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Research article

A non-conservative, non-local approximation of the Burgers equation

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Abstract: The analysis of non-local regularisations of scalar conservation laws is an active research program. Applications of such equations are found in modelling physical phenomena, such as traffic flow. In this paper, we propose an inviscid, non-local regularisation in a non-divergence form. The salient feature of our approach is that we can obtain sharp a priori estimates on the total variation (TV) and supremum norm and justify the singular limit for Lipschitz initial data up to the time of catastrophe. For generic conservation laws, this result was sharp, since we could demonstrate non-convergence when the initial data featured simple discontinuities. However, when the flux derivative was linear, such as for the Burgers equation, we obtained stronger limits on the singular limit. Therefore, we devoted special attention to regularisations of the Burgers equation, specifically the limiting behaviour of solutions to the Cauchy problems with fixed initial data.

Keywords: scalar conservation law; non-local equation; shock wave; entropy solution; convergence to the local model

1. Introduction

The Burgers equation is a canonical instance of a scalar conservation law. It is often used as a simple example to motivate the study of non-linear hyperbolic partial differential equations since it exhibits many of the characteristic features of such equations. In conservative form, it is cast as a Cauchy problem on the domain $[0, \infty) \times \mathbb{R}$ as follows:

$$\begin{aligned} \partial_t u + \partial_x \left(\frac{u^2}{2} \right) &= 0, \\ u(0, x) &= u_0(x), \end{aligned} \tag{1.1}$$

for $u_0 \in L^\infty(\mathbb{R})$. Generic solutions, even for smooth initial data, break down in finite time unless u_0 is monotone non-decreasing, and therefore motivate the notion of weak solutions to Eq (1.1), defined as

$u \in C([0,\infty); L^1_{loc}(\mathbb{R}))$ such that, for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$:

$$\int_0^\infty \int_{-\infty}^\infty u(t,x)\partial_t\varphi(t,x) + \frac{u(t,x)^2}{2}\partial_x\varphi(t,x)dxdt + \int_{-\infty}^\infty u_0(x)\varphi(0,x)dx = 0.$$

However, weak solutions to such initial value problems are not unique. A further condition is imposed on the class of weak solutions to any particular Cauchy problem to single out a unique solution. In the context of scalar conservation laws, the admissibility criterion is that the weak solution must satisfy a further inequality of the form

$$\partial_t n(u) + \partial_x q(u) \leq 0,$$

in the sense of distributions, where $n(u)$ is a C^1, convex function of its argument, and $q(u)$ is an antiderivative of $n'(u)u$ (for the Burgers equation). Such a pair of functions n, q is called an entropy-entropy flux pair with respect to the PDE. By taking $n(u) = \pm u$, we can see that every entropy solution is necessarily an admissible weak one as well. Another equivalent criterion is to insist that they arise as the 'vanishing viscosity' limits of certain parabolic equations [1]. In particular, consider the family of solutions, parametrised by $\epsilon > 0$, of the parabolic equations

$$\begin{aligned} \partial_t u^\epsilon + \partial_x\left(\frac{(u^\epsilon)^2}{2}\right) &= \epsilon\Delta u^\epsilon, \\ u^\epsilon(0,x) &= u_0(x). \end{aligned} \tag{1.2}$$

For each $\epsilon > 0$, solutions to Eq (1.2) are obtained by a fixed-point argument. The criterion now is that u must be a limit point of the sequence u^ϵ as $\epsilon \to 0^+$. Using a priori bounds on the total variation and uniform Lipschitz continuity in time with respect to the L^1 norm, the sequence u^ϵ is compact, and it is demonstrated that the limit is an entropy solution u of the Cauchy problem (1.1) with the given initial data [1]. The uniqueness of the limit, and therefore justification of the entropy/vanishing viscosity criterion, is proved in a separate argument.

This method uses a second-order parabolic PDE, known as the viscous Burgers equation, due to the Laplacian. Since this second-order term has a coefficient of ϵ that vanishes in the limit, this approach is known as the method of 'vanishing viscosity'. In this paper, we propose an alternative first-order approximation schema for the Burgers equation using non-linear transport equations. Our motivation is as follows: For smooth and uniformly Lipschitz initial data, a local-in-time solution for Eq (1.1) can be obtained by the method of characteristics. Applying the chain rule to the equation, we see that Lipschitz solutions satisfy

$$\partial_t u + u\partial_x u = 0.$$

Thus, the solution is constant along characteristics, which, in this case, are straight lines in spacetime with a slope determined by the value of u_0 at the base point (at $t = 0$). This can be done only locally because the characteristic curves for generic solutions will meet in finite time. Avoiding the formation of such discontinuities, or 'shocks', is the motivation for our approximation schema.

There are two ways to regularise the Burgers equation by convolution, depending on whether one uses the conservative or the transport formulation. Thus, the non-linear term could be replaced either by $\frac{1}{2}\partial_x([\eta_\epsilon * u]u)$ or by $(\eta_\epsilon * u)\partial_x u$. We choose the latter, for fixed $\epsilon > 0$, we analyse the following non-conservative, non-local (NN) Cauchy problem:

$$\begin{aligned} \partial_t u^\epsilon + (\eta_\epsilon * u^\epsilon)\partial_x u^\epsilon = 0; \\ u^\epsilon(0,x) = u_0(x). \end{aligned} \tag{NN}$$

The non-linear advective term $\eta_\epsilon * u^\epsilon$ represents the convolution of u^ϵ with a mollifier, thus making the equation non-local, while the form of (NN) makes it clear that it cannot be written in divergence form without adding a non-trivial source term, thus making it non-conservative. Here, and in the rest of this paper, η_ϵ is a symmetric family of mollifiers, parametrised by $\epsilon > 0$, with compact support $[-\epsilon, \epsilon]$. For instance, consider a smooth, symmetric, positive function $\eta(x)$ supported in $[-1, 1]$ such that $\|\eta\|_1 = 1$, and for any $\epsilon > 0$ let $\eta_\epsilon = \epsilon^{-1}\eta(\epsilon^{-1}x)$. For example, consider

$$\eta(x) = I^{-1} \exp\left(\frac{-1}{1-x^2}\right)\chi_{(-1,1)}(x),$$

where $\chi_{(-1,1)}(x)$ is the indicator function of the interval (-1.1) and I is a constant that ensures normalisation. We will consider such families $\{\eta_\epsilon\}$, though the particular form of η is not important as long as it is symmetric with compact support in $[-1, 1]$. For the results on well-posedness of the Cauchy problem, however, this symmetry condition can be relaxed, and any family of approximate identities can be considered. Symmetry of the mollifier plays only a crucial role in studying the non-local to local limit when the limiting entropy solution is not smooth. That is, we require only symmetry in Sections 3.2 and 3.3. For well-posedness of the non-local equation, and even in the singular limit in the smooth regime (Section 3.1), symmetry is not assumed. All we require is that as $\epsilon \to 0^+$, convolution with members of this family approximates identity, and that η is compactly supported in $[-1, 1]$.

Although (NN) is a regularisation of the Burgers equation, we will also briefly consider appropriate non-conservative, non-local regularisations of general scalar conservation laws in Section 3.1. As we shall see, however, the Burgers case is special with respect to discontinuous initial data, and therefore singled out for special attention. It is also easier to work with (NN), and since the well-posedness results trivially carry over, mutatis mutandis, to the appropriate non-conservative, non-local regularisation of any other scalar conservation law, we can work with this equation without any loss of generality.

We exploit the transport equation structure of (NN) to prove well-posedness by a fixed-point argument and obtain further a priori estimates on solutions. After demonstrating well-posedness for C^1 data by a fixed-point argument, we introduce the notion of weak solutions for (NN) and generalise the well-posedness result to BV (bounded variation) data, using the a priori estimates derived in the previous step. One advantage of our non-conservative regularisation is that we can prove that the L^∞ norm and BV seminorm are preserved for positive times, which greatly simplifies our analysis.

Since the paper of Zumbrun [2], there has been a steady output of results on non-local conservation and balance laws, where spatial mollification is used in place of small viscosity to gain and/or preserve regularity. In all nonlocal approximations of the local models, the main object of interest is to find out when the formal singular limit to the Dirac delta distribution can be justified rigorously. Investigations, both numerical and theoretical, have been saliently focused on models of traffic flow [3–5]. Other applications include sedimentation [6], opinion-formation [7], supply chains [8], and more. On the theoretical side, existence, uniqueness [9–11], and control problems [12, 13] have been studied.

The regularisation (NN) was first studied by Norgard and Mohseni, who dubbed it the "convective filtered Burgers equation" [14]. In another paper, they derived singular limit results for this equation for a class of continuously differentiable initial data [15]. The proof technique involved analysing the solutions as 'reparametrisations' of the initial data, based on the method of characteristics. For 'bell-shaped' and differentiable initial data, they show that solutions arising as reparametrisations must be entropy solutions, and thus justify the singular limit. In contrast, we work with more general initial data and follow the characteristics directly to prove (local-in-time) convergence to the entropy solution for smooth initial data. The regularisation of the Burgers equation in non-conservative form was also independently studied by Coron et al. [16], along with analogous regularisations of general scalar conservation laws. Our approach is closer to that of the latter, but differs slightly from both; we obtain the singular limit for a larger class of initial data and also generalise the framework to the multidimensional case. Non-local transport equations were also studied in higher dimensions by De Lellis et al. [17], but in their case, the non-locality was introduced through the source term, whereas our approach regularises the velocity field and introduces non-linearity by making the field depend on the solution.

In general, the differential equation remains in divergence form even at the non-local level, and the flux function is applied to the regularisation. In general, the non-locality does not behave well in the presence of shocks; one feature of our work is that we can work explicitly with discontinuous initial data, and justify the formal convergence to the entropy solution in the context of Burgers equation for a special class of initial data. Our framework also enables us to avoid a set of counter-examples to convergence (see Section 5), at least locally in time, that were obtained by Colombo et al. [18], where they worked with the nonlocal equation

$$\partial_t u + \partial_x((\eta_\epsilon * u)u)) = 0, \tag{1.3}$$

and proved non-convergence for some initial values. In Section 3.1, in contrast, we analyse the limit of solutions u^ϵ to (NN) with Lipschitz initial data and prove convergence to the entropy solution as $\epsilon \to 0$.The interplay of the viscous and non-local equations is explored in [19], where convergence to the entropy solution is recovered in the formal "double limit" where the parameters associated with viscosity as well as non-locality go to zero. While it is not the same as the singular limit as traditionally understood, it provides some insight into the behaviour of the inviscid limit.

More results were obtained in the case of a completely asymmetric, anisotropic convolution kernel in [20], where convergence to the local limit was obtained under the assumption of a one-sided Lipschitz condition on the local limit. In this paper, however, we will restrict ourselves to symmetric convolution kernels approximating identity for the singular limit problem with respect to the Burgers equation.

Recent work on systems of conservation laws include non-local regularisations of a non-strictly hyperbolic triangular system [21] and the generalised Aw-Rascle-Zhang model for traffic flow [22]. The latter includes a non-local equation of non-conservative transport form, analogous to (NN). In these works, unlike the regularisation considered by us for the isentropic Euler equations in Section 4, the non-local term is common to both equations. However, the system we consider is a decoupled pair of equations for the Riemann invariants.

Numerical schemes have historically played a crucial role in the analysis of hyperbolic equations. It is known that well-posedness results for scalar conservation laws can be obtained through convergent numerical schemes, as demonstrated in the classic text by Smoller [23, Theorem 16.1]. The classical theory of numerical schemes for these equations is vast, with its core foundations found in seminal texts such as [24–26]. Such schemes are especially important when analytic solutions are unknown, or indeed for equations whose well-posedness remains unestablished. Figures 1 and 2 illustrate simple algorithms applied to Eq (1.1) and (NN).

The Burgers equation and related non-linear models have gained popularity as of late. Since the non-linearity makes analytic solutions difficult to obtain, various numerical schemes have been proposed to solve the equations. We mention some recent works to illustrate the interest in these models. Wang et al. [27] considered a fourth order difference scheme for a generalised non-linear Burgers-type equation of the fourth order, with non-linearity of the form $u^\lambda u_x$ for $\lambda \geq 1$. Just as in the viscous model (1.2), the fourth-order term induces regularity in the analytic solution, though in this case, there are two viscosity coefficients corresponding to the second- and fourth-order terms, respectively. Wang et al. [28] proposed a novel third-order backward differentiation scheme for a similar second-order generalised viscous Burgers equation (GVBE) with the same non-linear term. Shi and Yang [29] considered a 'supergeneralised' viscous Burgers equation (SVB) with non-linearity of the form $u^p(1-u)^q$ with $p \geq 1, q \geq 0$.

Our model (NN) also satisfies the maximum principle and the total variation diminishing (TVD) property as shown in Theorem 2.1 and Lemma 2.2, respectively. These structural properties are useful in ensuring stability of numerical schemes even for general equations, such as in [30], where a 'discrete maximum principle' (DMP) ensures that spurious oscillations are absent from the non-linear finite volume (NFV) scheme for a two-dimensional sub-diffusion equation. The authors also employed an orthogonal Gauss collocation method (OGCM) for a similar sub-diffusion equation of the fourth order in [30].

1.1. Main results

Let us colloquially summarise our main results:

- For smooth data and fixed regularisation parameter $\epsilon > 0$, the non-conservative nonlocal Burgers equation has a unique global smooth solution (Theorem 2.1);
- For BV initial data, there exists a unique global weak solution, and no entropy condition is required for uniqueness (Theorem 2.2 and Lemma 2.6);
- As long as the entropy solution to the local conservation law remains Lipschitz, the non-conservative, non-local approximations with the same initial datum converge uniformly, as $\epsilon \to 0$, to the former (Theorem 3.1);
- For the Riemann problem, there are examples of convergence but also non-convergence of the approximations to the local entropy solution (Lemma 3.2).

Convergence in the Lipschitz regime (Theorem 3.1) most clearly brings out the improvement of our regularisation schema, where we can justify the formal singular limit with a simple proof, with much weaker assumptions on the regularity of the local limit. By contrast, the proof in Zumbrun [2] requires the local solution to be C^4 for the local-in-time convergence result. For the non-conservative regularisation in particular, this is also an improvement over the convergence obtained for quasi-concave initial data in [16], albeit only locally in time. Thus, in the Lipschitz regime where the weak solutions satisfy the partial differential equation point-wise almost everywhere, our results are sharp. We also do not require the special form of the Burgers equation for this result.

Another feature that stands out in our framework is the impossibility of total variation blow-up, as in [20]. Given the transport structure of our equation, the non-local solutions preserve the total variation of the initial data for all positive times (Lemma 2.2).

Generating total variation bounds from a transport structure is not new. For the conservative regularisation schema, this method was used to obtain bounds for the non-local term, which then translated to the compact embedding of the solution sequence under certain assumptions on the kernel [31], in particular, it is highly non-symmetric. One of the questions posed in the paper is precisely the convergence or lack thereof in the presence of a symmetric kernel. As shown in [32], conservative non-local regularisation with a symmetric kernel cannot, in general, satisfy the maximum principle, and total variation may blow up as in [20], but in our schema, both these possibilities are ruled out. Thus, if convergence must fail, it must fail in some other way.

For monotone initial data, we nearly recover the convergence results established in the conservative regularisation framework [32, 33]. If the data is non-decreasing and Lipschitz, or non-increasing and piece-wise constant, then we can justify the formal singular limit in arbitrary, compact intervals of time. Of course, this is true only with respect to regularisations of the Burgers equation or its rescaled variants; regarding counter-examples to convergence in the case of piece-wise constant, monotone decreasing data can be derived from Lemma 3.3 otherwise.

1.2. Structure of the paper

The structure of this paper is as follows: In Section 2.1, we demonstrate the well-posedness of the Cauchy problem for C^1 initial data with fixed $\epsilon > 0$ and derive a priori estimates on the L^∞ norm, along with the Lipschitz and Total Variation semi-norms. Then, in Section 2.2, we define weak solutions to (NN) and obtain well-posedness for less regular BV initial data using the a priori estimates derived before. We present the multidimensional non-conservative regularisation for scalar conservation laws with general flux in Section 2.3. In Section 3, we justify passage to the singular limit for some classes of initial data and also provide examples of data for which convergence to the entropy solution does not take place. Section 4 discusses a simple extension to the case of the isentropic Euler system with cubic pressure law, which is known to have some nice properties that simplify the analysis. Finally, in Section 5, we conclude the paper.

2. Well-posedness of the Cauchy problem

2.1. Smooth regime

In this section, we analyse the Cauchy problem for C^1 initial data and derive some a priori estimates on solutions. For $T > 0$, let $\Omega_T = [0, T] \times \mathbb{R}$; we will use this notation throughout the paper to denote such domains. By $\mathrm{Lip}(\mathbb{R})$ we denote the space of all functions in $L^\infty(\mathbb{R})$ that are globally Lipschitz continuous; in other words, $\mathrm{Lip}(\mathbb{R}) = W^{1,\infty}(\mathbb{R})$, the Sobolev space. We remark that no assumptions are made here on the symmetry of the mollifiers.

Theorem 2.1. *For $u_0 \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, the Cauchy problem* (NN) *has a unique C^1 solution in the domain Ω_T for arbitrary $T > 0$.*

Proof. Let $g \in L^\infty(\Omega_T), u_0 \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and consider the Cauchy problem for the following linear transport equation:

$$\begin{aligned} \partial_t u + (\eta_\epsilon * g)\partial_x u = 0,\\ u(0,x) = u_0(x). \end{aligned} \tag{2.1}$$

This Cauchy problem has a unique solution u. Define an operator Λ that maps $g \in L^\infty(\Omega_T)$ to the unique u solving Eq (2.1). For short enough time T^*, we can show that Λ is a contraction with respect to the L^∞ norm. Then, by the Banach fixed point theorem, we can prove the existence of a solution to the Cauchy problem (NN) with data $u_0 \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$.

Given any $g \in L^\infty(\Omega_T)$, Eq (2.1) becomes a linear hyperbolic initial value problem that can be solved by the method of characteristics. Furthermore, the spatial regularity of the initial data is preserved at positive times. Since $g \in L^\infty$, $(\eta_\epsilon * g)$ is uniformly Lipschitz in x at all non-negative times, and hence the characteristics, i.e., the integral curves of $(\eta_\epsilon * g)$, are well-defined. Thus, $\Lambda g = u$ at any given time level is some transportation of the initial data along suitable integral curves. In particular, let $(t,x) \in \Omega_T$. Let $y_{t,x} : [0,t] \to \mathbb{R}$ denote the solution of

$$\begin{aligned} &\dot{y}_{t,x}(s) = [\eta_\epsilon * g](s, y_{t,x}(s)),\\ &y_{t,x}(t) = x. \end{aligned} \tag{2.2}$$

Since $u(t,\cdot)$ is a transportation of the initial data along characteristic curves, we have that $u(t,x) = u_0(y_{t,x}(0))$, thus justifying our assertion that the spatial regularity of u_0 is preserved for positive times. More precisely, $\|\partial_x u\|_{L^\infty(\Omega_T)}$ is finite and depends only on $\|u_0\|_{C^1}$ and $\|g\|_{L^\infty(\Omega_T)}$.

Additionally, we have a maximum principle for Eq (2.1), which we will show also holds for (NN) later. This maximum principle enables us to restrict the operator Λ, viewing it as a map $\overline{B_R(0)} \to \overline{B_R(0)}$ for $R := \|u_0\|_{L^\infty}$, where the ball is measured in the norm of $L^\infty(\Omega_T)$.

Consider $g, h \in L^\infty(\Omega_T)$ and let $u = \Lambda g, v = \Lambda h$. Transport equations preserve the regularity of the initial data, so $u, v \in C^1(\Omega_T)$. If we define $w := u - v$, then w solves the differential equation

$$\partial_t w + (\eta_\epsilon * g)\partial_x w + (\eta_\epsilon * (g-h))\partial_x v = 0.$$

Hence, we have the a priori estimate

$$\left\|w(T^*,\cdot)\right\|_\infty \leq \left\|g-h\right\|_\infty \left\|\partial_x v\right\|_\infty T^*. \tag{2.3}$$

As noted before, $\left\|\partial_x v\right\|_\infty$ depends only on $\|h\|_\infty \leq R$ and $\|u_0\|_{C^1}$, both of which are fixed from the initial datum. Given v and for small enough $T^* > 0$, Eq (2.3) proves that Λ is a contraction in $L^\infty(\Omega_{T^*})$, so we can deduce the existence of a unique fixed point of Λ in $L^\infty(\Omega_{T^*})$, which exactly corresponds to a local-in-time solution of the Cauchy problem (NN) with initial data u_0.

Now, we need only an a priori bound on the Lipschitz constant of the solution at positive times to turn this into a global existence result, drawing on the well-known ‘continuous induction’ argument as elaborated in [34, Proposition 1.2.1, Chapter 1]. For a fixed $\epsilon > 0$ and $g \in L^\infty(\mathbb{R})$, $(\eta_\epsilon * g)$ is Lipschitz, and in particular by the mean value theorem,

$$\left|\eta_\epsilon * g(x_1) - \eta_\epsilon * g(x_2)\right| \le \int_{-\infty}^{\infty} \left|g(y)\right|\left|\eta_\epsilon(x_1 - y) - \eta_\epsilon(x_2 - y)\right| dy \le \left\|g\right\|_\infty C_\epsilon K_\epsilon |x_1 - x_2|\,,$$

for any $x_1, x_2 \in \mathbb{R}$, C_ϵ is the Lipschitz constant for η_ϵ and K_ϵ is the measure of the support of η_ϵ, which is assumed to be compact. Solutions are transported along characteristics, and by the maximum principle, $\eta_\epsilon * u$ is uniformly Lipschitz in x for all non-negative times, so $Y_T : \mathbb{R} \to \mathbb{R}$ defined as $Y_T(x) := y_{T,x}(0)$ via Eq (2.2) is Lipschitz with constant $\exp(\|u_0\|_\infty C_\epsilon T)$. Hence,

$$\begin{aligned} \left|u(T, x_1) - u(T, x_2)\right| &= \left|u_0(Y_T(x_1)) - u_0(Y_T(x_2))\right| \\ &\le \mathrm{Lip}(u_0) \exp(\|u_0\|_\infty C_\epsilon T)|x_1 - x_2|\,, \end{aligned}$$

which gives us the required a priori bound on the spatial Lipschitz constant of the solution for positive times. Hence, the spatial derivative of u cannot blow up in finite time, and therefore, we have a unique C^1 solution to the initial value problem (NN) in Ω_T for C^1, Lipschitz initial data. $T > 0$ is arbitrary, so we are done.

The following lemma will be useful later. Throughout this paper, we will slightly abuse notation and denote the total variation by $\|\cdot\|_{TV}$, even though it is only a semi-norm.

Lemma 2.2. *Let u be a solution to the Cauchy problem* (NN) *with $u_0 \in BV(\mathbb{R}) \cap C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. Then, for all $t \ge 0$: $u(t,\cdot) \in BV(\mathbb{R})$ and $\left\|u(t,\cdot)\right\|_{TV} = \|u_0\|_{TV}$, i.e., we have a Total Variation Preservation property.*

Proof. We exploit the fact that u is a fixed point of the mapping Λ as defined in Theorem 2.1, and therefore, the solution to a transport equation. Let the characteristics of u be denoted by the curves $y_{t,x}$ in Eq (2.2). Let $T > 0$, and consider any increasing sequence of real numbers $x_1, \ldots, x_n$. Since u is preserved along its characteristics, we have that

$$\sum_{i=1}^{n-1}\left|u(t, x_{i+1}) - u(t, x_i)\right| = \sum_{i=1}^{n-1}\left|u_0(y_{t,x_{i+1}}(0)) - u_0(y_{t,x_i}(0))\right| \le \|u_0\|_{TV}\,.$$

Taking the supremum over all such partitions yields

$$\left\|u(t,\cdot)\right\|_{TV} \le \|u_0\|_{TV}\,.$$

To prove the reverse inequality, choose $\delta > 0$ and let $x_1, \ldots, x_m$ be a strictly increasing sequence of real numbers such that

$$\|u_0\|_{TV} - \delta < \sum\left|u_0(x_{i+1}) - u_0(x_i)\right|.$$

The existence of such a partition is guaranteed by the definition of total variation. Let y_i denote the integral curves of $\eta_\epsilon * u$ that solve the corresponding initial value problems $y_i(0) = x_i$. Then, for $t > 0$, the sequence $y_1(t), \ldots, y_m(t)$ is a strictly increasing sequence of real numbers, and therefore

$$\begin{aligned} \left\|u(t,\cdot)\right\|_{TV} &\ge \sum_{i=1}^{m-1}\left|u(t, y_{i+1}(t)) - u(t, y_i(t))\right| \\ &= \sum_{i=1}^{m-1}\left|u_0(x_{i+1}) - u_0(x_i)\right| \\ &> \|u_0\|_{TV} - \delta. \end{aligned}$$

Since $\delta > 0$ can be arbitrarily chosen, we are done.

The solution u can also be shown uniformly Lipschitz continuous in time over compact intervals.

Lemma 2.3. *Solutions of the Cauchy problem* (NN) *with Lipschitz initial data, for all* $T > 0$*, are also uniformly Lipschitz on the domains* Ω_T.

Proof. It is enough to prove Lipschitz continuity with respect to time, at the initial time in terms of the spatial Lipschitz constant and L^∞ norm. Let $h > 0$, then by the mean value theorem, we have that for some $\lambda \in (0, 1)$:

$$\begin{aligned} \left|u(h, x) - u_0(x)\right| &= h\left|\partial_t u(\lambda h, x)\right| \\ &= \left|(\eta_\epsilon * u(\lambda h, x))\partial_x u(\lambda h, x)\right| h \\ &\leq \left\|\partial_x u(h, \cdot)\right\|_\infty \|u_0\|_\infty h. \end{aligned}$$

Hence,

$$\frac{\left|u(h, x) - u_0(x)\right|}{h} \leq \left\|\partial_x u(h, \cdot)\right\|_\infty \|u_0\|_\infty .$$

This concludes the proof.

Since we have a priori control on $\left\|\partial_x u(t, \cdot)\right\|_\infty$ for all positive times by Eq (2.3), we can conclude that u is uniformly Lipschitz in compact intervals of time. Using this, we prove Lipschitz continuity with respect to the L^1 norm. Importantly, this is established independent of ϵ.

Lemma 2.4. *For any fixed* $T > 0$*, the solution* u *of the initial value problem* (NN) *with* C^1, BV *data* u_0 *is uniformly Lipschitz in time with respect to the* L^1 *norm, i.e., there exists* K *such that for all* $t_1, t_2 \leq T$*:*

$$\left\|u(t_2, \cdot) - u(t_1, \cdot)\right\|_1 \leq K(t_2 - t_1).$$

Proof. By Lemma 2.3, u is Lipschitz in time, so by the fundamental theorem of calculus

$$u(t_2, x) - u(t_1, x) = \int_{t_1}^{t_2} \partial_t u(s, x) ds.$$

From (NN) and Lemma 2.2, we can bound the L^1 norm of this difference as follows:

$$\begin{aligned} \int_{\mathbb{R}} \left|u(t_2, x) - u(t_1, x)\right| dx &= \int_{\mathbb{R}} \left| \int_{t_1}^{t_2} \partial_t u(s, x) ds \right| dx \\ &\leq \int_{\mathbb{R}} \int_{t_1}^{t_2} \left|(\eta_\epsilon * u)\partial_x u\right| dx \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}} \left|(\eta_\epsilon * u)\partial_x u\right| dx \\ &\leq \int_{t_1}^{t_2} \left\|u(s, \cdot)\right\|_\infty \left\|\partial_x u(s, \cdot)\right\|_1 ds \\ &\leq \|u_0\|_\infty \|u_0\|_{TV} (t_2 - t_1), \end{aligned}$$

which completes the proof, with $K = \|u_0\|_\infty \|u_0\|_{TV}$.

Thus, a uniform Lipschitz property, and therefore equicontinuity in time with respect to the L^1 norm is established independent of ϵ. This helps us in the next section to establish the well-posedness of (NN) for less regular initial data. In higher dimensions, however, the total variation estimate no longer holds independently of ϵ, and hence, we cannot justify the passage to the limit, although the solution remains BV for BV initial data.

2.2. Weak solutions

Smooth solutions to the nonlocal equation exist globally in time for smooth initial data. However, we also want to work with less regular initial data, and unlike the case of parabolic approximations, our solutions do not gain any regularity. Thus, we need to define admissible weak solutions. We say that $u \in C([0,\infty); L^1_{loc}(\mathbb{R}))$ is a weak solution to the Cauchy problem (NN) with initial data u_0 if, for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$:

$$\iint_{\mathbb{R}^2_+} u\left[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u))\right]dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0.$$

Since $\eta_\epsilon * u$ is always smooth regardless of the regularity of u, this integral is always well-defined, and every classical solution of the initial value problem is also a weak one. From the a priori estimates on the total variation and the uniform Lipschitz time-regularity with respect to the L^1 norm, we can now deduce the existence of weak solutions for BV initial data by a diagonal argument.

Theorem 2.5. *The Cauchy problem* (NN) *has a weak solution* $u \in C([0,T]; L^1_{loc}(\mathbb{R}))$ *for all* $T > 0$*, such that* $\|u\|_\infty \le \|u_0\|_\infty$*, and also for all* $t_1, t_2 \ge 0$:

$$\left\|u(t_1,\cdot) - u(t_2,\cdot)\right\|_{L^1} \le \|u_0\|_{BV}\,\|u_0\|_\infty|t_1 - t_2|\,.$$

Proof. Given $u_0 \in BV(\mathbb{R})$, let $u_0^n(x) = \eta_{1/n} * u_0(x)$. Then, for all n, $u_0^n \in BV(\mathbb{R}) \cap C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. By Theorem 2.1, the Cauchy problem (NN) with initial data u_0^n has a classical solution for each $n \in \mathbb{N}$, say u^n, defined globally. Furthermore, $\left\|u_0^n\right\|_\infty \le \|u_0\|_\infty$ and $\left\|u_0^n\right\|_{TV} \le \|u_0\|_{TV}$, so we have that for all $t > 0$:

$$\left\|u^n(t,\cdot)\right\|_{TV} = \left\|u_0^n\right\|_{TV} \le \|u_0\|_{TV}\,,$$

and for all $t_1, t_2 > 0$:

$$\left\|u^n(t_1,\cdot) - u^n(t_2,\cdot)\right\|_{L^1} \le \left\|u_0^n\right\|_\infty \left\|u_0^n\right\|_{TV}|t_1 - t_2| \le \|u_0\|_\infty\,\|u_0\|_{TV}|t_1 - t_2|\,.$$

The sequence $\{u^n\}$ is comprised of weak solutions with initial data converging to u_0. By Helly's theorem, $\{u^n(t,\cdot)\}$ is uniformly bounded in BV for all $t > 0$ and therefore compact in $L^1([-K,K])$ for any $K > 0$.

Thus, by a diagonalisation argument along the lines of [35], we can extract a subsequence u^{n_k} that converges to some $u \in L^1_{loc}([0,\infty)\times\mathbb{R})$ pointwise almost everywhere and in L^1 on compact sets. A brief sketch of the argument is as follows: Let $\{t_i\}$ be a countable enumeration of all rational times in $[0,T]$. For every rational time t_i, we can extract successive convergent subsequences by the compactness theorem, say $u^{n_{i,k}}$. Define the subsequence $u^k = u^{n_{k,k}}$, which by construction converges at all rational times and thus by density and Lemma 2.4 converges at all times. The limit u also inherits the Lipschitz time-regularity with respect to the L^1 norm. Hence, for any $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ with support (say) K:

$$\begin{aligned}
&\iint_K u[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u))]dxdt \\
&= \iint_K u[\partial_t\varphi + \partial_x\varphi(\eta_\epsilon * u) + \varphi(\eta'_\epsilon * u)]dxdt \\
&= \lim_{n\to\infty} \iint_K u^n[\partial_t\varphi + \partial_x\varphi(\eta_\epsilon * u^n) + \varphi(\eta'_\epsilon * u^n)]dxdt \\
&= \lim_{n\to\infty} \iint_K u^n[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u^n))]dxdt = 0,
\end{aligned}$$

by an application of Lebesgue's dominated convergence theorem, since for any smooth function κ with compact support and $u^n \to u$ in L^1, $\kappa * u^n \to \kappa * u$ pointwise almost everywhere, and we can simply take the dominating function to be $\|u_0\|_\infty$ by the maximum principle because we can restrict ourselves to a compact domain K. This completes the proof; by time-regularity with respect to the L^1 norm, we can see that the initial condition is also satisfied, and thus, a weak solution to (NN) exists within the postulated class.

We can now prove the uniqueness of this limit; more precisely, we will prove stability with respect to L^1 perturbations, from which uniqueness will trivially follow. As is common in the nonlocal literature, we do not require additional criteria such as entropy conditions to ensure the uniqueness of weak solutions. However, as we shall see, this stability is not independent of ϵ, and thus cannot naïvely help us pass to the limit as $\epsilon \to 0$.

Lemma 2.6. *Let $u, v \in L^\infty(\Omega_T) \cap BV(\Omega_T)$ be two weak solutions of* (NN) *with initial data u_0, v_0, respectively, such that $(u_0 - v_0) \in L^1(\mathbb{R})$. Then, there exists a constant $C_\epsilon > 0$ depending on ϵ, such that*

$$\left\|(u-v)(t,\cdot)\right\|_1 \le e^{C_\epsilon t}\|u_0 - v_0\|_1$$

Note that if $u_0 - v_0 \notin L^1(\mathbb{R})$, then the right-hand side is infinite, so the inequality would trivially hold.

Proof. Since u, v are of bounded variation, their derivatives are Radon measures. Additionally, $w = |u-v|$ is also of bounded variation, and

$$\begin{aligned}
\partial_t w &= \operatorname{sgn}(u-v)\partial_t(u-v) \\
&= \operatorname{sgn}(v-u)((\eta_\epsilon * u)\partial_x(u-v) + (\eta_\epsilon * (u-v))\partial_x v) \\
&= -(\eta_\epsilon * u)\partial_x w + \operatorname{sgn}(v-u)(\eta_\epsilon * (u-v))\partial_x v.
\end{aligned}$$

Hence,

$$\partial_t w + \partial_x((\eta_\epsilon * u)w(t,x)) = -\operatorname{sgn}(u-v)(\eta_\epsilon * (u-v))\partial_x v + w\partial_x(\eta_\epsilon * u),$$

and so, by the Gauss-Green theorem for BV functions [36, Theorems 3.84–3.86; Eq (3.85)],

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|w(t,\cdot)\right\|_1 \le \left\|w(t,\cdot)\right\|_1 \left\|\eta_\epsilon\right\|_\infty (\|u\|_{TV} + \|v\|_{TV}),$$

after which the lemma follows from Grönwall's inequality, with

$$C_\epsilon = \left\|\eta_\epsilon\right\|_\infty (\|u\|_{TV} + \|v\|_{TV}),$$

completing the proof.

We can also prove the existence of weak solutions for general L^∞ data by taking a subsequence in the weak-$*$ topology of L^∞, but uniqueness is not guaranteed as in the BV case. We do not pursue this matter further here.

As a corollary of the uniqueness result, and by analogy with the linear transport equation, we also have a finite speed of propagation for the non-local equation for mass if not information.

Corollary 2.6.1. *Let $u_0 \in BV(\mathbb{R})$. Then, letting $y_{t,x}$ denote the integral curves of the (smooth) velocity field $\eta_\epsilon * u$, we have that $u(t,x) = u_0(y_{t,x}(0))$ almost everywhere, i.e., at all points of continuity. Furthermore, for compactly supported initial data, the solution u is is also compactly supported in space at all times.*

Proof. Let u denote the (unique) weak solution to the Cauchy problem (NN) with BV initial data u_0. Denote the non-local field as $\eta_\epsilon * u(x,t) = b(x,t)$, then we can consider the transport equation

$$\partial_t u + b(t,x)\partial_x u = 0,$$

with initial data u_0 as a Cauchy problem, which also has solution $u(x,t)$. Hence, we must have that $u(x,t) = u_0(y_{t,x}(0))$ as claimed. For compactly supported initial data, since u obeys a maximum principle, the integral curves have a uniform Lipschitz bound and therefore propagate with finite speed depending only on $\|u_0\|_\infty$.

2.3. The multidimensional problem

Consider the multi-dimensional scalar conservation law

$$\partial_t u + \sum_{i=1}^{n} \partial_i f_i(u) = 0 \text{ in } [0,T] \times \mathbb{R}^d, \tag{2.4}$$

which can also be written in quasilinear form

$$\partial_t u + \sum_{i=1}^{n} f_i'(u)\partial_i u = 0,$$

for which we can apply a non-local velocity-regularisation schema and obtain

$$\partial_t u^\epsilon + (\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon = 0, \tag{2.5}$$

where

$$F'(u) = (f_1'(u), \ldots, f_n'(u)).$$

We will always assume that F is locally Lipschitz. This is still a non-linear transport equation with convolution as before, and if we define characteristics starting from time zero by $y^\epsilon(t;x)$, then they satisfy the differential equation

$$\dot{y}^\epsilon(t;x) = [\eta_\epsilon * (F' \circ u^\epsilon)](y^\epsilon(t;x),t) \text{ with } y(0,x) = x.$$

By taking one derivative and letting $v_i^\epsilon = \partial_i u^\epsilon$, we get the transport equation for the derivative as before

$$\partial_i \left(\partial_t u^\epsilon + (\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon\right) = \partial_t(\partial_i u^\epsilon) + (\eta_\epsilon * F'(u^\epsilon)) \cdot \partial_i \left(\nabla u^\epsilon\right) + \partial_i(\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon$$

$$= \partial_t v_i^\epsilon + (\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla v_i^\epsilon + \partial_i(\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon,$$

and hence

$$\partial_t v_i^\epsilon + (\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla v_i^\epsilon = -\partial_i(\eta_\epsilon * F'(u^\epsilon)) \cdot \nabla u^\epsilon.$$

Hence, the derivative does not blow up, and the fixed point argument for small time can be extended to a global existence result for the nonlocal equation.

The same method of characteristics/fixed point arguments suffice to conclude the well-posedness of the multidimensional non-local equation. The only difference is the total variation estimate, which cannot be obtained in the same way. However, we still have an estimate of the form given below.

Lemma 2.7. *The total variation of $u^\epsilon(\cdot, t)$ at each time level is bounded.*

Proof. Unlike the 1D case, we cannot use the supremum-over-partitions definition to easily conclude. Instead, we use the dual formulation of the total variation seminorm:

$$\|u\|_{TV} = \sup\left\{\int u \operatorname{div} \varphi dx; \varphi \in C_c^\infty, \left\|\varphi\right\|_\infty \leq 1\right\}.$$

Let $Y_\epsilon(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ denote the backward transport function of the nonlocal equation, i.e., $Y_\epsilon(x,t) = y_x^\epsilon(0)$, where $y_x^\epsilon(s)$ solves the terminal value problem

$$\begin{aligned} \dot{y}_x^\epsilon(s) &= \eta_\epsilon * F'(u^\epsilon)(y_x^\epsilon(s), s), \\ y_x^\epsilon(t) &= x, \end{aligned}$$

in the domain $[0,t]$. Now, by the continuous dependence properties of ODEs, $Y^\epsilon(\cdot, t)$ is a bilipschitz diffeomorphism of $\mathbb{R}^n$ onto itself. Furthermore, we know that the solution u^ϵ is given as

$$u^\epsilon(x,t) = u_0(Y^\epsilon(x,t)).$$

Since we are no longer in one spatial dimension, the total variation of u^ϵ is no longer (necessarily) conserved but bounded by

$$\left\|u^\epsilon(\cdot, x)\right\|_{TV} \leq \left\|DY^\epsilon(\cdot, t)\right\|_\infty^{d-1} \|u_0\|_{TV}.$$

Since Y^ϵ is derived from a flow corresponding to the vector field $\eta_\epsilon * F'(u^\epsilon)$, we can estimate its Lipschitz norm by the Grönwall inequality in terms of the Lipschitz constant of $\eta_\epsilon * u$, which in turn is given by $C_\epsilon \left\|F'(u_0)\right\|_\infty$, where C_ϵ is the L^1 norm of $\nabla\eta_\epsilon$. Thus, we have that

$$\left\|u^\epsilon(\cdot, x)\right\|_{TV} \leq \left(\exp\left(tC_\epsilon \left\|F'\right\|_\infty \|u_0\|_\infty\right)\right)^{d-1} \|u_0\|_{TV}.$$

Thus, solutions to the nonlocal conservation law remain in BV but we do not have a uniform bound independent of ϵ to establish pre-compactness. However, we can obtain convergence to the appropriate limit in other ways, as long as the entropy solution is smooth. Analogous results hold for a 'flux-regularised' non-local equation of the form

$$\partial_t u^\epsilon + F'(\eta_\epsilon * u) \cdot \nabla u^\epsilon = 0 \tag{2.6}$$

as well. In particular, the vector field determining the flow is now given by

$$\dot{y}_x^\epsilon(s) = F' \circ (\eta_\epsilon * u^\epsilon)(y_x^\epsilon(s), s)$$

instead, and thus the final estimate changes slightly due to the order of operations. However, we require here that F' also be locally Lipschitz. In particular, the Lipschitz bound on the bilipschitz diffeomorphism $Y^\epsilon(\cdot, t)$ is given by

$$\left\|Y^\epsilon(\cdot, t)\right\|_\infty \leq \exp\left(tC_\epsilon \left\|F''\right\|_\infty \|u_0\|_\infty\right),$$

by the chain rule.

For fixed $\epsilon > 0$, the solution of the non-local equation(s) remains of bounded variation if the initial data is of bounded variation as well. Hence, the existence and uniqueness of weak solutions, Lipschitz continuity with respect to the L^1 norm, and similar properties remain as in the scalar case. However, we can no longer assume a priori that the sequence of solutions as $\epsilon \to 0$ is pre-compact since the total variation estimate now depends on ϵ.

3. The non-local to local limit

We know from the theory of conservation laws that for Lipschitz initial data, the Cauchy problem for the Burgers equation has a Lipschitz solution for a short amount of time. The exact time of blow-up t^* for the Lipschitz constant can be precisely calculated as

$$t^* = \left[\operatorname*{ess\,sup}_{y\in\mathbb{R}} \{-\partial_x u_0(y)\}\right]^{-1},$$

with $t^* = \infty$ if u_0 is monotone non-decreasing. In the following subsection, we justify the non-local to local limit in the smooth regime, i.e., when the entropy solution of the Burgers equation is known to be Lipschitz. The results we obtain are not specific to the Burgers equation and work for any scalar conservation law, appropriately regularised, and therefore, we will present our results in this slightly more general setting, which includes also the multidimensional case.

In their paper, Coron et al. [16] demonstrated that every convergent subsequence u^ϵ converges to a weak solution of Eq (1.1), but not necessarily the entropy solution. Passage to the unique entropy solution was also demonstrated for quasi-concave initial data, satisfying a one-sided Lipschitz condition. In this special case, Coron et al. proved that (right-continuous representatives of) limit points u of sequences u^ϵ as $\epsilon \to 0$ also satisfy Oleinik's 'Condition E', i.e., the one-sided Lipschitz inequality $u(x + h, t) - u(x, t) \leq Ch$ for $h > 0$ [16, Theorem 3.10]. The proof was carried out by a density argument, where any given quasi-concave initial data u_0 satisfying the Oleinik condition is smoothed out by convolution with a Gaussian kernel, and it is shown that the quasi-concavity transfers to the solution as well. By tracking the points of maximality for u^ϵ and $\eta_\epsilon * u^\epsilon$ over time, unique due to the quasi-concavity, it is shown that the Oleinik condition holds for u^ϵ. However, if u_0 is not quasi-concave, then we cannot obtain such bounds. For instance, if u_0 has two points of (local) maximality whose corresponding characteristic curves approach each other in the limit, then, by the mean value theorem, the derivative $\partial_x u^\epsilon$ is not bounded above; as the twin peaks "squash" the lower intermediate values, the derivative $\partial_x u^\epsilon$ must grow unboundedly. In this section, we expand this convergence result for a broader class of initial data.

3.1. Smooth regime

Let $F \in C^2, F : \mathbb{R} \to \mathbb{R}^n$ be such that F' is uniformly Lipschitz with constant M, and consider the following scalar conservation law:

$$\partial_t u + \nabla_x \cdot F(u) = 0. \tag{3.1}$$

Note that this is not a restrictive assumption, since the Cauchy problems we are working with satisfy a maximum principle with respect to the initial data. When the entropy solution is Lipschitz, we can apply the chain rule to get

$$\partial_t u + F'(u) \cdot \nabla u = 0. \tag{3.2}$$

Now, there are two possible regularisations of Eq (3.1) matching our 'nonlocal schema' for the Burgers equation, as follows:

$$\partial_t u^{\epsilon,1} + (\eta_\epsilon * F'(u^{\epsilon,1})) \cdot \nabla u^{\epsilon,1} = 0, \tag{3.3}$$

$$\partial_t u^{\epsilon,2} + F'(\eta_\epsilon * u^{\epsilon,2}) \cdot \nabla u^{\epsilon,2} = 0, \tag{3.4}$$

and the well-posedness theory for the nonlocal Burgers equation illustrated before holds mutatis mutandis. Note that both regularisations are equivalent to (NN) for the Burgers equation, where f' is linear. Also note that Eqs (3.3) and (3.4) are the cases of Eqs (2.5) and (2.6), respectively.

We will show that, locally in time, smooth solutions of (NN) converge to the entropy solution of Eq (3.1) with the same initial data as $\epsilon \to 0^+$. The following theorem is independent of spatial dimension since we analyse only the deviation along characteristics.

Theorem 3.1. *Consider $u_0 \in \mathrm{Lip}(\mathbb{R}^n), F : \mathbb{R} \to \mathbb{R}^n$ such that $F' \in \mathrm{Lip}(\mathbb{R})$, and let $\tau > 0$ be the maximal time such that the solution u to Eq (3.1) with initial data u_0 is uniformly Lipschitz on Ω_T for all $T < \tau$, with Ω_T now denoting the domain $[0, T] \times \mathbb{R}^n$. Let u be the solution to Eq (1.1) with this initial data. Then, the family of solutions to the nonlocal equations (3.3) and (3.4) with the same initial data, say $u^{\epsilon,1}, u^{\epsilon,2}$, converge uniformly to u as $\epsilon \to 0$ on domains Ω_T when $T < \tau$. On Ω_τ, in the typical case where τ is finite, we thus have convergence in L^1 on compact sets.*

Proof. For $i = 1, 2$, define the sequences of functions $w^{\epsilon,i}$ on the domain Ω_T by

$$w^{\epsilon,i} = u - u^{\epsilon,i}.$$

Let us consider $i = 1$; the other case is quite similar. Let L be the uniform Lipschitz constant for u on Ω_T. From Eqs (3.3) and (3.2):

$$\begin{aligned}\partial_t w^{\epsilon,1} + (\eta_\epsilon * F'(u^{\epsilon,1})) \cdot \nabla w^{\epsilon,1} =& (\eta_\epsilon * F'(u^{\epsilon,1}) - F'(u)) \cdot \nabla u \\ =& (\eta_\epsilon * (F'(u^{\epsilon,1}) - F'(u))) \cdot \nabla u \\ &+ (\eta_\epsilon * F'(u) - F'(u)) \cdot \nabla u.\end{aligned}$$

The last equality is simply a transport equation with a source term, due to our regularity assumptions on the entropy solution. Since f' is uniformly Lipschitz, for $t < T$ we have

$$\left\| w^{\epsilon,1}(\cdot, t) \right\|_\infty \le \left\| w^{\epsilon,1}(\cdot, 0) \right\|_\infty + \int_0^t LM \left\| w^{\epsilon,1}(\cdot, s) \right\|_\infty ds + \epsilon L^2 M t,$$

where M is the Lipschitz constant of f'. Hence, by Grönwall's inequality, we have that on the domain Ω_T:

$$\left\|w^{\epsilon,1}\right\|_\infty \leq \epsilon L^2 M T e^{LMT}.$$

Thus, the nonlocal approximations converge uniformly to the Lipschitz entropy solution as the parameter ϵ goes to zero. By considering a monotonically increasing sequence of times T_n converging to τ and uniform convergence on the domains Ω_{T_n}, we also obtain the L^1_{loc} convergence of non-local solutions to the local limit by Lebesgue's dominated convergence theorem. That is to say, for compact spatial domains $K \subset \mathbb{R}^n$, the nonlocal solutions converge to the local solution in $L^1([0,T] \times K)$. Thus, we can justify the formal singular limit up to the time of catastrophe, i.e., gradient blow-up.

Once the entropy solution forms discontinuities, however, this argument based on the method of characteristics can no longer be extended. This motivates our analysis of solutions with discontinuous initial data. The simplest such case is the Riemann problem, which we shall turn to next.

3.2. The Riemann problem

We now arrive at our first main result tackling discontinuities, which shows both convergence and non-convergence to the local limit depending on the form of u_0. Furthermore, we shall see that the Burgers case really is special in enabling nonlocal-to-local convergence to the entropy solution in the presence of discontinuities.

Lemma 3.2. *Let $u_0(x)$ be a piece-wise constant function taking on exactly two values, i.e., suppose*

$$u_0(x) = \begin{cases} u_L \text{ if } x \leq 0, \\ u_R \text{ if } x > 0. \end{cases} \tag{3.5}$$

Then, the unique BV solution u^ϵ to (NN) *with initial data given by Eq* (3.5) *is independent of ϵ and given by*

$$u(t,x) = \begin{cases} u_L \text{ if } x \leq \sigma t, \\ u_R \text{ if } x > \sigma t, \end{cases} \tag{3.6}$$

where $\sigma = \frac{1}{2}(u_L + u_R)$ is the Rankine-Hugoniot shock speed for the Burgers equation. In particular, we have trivial convergence to the entropic weak solution for shock-type Riemann data ($u_L > u_R$) and non-convergence for rarefaction-type initial data ($u_L < u_R$).

Proof. If $u_L = u_R$, there is nothing to prove, so we assume that we are not dealing with the trivial case. As an ansatz, define $u(t,x)$ as in the theorem; we simply have to show that it is a weak solution. Let $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R})$. Then, let $\Omega_L = \{(t,x) : x \leq \sigma t\}, \Omega_R = \{(t,x) : x > \sigma t\}$; note that they partition

$\Omega = [0,\infty) \times \mathbb{R}$. Also note that, by the symmetry of the mollifier, $\eta_\epsilon * u(t, \sigma t) = \sigma$. Hence,

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty u[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u))]dxdt + \int_{-\infty}^\infty u_0(x)\varphi(0,x)dx \\
&= \iint_\Omega u[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u))]dxdt + \int_{-\infty}^\infty u_0(x)\varphi(0,x)dx \\
&= \iint_{\Omega_L} u_L(\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u)))dxdt \\
&\quad + \iint_{\Omega_R} u_R(\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u)))dxdt \\
&\quad + \int_{-\infty}^\infty u_0(x)\varphi(0,x)dx,
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty u[\partial_t\varphi + \partial_x(\varphi(\eta_\epsilon * u))]dxdt + \int_{-\infty}^\infty u_0(x)\varphi(0,x)dx \\
=&\frac{1}{\sqrt{1+\sigma^2}} \int_0^\infty u_L[\sigma\varphi(t,\sigma t) - \varphi(t,\sigma t)(\eta_\epsilon * u(t,\sigma t))]dxdt \\
&+ \frac{1}{\sqrt{1+\sigma^2}} \int_0^\infty u_R[-\sigma\varphi(t,\sigma t) + \varphi(t,\sigma t)(\eta_\epsilon * u(t,\sigma t))](t,x)dt \\
=&\frac{1}{\sqrt{1+\sigma^2}} \int_0^\infty (u_L - u_R)\varphi(t,\sigma t)[\sigma - \eta_\epsilon * u(t,\sigma t)]dt \\
=&0.
\end{aligned}
$$

Since φ is arbitrary, this shows that the solution to (NN) with initial data Eq (3.5) is indeed Eq (3.6), and it is the unique BV solution by Lemma 2.6.

Note that the particular form of the Burgers flux was crucial to the proof, and it is easy to see that convergence to the entropy solution does not hold in general even for the Riemann problem if we consider regularisations like Eq (3.3) or (3.4). Thus, we have the following lemma for the generic Riemann problem with respect to our non-conservative, non-local equation.

Lemma 3.3. *Let $u^{\epsilon,1}, u^{\epsilon,2}$ denote solutions to the Cauchy problems* (3.3) *and* (3.4)*, respectively, with initial data*

$$
u_0(x) = \begin{cases} u_L \text{ if } x \le 0, \\ u_R \text{ if } x > 0. \end{cases}
$$

Then, the solutions are independent of $\epsilon > 0$ and given by

$$
u^{\epsilon,i}(t,x) = \begin{cases} u_L \text{ if } x \le \sigma_i t, \\ u_R \text{ if } x > \sigma_i t, \end{cases}
$$

where

$$
\sigma_i = \begin{cases} \dfrac{f'(u_L) + f'(u_R)}{2} \text{ if } i = 1, \\ f'\left(\dfrac{u_L + u_R}{2}\right) \text{ if } i = 2. \end{cases}
$$

The proof is trivially similar to that of Lemma 3.2, and therefore not be repeated. These two lemmas suggest the following:

(a) For fluxes f such that f' is non-linear, we cannot expect to justify the singular limit of either Eq (3.3) or (3.4) outside domains where the entropy solution is smooth. Consider the Riemann data: The Rankine-Hugoniot speed for shock-type data is given by

$$\sigma_{RH} = \frac{f(u_L) - f(u_R)}{u_L - u_r},$$

hence, if it is to match either of σ_i, we must have that

$$\frac{f(u_L) - f(u_R)}{u_L - u_r} = \frac{f'(u_L) + f'(u_R)}{2},$$
$$\text{or } \frac{f(u_L) - f(u_R)}{u_L - u_r} = f'\left(\frac{u_L + u_R}{2}\right).$$

Unless f is of the form $f(y) = a^2 y + by + c$, neither equality can hold for all values u_L, u_R. Note that if $f = u^2/2$ as in Lemma 3.2, then $f'(y) = y$, and thus both $\sigma_1 = \sigma_2 = \sigma_{RH}$.

(b) Even for linear fluxes (i.e., the Burgers equation and its rescaled variants), we cannot expect convergence if the initial data contains a positive jump discontinuity, since the entropy conditions are violated across the jump.

3.3. *Piece-wise Lipschitz increasing data*

Coron et al. [16] proved, for (NN), that (convergent subsequences of) non-local solutions converge to *weak* solutions of the Burgers equation with the same initial data. This holds only as long as the mollifier is symmetric and the flux is quadratic, i.e., the flux derivative is linear. Lemma 3.3 demonstrates that this condition is sharp. Thus, we assume compactly supported, symmetric mollifiers η_ϵ. For C^1, Lipschitz initial data, we can differentiate (NN) to obtain the following transport equation for the spatial derivative $\partial_x u^\epsilon = v^\epsilon$:

$$\partial_t v^\epsilon + (\eta_\epsilon * u^\epsilon)\partial_x v^\epsilon = -v^\epsilon \partial_x(\eta_\epsilon * u^\epsilon). \tag{3.7}$$

Now, if $u^\epsilon(x + \delta, 0) > u^\epsilon(x, 0)$ for all $\delta > 0$ is small enough, then we also have that $u^\epsilon(x + \delta', t) > u^\epsilon(x, t)$ for all $\delta' > 0$ is small enough. That is to say, v^ϵ does not change sign. Hence, along a characteristic $\gamma^\epsilon(t)$, we have that $v^\epsilon(t, \gamma^\epsilon(t))$ is decreasing in time if it is positive, unless $\partial_x(\eta_\epsilon * u^\epsilon)(t, \gamma^\epsilon(t)) = \eta_\epsilon * v^\epsilon(t, \gamma^\epsilon(t)) < 0$.

In particular, for Lipschitz monotone increasing initial data, the Oleinik condition is satisfied and the non-local solutions u^ϵ converge to the entropy solution in L^1 on all compact subsets of spacetime. However, we saw this in Theorem 3.1. Let us now consider the case of piece-wise Lipschitz increasing data to generalise this idea.

Consider initial data $u_0 \in BV(\mathbb{R})$ such that, for $a_1, a_2 \in \mathbb{R}, a_1 < a_2$:

$$u_0(x) = \begin{cases} v_0(x), \text{ if } x < a_1, \\ v_1(x) \text{ if } x \in [a_1, a_2), \\ v_2(x) \text{ if } x \geq a_2, \end{cases}$$

where $v_i(x)$ are uniformly Lipschitz increasing functions with constant C, and the jump discontinuities to satisfy

$$u_0(a_1-) > u_0(a_1+), u_0(a_2-) > u_0(a_2+),$$

with $u_0(x\pm)$ denoting the left and right limits, respectively. Note that if the inequalities are not strict, then we are reduced to the Lipschitz increasing case we already covered earlier. In the case where one of the inequalities is strict but not the other, we can redefine the other point to be much further away without losing generality, and for each finite time horizon, perform the following analysis trivially, as we shall see.

Let $\gamma_j^\epsilon(t)$ denote the characteristic curves of the Cauchy problem (NN) with initial data u_0 as above. Then, from Eq (3.7), we have that $u^\epsilon(\cdot, t)$ is monotone, increasing for values in each of the intervals $(-\infty, \gamma_1^\epsilon(t)), (\gamma_1^\epsilon(t), \gamma_2^\epsilon(t)), (\gamma_2^\epsilon(t), \infty)$ separately.

Recall that u^ϵ for any sequence of $\epsilon \to 0$ is pre-compact in L^1_{loc}, and every convergent subsequence converges to a weak solution. Furthermore, the curves γ_i^ϵ are uniformly Lipschitz, hence we can extract subsequences converging uniformly on compact intervals of time. Fix $T > 0$, and consider a sequence, denoted by ϵ, such that $\gamma_i^\epsilon \to \gamma_j$ uniformly on $[0, T]$. We can choose T small enough such that the curves γ_i do not meet.

Since the initial jumps are entropic, we need only show that the increasing parts of the limiting weak solution u between the two curves γ_i satisfies the Oleinik condition

$$\partial_x u \leq C.$$

To prove this, we use the transport formulation for the derivative at the non-local level and transfer the property onto the limit.

Let $\delta > 0$, with $\delta < \frac{1}{2} \min \gamma_2 - \gamma_1$, where the minimum is taken over $[0, T]$. It is enough to show that for each such $\delta > 0$, the non-local solution u^ϵ satisfies $\partial_x u^\epsilon(x, t) \leq C$ for x such that $\min\left|x - \gamma_i(t)\right| > \delta$. Without the loss of generality, it is enough to show this for values of x lying between the curves γ_i. Thus, let $\epsilon > 0$ be small enough such that $\epsilon < \frac{\delta}{4}$, and

$$\left\|\gamma_i - \gamma_i^\epsilon\right\|_{L^\infty(0,T)} < \frac{\delta}{4},$$

for $i = 1, 2$. Define the domain

$$\mathcal{D}_\delta^\epsilon = \{(x, t) | t \in [0, T], \gamma_1^\epsilon(t) + \epsilon \leq x \leq \gamma_2^\epsilon(t) - \epsilon\},$$

which, by our choice of ϵ, contains our set of interest

$$\mathcal{D}_\delta = \{(x, t) | t \in [0, T], \gamma_1(t) + \delta \leq x \leq \gamma_2(t) - \delta\}.$$

Since the support of the mollifier η_ϵ is $[-\epsilon, \epsilon]$, we have that

$$\partial_x(\eta_\epsilon * u^\epsilon) = \eta_\epsilon * (\partial_x u^\epsilon) > 0,$$

on the set $\mathcal{D}_\delta^\epsilon$. Hence, as long as a characteristic stays in $\mathcal{D}_\delta^\epsilon$, the spatial derivative along the characteristic is decreasing. Since the initial value of the spatial derivative is bounded above by C, we are done if we can show that characteristics do not enter the domain at positive time, since characteristics 'fill up' the spacetime domain.

Note that the lateral boundaries of $\mathcal{D}_\delta^\epsilon$ are given by the curves

$$\theta_i^\epsilon(t) = \gamma_i^\epsilon(t) - (-1)^i \epsilon,$$

for $i = 1, 2$. Hence, we have that

$$\dot{\theta}_i^\epsilon(t) = \dot{\gamma}_i^\epsilon(t),$$

for $i = 1, 2$. However,

$$\eta_\epsilon * u^\epsilon(\gamma_1^\epsilon(t) + \epsilon, t) < \eta_\epsilon * u^\epsilon(\gamma_1^\epsilon(t), t) = \dot{\gamma}_1^\epsilon(t),$$

since the jump is entropic, and the term on the left is nothing but the speed of the characteristic passing through the point $(\gamma_1^\epsilon(t) + \epsilon)$. Hence, the characteristic cannot enter the domain $\mathcal{D}_\delta^\epsilon$ at any positive time. A similar analysis can be carried out for the other boundary curve, which concludes the proof.

Note that for values of x outside region $\mathcal{D}_\delta$, we can carry out a similar analysis looking only at one boundary. Since this is essentially a local result based on analysis of characteristic curves, we can generalise the convergence result to include all initial data that are piece-wise Lipschitz increasing, by extracting further subsequences if necessary and as long as the curves of discontinuity are uniformly apart from each other. Thus, we have the following theorem.

Theorem 3.4. *Let $u_0(x) \in BV(\mathbb{R})$ be such that, for real numbers $a_1 < a_2 < \ldots < a_n$ and uniformly Lipschitz increasing functions $v_0, \ldots, v_n$ with $v_k(a_{k+1}) > v_{k+1}(a_{k+1})$:*

$$u_0(x) = \begin{cases} v_0(x), & \text{if } x < a_1, \\ v_i(x), & \text{if } x \in [a_i, a_{i+1}), 1 \le i \le n-1, \\ v_n(x) & \text{if } x \ge a_n. \end{cases}$$

Then, there is a positive time T^* depending on $\{a_k\}, \|u_0\|_\infty$ such that the non-local solutions u^ϵ of the Cauchy problem (NN) *with initial data u_0 converge to the entropy solution u of Eq* (1.1) *with the same initial data.*

Proof. Since the characteristics propagate with finite speed, the curves of discontinuity stay apart for some finite time. In particular, we can bound this 'secondary catastrophe' time from below by $\frac{1}{2} \|u_0\|_\infty D$, where

$$D = \min\{a_k - a_{k-1}; k = 2, \ldots, n\}.$$

This follows from the finite speed of propagation of characteristics, which is bounded by $\|u_0\|_\infty$. In particular, two curves $\gamma_1(t),\ \gamma_2(t)$ with Lipschitz constant L and starting from $x_1,\ x_2$, respectively cannot intersect before $t = \frac{1}{2}L$, else one of them would have to violate the Lipschitz bound by the triangle inequality.

For every convergent subsequence of u^ϵ, we can, by extracting further subsequences if necessary, obtain a sequence whose limit satisfies the Oleinik condition by the above argument. Furthermore, we know that the limit must be a weak solution of Eq (1.1). Hence, in particular, every subsequence has a further subsequence converging to the unique entropy solution of Eq (1.1) with initial data u_0, and thus, we conclude that every sequence of u^ϵ as $\epsilon \to 0$ converges to the entropy solution.

Even beyond this 'secondary catastrophe' time, the results of Coron et al. [16] showed that every converging subsequence converges to a weak solution, but it is open as to whether this limit is the entropy solution, or whether a unique limit exists.

4. Extension to the isentropic Euler system

The isentropic Euler equations with pressure law $p(\rho) = \rho^3/3$ enable for a natural extension of our inviscid regularisation schema due to the structure of the associated Riemann invariants and eigenvalues. The equations of gas dynamics for a pressure law as above can be written in conservative form as follows:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0, \\ \partial_t(\rho v) + \partial_x\left(\rho v^2 + \frac{\rho^3}{3}\right) &= 0. \end{aligned} \tag{4.1}$$

The eigenvalues of this system are given by $v \pm \rho$, and the Riemann invariants are given by $\rho \pm v$, and hence, for classical solutions, we have that

$$\begin{aligned} \partial_t(\rho + v) + (\rho + v)\partial_x(\rho + v) &= 0, \\ \partial_t(\rho - v) - (\rho - v)\partial_x(\rho - v) &= 0. \end{aligned} \tag{4.2}$$

Note that this is similar in form to Burgers equation in non-conservative form. Thus, let $\mu = \rho + u$, $\lambda = \rho - \mu$, and consider the analogous inviscid regularisation of these equations:

$$\begin{aligned} \partial_t \mu^\epsilon + (\eta_\epsilon * \mu^\epsilon)\partial_x \mu^\epsilon &= 0, \\ \partial_t \lambda^\epsilon - (\eta_\epsilon * \lambda^\epsilon)\partial_x \lambda^\epsilon &= 0. \end{aligned} \tag{4.3}$$

Since we have decomposed the system of conservation laws into a pair of decoupled scalar equations, we can extend Theorem 3.1 to the isentropic system (4.1). Note that as we are assuming a classical regime, we can equivalently work with Riemann invariants instead of the conserved quantities.

5. Conclusions

In contrast to the standard conservative method for non-local regularisation of scalar conservation laws, we have demonstrated here the merits of an explicitly non-conservative approach, showcasing the strong maximum principle and total variation-preserving properties of (NN). In the smooth regime, and therefore locally-in-time for any Lipschitz initial data, our a priori estimates help us justify the formal limit of the non-local equations as $\epsilon \to 0^+$. In a sense, this result is also sharp, since for even the simplest cases of discontinuous initial data, convergence to the entropy solution does not hold in general, except in particular for regularisations of the Burgers equation.

One interesting question that we were unfortunately not able to resolve was that of the singular limit for Lipschitz initial data beyond the catastrophe time. Since non-entropic shocks cannot form at positive times, the Riemann-type counter-examples do not pose an obstacle to convergence. However, we were not able to justify the positive result. Of course, by total variation preservation, the sequence

u^ϵ has an L^1-convergent subsequence on compact subsets of Ω_T for every T, but it is not clear whether the limit is always the appropriate entropy solution.

It is easy to see that for any initial data with a rarefaction-type discontinuity, the nonlocal-to-local limit does not hold. However, this failure of convergence is not so obvious if the data does not have any positive jumps. Adapting a counter-example from [18], we can show that our schema (NN) satisfies the non-local to local limit in some cases where the conservative regularisation Eq (1.3) fails to do so. In particular, consider initial data u_0 such that

$$u_0(x) = \begin{cases} 0 & \text{if } |x| \geq 2, \\ -\operatorname{sgn}(x) & \text{if } |x| \leq 1, \end{cases}$$

and Lipschitz monotone non-decreasing on $(-\infty, 0) \cup (0, \infty)$. Note that such a function u_0 cannot be quasiconcave either. For the conservative regularisation Eq (1.3), the nonlocal-to-local limit does not hold, even weakly, for any time interval. However, by construction, u_0 is a piece-wise Lipschitz increasing function with entropic discontinuity. Hence, by Theorem 3.4, we have convergence to the entropy solution in the singular limit with respect to the non-conservative non-local regularisation (NN). Since there is only one discontinuity, this convergence is global in time.

It would be interesting to demonstrate that the nonlocal-to-local limit always holds, provided the initial data does not contain any positive jumps, or construct a counter-example to convergence. However, the standard counter-examples for symmetric convolutional kernels from the theory of conservative nonlocal equations do not carry over. Furthermore, even in the presence of positive jumps, one could have convergence to a unique non-entropic weak solution, as the Riemann data shows. A precise characterisation of the singular limits would be quite valuable.

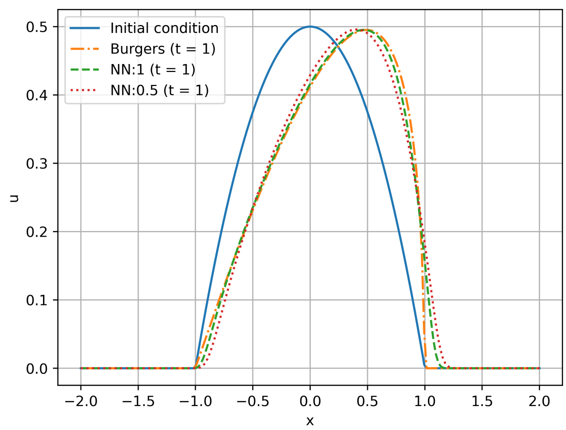

Figure 1. Lipschitz initial data.

Figure 1 shows the profiles of solutions at $t = 1$ for the Cauchy problem (1.1) and (NN) with the same (Lipschitz) initial data, and varying non-local parameter; we graph the solution for $\epsilon = 1, \epsilon = 0.5$. The initial data is supported in $[-1, 1]$ and given by $u_0(x) = \frac{1}{2}(1 - x^2)$. Since $u_0 \geq 0$, an upwind scheme was used to generate all the profiles. The support thus remains within $[-1, 1]$ for (1.1), while it is shifted slightly for (NN). At $t = 1$, the Lipschitz continuity of the entropy solution to Eq (1.1) breaks down and a shock forms at $(1, 1)$.

The classical upwind scheme can be found in Godlewski and Raviart's seminal text [24, Example 1.3, Chapter IV]. It works based on the direction of wave propagation (hence "upwind"). Since (NN) is in quasi-linear form, the upwind scheme can be directly applied. However, the scheme's stability and accuracy depend on the sign and magnitude of the characteristic speeds, and since the convolution needs to be computed at each time step, the scheme is computationally expensive for (NN) relative to Eq (1.1).

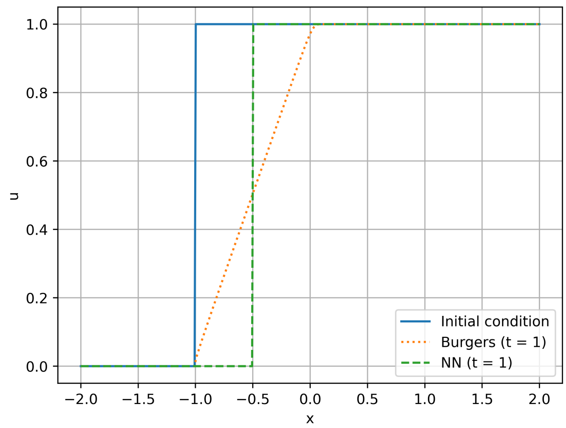

Figure 2. Discontinuous initial data.

Figure 2 shows the profiles of solutions at $t = 1$ for the Cauchy problem (1.1) and (NN) with the same (discontinuous) initial data. The solution of (NN) is independent of the non-local parameter in this case, since the initial data is the Heaviside function $u_0(x) = \chi_{[0,\infty)}(x)$. The discontinuity in (NN) merely travels at Rankine-Hugoniot speed, but the initial jump is positive, so the entropy solution of Eq (1.1) for this initial data is a rarefaction fan. A finite-volume scheme was used to generate the rarefaction profile for Eq (1.1), while the solution at $t = 1$ for (NN) was graphed directly as it can be computed to be the shifted profile pictured above.

The Godunov scheme used to solve the Burgers equation rarefaction wave in Figure 2 is a classical finite volume scheme [24, Section 2, Chapter IV]. It works by solving Riemann problems at cell interfaces, then reconstructing a piecewise constant solution at the next time step by spatially averaging the exact solution over each cell. This method handles discontinuities and rarefaction waves more accurately than the upwind scheme, as demonstrated in Figure 2. However, the Godunov scheme is not applicable to equation (NN) since it lacks the conservative form required for finite volume methods.

Authors contribution

All authors contributed equally to the framing, analysis, and writing of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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