$, *then* $h_\infty < \infty$.

Proof. By $a_{12}G'(0) \le a_{11}a_{22}$, we have, for any $n \in \mathbb{N}[0, +\infty)$,

$$\begin{aligned}
&\int_0^{h_{n+1}} \left[u(n+1,x) + \frac{a_{12}}{a_{22}} v(n+1,x)\right] dx\\
=& \int_0^{h_{n+1}} \left[du_{xx}(n+1,x) + (1-a_{11})u(n,x) + a_{12}v(n,x) + \frac{a_{12}}{a_{22}}(1-a_{22})v(n,x) + \frac{a_{12}}{a_{22}} G(u(n,x))\right] dx\\
=& \int_0^{h_{n+1}} \left[du_{xx}(n+1,x) + u(n,x) + \frac{a_{12}}{a_{22}} v(n,x) + \frac{a_{12}}{a_{22}} G(u(n,x)) - a_{11}u(n,x)\right] dx\\
\le& du_x(n+1,h_{n+1}) - du_x(n+1,0) + \int_0^{h_{n+1}} \left[u(n,x) + \frac{a_{12}}{a_{22}} v(n,x) + \left(\frac{a_{12}}{a_{22}} G'(0) - a_{11}\right) u(n,x)\right] dx\\
\le& \frac{d}{\mu}(h_{n+1} - h_{n+2}) + \int_0^{h_{n+1}} \left[u(n,x) + \frac{a_{12}}{a_{22}} v(n,x)\right] dx\\
=& \frac{d}{\mu}(h_{n+1} - h_{n+2}) + \int_0^{h_n} \left[u(n,x) + \frac{a_{12}}{a_{22}} v(n,x)\right] dx.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_0^{h_{n+1}} \left[u(n+1,x) + \frac{a_{12}}{a_{22}} v(n+1,x)\right] dx\\
\le& \frac{d}{\mu}(h_1 - h_{n+2}) + \int_0^{h_0} \left[u(0,x) + \frac{a_{12}}{a_{22}} v(0,x)\right] dx,
\end{aligned}$$

which implies that

$$h_{n+2} \le h_1 + \frac{\mu}{d} \int_0^{h_0} \left[u(0,x) + \frac{a_{12}}{a_{22}} v(0,x)\right] dx.$$

By letting $n \to \infty$, we have $h_\infty < \infty$.

In the following, we always assume that $a_{12}G'(0) > a_{11}a_{22}$ and $a_{12}G'(0) \neq (1-a_{11})(1-a_{22})$.

Lemma 3.6. *Let* $(u, v; h_n)$ *be a solution of Eq* (1.5). *If* $h_\infty < +\infty$*, then*

$$h_\infty \leq L^*,$$

where L^* *is defined in Eq* (2.13).

Proof. According to Theorem 3.4, it follows from $h_\infty < +\infty$ that

$$\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0, \ \lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0.$$

On the contrary, we assume that $h_\infty > L^*$. Then, there exists a sufficiently large $T > 0$ such that $h_n > L^*$ for $n \in \mathbb{N}[T, +\infty)$. Combined with Proposition 2.7, we can obtain Eq (2.12) with $L = h_T$ has a principal eigenvalue $\lambda_1 > 1$ and corresponding eigenfunction is $(\phi(x), \psi(x))$. Let

$$\underline{u}(n, x) = \epsilon\phi(x), \ \underline{v}(n, x) = \epsilon\psi(x), \ x \in [0, L],$$

where the constant $\epsilon > 0$ will be determined later. Then, for $(n, x) \in \mathbb{N}[T, +\infty) \times (0, L)$, direct calculations give

$$\begin{aligned}
&\underline{u}(n+1, x) - d\underline{u}_{xx}(n+1, x) - (1-a_{11})\underline{u}(n, x) - a_{12}\underline{v}(n, x) \\
&= \epsilon\phi(x) - d\epsilon\phi''(x) - \epsilon(1-a_{11})\phi(x) - \epsilon a_{12}\psi(x) \\
&= \epsilon\phi(x) - \epsilon\phi(x) + \frac{\epsilon(1-a_{11})\phi(x)}{\lambda_1} + \frac{\epsilon a_{12}\psi(x)}{\lambda_1} - \epsilon(1-a_{11})\phi(x) - \epsilon a_{12}\psi(x) \\
&= \epsilon\left[(1-a_{11})\phi(x) + a_{12}\psi(x)\right](\frac{1}{\lambda_1} - 1) \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\underline{v}(n+1, x) - (1-a_{22})\underline{v}(n, x) - G(\underline{u}(n, x)) \\
&= \epsilon\psi(x) - (1-a_{22})\epsilon\psi(x) - G'(\xi(n, x))\epsilon\phi(x) \\
&= a_{22}\epsilon\psi(x) - G'(\xi(n, x))\epsilon\phi(x) \\
&= \epsilon\psi(x)(1-\lambda_1) + \epsilon\phi(x)\left[G'(0) - G'(\xi(n, x))\right] \\
&= \epsilon\phi(x)\left[\frac{G'(0)}{\lambda_1 - (1-a_{22})}(1-\lambda_1) + G'(0) - G'(\xi(n, x))\right],
\end{aligned}$$

where $\xi \in (0, \epsilon\phi)$. We choose some sufficiently small ϵ such that

$$\frac{G'(0)}{\lambda_1 - (1-a_{22})}(1-\lambda_1) + G'(0) - G'(\xi(n, x)) \leq 0.$$

By

$$u(T, x) \geq \underline{u}(T, x) = \epsilon\phi(x), \ v(T, x) \geq \underline{v}(T, x) = \epsilon\psi(x) \text{ for } x \in [0, L],$$

and

$$\underline{u}_x(n, 0) = \underline{u}(n, L) = \underline{v}_x(n, 0) = \underline{v}(n, L) = 0 \text{ for } n \in \mathbb{N}[T, +\infty),$$

it follows from Lemma 2.5 that

$$u(n,x) \geq \underline{u}(n,x) > 0,\ v(n,x) \geq \underline{v}(n,x) > 0 \text{ for } (n,x) \in \mathbb{N}[T,+\infty) \times (0,L),$$

which is a contradiction to the fact that $\lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0$ and $\lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0$. Hence, $h_\infty \leq L^*$.

Lemma 3.7. *Let* $(u, v; h_n)$ *be a solution of Eq* (1.5). *If* $h_\infty = +\infty$, *then*

$$\lim_{n\to+\infty} u(n,x) = u^*,\ \lim_{n\to+\infty} v(n,x) = v^* \textit{ locally uniformly in } [0,+\infty).$$

Proof. Applying the comparison principle, we have

$$u(n,x) \leq u_n,\ v(n,x) \leq v_n \text{ for } n > 0 \text{ and } x \in [0,h_n],$$

where (u_n, v_n) is the solution of Eq (2.17). By $a_{12}G'(0) > a_{11}a_{22}$, it follows from Theorem 2.8 that

$$\lim_{n\to+\infty} u_n = u^*,\ \lim_{n\to+\infty} v_n = v^*.$$

Therefore,

$$\limsup_{n\to+\infty} u(n,x) \leq u^*,\ \limsup_{n\to+\infty} v(n,x) \leq v^* \text{ uniformly for } x \in [0,+\infty). \tag{3.8}$$

On the other hand, for any $l > \max\{h_0, L^*\}$, there exists $n_l \in \mathbb{N}(0,+\infty)$ such that $h_{n_l} \geq l$. By the comparison principle, we have

$$u(n,x) \geq u_l(n,x),\ v(n,x) \geq v_l(n,x) \text{ for } (n,x) \in \mathbb{N}(n_l,+\infty) \times (0,l),$$

where $(u_l(n,x), v_l(n,x))$ is the solution of the following problem:

$$\begin{cases} u_l(n+1,x) = d(u_l)_{xx}(n+1,x) + (1-a_{11})u_l(n,x) + a_{12}v_l(n,x), & n \geq n_l,\ x \in (0,l), \\ v_l(n+1,x) = (1-a_{22})v_l(n,x) + G(u_l(n,x)), & n \geq n_l,\ x \in (0,l), \\ (u_l)_x(n,0) = u_l(n,l) = 0,\ (v_l)_x(n,0) = v_l(n,l) = 0, & n \geq n_l, \\ u_l(n_l,x) = u(n_l,x),\ v_l(n_l,x) = v(n_l,x), & x \in [0,l]. \end{cases}$$

Since $l > L^*$, it follows from Theorem 2.9 that

$$\lim_{n\to+\infty} u_l(n,x) = u_l^*(x),\ \lim_{n\to+\infty} v_l(n,x) = v_l^*(x) \text{ uniformly in } [0,l],$$

where $(u_l^*(x), v_l^*(x))$ is the unique positive solution of the following problem:

$$\begin{cases} -du_l''(x) = -a_{11}u_l(x) + a_{12}v_l(x), & x \in (-l,l), \\ 0 = -a_{22}v_l(x) + G(u_l(x)), & x \in (-l,l), \\ u_l(-l) = u_l(l) = 0,\ v_l(-l) = v_l(l) = 0. \end{cases}$$

Thus, we have

$$\liminf_{n\to+\infty} u(n,x) \geq u_l^*(x),\ \liminf_{n\to+\infty} v(n,x) \geq v_l^*(x) \text{ uniformly in } [0,l].$$

Using similar arguments to those in the proof of [10, Lemma 3.2], we can obtain that

$$\lim_{l\to+\infty} u_l^*(x) = u^*,\ \lim_{l\to+\infty} v_l^*(x) = v^* \text{ locally uniformly in } [0,+\infty).$$

Therefore,

$$\liminf_{n\to+\infty} u(n,x) \geq u^*,\ \liminf_{n\to+\infty} v(n,x) \geq v^* \text{ locally uniformly in } [0,+\infty).$$

Combining with Eq (3.8), we complete the proof.

According to Lemmas 3.4 and 3.7, we immediately obtain the following spreading-vanishing dichotomy.

Theorem 3.8. *Let* $(u, v; h_n)$ *be the solution of Eq* (1.5)*. Then, one of the following alternatives must happen:*

(i) Spreading:

$$h_\infty = +\infty, \text{ and } \lim_{n\to+\infty} u(n,x) = u^*,\ \lim_{n\to+\infty} v(n,x) = v^* \text{ locally uniformly in } [0,+\infty).$$

(ii) Vanishing:

$$h_\infty \leq L^*, \text{ and } \lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0,\ \lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0.$$

Next, we will give the criteria for spreading and vanishing.

For $h_0 \geq L^*$, due to $h_{n+1} = h_n - \mu u_x(n,h_n)$ and $u_x(n,h_n) < 0$, we can easily conclude that $h_\infty > L^*$. Hence, Lemma 3.6 implies the following corollary.

Corollary 3.9. *If* $h_0 \geq L^*$*, then* $h_\infty = +\infty$.

For $h_0 < L^*$, we have the following two lemmas.

Lemma 3.10. *Suppose that* $h_0 < L^*$*. Then,* $h_\infty = +\infty$ *for sufficiently large* μ.

Proof. For any given $u_0(x)$, we can choose some sufficiently large μ such that

$$\mu \geq \frac{h_0 - L^*}{u_0'(h_0)} =: \mu^*.$$

Then, $h_1 = h_0 - \mu u_x(0,h_0) \geq L^*$. By Corollary 3.9, we can obtain that $h_\infty = +\infty$ for $\mu \geq \mu^*$.

Lemma 3.11. *Suppose that* $h_0 < L^*$*. Then,* $h_\infty < +\infty$ *for sufficiently small* μ.

Proof. We will construct a suitable upper solution to Eq (1.5) and then apply Lemma 3.2. Inspired by [27, Theorem 3.10], we define

$$\sigma_n = h_0\left[1+\frac{\delta}{2}(1-e^{-\alpha n})\right],\ n\in\mathbb{N}[0,+\infty),$$
$$\overline{u}(n,x)=\begin{cases} M\beta^n\cos(\frac{\pi x}{2\sigma_n}), & n\in\mathbb{N}[0,+\infty),\ x\in[0,\sigma_n],\\ 0, & n\in\mathbb{N}[0,+\infty),\ x\in(\sigma_n,+\infty),\end{cases}$$
$$\overline{v}(n,x)=\begin{cases} KM\beta^n\cos(\frac{\pi x}{2\sigma_n}), & n\in\mathbb{N}[0,+\infty),\ x\in[0,\sigma_n],\\ 0, & n\in\mathbb{N}[0,+\infty),\ x\in(\sigma_n,+\infty),\end{cases}$$

where the positive constants $\delta,\ \alpha,\ M,\ \beta$, and K will be determined later.

Since $h_0 < L^*$, we can choose some sufficiently small δ such that $h_0(1+\frac{\delta}{2}) < L^*$. For the above δ, we choose suitable β and K such that

$$\begin{cases} \beta\left(1+d(\frac{\pi}{2h_0(1+\frac{\delta}{2})})^2\right)-(1-a_{11}+a_{12}K)=0,\\ K\beta-(1-a_{22})K-G'(0)=0,\end{cases}$$

and then we can obtain

$$K=\frac{(1-a_{22})p-(1-a_{11})+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4a_{12}pG'(0)}}{2a_{12}},$$
$$\beta=\frac{1-a_{11}+a_{12}K}{p},$$

where $p=1+d(\frac{\pi}{2h_0\left(1+\frac{\delta}{2}\right)})^2$. It is easy to check that

$$\begin{aligned} &1-a_{11}+a_{12}K\\ =&1-a_{11}+\frac{(1-a_{22})p-(1-a_{11})+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4a_{12}pG'(0)}}{2}\\ =&\frac{(1-a_{22})p+(1-a_{11})+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4a_{12}pG'(0)}}{2}>0.\end{aligned}$$

By $h_0(1+\frac{\delta}{2}) < L^*$, we can have

$$\begin{aligned} &1-a_{11}+a_{12}K-p\\ =&1-a_{11}+\frac{(1-a_{22})p-(1-a_{11})+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4pa_{12}G'(0)}}{2}-p\\ =&\frac{1}{2}[1-a_{11}-(1+a_{22})p+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4pa_{12}G'(0)}]\\ =&\frac{-[1-a_{11}-(1+a_{22})p]^2+[(1-a_{22})p-(1-a_{11})]^2+4pa_{12}G'(0)}{2[-(1-a_{11})+(1+a_{22})p+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4pa_{12}G'(0)}]}\\ =&\frac{4pa_{22}[-p+(1-a_{11})+\frac{a_{12}}{a_{22}}G'(0)]}{2[-(1-a_{11})+(1+a_{22})p+\sqrt{[(1-a_{22})p-(1-a_{11})]^2+4pa_{12}G'(0)}]}\end{aligned}$$

$$= \frac{4pa_{22}[-a_{11} + \frac{a_{12}G'(0)}{a_{22}} - \frac{d\pi^2}{4h_0^2(1+\frac{\delta}{2})^2}]}{2[-(1-a_{11}) + (1+a_{22})p + \sqrt{[(1-a_{22})p - (1-a_{11})]^2 + 4pa_{12}G'(0)}]} < 0,$$

and then we have

$$0 < \beta < 1.$$

Noting that $\sigma_n < \sigma_{n+1} \leq h_0(1 + \frac{\delta}{2})$, we can obtain that

$$\cos(\frac{\pi x}{2\sigma_n}) \leq \cos(\frac{\pi x}{2\sigma_{n+1}}) \text{ for } x \in (0, \sigma_{n+1}).$$

Combined with this result, we can have, for $(n, x) \in \mathbb{N}[0, +\infty) \times (0, \sigma_{n+1})$,

$$\begin{aligned}
&\overline{u}(n+1, x) - d\overline{u}_{xx}(n+1, x) - (1-a_{11})\overline{u}(n, x) - a_{12}\overline{v}(n, x) \\
&= M\beta^{n+1}\cos(\frac{\pi x}{2\sigma_{n+1}}) + dM\beta^{n+1}(\frac{\pi}{2\sigma_{n+1}})^2\cos(\frac{\pi x}{2\sigma_{n+1}}) \\
&\quad - (1-a_{11})M\beta^n\cos(\frac{\pi x}{2\sigma_n}) - a_{12}KM\beta^n\cos(\frac{\pi x}{2\sigma_n}) \\
&\geq M\beta^n\cos(\frac{\pi x}{2\sigma_{n+1}})\left[\beta\left(1 + d(\frac{\pi}{2\sigma_{n+1}})^2\right) - (1 - a_{11} + a_{12}K)\right] \\
&\geq M\beta^n\cos(\frac{\pi x}{2\sigma_{n+1}})\left[\beta\left(1 + d(\frac{\pi}{2h_0(1+\frac{\delta}{2})})^2\right) - (1 - a_{11} + a_{12}K)\right] = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{v}(n+1, x) - (1-a_{22})\overline{v}(n, x) - G(\overline{u}(n, x)) \\
&= KM\beta^{n+1}\cos(\frac{\pi x}{2\sigma_{n+1}}) - (1-a_{22})KM\beta^n\cos(\frac{\pi x}{2\sigma_n}) - G(M\beta^n\cos(\frac{\pi x}{2\sigma_n})) \\
&\geq KM\beta^{n+1}\cos(\frac{\pi x}{2\sigma_{n+1}}) - (1-a_{22})KM\beta^n\cos(\frac{\pi x}{2\sigma_n}) - G'(0)M\beta^n\cos(\frac{\pi x}{2\sigma_n}) \\
&\geq M\beta^n\cos(\frac{\pi x}{2\sigma_{n+1}})\left[K\beta - (1-a_{22})K - G'(0)\right] = 0,
\end{aligned}$$

where we use the fact $G(z) \leq G'(0)z$ for $z > 0$ by assumption **(G2)**.

Moreover, we have

$$\overline{u}_x(n, 0) = \overline{v}_x(n, 0) = 0 \text{ for } n \in \mathbb{N}[0, +\infty)$$

and

$$\overline{u}(n, x) = \overline{v}(n, x) = 0 \text{ for } x \geq \sigma_n.$$

Now we can choose some sufficiently large M such that

$$u_0(x) \leq \overline{u}(0, x) = M\cos\left(\frac{\pi x}{2\sigma_0}\right),\ v_0(x) \leq \overline{v}(0, x) = KM\cos\left(\frac{\pi x}{2\sigma_0}\right) \text{ for } x \in [0, h_0].$$

Then, we have

$$\overline{u}_x(0, \sigma_0) \leq u_x(0, h_0),\ \overline{v}_x(0, \sigma_0) \leq v_x(0, h_0).$$

If we take $\alpha = -\ln\beta$, then, for $n \in \mathbb{N}[0, +\infty)$,

$$\sigma_{n+1} - \sigma_n = \frac{\delta h_0 e^{-\alpha n}(1-e^{-\alpha})}{2} = \frac{\delta h_0 \beta^n (1-e^{-\alpha})}{2} \geq \mu \frac{\pi M \beta^n}{2h_0} \geq \mu \frac{\pi M \beta^n}{2\sigma_n} = -\mu \overline{u}_x(n, \sigma_n)$$

provided that

$$0 < \mu \leq \mu_* = \min\left\{\frac{d}{\gamma}, \frac{\delta h_0^2(1-e^{-\alpha})}{\pi M}\right\}.$$

Applying Lemma 3.2, we can obtain

$$h_n \leq \sigma_n,\ u(n,x) \leq \overline{u}(n,x),\ v(n,x) \leq \overline{v}(n,x) \text{ for } (n,x) \in \mathbb{N}(0,+\infty) \times (0, h_n),$$

which implies

$$h_\infty \leq \lim_{n\to+\infty} \sigma_n = h_0(1 + \frac{\delta}{2}) < +\infty \text{ for } 0 < \mu \leq \mu_*.$$

4. Discussion

In this paper, following the approach developed in [27] where a single-species time-discrete free boundary model was treated, we consider the time-discrete version of the time-continuous free boundary model in [4] to describe the spreading of some fecally-orally transmitted diseases such as cholera. We extend the results in [27] to the case of a system with free boundary, and obtain some initial results on Eq (1.5).

More precisely, under the assumptions **(A)**, **(G1)**, and **(G2)**, we show that Eq (1.5) has a unique solution defined for all integers $n \geq 1$. For the long-time dynamics, we obtain the spreading-vanishing dichotomy, namely, the diseases will spread,

$$h_\infty = +\infty,\ \textit{and} \lim_{n\to+\infty} u(n,x) = u^*,\ \lim_{n\to+\infty} v(n,x) = v^* \text{ locally uniformly in } [0,+\infty)$$

or vanish,

$$h_\infty \leq \infty,\ \textit{and} \lim_{n\to+\infty} \|u(n,\cdot)\|_{C([0,h_n])} = 0,\ \lim_{n\to+\infty} \|v(n,\cdot)\|_{C([0,h_n])} = 0.$$

Moreover, sufficient conditions are given for vanishing and spreading to happen, namely, the following conclusions hold:

(i) if $a_{12}G'(0) \leq a_{11}a_{22}$, then $h_\infty < \infty$;

(ii) if $a_{12}G'(0) > a_{11}a_{22}$ and $a_{12}G'(0) \neq (1-a_{11})(1-a_{22})$, then

  (a) if $h_0 \geq L^*$, then $h_\infty = +\infty$;

  (b) if $h_0 < L^*$, then $h_\infty = +\infty$ for sufficiently large μ;

  (c) if $h_0 < L^*$, then $h_\infty < +\infty$ for sufficiently small μ.

Since the comparison principle in this paper holds only when μ is sufficiently small, we can not obtain a sharp value μ^* such that spreading will happen for $\mu \geq \mu^*$ and vanishing will happen for $\mu \leq \mu^*$.

Above results tell us that increasing the removed rate of the bacteria and the infective human population can decrease the chance of the spreading of the diseases.

The above results show that the diseases will not always spread for the basic reproduction number is larger than 1. Compared with the fixed boundary problem and the Cauchy problem, the free boundary problem (1.5) can describe the spreading of the diseases better and tell us where the spreading front is. On the other hand, compared with the continuous-time model, the discrete-time model in this paper can better fit the reality that the statistical data for scientific research are collected in discrete time.

However, we only obtain some initial results on problem (1.5). Since we introduce the condition $\mu\gamma \le d$ to obtain conventional comparative result in Lemma 3.2, the comparison principle in this paper only hold for sufficiently small $\mu \in (0, d/\gamma]$. Besides, we add an additional condition $\overline{u}_x(0, \overline{h}_0) \le u_x(0, h_0)$, which make it difficult to construct the suitable upper and lower solution considering the spreading speed of the free boundary. We will try to obtain more complete results without the above limitations in the future.

Author contributions

Jian Feng wrote the original draft. Meng Zhao did conceptualization, supervision, funding acquisition, revised the draft and edited the article. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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