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Theory article

Projective synchronization for quaternion-valued memristor-based neural networks under time-varying delays

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Abstract: In this paper, the projective synchronization of quaternion-valued memristor-based neural networks with time-varing delays was studied. First, by utilizing set-valued map and differential inclusion theories, we reformulated the networks as an uncertain system with interval parameters. Then, through designing a novel controller and utilizing Lyapunov function and Young's inequality, several new synchronization conditions for projection synchronization of quaternion-valued memristor-based neural networks were obtained. Finally, the effectiveness of this method was demonstrated through a numerical example, underscoring its practical applicability.

Keywords: projective synchronization; quaternion-valued; time-varying delays; memristor-based neural networks

Notations: Throughout this paper, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ denote the real, complex, and quaternion fields, respectively. The notation adheres to standard mathematical conventions.

1. Introduction

The memristor, a pioneering circuit component that encapsulates the interplay between magnetic flux and electric charge, made its debut in scientific literature through the visionary work of Chua [1] in 1971. This innovative element distinguishes itself from traditional circuit components due to its nonlinear resistance and inherent memory capabilities. Its mnemonic attribute bears a striking resemblance to the synaptic plasticity observed in the neural connections of the human brain. Leveraging this distinctive feature, the memristor has transcended its role as an electrical resistor to become a cornerstone in simulating the cognitive functions of the human brain. Consequently, the exploration of memristor-based neural networks (MBNNs) has emerged as a vibrant field of study, with researchers delving into the intricate dynamics that these networks exhibit [2–4]. MBNNs are characterized as state-dependent dynamical systems, with coefficients that are inherently linked to their instantaneous state. The discontinuous nature of these systems can give rise to a spectrum of complex nonlinear phenomena, including but not limited to chaos, oscillations, and instability. The investigation of these dynamical traits is not merely an academic pursuit; it holds profound significance in both theoretical exploration and practical applications, paving the way for advancements in neuromorphic engineering and computational neuroscience.

The quaternion, a mathematical construct first articulated by the Irish mathematician Sir William Rowan Hamilton in 1843, stands as an innovative extension of the real and complex number systems. Unlike their simpler counterparts, quaternions introduce a non-commutative multiplication, a feature that has historically posed challenges and led to a period of relative dormancy in their study. However, in recent years, there has been a resurgence of interest and a broadening scope of applications for quaternions. They have demonstrated remarkable utility across a spectrum of disciplines, including artificial intelligence [5], image processing [6], quantum mechanics [7], and aerospace technology [8]. These applications have not only reinvigorated the study of quaternions but also unveiled their potential to address complex problems in a multidimensional context. Especially in image processing, the nuanced representational power of quaternions has been harnessed. They are used to encode the three primary color channels within the imaginary components of a quaternion, while the real part is often reserved for the alpha channel or other metadata. This elegant mapping of color images to pure quaternions has opened new avenues for the representation and manipulation of visual data.

Quaternion-valued neural networks (QVNNs) distinguish themselves from their complex-numbered counterparts by employing quaternions in every aspect of their architecture—states, connection weights, and activation functions. This holistic approach to quaternion integration endows QVNNs with a unique capacity for handling multidimensional data representations. In recent scholarly discourse, a surge of interest has been directed towards the dynamics of QVNNs, as evidenced by a burgeoning body of literature [9–14]. Notably, the exploration of robust stability within the fractional-order realm of QVNNs has garnered significant attention, as demonstrated by the contributions of [9]. Additionally, innovative control methodologies, such as sampled-data approaches, have been applied to stabilize QVNNs, as explored in [10]. The investigation into the robust stability of these networks continues to evolve, with offering fresh insights into the fractional-order context [13]. Furthermore, the stability analysis of quaternion-valued memristive neural networks has been enriched by the application of Lagrangian mechanics, as discussed in [14], and so on [15–19]. These studies collectively contribute to a deeper understanding of the intricate dynamics that govern QVNNs, paving the way for advancements in network stability and synchronization.

The recent research on quaternion-valued memristive neural networks (QVMNNs) has attempted to understand their periodic solutions, as highlighted in [15]. This work has been complemented by an examination of dissipativity in neutral-type memristor-based networks, which provides a fresh perspective on stability and energy dynamics, as discussed in [16]. Moreover, the exploration of finite-time stabilization through implicit function methods, as introduced in [17], has opened new avenues for the rapid and reliable control of these networks, underscoring the growing sophistication in the field of QVNNs.

Synchronization, a fundamental and pivotal phenomenon in the realm of neural networks, plays a crucial role in managing and orchestrating the inherent chaotic dynamics that are often observed in natural systems. It serves as a powerful tool for harnessing order from chaos, offering a mechanism to regulate and predict the behavior of complex neural systems. Through synchronization, we can explore unknown dynamical systems from known ones. So far, the synchronization of neural networks includes quasi-uniform synchronization [20–22], anti-synchronization [23,24], finite-time synchronization [25–27], projection synchronization [28,29], exponential synchronization [30,31], and global Mittag-Leffler synchronization [32,33], and others [34,35].

Among the spectrum of synchronization phenomena, projection synchronization emerges as an inclusive and versatile approach within the domain of neural network systems. This method, underpinned by proportional dynamics, facilitates accelerated communication pathways, a feature that sets it apart from other synchronization modalities [36–38]. In this work, we extend the discourse to the projective synchronization of quaternion-valued memristor-based neural networks, a topic that gains relevance amidst time-varying delays. Our exploration is anchored in the following pivotal contributions:

(i) QVMNNs offer distinctive benefits over traditional real- and complex-valued neural networks, particularly in their adept handling of multi-dimensional data through low-dimensional constructs and enhanced computational efficiency.

(ii) The synchronization criteria established within this study are not only capable of orchestrating complete synchronization and anti-synchronization but also of embracing the general projection synchronization paradigm. These findings are posited as universally applicable and representative of the broader scope of neural network synchronization.

(iii) By designing a novel controller and using Lyapunov function and Young's inequality, some new synchronization conditions for projection synchronization of quaternion-valued memristor-based neural networks are obtained.

2. Problem description and preliminaries

The quaternion constitutes a realm of hypercomplex numbers, encapsulating a singular real component alongside three imaginary elements, thereby extending the numerical system beyond the conventional real and complex planes. The quaternion $m \in \mathbb{Q}$ can be described as

$$m = m^R + m^I i + m^J j + m^K k,$$

where $m^R, m^I, m^J, m^K \in \mathbb{R}$, the imaginary parts $i,\ j,\ k$ obey the Hamilton rule:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Remark 1. Quaternions, a class of hypercomplex numbers, diverge from the algebraic properties of real and complex numbers. Unlike their commutative counterparts, the multiplication of any two quaternions, denoted as $m,\ n \in \mathbb{Q}$, does not necessarily adhere to the associative law $mn = nm$. This departure signifies that several principles governing real and complex number systems are inapplicable to quaternions. Consequently, it mandates the innovation of novel methodologies and theoretical frameworks to effectively harness and comprehend the intricacies of quaternion arithmetic.

For the quaternion m and n, where $n = n^R + n^I i + n^J j + n^K k$, then we can denote $m + n$ as follows:

$$m + n = m^R + n^R + (m^I + n^I)i + (m^J + n^J)j + (m^K + n^K)k.$$

By Hamilton rule, we can denote mn as

$$\begin{aligned} mn &= (m^R n^R - m^I n^I - m^J n^J - m^K n^K) + (m^R n^I + m^I n^R + m^J n^K - m^K n^J)i \\ &\quad +(m^R n^J + m^J n^R + m^K n^J - m^I n^K)j + (m^R n^K + m^K n^R + m^I n^J - m^J n^I)k. \end{aligned}$$

The modulus of m is written as

$$|m| = \sqrt{m\overline{m}} = \sqrt{(m^R)^2 + (m^I)^2 + (m^I)^2 + (m^K)^2}.$$

Furthermore, for $m = (m_1, m_2, \ldots, m_n)^T$, $\|m\| = \sum_{i=1}^{n} |m_i|$ denotes the norm of m.

In this study, we delve into the dynamics of a memristor-driven neural network enriched with quaternion-valued parameters and subject to time-varying delays. The system's evolution is intricately captured by a set of differential equations that account for these temporal lags.

$$\begin{aligned} \dot{m}_q(t) &= -c_q(t)m_q(t) + \sum_{s=1}^{n} b_{qs}(m_q(t))f_s(m_s(t)) + \sum_{s=1}^{n} d_{qs}(m_q(t))f_s(m_s(t-\tau(t))) + I_q, \\ m_q(s) &= \psi_q(s), s \in [-\tau, 0], \end{aligned} \tag{2.1}$$

with each neuron's state vector represented by the components $m(t) = (m_1(t), m_2(t), \cdots, m_n(t))^T \in \mathbb{Q}^n$, $q = 1, 2, \cdots, n$, and $m_q(t) \in \mathbb{Q}$ corresponds to the individual neuron's state. The positive self-feedback coefficient is denoted by $c_q > 0$, and the quaternion-valued connection weights, stemming from memristor dynamics, are given by $b_{qs}(m_q(t))$ and $d_{qs}(m_q(t))$. The vector $f_s(m_s(t)) = (f_1(m_1(t)), f_2(m_2(t)), \cdots f_n(m_n(t)))$ encapsulates the activation functions that govern the neurons' firing patterns. The external stimuli to the network are captured by the input vector $I_q = (I_1, I_2, \cdots, I_n)^T \in \mathbb{Q}^n$. Additionally, the network's signal transmission is subject to delays characterized by $\tau(t)$, which are constrained to be non-negative and less than a maximum delay, satisfying $0 \le \tau(t) < \tau$. For the initial setup of the system (2.1), we select an initial condition that is continuously differentiable over the interval $[-\tau,\ 0]$, This is mathematically expressed as $m_q(s) = \psi_q(s) = (\psi_1(s), \psi_2(s), \cdots, \psi_n(s)) \in C^{(1)}([-\tau,\ 0], \mathbb{Q}^n), -\tau \le s \le 0$, laying down a well-defined starting point for the system's trajectory.

Incorporating the framework of differential inclusion and set-valued mappings, coupled with our preceding discourse, the representation of system (2.1) is delineated as:

$$\dot{m}_q(t) \in -c_q m_q(t) + \sum_{s=1}^{n} co(b_{qs}^-, b_{qs}^+)f_s(m_s(t)) + \sum_{s=1}^{n} co(d_{qs}^-, d_{qs}^+)f_s(m_s(t-\tau(t))) + I_q, \tag{2.2}$$

where $b_{qs}^- = min\{\acute{b}_{qs}, \grave{b}_{qs}\}, b_{qs}^+ = max\{\acute{b}_{qs}, \grave{b}_{qs}\}$, $d_{qs}^- = min\{\acute{d}_{qs}, \grave{d}_{qs}\}, d_{qs}^+ = max\{\acute{d}_{qs}, \grave{d}_{qs}\}$. The essence of differential inclusion encapsulates the existence of a collection of differential equations where the terms $b_{qs}(t) \in co(b_{qs}^-, b_{qs}^+)$, $d_{qs}(t) \in co(d_{qs}^-, d_{qs}^+)$, such that

$$\dot{m}_q(t) = -c_q m_q(t) + \sum_{s=1}^{n} b_{qs}(t)f_s(m_s(t)) + \sum_{s=1}^{n} d_{qs}(t)f_s(m_s(t-\tau(t))) + I_q. \tag{2.3}$$

Assumption 1. Let $m = m^R + m^I i + m^J j + m^K k$, where $m^R,\ m^I,\ m^J,\ m^K \in \mathbb{R}$. $f_s(m)$ can be decomposed into its real and imaginary constituents, denoted by $f_s(m) = f_s^R(m^R) + f_s^I(m^I)i + f_s^J(m^J)j + f_s^K(m^K)k$.
Assumption 2. For any $m_q^\epsilon(t), n_q^\epsilon(t) \in \mathbb{R}^n$, we identify the existence of certain positive constants $\epsilon = R,\ I,\ J,\ K,\ q = 1,\ 2,\ \cdots,\ n$, such that

$$|f_q^\epsilon(n_q^\epsilon) - f_q^\epsilon(m_q^\epsilon)| \le l_q^\epsilon |n_q^\epsilon - m_q^\epsilon|.$$

Under Assumption 1, we divide the network (2.1) into one real part and three imaginary parts, respectively.

$$\begin{aligned} \dot{m}_q^R(t) &= -c_q m_q^R(t) + \sum_{s=1}^n b_{qs}^R(t) f_s^R(m_s^R(t)) - \sum_{s=1}^n b_{qs}^I(t) f_s^I(m_s^I(t)) - \sum_{s=1}^n b_{qs}^J(t) f_s^J(m_s^J(t)) \\ &\quad - \sum_{s=1}^n b_{qs}^K(t) f_s^K(m_s^K(t)) + \sum_{s=1}^n d_{qs}^R(t) f_s^R(m_s^R(t-\tau(t))) - \sum_{s=1}^n d_{qs}^I(t) f_s^I(m_s^I(t-\tau(t))) \\ &\quad - \sum_{s=1}^n d_{qs}^J(t) f_s^J(m_s^J(t-\tau(t))) - \sum_{s=1}^n d_{qs}^K(t) f_s^K(m_s^K(t-\tau(t))) + I_q^R, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \dot{m}_q^I(t) &= -c_q m_q^I(t) + \sum_{s=1}^n b_{qs}^R(t) f_s^I(m_s^I(t)) + \sum_{s=1}^n b_{qs}^I(t) f_s^R(m_s^R(t)) + \sum_{s=1}^n b_{qs}^J(t) f_s^K(m_s^K(t)) \\ &\quad - \sum_{s=1}^n b_{qs}^K(t) f_s^J(m_s^J(t)) + \sum_{s=1}^n d_{qs}^R(t) f_s^I(m_s^I(t-\tau(t))) + \sum_{s=1}^n d_{qs}^I(t) f_s^R(m_s^R(t-\tau(t))) \\ &\quad + \sum_{s=1}^n d_{qs}^J(t) f_s^K(m_s^K(t-\tau(t))) - \sum_{s=1}^n d_{qs}^K(t) f_s^J(m_s^J(t-\tau(t))) + I_q^I, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \dot{m}_q^J(t) &= -c_q m_q^J(t) + \sum_{s=1}^n b_{qs}^R(t) f_s^J(m_s^J(t)) + \sum_{s=1}^n b_{qs}^J(t) f_s^R(m_s^R(t)) + \sum_{s=1}^n b_{qs}^K(t) f_s^I(m_s^I(t)) \\ &\quad - \sum_{s=1}^n b_{qs}^I(t) f_s^K(m_s^K(t)) + \sum_{s=1}^n d_{qs}^R(t) f_s^J(m_s^J(t-\tau(t))) + \sum_{s=1}^n d_{qs}^J(t) f_s^R(m_s^R(t-\tau(t))) \\ &\quad + \sum_{s=1}^n d_{qs}^K(t) f_s^I(m_s^I(t-\tau(t))) - \sum_{s=1}^n d_{qs}^I(t) f_s^K(m_s^K(t-\tau(t))) + I_q^J, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \dot{m}_q^K(t) &= -c_q m_q^K(t) + \sum_{s=1}^n b_{qs}^R(t) f_s^K(m_s^K(t)) + \sum_{s=1}^n b_{qs}^K(t) f_s^R(m_s^R(t)) + \sum_{s=1}^n b_{qs}^I(t) f_s^J(m_s^J(t)) \\ &\quad - \sum_{s=1}^n b_{qs}^J(t) f_s^I(m_s^I(t)) + \sum_{s=1}^n d_{qs}^R(t) f_s^K(m_s^K(t-\tau(t))) + \sum_{s=1}^n d_{qs}^K(t) f_s^R(m_s^R(t-\tau(t))) \\ &\quad + \sum_{s=1}^n d_{qs}^I(t) f_s^J(m_s^J(t-\tau(t))) - \sum_{s=1}^n d_{qs}^J(t) f_s^I(m_s^I(t-\tau(t))) + I_q^K. \end{aligned} \tag{2.7}$$

Based on the characteristics of the memristor, the quaternion-valued memristive connective weights are defined as

$$b_{qs}(\cdot)=\begin{cases}\acute{b}_{qs}=b^R_{1qs}+b^I_{1qs}i+b^J_{1qs}j+b^K_{1qs}k & |\cdot|<T_q,\\ \grave{b}_{qs}=b^R_{2qs}+b^I_{2qs}i+b^J_{2qs}j+b^K_{2qs}k & |\cdot|\geq T_q,\end{cases}$$

$$d_{qs}(\cdot)=\begin{cases}\acute{d}_{qs}=d^R_{1qs}+d^I_{1qs}i+d^J_{1qs}j+d^K_{1qs}k & |\cdot|<T_q,\\ \grave{d}_{qs}=d^R_{2qs}+d^I_{2qs}i+d^J_{2qs}j+d^K_{2qs}k & |\cdot|\geq T_q,\end{cases}$$

where the switching jump $T_q>0$.

Consider the system (2.1) as the drive system; then, the response system is given as

$$\begin{aligned}\dot{n}_q(t) &= -c_q n_q(t)+\sum_{s=1}^{n} b_{qs}(n_q(t))f_s(n_s(t))+\sum_{s=1}^{n} d_{qs}(n_q(t))f_s(n_s(t-\tau(t)))+I_q+u_q(t),\\ n_q(s) &= \phi_q(s),\ s\in[-\tau,\ 0],\end{aligned}\tag{2.8}$$

where $q=1,2,\cdots,n$; $n(t)=(n_1(t),n_2(t),\cdots,n_n(t))^T\in\mathbb{Q}^n$. $n_q(t)\in\mathbb{Q}$ stand for the state vector of the neuron. The initial condition of system (2.4) is chosen to be $n_q(s)=\phi_q(s)=(\phi_1(s),\ \phi_2(s),\ \cdots,\ \phi_n(s))\in\mathbb{C}^{(1)}([-\tau,\ 0],\ \mathbb{Q}^n)$, $-\tau\leq s\leq 0$. $u_q(t)$ is the controller.

Based on the theory of differential inclusion set-valued map, it yields from Eq (2.8) that

$$\dot{n}_q(t)\in -c_q n_q(t)+\sum_{s=1}^{n} co(b^-_{qs},b^+_{qs})f_s(n_s(t))+\sum_{s=1}^{n} co(d^-_{qs},d^+_{qs})f_s(n_s(t-\tau(t)))+I_q+u_q(t).\tag{2.9}$$

Differential inclusion means that there exist $b^*_{qs}(t)\in co(b^-_{qs},b^+_{qs})$, $d^*_{qs}(t)\in co(d^-_{qs},d^+_{qs})$ such that

$$\dot{n}_q(t)=-c_q n_q(t)+\sum_{s=1}^{n} b^*_{qs}(t)f_s(n_s(t))+\sum_{s=1}^{n} d^*_{qs}(t)f_s(n_s(t-\tau(t)))+I_q+u_q(t).\tag{2.10}$$

Similarly, we divide the network (2.10) into one real part and three imaginary parts, respectively

$$\begin{aligned}\dot{n}^R_q(t) &= -c_q n^R_q(t)+\sum_{s=1}^{n} b^{*R}_{qs}(t)f^R_s(n^R_s(t))-\sum_{s=1}^{n} b^{*I}_{qs}(t)f^I_s(n^I_s(t))-\sum_{s=1}^{n} b^{*J}_{qs}(t)f^J_s(n^J_s(t))\\ &\quad -\sum_{s=1}^{n} b^{*K}_{qs}(t)f^K_s(n^K_s(t))+\sum_{s=1}^{n} d^{*R}_{qs}(t)f^R_s(n^R_s(t-\tau(t)))-\sum_{s=1}^{n} d^{*I}_{qs}(t)f^I_s(n^I_s(t-\tau(t)))\\ &\quad -\sum_{s=1}^{n} d^{*J}_{qs}(t)f^J_s(n^J_s(t-\tau(t)))-\sum_{s=1}^{n} d^{*K}_{qs}(t)f^K_s(n^K_s(t-\tau(t)))+I^R_q+u^R_q(t),\end{aligned}\tag{2.11}$$

$$\begin{aligned}\dot{n}^I_q(t) &= -c_q n^I_q(t)+\sum_{s=1}^{n} b^{*R}_{qs}(t)f^I_s(n^I_s(t))+\sum_{s=1}^{n} b^{*I}_{qs}(t)f^R_s(n^R_s(t))+\sum_{s=1}^{n} b^{*J}_{qs}(t)f^K_s(n^K_s(t))\\ &\quad -\sum_{s=1}^{n} b^{*K}_{qs}(t)f^J_s(n^J_s(t))+\sum_{s=1}^{n} d^{*R}_{qs}(t)f^I_s(n^I_s(t-\tau(t)))+\sum_{s=1}^{n} d^{*I}_{qs}f^R_s(n^R_s(t-\tau(t)))\end{aligned}\tag{2.12}$$

$$+\sum_{s=1}^{n} d_{qs}^{*J}(t) f_s^K(n_s^K(t-\tau(t))) - \sum_{s=1}^{n} d_{qs}^{*K}(t) f_s^J(n_s^J(t-\tau(t))) + I_q^I + u_q^I(t),$$

$$\begin{aligned}
\dot{n}_q^J(t) &= -c_q n_q^J(t) + \sum_{s=1}^{n} b_{qs}^{*R}(t) f_s^J(n_s^J(t)) + \sum_{s=1}^{n} b_{qs}^{*J}(t) f_s^R(n_s^R(t)) + \sum_{s=1}^{n} b_{qs}^{*K}(t) f_s^I(n_s^I(t)) \\
&\quad - \sum_{s=1}^{n} b_{qs}^{*I}(t) f_s^K(n_s^K(t)) + \sum_{s=1}^{n} d_{qs}^{*R}(t) f_s^J(n_s^J(t-\tau(t))) + \sum_{s=1}^{n} d_{qs}^{*J}(t) f_s^R(n_s^R(t-\tau(t))) \\
&\quad + \sum_{s=1}^{n} d_{qs}^{*K}(t) f_s^I(n_s^I(t-\tau(t))) - \sum_{s=1}^{n} d_{qs}^{*I}(t) f_s^K(n_s^K(t-\tau(t))) + I_q^J + u_q^J(t),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\dot{n}_q^K(t) &= -c_q n_q^K(t) + \sum_{s=1}^{n} b_{qs}^{*R}(t) f_s^K(n_s^K(t)) + \sum_{s=1}^{n} b_{qs}^{*K}(t) f_s^R(n_s^R(t)) + \sum_{s=1}^{n} b_{qs}^{*I}(t) f_s^J(n_s^J(t)) \\
&\quad - \sum_{s=1}^{n} b_{qs}^{*J}(t) f_s^I(n_s^I(t)) + \sum_{s=1}^{n} d_{qs}^{*R}(t) f_s^K(n_s^K(t-\tau(t))) + \sum_{s=1}^{n} d_{qs}^{*K}(t) f_s^R(n_s^R(t-\tau(t))) \\
&\quad + \sum_{s=1}^{n} d_{qs}^{*I}(t) f_s^J(n_s^J(t-\tau(t))) - \sum_{s=1}^{n} d_{qs}^{*J}(t) f_s^I(n_s^I(t-\tau(t))) + I_q^K + u_q^K(t).
\end{aligned} \tag{2.14}$$

Let $e(t) = (e_1(t), e_2(t), \cdots, e_n(t))$ be the synchronization error. The synchronization error between the drive system (2.2) and the response system (2.9) is defined as $e_q(t) = n_q(t) - \beta m_q(t)$, expressed as one real part and three imaginary parts

$$\begin{cases}
e_q^R(t) = n_q^R(t) - \beta m_q^R(t), \\
e_q^I(t) = n_q^I(t) - \beta m_q^I(t), \\
e_q^J(t) = n_q^J(t) - \beta m_q^J(t), \\
e_q^K(t) = n_q^K(t) - \beta m_q^K(t),
\end{cases} \tag{2.15}$$

with the initial value $\phi_q(s) - \psi_q(s)$, $-\tau \le s \le 0$.

In order to synchronize the drive system and the response system, we choose the following controller

$$\begin{aligned}
u_q^R(t) &= -k_i(n_q^R(t) - \beta m_q^R(t)) + \sum_{s=1}^{n} \Big(b_{qs}^R(t) \beta f_s^R(m_s^R(t)) - b_{qs}^{*R}(t) f_s^R(\beta m_s^R(t)) \Big) \\
&\quad + \sum_{s=1}^{n} \Big(b_{qs}^{*I}(t) f_s^I(\beta m_s^I(t)) - b_{qs}^I(t) \beta f_s^I(m_s^I(t)) \Big) + \sum_{s=1}^{n} \Big(b_{qs}^{*J}(t) f_s^J(\beta m_s^J(t)) - b_{qs}^J(t) \beta f_s^J(m_s^J(t)) \Big) \\
&\quad + \sum_{s=1}^{n} \Big(b_{qs}^{*K}(t) f_s^K(\beta m_s^K(t)) - b_{qs}^K(t) \beta f_s^K(m_s^K(t)) \Big) + \sum_{s=1}^{n} \Big(d_{qs}^R(t) \beta f_s^R(m_s^R(t-\tau(t))) \\
&\quad - d_{qs}^{*R}(t) f_s^R(\beta m_s^R(t-\tau(t))) \Big) + \sum_{s=1}^{n} \Big(d_{qs}^{*I}(t) f_s^I(\beta m_s^I(t-\tau(t))) \\
&\quad - d_{qs}^I(t) \beta f_s^I(m_s^I(t-\tau(t))) \Big) + \sum_{s=1}^{n} \Big(d_{qs}^{*J}(t) f_s^J(\beta m_s^J(t-\tau(t)))
\end{aligned}$$

$$
\begin{aligned}
&-d^{J}_{qs}(t)\beta f^{J}_{s}(m^{J}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{*K}_{qs}(t)f^{K}_{s}(\beta m^{K}_{s}(t-\tau(t)))\\
&-d^{K}_{qs}(t)\beta f^{K}_{s}(m^{K}_{s}(t-\tau(t)))\Big)+(1-\beta)I^{R}_{q},\\
u^{I}_{q}(t) \;=\;& -k_i(n^{I}_{q}(t)-\beta m^{I}_{q}(t))+\sum_{s=1}^{n}\Big(b^{R}_{qs}(t)\beta(f^{I}_{s}(m^{I}_{s}(t))-b^{*R}_{qs}(t)f^{I}_{s}(\beta m^{I}_{s}(t))\Big)\\
&+\sum_{s=1}^{n}\Big(b^{I}_{qs}(t)\beta(f^{R}_{s}(m^{R}_{s}(t))-b^{*I}_{qs}(t)f^{R}_{s}(\beta m^{R}_{s}(t))\Big)+\sum_{s=1}^{n}\Big(b^{*K}_{qs}(t)(f^{J}_{s}(\beta m^{J}_{s}(t))-b^{K}_{qs}(t)\beta f^{J}_{s}(m^{J}_{s}(t))\Big)\\
&+\sum_{s=1}^{n}\Big(b^{J}_{qs}(t)\beta(f^{K}_{s}(m^{K}_{s}(t))-b^{*J}_{qs}(t)f^{K}_{s}(\beta m^{K}_{s}(t))\Big)+\sum_{s=1}^{n}\Big(d^{R}_{qs}(t)\beta(f^{I}_{s}(m^{I}_{s}(t-\tau(t)))\\
&-d^{*R}_{qs}(t)f^{I}_{s}(\beta m^{I}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{I}_{qs}(t)\beta(f^{R}_{s}(m^{R}_{s}(t-\tau(t)))\\
&-d^{*I}_{qs}(t)f^{R}_{s}(\beta m^{R}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{*K}_{qs}(t)(f^{J}_{s}(\beta m^{J}_{s}(t-\tau(t)))\\
&-d^{K}_{qs}(t)\beta f^{J}_{s}(m^{J}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{J}_{qs}(t)\beta(f^{K}_{s}(m^{K}_{s}(t-\tau(t)))\\
&-d^{*J}_{qs}(t)f^{K}_{s}(\beta m^{K}_{s}(t-\tau(t)))\Big)+(1-\beta)I^{I}_{q},\\
u^{J}_{q}(t) \;=\;& -k_i(n^{J}_{q}(t)-\beta m^{J}_{q}(t))+\sum_{s=1}^{n}\Big(b^{R}_{qs}(t)\beta(f^{J}_{s}(m^{J}_{s}(t))-b^{*R}_{qs}(t)f^{J}_{s}(\beta m^{J}_{s}(t))\Big)\\
&+\sum_{s=1}^{n}\Big(b^{*I}_{qs}(t)(f^{K}_{s}(\beta m^{K}_{s}(t))-b^{I}_{qs}(t)\beta f^{K}_{s}(m^{K}_{s}(t))\Big)+\sum_{s=1}^{n}\Big(b^{K}_{qs}(t)\beta(f^{I}_{s}(m^{I}_{s}(t))-b^{*K}_{qs}(t)f^{I}_{s}(\beta m^{I}_{s}(t))\Big)\\
&+\sum_{s=1}^{n}\Big(b^{J}_{qs}(t)\beta(f^{R}_{s}(m^{R}_{s}(t))-b^{*J}_{qs}(t)f^{R}_{s}(\beta m^{R}_{s}(t))\Big)+\sum_{s=1}^{n}\Big(d^{R}_{qs}(t)\beta(f^{J}_{s}(m^{J}_{s}(t-\tau(t)))\\
&-d^{*R}_{qs}(t)f^{J}_{s}(\beta m^{J}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{*I}_{qs}(t)(f^{K}_{s}(\beta m^{K}_{s}(t-\tau(t)))\\
&-d^{I}_{qs}(t)\beta f^{K}_{s}(m^{K}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{K}_{qs}(t)\beta(f^{I}_{s}(m^{I}_{s}(t-\tau(t)))\\
&-d^{*K}_{qs}(t)f^{I}_{s}(\beta m^{I}_{s}(t-\tau(t)))\Big)+\sum_{s=1}^{n}\Big(d^{J}_{qs}(t)\beta(f^{R}_{s}(m^{R}_{s}(t-\tau(t)))\\
&-d^{*J}_{qs}(t)f^{R}_{s}(\beta m^{R}_{s}(t-\tau(t)))\Big)+(1-\beta)I^{J}_{q},\\
u^{K}_{q}(t) \;=\;& -k_i(n^{K}_{q}(t)-\beta m^{K}_{q}(t))+\sum_{s=1}^{n}\Big(b^{K}_{qs}(t)\beta(f^{R}_{s}(m^{R}_{s}(t))-b^{*K}_{qs}(t)f^{R}_{s}(\beta m^{R}_{s}(t))\Big)
\end{aligned}
$$

$$+\sum_{s=1}^{n}\Big(b_{qs}^{R}(t)\beta(f_s^K(m_s^K(t)) - b_{qs}^{*R}(t)f_s^K(\beta m_s^K(t))\Big) + \sum_{s=1}^{n}\Big(b_{qs}^{I}(t)\beta(f_s^J(m_s^J(t)) - b_{qs}^{*I}(t)f_s^J(\beta m_s^J(t))\Big)$$
$$+\sum_{s=1}^{n}\Big(b_{qs}^{*J}(t)(f_s^I(\beta m_s^I(t)) - b_{qs}^{J}(t)\beta f_s^I(m_s^I(t))\Big) + \sum_{s=1}^{n}\Big(d_{qs}^{K}(t)\beta(f_s^R(m_s^R(t-\tau(t))) \tag{2.16}$$
$$-d_{qs}^{*K}(t)f_s^R(\beta m_s^R(t-\tau(t)))\Big) + \sum_{s=1}^{n}\Big(d_{qs}^{R}(t)\beta(f_s^K(m_s^K(t-\tau(t)))$$
$$-d_{qs}^{*R}(t)f_s^K(\beta m_s^K(t-\tau(t)))\Big) + \sum_{s=1}^{n}\Big(d_{qs}^{I}(t)\beta(f_s^J(m_s^J(t-\tau(t)))$$
$$-d_{qs}^{*I}(t)f_s^J(\beta m_s^J(t-\tau(t)))\Big) + \sum_{s=1}^{n}\Big(d_{qs}^{*J}(t)(f_s^I(\beta m_s^I(t-\tau(t)))$$
$$-d_{qs}^{J}(t)\beta f_s^I(m_s^I(t-\tau(t)))\Big) + (1-\beta)I_q^K.$$

Let $e_q(t) = e_q^R(t) + e_q^I(t)i + e_q^J(t)j + e_q^K(t)k$; then, according to the controller (2.16), the error system (2.15) can be separated into four real parts as below

$$\begin{aligned}\dot{e}_q^R(t) = & -c_q e_q^R(t) - k_q e_q^R(t)\sum_{s=1}^{n} b_{qs}^{*R}(t)\tilde{f}_s^R(e_s^R(t)) - \sum_{s=1}^{n} b_{qs}^{*I}(t)\tilde{f}_s^I(e_s^I(t)) \\ & - \sum_{s=1}^{n} b_{qs}^{*J}(t)\tilde{f}_s^J(e_s^J(t)) - \sum_{s=1}^{n} b_{qs}^{*K}(t)\tilde{f}_s^K(e_s^K(t)) + \sum_{s=1}^{n} d_{qs}^{*R}(t)\tilde{f}_s^R(e_s^R(t-\tau(t))) \\ & - \sum_{s=1}^{n} d_{qs}^{*I}(t)\tilde{f}_s^I(e_s^I(t-\tau(t))) - \sum_{s=1}^{n} d_{qs}^{*J}(t)\tilde{f}_s^J(e_s^J(t-\tau(t))) \\ & - \sum_{s=1}^{n} d_{qs}^{*K}(t)\tilde{f}_s^K(e_s^K(t-\tau(t))),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\dot{e}_q^I(t) = & -c_q e_q^I(t) - k_q e_q^I(t) + \sum_{s=1}^{n} b_{qs}^{*R}(t)\tilde{f}_s^I(e_s^I(t)) + \sum_{s=1}^{n} b_{qs}^{*I}(t)\tilde{f}_s^R(e_s^R(t)) \\ & - \sum_{s=1}^{n} b_{qs}^{*K}(t)\tilde{f}_s^J(e_s^J(t)) + \sum_{s=1}^{n} b_{qs}^{*J}(t)\tilde{f}_s^K(e_s^K(t)) + \sum_{s=1}^{n} d_{qs}^{*R}(t)\tilde{f}_s^I(e_s^I(t-\tau(t))) \\ & + \sum_{s=1}^{n} d_{qs}^{*I}(t)\tilde{f}_s^R(e_s^R(t-\tau(t))) - \sum_{s=1}^{n} d_{qs}^{*K}(t)\tilde{f}_s^J(e_s^J(t-\tau(t))) \\ & + \sum_{s=1}^{n} d_{qs}^{*J}(t)\tilde{f}_s^K(e_s^K(t-\tau(t))),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\dot{e}_q^J(t) = & -c_q e_q^J(t) - k_q e_q^J(t) + \sum_{s=1}^{n} b_{qs}^{*R}(t)\tilde{f}_s^J(e_s^J(t)) - \sum_{s=1}^{n} b_{qs}^{*I}(t)\tilde{f}_s^K(e_s^K(t)) \\ & + \sum_{s=1}^{n} b_{qs}^{*K}(t)\tilde{f}_s^I(e_s^I(t)) + \sum_{s=1}^{n} b_{qs}^{*J}(t)\tilde{f}_s^R(e_s^R(t)) + \sum_{s=1}^{n} d_{qs}^{*R}(t)\tilde{f}_s^J(e_s^J(t-\tau(t)))\end{aligned} \tag{2.19}$$

$$
\begin{aligned}
& -\sum_{s=1}^{n} d_{qs}^{*I}(t)\tilde{f}_s^K(e_s^K(t-\tau(t))) + \sum_{s=1}^{n} d_{qs}^{*K}(t)\tilde{f}_s^I(e_s^I(t-\tau(t))) \\
& + \sum_{s=1}^{n} d_{qs}^{*J}(t)\tilde{f}_s^R(e_s^R(t-\tau(t))), \\
\dot{e}_q^K(t) \; = \; & -c_q e_q^K(t) - k_q e_q^K(t) + \sum_{s=1}^{n} b_{qs}^{*K}(t)\tilde{f}_s^R(e_s^R(t)) + \sum_{s=1}^{n} b_{qs}^{*R}(t)\tilde{f}_s^K(e_s^K(t)) \\
& + \sum_{s=1}^{n} b_{qs}^{*I}(t)\tilde{f}_s^J(e_s^J(t)) - \sum_{s=1}^{n} b_{qs}^{*J}(t)\tilde{f}_s^I(e_s^I(t)) + \sum_{s=1}^{n} d_{qs}^{*K}(t)\tilde{f}_s^R(e_s^R(t-\tau(t))) \\
& + \sum_{s=1}^{n} d_{qs}^{*R}(t)\tilde{f}_s^K(e_s^K(t-\tau(t))) + \sum_{s=1}^{n} d_{qs}^{*I}(t)\tilde{f}_s^J(e_s^J(t-\tau(t))) \\
& - \sum_{s=1}^{n} d_{qs}^{*J}(t)\tilde{f}_s^I(e_s^I(t-\tau(t))),
\end{aligned} \tag{2.20}
$$

where $\tilde{f}_s^\epsilon(e_s^\epsilon(t)) = f_s(n_s^\epsilon(t)) - f_s(\beta m_s^\epsilon(t))$, $\tilde{f}_s^\epsilon(e_s^\epsilon(t-\tau(t))) = f_s(n_s^\epsilon(t-\tau(t))) - f_s(\beta m_s^\epsilon(t-\tau(t)))$, $\epsilon = R,\ I,\ J,\ K$.

The following notations will be used:

$|b_{qs}^R| = \sup_{t\geq 0}|b_{qs}^{*R}(t)|$, $|b_{qs}^I| = \sup_{t\geq 0}|b_{qs}^{*I}(t)|$, $|b_{qs}^J| = \sup_{t\geq 0}|b_{qs}^{*J}(t)|$, $|b_{qs}^K| = \sup_{t\geq 0}|b_{qs}^{*K}(t)|$,

$|d_{qs}^R| = \sup_{t\geq 0}|d_{qs}^{*R}(t)|$, $|d_{qs}^I| = \sup_{t\geq 0}|d_{qs}^{*I}(t)|$, $|d_{qs}^J| = \sup_{t\geq 0}|d_{qs}^{*J}(t)|$, $|d_{qs}^K| = \sup_{t\geq 0}|d_{qs}^{*K}(t)|$.

Before deriving the result, the definitions and lemmas are given to facilitate the subsequent derivation.

Definition 2.1. [38] The driving network (2.1) and the response network (2.8) are said to achieve projection synchronization if

$$\lim_{t\to\infty}\|n_q(t) - \beta m_q(t)\| = 0, \qquad q = 1, 2, \cdots, n,$$

where $\beta \in \mathbb{R}$ is a nonzero constant.

Remark 2. When the projection factor $\beta = 1$, complete synchronization is achieved. When the projection factor $\beta = -1$, anti-synchronization is obtained.

Lemma 2.1. [28] Let $m > 0$, $n > 0$, $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Then, the following inequality holds

$$mn \leq \frac{1}{r}m^r + \frac{1}{s}n^s.$$

Lemma 2.2. [39] Suppose that function $V(t)$ is non-negative when $t \in (-\tau, \infty)$ and satisfies the following inequality

$$\dot{V}(t) \leq -aV(t) - bV(t-\tau(t)), t \geq 0,$$

where a and b are positive constants with $a > b$. Then,

$$V(t) \leq \sup_{-\tau\leq s\leq 0} V(s)e^{-rt},$$

where r is the unique positive solution of the following equation

$$a - be^{-r\tau} = r.$$

3. Main results

Theorem 3.1. Under the controller (2.16), if Assumptions 1 and 2 hold, and

$$\lambda > \zeta > 0. \tag{3.1}$$

Then the projection synchronization of quaternion-valued memristor-based neural networks (2.1) and (2.8) is obtained, where $\lambda = min\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\}$, $\zeta = max\{\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4\}$ and

$$\begin{aligned}
\lambda_1 &= \min_{1\le q\le n}\Big\{rc_q + rk_q - \sum_{s=1}^{n}(l_s^R|b_{qs}^R| + l_s^R|b_{qs}^I| + l_s^R|b_{qs}^J| + l_s^R|b_{qs}^K| \\
&\quad +(r-1)l_s^R|b_{qs}^R| + (r-1)l_s^I|b_{qs}^I| + (r-1)l_s^J|b_{qs}^J| + (r-1)l_s^K|b_{qs}^K| \\
&\quad +(r-1)l_s^R|d_{qs}^R| + (r-1)l_s^I|d_{qs}^I| + (r-1)l_s^J|d_{qs}^J| + (r-1)l_s^K|d_{qs}^K|)\Big\}, \\
\lambda_2 &= \min_{1\le q\le n}\Big\{rc_q + rk_q - \sum_{s=1}^{n}(l_s^I|b_{qs}^R| + l_s^I|b_{qs}^I| + l_s^I|b_{qs}^J| + l_s^I|b_{qs}^K| \\
&\quad +(r-1)l_s^I|b_{qs}^R| + (r-1)l_s^R|b_{qs}^I| + (r-1)l_s^K|b_{qs}^J| + (r-1)l_s^J|b_{qs}^K| \\
&\quad +(r-1)l_s^I|d_{qs}^R| + (r-1)l_s^R|d_{qs}^I| + (r-1)l_s^K|d_{qs}^J| + (r-1)l_s^J|d_{qs}^K|)\Big\}, \\
\lambda_3 &= \min_{1\le q\le n}\Big\{rc_q + rk_q - \sum_{s=1}^{n}(l_s^J|b_{qs}^R| + l_s^J|b_{qs}^I| + l_s^J|b_{qs}^J| + l_s^J|b_{qs}^K| \\
&\quad +(r-1)l_s^J|b_{qs}^R| + (r-1)l_s^R|b_{qs}^J| + (r-1)l_s^K|b_{qs}^I| + (r-1)l_s^I|b_{qs}^K| \\
&\quad +(r-1)l_s^J|d_{qs}^R| + (r-1)l_s^R|d_{qs}^J| + (r-1)l_s^K|d_{qs}^I| + (r-1)l_s^I|d_{qs}^K|)\Big\}, \\
\lambda_4 &= \min_{1\le q\le n}\Big\{rc_q + rk_q - \sum_{s=1}^{n}(l_s^K|b_{qs}^R| + l_s^K|b_{qs}^I| + l_s^K|b_{qs}^J| + l_s^K|b_{qs}^K| \\
&\quad +(r-1)l_s^K|b_{qs}^R| + (r-1)l_s^R|b_{qs}^K| + (r-1)l_s^J|b_{qs}^I| + (r-1)l_s^I|b_{qs}^J| \\
&\quad +(r-1)l_s^K|d_{qs}^R + (r-1)l_s^R|d_{qs}^K| + (r-1)l_s^J|d_{qs}^I| + (r-1)l_s^I|d_{qs}^J|)\Big\}.
\end{aligned}$$

$$\begin{aligned}
\zeta_1 &= \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^R|d_{qs}^R| + l_s^R|d_{qs}^I| + l_s^R|d_{qs}^J| + l_s^R|d_{qs}^K|), \\
\zeta_2 &= \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^I|d_{qs}^R| + l_s^I|d_{qs}^I| + l_s^I|d_{qs}^J| + l_s^I|d_{qs}^K|), \\
\zeta_3 &= \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^J|d_{qs}^R| + l_s^J|d_{qs}^I| + l_s^J|d_{qs}^J| + l_s^J|d_{qs}^K|), \\
\zeta_1 &= \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^K|d_{qs}^R| + l_s^K|d_{qs}^I| + l_s^K|d_{qs}^J| + l_s^K|d_{qs}^K|).
\end{aligned}$$

Proof. Considering the following Lyapunov function

$$V(e(t)) = \frac{1}{r}\sum_{q=1}^{n}|e_q^R(t)|^r + \frac{1}{r}\sum_{q=1}^{n}|e_q^I(t)|^r + \frac{1}{r}\sum_{q=1}^{n}|e_q^J(t)|^r + \frac{1}{r}\sum_{q=1}^{n}|e_q^K(t)|^r. \tag{3.2}$$

Computing the time derivative of $V(e(t))$ along the trajectory (2.17)–(2.20), from Lemma 2.1, we have

$$\begin{aligned}
\dot{V}(e(t) \leq\ & \sum_{q=1}^{n}|e_q^R(t)|^{r-1}\Big[-c_q|e_q^R(t)| + \sum_{s=1}^{n}|b_{qs}^{*R}(t)||\tilde{f}_s^R(e_s^R(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*I}(t)||\tilde{f}_s^I(e_s^I(t))| + \sum_{s=1}^{n}|b_{qs}^{*J}(t)||\tilde{f}_s^J(e_s^J(t))| + \sum_{s=1}^{n}|b_{qs}^{*K}(t)||\tilde{f}_s^K(e_s^K(t))| \\
&+ \sum_{s=1}^{n}|d_{qs}^{*R}(t)||\tilde{f}_s^R(e_s^R(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*I}(t)||\tilde{f}_s^I(e_s^I(t-\tau(t)))| \\
&+ \sum_{s=1}^{n}|d_{qs}^{*J}(t)||\tilde{f}_s^J(e_s^J(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*K}(t)||\tilde{f}_s^K(e_s^K(t-\tau(t)))| - k_q|e_q^R(t)|\Big] \\
&+ \sum_{q=1}^{n}|e_q^I(t)|^{r-1}\Big[-c_q|e_q^I(t)| + \sum_{s=1}^{n}|b_{qs}^{*R}(t)||\tilde{f}_s^I(e_s^I(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*I}(t)||\tilde{f}_s^R(e_s^R(t))| + \sum_{s=1}^{n}|b_{qs}^{*K}(t)||\tilde{f}_s^J(e_s^J(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*J}(t)||\tilde{f}_s^K(e_s^K(t))| + \sum_{s=1}^{n}|d_{qs}^{*R}(t)||\tilde{f}_s^I(e_s^I(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*I}(t)||\tilde{f}_s^R(e_s^R(t-\tau(t)))| \\
&+ \sum_{s=1}^{n}|d_{qs}^{*K}(t)||\tilde{f}_s^J(e_s^J(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*J}(t)||\tilde{f}_s^K(e_s^K(t-\tau(t)))| - k_q|e_q^I(t)|\Big] \qquad (3.3) \\
&+ \sum_{q=1}^{n}|e_q^J(t)|^{r-1}\Big[-c_q|e_q^J(t)| + \sum_{s=1}^{n}|b_{qs}^{*R}(t)||\tilde{f}_s^J(e_s^J(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*I}(t)||\tilde{f}_s^K(e_s^K(t))| + \sum_{s=1}^{n}|b_{qs}^{*K}(t)||\tilde{f}_s^I(e_s^I(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*J}(t)||\tilde{f}_s^R(e_s^R(t))| + \sum_{s=1}^{n}|d_{qs}^{*R}(t)||\tilde{f}_s^J(e_s^J(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*I}(t)||\tilde{f}_s^K(e_s^K(t-\tau(t)))| \\
&+ \sum_{s=1}^{n}|d_{qs}^{*K}(t)||\tilde{f}_s^I(e_s^I(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*J}(t)||\tilde{f}_s^R(e_s^R(t-\tau(t)))| - k_q|e_q^J(t)|\Big] \\
&+ \sum_{q=1}^{n}|e_q^K(t)|^{r-1}\Big[-c_q|e_q^K(t)| + \sum_{s=1}^{n}|b_{qs}^{*R}(t)||\tilde{f}_s^K(e_s^K(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*K}(t)||\tilde{f}_s^R(e_s^R(t))| + \sum_{s=1}^{n}|b_{qs}^{*I}(t)||\tilde{f}_s^J(e_s^J(t))| \\
&+ \sum_{s=1}^{n}|b_{qs}^{*J}(t)||\tilde{f}_s^I(e_s^I(t))| + \sum_{s=1}^{n}|d_{qs}^{*R}(t)||\tilde{f}_s^K(e_s^K(t-\tau(t)))| + \sum_{s=1}^{n}|d_{qs}^{*I}(t)||\tilde{f}_s^J(e_s^J(t-\tau(t)))|
\end{aligned}$$

$$+\sum_{s=1}^{n}|d_{qs}^{*K}(t)||\tilde{f}_s^{R}(e_s^{R}(t-\tau(t)))|+\sum_{s=1}^{n}|d_{qs}^{*J}(t)||\tilde{f}_s^{I}(e_s^{I}(t-\tau(t)))|-k_q|e_q^{K}(t)|\Big].$$

According to Assumption 2 and the notations defined previously, we can get

$$\begin{aligned}
\dot{V}(e(t) \leq\ & \sum_{q=1}^{n}|e_q^{R}(t)|^{r-1}\Big[-c_q|e_q^{R}(t)|+\sum_{s=1}^{n}|b_{qs}^{R}|l_s^{R}|e_s^{R}(t)| \\
&+\sum_{s=1}^{n}|b_{qs}^{I}|l_s^{I}|e_s^{I}(t)|+\sum_{s=1}^{n}|b_{qs}^{J}|l_s^{J}|e_s^{J}(t)|+\sum_{s=1}^{n}|b_{qs}^{K}|l_s^{K}|e_s^{K}(t)|+\sum_{s=1}^{n}|d_{qs}^{R}|l_s^{R}|e_s^{R}(t-\tau(t))| \\
&+\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{I}|e_s^{I}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{J}|e_s^{J}(t-\tau(t))| \\
&+\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{K}|e_s^{K}(t-\tau(t))|-k_q|e_q^{R}(t)|\Big]+\sum_{q=1}^{n}|e_q^{I}(t)|^{r-1}\Big[-c_q|e_q^{I}(t)|+\sum_{s=1}^{n}|b_{qs}^{R}|l_s^{I}|e_s^{I}(t)| \\
&+\sum_{s=1}^{n}|b_{qs}^{I}|l_s^{R}|e_s^{R}(t)|+\sum_{s=1}^{n}|b_{qs}^{K}|l_s^{J}|e_s^{J}(t)|+\sum_{s=1}^{n}|b_{qs}^{J}|l_s^{K}|e_s^{K}(t)|+\sum_{s=1}^{n}|d_{qs}^{R}|l_s^{I}|e_s^{I}(t-\tau(t))| \\
&+\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{R}|e_s^{R}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{K}|e_s^{K}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{J}|e_s^{J}(t-\tau(t))|-k_q|e_q^{I}(t)|\Big] \\
&+\sum_{q=1}^{n}|e_q^{J}(t)|^{r-1}\Big[-c_q|e_q^{J}(t)|+\sum_{s=1}^{n}|b_{qs}^{R}|l_s^{J}|e_s^{J}(t)|+\sum_{s=1}^{n}|b_{qs}^{I}|l_s^{K}|e_s^{K}(t)|+\sum_{s=1}^{n}|b_{qs}^{K}|l_s^{I}|e_s^{I}(t)| \qquad (3.4)\\
&+\sum_{s=1}^{n}|b_{qs}^{J}|l_s^{R}|e_s^{R}(t)|+\sum_{s=1}^{n}|d_{qs}^{R}|l_s^{J}|e_s^{J}(t-\tau(t))| \\
&+\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{K}|e_s^{K}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{I}|e_s^{I}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{R}|e_s^{R}(t-\tau(t))|-k_q|e_q^{J}(t)|\Big] \\
&+\sum_{q=1}^{n}|e_q^{K}(t)|^{r-1}\Big[-c_q|e_q^{K}(t)|+\sum_{s=1}^{n}|b_{qs}^{R}|l_s^{K}|e_s^{K}(t)|+\sum_{s=1}^{n}|b_{qs}^{K}|l_s^{R}|e_s^{R}(t)|+\sum_{s=1}^{n}|b_{qs}^{I}|l_s^{J}|e_s^{J}(t)| \\
&+\sum_{s=1}^{n}|b_{qs}^{J}|l_s^{I}|e_s^{I}(t)|+\sum_{s=1}^{n}|d_{qs}^{R}|l_s^{K}|e_s^{K}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{I}|e_s^{I}(t-\tau(t))|-k_q|e_q^{K}(t)| \\
&+\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{R}|e_s^{R}(t-\tau(t))|+\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{J}|e_s^{J}(t-\tau(t))|\Big].
\end{aligned}$$

Then, according to Lemma 2.2, we have

$$|e_q^{R}(t)|^{r-1}|e_s^{R}(t)|\leq\frac{1}{r}|e_s^{R}(t)|^{r}+\frac{r-1}{r}|e_q^{R}(t)|^{r},\ \ |e_q^{R}(t)|^{r-1}|e_s^{I}(t)|\leq\frac{1}{r}|e_s^{I}(t)|^{r}+\frac{r-1}{r}|e_q^{R}(t)|^{r},$$
$$|e_q^{R}(t)|^{r-1}|e_s^{J}(t)|\leq\frac{1}{r}|e_s^{J}(t)|^{r}+\frac{r-1}{r}|e_q^{R}(t)|^{r},\ \ |e_q^{R}(t)|^{r-1}|e_s^{K}(t)|\leq\frac{1}{r}|e_s^{K}(t)|^{r}+\frac{r-1}{r}|e_q^{R}(t)|^{r},$$
$$|e_q^{R}(t)|^{r-1}|e_s^{R}(t-\tau(t))|\leq\frac{1}{r}|e_s^{R}(t-\tau(t))|^{r}+\frac{r-1}{r}|e_q^{R}(t)|^{r},$$

$$|e_q^R(t)|^{r-1}|e_s^I(t-\tau(t))| \le \frac{1}{r}|e_s^I(t-\tau(t))|^r + \frac{r-1}{r}|e_q^R(t)|^r,$$
$$|e_q^R(t)|^{r-1}|e_s^J(t-\tau(t))| \le \frac{1}{r}|e_s^J(t-\tau(t))|^r + \frac{r-1}{r}|e_q^R(t)|^r,$$
$$|e_q^R(t)|^{r-1}|e_s^K(t-\tau(t))| \le \frac{1}{r}|e_s^K(t-\tau(t))|^r + \frac{r-1}{r}|e_q^R(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^I(t)| \le \frac{1}{r}|e_s^I(t)|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^R(t)| \le \frac{1}{r}|e_s^R(t)|^r + \frac{r-1}{r}|e_q^I(t)|^r, \quad |e_q^I(t)|^{r-1}|e_s^J(t)| \le \frac{1}{r}|e_s^J(t)|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^K(t)| \le \frac{1}{r}|e_s^K(t)|^r + \frac{r-1}{r}|e_q^I(t)|^r, \quad |e_q^I(t)|^{r-1}|e_s^R(t-\tau(t))| \le \frac{1}{r}|e_s^R(t-\tau(t))|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^I(t-\tau(t))| \le \frac{1}{r}|e_s^I(t-\tau(t))|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^J(t-\tau(t))| \le \frac{1}{r}|e_s^J(t-\tau(t))|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^I(t)|^{r-1}|e_s^K(t-\tau(t))| \le \frac{1}{r}|e_s^K(t-\tau(t))|^r + \frac{r-1}{r}|e_q^I(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^I(t)| \le \frac{1}{r}|e_s^I(t)|^r + \frac{r-1}{r}|e_q^J(t)|^r, \quad |e_q^J(t)|^{r-1}|e_s^R(t)| \le \frac{1}{r}|e_s^R(t)|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^J(t)| \le \frac{1}{r}|e_s^J(t)|^r + \frac{r-1}{r}|e_q^J(t)|^r, \quad |e_q^J(t)|^{r-1}|e_s^K(t)| \le \frac{1}{r}|e_s^K(t)|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^R(t-\tau(t))| \le \frac{1}{r}|e_s^R(t-\tau(t))|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^I(t-\tau(t))| \le \frac{1}{r}|e_s^I(t-\tau(t))|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^J(t-\tau(t))| \le \frac{1}{r}|e_s^J(t-\tau(t))|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^J(t)|^{r-1}|e_s^K(t-\tau(t))| \le \frac{1}{r}|e_s^K(t-\tau(t))|^r + \frac{r-1}{r}|e_q^J(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^I(t)| \le \frac{1}{r}|e_s^I(t)|^r + \frac{r-1}{r}|e_q^K(t)|^r, \quad |e_q^K(t)|^{r-1}|e_s^R(t)| \le \frac{1}{r}|e_s^R(t)|^r + \frac{r-1}{r}|e_q^K(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^J(t)| \le \frac{1}{r}|e_s^J(t)|^r + \frac{r-1}{r}|e_q^K(t)|^r, \quad |e_q^K(t)|^{r-1}|e_s^K(t)| \le \frac{1}{r}|e_s^K(t)|^r + \frac{r-1}{r}|e_q^K(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^R(t-\tau(t))| \le \frac{1}{r}|e_s^R(t-\tau(t))|^r + \frac{r-1}{r}|e_q^K(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^I(t-\tau(t))| \le \frac{1}{r}|e_s^I(t-\tau(t))|^r + \frac{r-1}{r}|e_q^K(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^J(t-\tau(t))| \le \frac{1}{r}|e_s^J(t-\tau(t))|^r + \frac{r-1}{r}|e_q^K(t)|^r,$$
$$|e_q^K(t)|^{r-1}|e_s^K(t-\tau(t))| \le \frac{1}{r}|e_s^K(t-\tau(t))|^r + \frac{r-1}{r}|e_q^K(t)|^r.$$

Therefore, one has

$$
\begin{aligned}
\dot{V}(e(t) \quad \leq \quad & -\sum_{q=1}^{n}(c_q+k_q)|e_q^R(t)|^r-\sum_{q=1}^{n}(c_q+k_q)|e_q^I(t)|^r-\sum_{q=1}^{n}(c_q+k_q)|e_q^J(t)|^r-\sum_{q=1}^{n}(c_q+k_q)|e_q^K(t)|^r \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^R|l_s^R(\frac{1}{r}|e_s^R(t)|^r+\frac{r-1}{r}|e_q^R(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^I|l_s^I(\frac{1}{r}|e_s^I(t)|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^J|l_s^J(\frac{1}{r}|e_s^J(t)|^r+\frac{r-1}{r}|e_q^R(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^K|l_s^K(\frac{1}{r}|e_s^K(t)|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^R|l_s^R(\frac{1}{r}|e_s^R(t-\tau(t))|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^I|l_s^I(\frac{1}{r}|e_s^I(t-\tau(t))|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^J|l_s^J(\frac{1}{r}|e_s^J(t-\tau(t))|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^K|l_s^K(\frac{1}{r}|e_s^K(t-\tau(t))|^r+\frac{r-1}{r}|e_q^R(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^R|l_s^I(\frac{1}{r}|e_s^I(t)|^r+\frac{r-1}{r}|e_q^I(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^I|l_s^R(\frac{1}{r}|e_s^R(t)|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^J|l_s^K(\frac{1}{r}|e_s^K(t)|^r+\frac{r-1}{r}|e_q^I(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^K|l_s^J(\frac{1}{r}|e_s^J(t)|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^R|l_s^I(\frac{1}{r}|e_s^I(t-\tau(t))|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^I|l_s^R(\frac{1}{r}|e_s^R(t-\tau(t))|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^K|l_s^J(\frac{1}{r}|e_s^J(t-\tau(t))|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^J|l_s^K(\frac{1}{r}|e_s^K(t-\tau(t))|^r+\frac{r-1}{r}|e_q^I(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^R|l_s^J(\frac{1}{r}|e_s^J(t)|^r+\frac{r-1}{r}|e_q^J(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^I|l_s^K(\frac{1}{r}|e_s^K(t)|^r+\frac{r-1}{r}|e_q^J(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^K|l_s^I(\frac{1}{r}|e_s^I(t)|^r+\frac{r-1}{r}|e_q^J(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^J|l_s^R(\frac{1}{r}|e_s^R(t)|^r+\frac{r-1}{r}|e_q^J(t)|^r) \\
&+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^R|l_s^J(\frac{1}{r}|e_s^J(t-\tau(t))|^r+\frac{r-1}{r}|e_q^J(t)|^r)
\end{aligned}
$$

$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{K}(\frac{1}{r}|e_s^{K}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{J}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{I}(\frac{1}{r}|e_s^{I}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{J}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{R}(\frac{1}{r}|e_s^{R}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{J}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^{R}|l_s^{K}(\frac{1}{r}|e_s^{K}(t)|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^{K}|l_s^{R}(\frac{1}{r}|e_s^{R}(t)|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^{I}|l_s^{J}(\frac{1}{r}|e_s^{J}(t)|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)+\sum_{q=1}^{n}\sum_{s=1}^{n}|b_{qs}^{J}|l_s^{I}(\frac{1}{r}|e_s^{I}(t)|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{R}|l_s^{K}(\frac{1}{r}|e_s^{K}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{K}|l_s^{R}(\frac{1}{r}|e_s^{R}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{I}|l_s^{J}(\frac{1}{r}|e_s^{J}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{K}(t)|^r)$$
$$+\sum_{q=1}^{n}\sum_{s=1}^{n}|d_{qs}^{J}|l_s^{I}(\frac{1}{r}|e_s^{I}(t-\tau(t))|^r+\frac{r-1}{r}|e_q^{K}(t)|^r).$$

Then, we have

$$\begin{aligned}
\dot{V}(e(t)) \leq\ & \sum_{q=1}^{n} r\Big(-c_q-k_q+\frac{1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{R}|+\frac{1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{I}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{I}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{J}|b_{qs}^{J}| \\
&+\frac{1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{J}|+\frac{1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{K}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{R}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{K}|b_{qs}^{K}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{R}|d_{qs}^{R}| \\
&+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{I}|d_{qs}^{I}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{J}|d_{qs}^{J}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{K}|d_{qs}^{K}|\Big)\frac{1}{r}|e_q^{R}(t)|^r \\
&+\sum_{q=1}^{n} r\Big(-c_q-k_q+\frac{1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{R}|+\frac{1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{I}| \\
&+\frac{1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{J}|+\frac{1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{K}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{I}|b_{qs}^{R}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{R}|b_{qs}^{I}| \\
&+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{K}|b_{qs}^{J}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{J}|b_{qs}^{K}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{I}|d_{qs}^{R}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{R}|d_{qs}^{I}| \qquad (3.5)\\
&+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{K}|d_{qs}^{J}|+\frac{r-1}{r}\sum_{s=1}^{n}l_s^{J}|d_{qs}^{K}|\Big)\frac{1}{r}|e_q^{I}(t)|^r
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{n} r\Big(-c_q-k_q+\frac{1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^R|+\frac{1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^I|+\frac{1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^J|+\frac{1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^K|\\
&+\frac{r-1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^R|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^R|b_{qs}^J|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^I|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^I|b_{qs}^K|\\
&+\frac{r-1}{r}\sum_{s=1}^{n} l_s^J|d_{qs}^R|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^R|d_{qs}^J|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^K|d_{qs}^I|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^I|d_{qs}^K|\Big)\frac{1}{r}|e_q^J(t)|^r\\
&+\sum_{q=1}^{n} r\Big(-c_q-k_q+\frac{1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^R|+\frac{1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^I|+\frac{1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^J|+\frac{1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^K|\\
&+\frac{r-1}{r}\sum_{s=1}^{n} l_s^K|b_{qs}^R|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^R|b_{qs}^K|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^J|b_{qs}^I|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^I|b_{qs}^J|\\
&+\frac{r-1}{r}\sum_{s=1}^{n} l_s^K|d_{qs}^R|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^R|d_{qs}^K|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^J|d_{qs}^I|+\frac{r-1}{r}\sum_{s=1}^{n} l_s^I|d_{qs}^J|\Big)\frac{1}{r}|e_q^K(t)|^r\\
&+\sum_{q=1}^{n}\sum_{s=1}^{n} r\Big(\frac{1}{r}l_s^R|d_{qs}^R|+\frac{1}{r}l_s^R|d_{qs}^I|+\frac{1}{r}l_s^R|d_{qs}^J|+\frac{1}{r}l_s^R|d_{qs}^K|\Big)\frac{1}{r}|e_q^R(t-\tau(t))|^r\\
&+\sum_{q=1}^{n}\sum_{s=1}^{n} r\Big(\frac{1}{r}l_s^I|d_{qs}^R|+\frac{1}{r}l_s^I|d_{qs}^I|+\frac{1}{r}l_s^I|d_{qs}^J|+\frac{1}{r}l_s^I|d_{qs}^K|\Big)\frac{1}{r}|e_q^I(t-\tau(t))|^r\\
&+\sum_{q=1}^{n}\sum_{s=1}^{n} r\Big(\frac{1}{r}l_s^J|d_{qs}^R|+\frac{1}{r}l_s^J|d_{qs}^I|+\frac{1}{r}l_s^J|d_{qs}^J|+\frac{1}{r}l_s^J|d_{qs}^K|\Big)\frac{1}{r}|e_q^J(t-\tau(t))|^r\\
&+\sum_{q=1}^{n}\sum_{s=1}^{n} r\Big(\frac{1}{r}l_s^K|d_{qs}^R|+\frac{1}{r}l_s^K|d_{qs}^I|+\frac{1}{r}l_s^K|d_{qs}^J|+\frac{1}{r}l_s^K|d_{qs}^K|\Big)\frac{1}{r}|e_q^K(t-\tau(t))|^r.
\end{aligned}
$$

We suppose that

$$
\begin{aligned}
\lambda_1 &= \min_{1\le q\le n}\Big\{rc_q+rk_q-\sum_{s=1}^{n}(l_s^R|b_{qs}^R|+l_s^R|b_{qs}^I|+l_s^R|b_{qs}^J|+l_s^R|b_{qs}^K|+(r-1)l_s^R|b_{qs}^R|+(r-1)l_s^I|b_{qs}^I|\\
&\quad+(r-1)l_s^J|b_{qs}^J|+(r-1)l_s^K|b_{qs}^K|+(r-1)l_s^R|d_{qs}^R|+(r-1)l_s^I|d_{qs}^I|+(r-1)l_s^J|d_{qs}^J|+(r-1)l_s^K|d_{qs}^K|)\Big\},\\
\lambda_2 &= \min_{1\le q\le n}\Big\{rc_q+rk_q-\sum_{s=1}^{n}(l_s^I|b_{qs}^R|+l_s^I|b_{qs}^I|+l_s^I|b_{qs}^J|+l_s^I|b_{qs}^K|+(r-1)l_s^I|b_{qs}^R|+(r-1)l_s^R|b_{qs}^I|\\
&\quad+(r-1)l_s^K|b_{qs}^J|+(r-1)l_s^J|b_{qs}^K|+(r-1)l_s^I|d_{qs}^R|+(r-1)l_s^R|d_{qs}^I|+(r-1)l_s^K|d_{qs}^J|+(r-1)l_s^J|d_{qs}^K|)\Big\},\\
\lambda_3 &= \min_{1\le q\le n}\Big\{rc_q+rk_q-\sum_{s=1}^{n}(l_s^J|b_{qs}^R|+l_s^J|b_{qs}^I|+l_s^J|b_{qs}^J|+l_s^J|b_{qs}^K|+(r-1)l_s^J|b_{qs}^R|+(r-1)l_s^R|b_{qs}^J|\\
&\quad+(r-1)l_s^K|b_{qs}^I|+(r-1)l_s^I|b_{qs}^K|+(r-1)l_s^J|d_{qs}^R|+(r-1)l_s^R|d_{qs}^J|+(r-1)l_s^K|d_{qs}^I|+(r-1)l_s^I|d_{qs}^K|)\Big\},\\
\lambda_4 &= \min_{1\le q\le n}\Big\{rc_q+rk_q-\sum_{s=1}^{n}(l_s^K|b_{qs}^R|+l_s^K|b_{qs}^I|+l_s^K|b_{qs}^J|+l_s^K|b_{qs}^K|+(r-1)l_s^K|b_{qs}^R|+(r-1)l_s^R|b_{qs}^K|
\end{aligned}
$$

$$+(r-1)l_s^J|b_{qs}^I|+(r-1)l_s^I|b_{qs}^J|+(r-1)l_s^K|d_{qs}^R|+(r-1)l_s^R|d_{qs}^K|+(r-1)l_s^J|d_{qs}^I|+(r-1)l_s^I|d_{qs}^J|)\Big\},$$

$$\zeta_1 = \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^R|d_{qs}^R|+l_s^R|d_{qs}^I|+l_s^R|d_{qs}^J|+l_s^R|d_{qs}^K|),\ \ \zeta_2=\max_{1\le q\le n}\sum_{s=1}^{n}(l_s^I|d_{qs}^R|+l_s^I|d_{qs}^I|+l_s^I|d_{qs}^J|+l_s^I|d_{qs}^K|),$$
$$\zeta_3 = \max_{1\le q\le n}\sum_{s=1}^{n}(l_s^J|d_{qs}^R|+l_s^J|d_{qs}^I|+l_s^J|d_{qs}^J|+l_s^J|d_{qs}^K|),\ \ \zeta_1=\max_{1\le q\le n}\sum_{s=1}^{n}(l_s^K|d_{qs}^R|+l_s^K|d_{qs}^I|+l_s^K|d_{qs}^J|+l_s^K|d_{qs}^K|).$$

Then,

$$\begin{aligned}\dot{V}(e(t) &\le -\lambda_1\frac{1}{r}|e_q^R(t)|^r-\lambda_2\frac{1}{r}|e_q^I(t)|^r-\lambda_3\frac{1}{r}|e_q^J(t)|^r-\lambda_4\frac{1}{r}|e_q^K(t)|^r+\zeta_1\frac{1}{r}|e_q^R(t-\tau(t))|^r\\ &\quad+\zeta_2\frac{1}{r}|e_q^I(t-\tau(t))|^r+\zeta_3\frac{1}{r}|e_q^J(t-\tau(t))|^r+\zeta_4\frac{1}{r}|e_q^K(t-\tau(t))|^r\\ &\le -\lambda V(e(t))+\zeta V(e(t-\tau(t))),\end{aligned}\tag{3.6}$$

where $\lambda = min\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, $\zeta = max\{\zeta_1,\zeta_2,\zeta_3,\zeta_4\}$.

Therefore, according to Lemma 2.4 and Theorem 3.1, the projection synchronization between drive system (2.1) and response system (2.8) can be achieved.

Remark 3. In recent years, the research of real-valued fractional-order neural networks and complex-valued fractional-order neural networks in projection synchronization [28, 29, 36, 37] has also achieved excellent results. Compared with previous studies, QVNN is superior in handling multi-dimensional problems. Therefore, our results are more general.

4. Numerical example

Considering the two dimensional quaternion-valued memristor-based neural networks model of Eq (2.1) with the memristive connection weights are

$$b_{11}(m_1(t))=\begin{cases}2.3-1.6i+2.3j-1.5k, |m_1(t)|<1,\\ 2.0-2.7i+2.0j-0.7k, |m_1(t)|\ge 1,\end{cases}$$

$$b_{12}(m_1(t))=\begin{cases}-0.5-0.4i-0.5j-0.7k, |m_1(t)|<1,\\ -0.1-0.9i-0.1j-0.3k, |m_1(t)|\ge 1,\end{cases}$$

$$b_{21}(m_2(t))=\begin{cases}1.1+0.7i+1.0j+0.6k, |m_2(t)|<1,\\ 1.6-0.3i+1.5j-0.4k, |m_2(t)|\ge 1,\end{cases}$$

$$b_{22}(m_2(t))=\begin{cases}-1.2-0.3i-0.7j-0.3k, |m_2(t)|<1,\\ -0.8-0.1i-1.3j-0.2k, |m_2(t)|\ge 1,\end{cases}$$

$$d_{11}(m_1(t))=\begin{cases}-1.4+3.1i-1.4j+3.0k, |m_1(t)|<1,\\ -1.5+2.6i-1.5j+2.3k, |m_1(t)|\ge 1,\end{cases}$$

$$d_{12}(m_1(t))=\begin{cases}-0.5-1.5i-0.5j-1.6k, |m_1(t)|<1,\\ -0.1-0.9i-0.1j-0.6k, |m_1(t)|\ge 1,\end{cases}$$

$$d_{21}(m_2(t)) = \begin{cases} -0.8 - 0.2i - 0.6j - 0.1k, |m_2(t)| < 1, \\ -1.2 - 1.1i - 1.3j - 1.3k, |m_2(t)| \geq 1, \end{cases}$$

$$d_{22}(m_2(t)) = \begin{cases} 0.5 - 0.8i + 0.4j - 0.7k, |m_2(t)| < 1, \\ 1.3 - 0.5i + 1.2j - 0.4k, |m_2(t)| \geq 1. \end{cases}$$

The response system of (2.8) is

$$\begin{aligned} \dot{n}_1(t) &= -c_1 n_1(t) + \sum_{s=1}^{2} b_{1s}(n_1(t)) f_s(n_s(t)) + \sum_{s=1}^{2} d_{1s}(n_1(t)) f_s(n_s(t-\tau(t))) + I_1 + u_1(t), \\ \dot{n}_2(t) &= -c_2 n_2(t) + \sum_{s=1}^{2} b_{2s}(n_2(t)) f_s(n_s(t)) + \sum_{s=1}^{2} d_{2s}(n_2(t)) f_s(n_s(t-\tau(t))) + I_2 + u_2(t). \end{aligned}$$

Take the time delay $\tau(t) = 0.75 - 0.25cos(t)$ such that $\tau = 1$ and the activation function $f(m_q(t)) = 0.23 * (|m_q^R(t) + 1| - |m_q^R(t) - 1|) + 0.23 * (|m_q^I(t) + 1| - |m_q^I(t) - 1|)i + 0.23 * (|m_q^J(t) + 1| - |m_q^J(t) - 1|)j + 0.23 * (|m_q^K(t) + 1| - |m_q^K(t) - 1|)k$ $(q = 1, 2)$, and external inputs $I_1 = I_2 = 0$. According to the Assumption 2, we can get $l_q^\varepsilon = 0.46$, where $q = 1, 2$. The control gain parameter is taken as $k_1 = k_2 = 40$ and the connection weights are taken as $c_1 = c_2 = 1$. Then, the trajectories of error systems with controllers and without controllers are obtained.

Besides, we take $r = 2$. According to the above parameters, we can directly calculate the direct result of the condition in Theorem 3.1: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 53.526$ and $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 10.166$. Then, we can obtain $\lambda = 53.526 > \zeta = 10.166$.

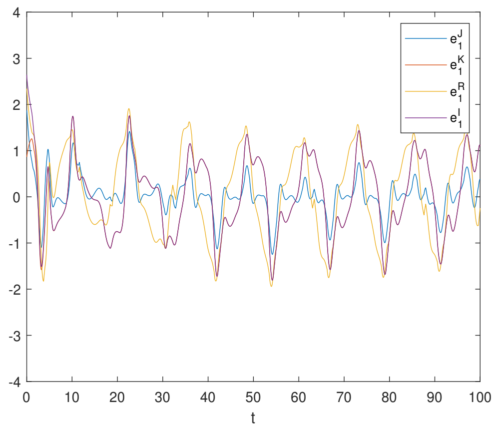

Figure 1. The errors e_1 with $\beta = 0.8$ without the controller.

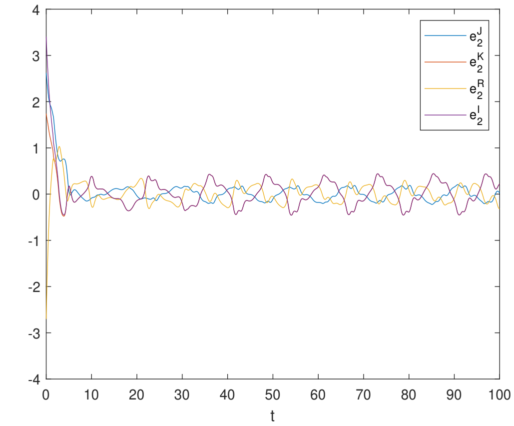

Figure 2. The errors e_2 with $\beta = 0.8$ without the controller.

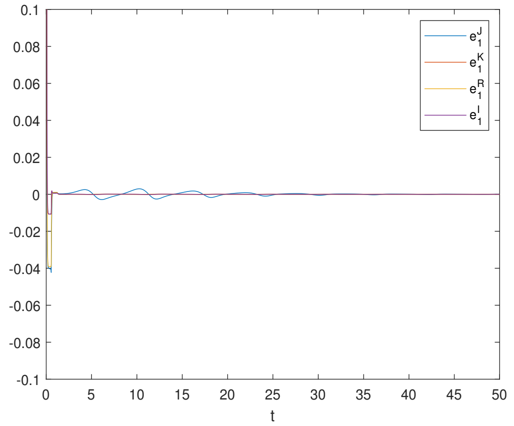

Figure 3. The errors e_1 with $\beta = 0.8$ under the controller.

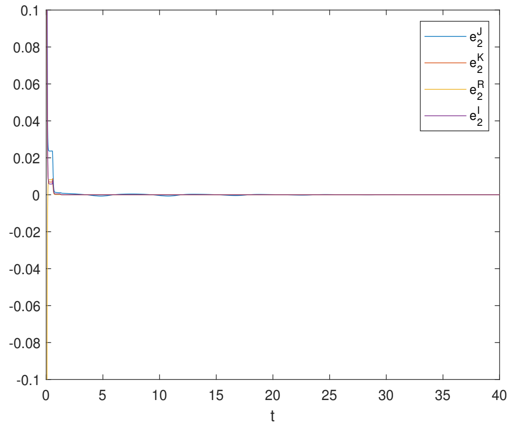

Figure 4. The errors e_2 with $\beta = 0.8$ under the controller.

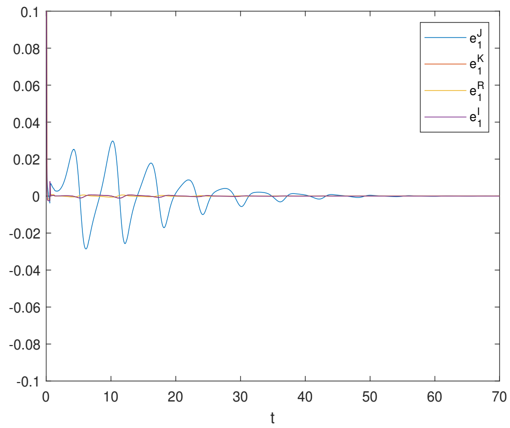

Figure 5. The errors e_1 with $\beta = -1$ under the controller.

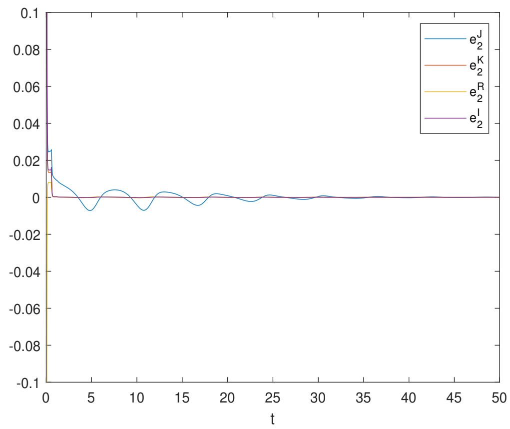

Figure 6. The errors e_2 with $\beta = -1$ under the controller.

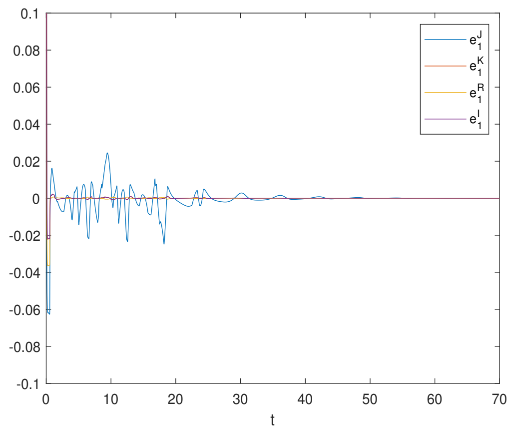

Figure 7. The errors e_1 with $\beta = 2$ under the controller.

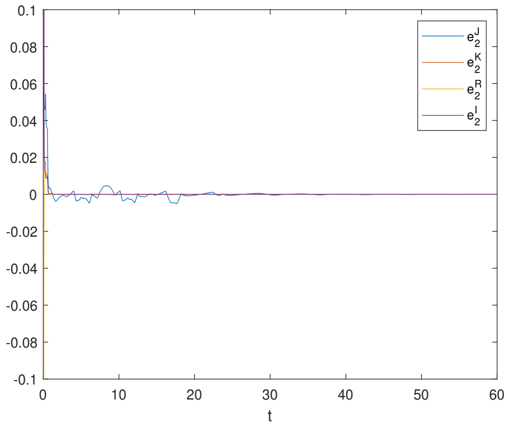

Figure 8. The errors e_2 with $\beta = 2$ under the controller.

The above result satisfies the condition in Theorem 3.1, so systems (2.1) and (2.8) achieve projective synchronization. Figures 1 and 2 show the synchronization error curves e_q $(q = 1, 2)$ for $\beta = 0.8$ without the controller. Figures 3 and 4 show the synchronization error curves e_q $(q = 1, 2)$ for $\beta = 0.8$ with the controller (2.16). Figures 5 and 6 show the synchronization error curves e_q $(q = 1, 2)$ for $\beta = -1$ with the controller (2.16). Figures 7 and 8 show the synchronization error curves e_q $(q = 1, 2)$ for $\beta = 2$ with the controller (2.16). From the above simulation results, we know that the derive system (2.1) and the response system (2.8) are synchronized, which verifies the effectiveness of Theorem 3.1.

5. Conclusions

This study delves into the dynamics of projective synchronization within the realm of quaternion-valued memristor-based neural networks, which are subject to time-varying delays. Employing the theoretical underpinnings of set-valued mappings and differential inclusion, we formulate a hybrid control approach to dissect the projection synchronization dilemma of the network. By harnessing the stability assurances of a Lyapunov function and the quantitative bounds provided by Young's inequality, we formulate a novel criterion for synchronization. This leads to the achievement of projective synchronization in the context of the aforementioned neural networks. The efficacy and practicality of our proposed strategy are substantiated through rigorous numerical simulations.

Author contribution

Jun Guo, Yanchao Shi and Yanzhao Cheng wrote the main manuscript text. All authors reviewed the manuscript.

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Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. L. O. Chua, Memristor-the missing circuit element, *IEEE Trans. Circuit Theory*, **18** (1971), 507–519. https://doi.org/10.1109/TCT.1971.1083337
2. H. Bao, Y. Zhang, W. Liu, B. Bao, Memristor synapse-coupled memristive neuron network: Synchronization transition and occurrence of chimera, *Nonlinear Dyn.*, **100** (2020), 937–950. https://doi.org/10.1007/s11071-020-05529-2
3. F. Wei, G. Chen, W. Wang, Finite-time synchronization of memristor neural networks via interval matrix method, *Neural Networks*, **127** (2020), 7–18. https://doi.org/10.1016/j.neunet.2020.04.003
4. D. Ding, Z. You, Y. Hu, Z. Yang, L. Ding, Finite-time synchronization for fractional-order memristor-based neural networks with discontinuous activations and multiple delays, *Mod. Phys. Lett. B*, **34** (2020), 2050162. https://doi.org/10.1142/S0217984920501626
5. U. Bhatti, Z. Yu, L. Yuan, Z. Zeeshan, M. Bhatti, M. Anum, et al., Geometric algebra applications in geospatial artificial intelligence and remote sensing image processing, *IEEE Access*, **8** (2020), 155783–155796. https://doi.org/10.1109/ACCESS.2020.3018544
6. L. Hua, Y. Qiang, J. Gu, L. Chen, X. Zhang, H. Zhu, Mechanical fault diagnosis using color image recognition of vibration spectrogram based on quaternion invariable moment, *Math. Probl. Eng.*, **2015** (2015), 1–11. https://doi.org/10.1155/2015/702760
7. M. Hasan, B. P. Mandal, New scattering features of quaternionic point interaction in non-Hermitian quantum mechanics, *J. Math. Phys.*, **61** (2020), 032104. https://doi.org/10.1063/1.5117873
8. R. J. Goodman, Digital simulation of aerospace vehicle flight path dynamics using quaternions, in *Prague International Astronautical Federation Congress*, 1977.
9. H. Wang, G. Wei, S. Wen, T. Huang, Impulsive disturbance on stability analysis of delayed quaternion-valued neural networks, *Neurocomputing*, **390** (2021), 125680. https://doi.org/10.1016/j.amc.2020.125680
10. J. Shu, B. Wu, L. Xiong, T. Wu, H. Zhang, Stochastic stabilization of Markov jump quaternion-valued neural network using sampled-data control, *Appl. Math. Comput.*, **400** (2021), 1260414. https://doi.org/10.1016/j.amc.2021.126041
11. Y. Zhang, L. Zhou, Novel global polynomial stability criteria of impulsive complex-valued neural networks with multi-proportional delays, *Neural Comput. Appl.*, **34** (2022), 2913–2924. https://doi.org/10.1007/s00521-021-06555-w

12. S. Wang, Y. Shi, J. Guo, Exponential stability of a class of quaternion-valued memristor-based neural network with time-varying delay via M-matrix, *Math. Methods Appl. Sci.*, 2024. https://doi.org/10.1002/mma.10486
13. Q. Song, Y. Chen, Z. Zhao, Y. Liu, F. Alsaadi, Robust stability of fractional-order quaternion-valued neural networks with neutral delays and parameter uncertainties, *Neurocomputing*, **420** (2021), 70–81. https://doi.org/10.1016/j.neucom.2020.08.059
14. R. Li, J. Cao, N. Li, Stop and go strategy for Lagrange stability of quaternion-valued memristive neural networks, *Math. Methods Appl. Sci.*, **46** (2023), 6578–6589. https://doi.org/10.1002/mma.8926
15. W. Liu, J. Huang, Q. Yao, Stability analysis for quaternion-valued inertial memristor-based neural networks with time delays, *Neurocomputing*, **448** (2021), 67–81. https://doi.org/10.1016/j.neucom.2021.03.106
16. Y. Shi, X. Chen, P. Zhu, Dissipativity for a class of quaternion-valued memristor-based neutral-type neural networks with time-varying delays, *Math. Methods Appl. Sci.*, **46** (2023), 18166–18184. https://doi.org/10.1002/mma.9551
17. T. Peng, J. Lu, Z. Tu, J. Lou, Finite-time stabilization of quaternion-valued neural networks with time delays: An implicit function method, *Inf. Sci.*, **613** (2022), 747–762. https://doi.org/10.1016/j.ins.2022.09.014
18. G. Tan, Z. Wang, Z. Shi, Proportional-integral state estimator for quaternion-valued neural networks with time-varying delays, *IEEE Trans. Neural Networks Learn. Syst.*, **34** (2023), 1074–1079. https://doi.org/10.1109/TNNLS.2021.3103979
19. J. Hu, G. Tan, L. Liu, A new result on H∞ state estimation for delayed neural networks based on an extended reciprocally convex inequality, *IEEE Trans. Circuits Syst. II Express Briefs*, **71** (2024), 1181–1185. https://doi.org/10.1109/TCSII.2023.3323834
20. J. Cai, J. Feng, J. Wang, Y. Zhao, Quasi-synchronization of neural networks with diffusion effects via intermittent control of regional division, *Neurocomputing*, **409** (2020), 146–156. https://doi.org/10.1016/j.neucom.2020.05.037
21. R. Li, X. Gao, J. Cao, Quasi-state estimation and quasi-synchronization control of quaternion-valued fractional-order fuzzy memristive neural networks: Vector ordering approach, *Appl. Math. Comput.*, **362** (2019), 124572. https://doi.org/10.1016/j.amc.2019.124572
22. X. Song, X. Li, S. Song, Y. Zhang, Z. Ning, Quasi-synchronization of coupled neural networks with reaction-diffusion terms driven by fractional brownian motion, *J. Franklin Inst.*, **358** (2021), 2482–2499. https://doi.org/10.1016/j.jfranklin.2021.01.023
23. Z. Zhang, T. Zheng, S. Yu, Finite-time anti-synchronization of neural networks with time-varying delays via inequality skills, *Neurocomputing*, **356** (2019), 60–68. https://doi.org/10.1016/j.neucom.2019.05.012
24. X. Liu, Z. Li, Finite time anti-synchronization of complex-valued neural networks with bounded asynchronous time-varying delays, *Neurocomputing*, **387** (2020), 129–138. https://doi.org/10.1016/j.neucom.2020.01.035

25. Y. Qiao, H. Yan, L. Duan, J. Miao, Finite-time synchronization of fractional-order gene regulatory networks with time delay, *Neural Networks*, **126** (2020), 1–10. https://doi.org/10.1016/j.neunet.2020.02.004

26. T. Peng, J. Zhong, Z. Tu, J. Lu, J. Lou, Finite-time synchronization of quaternion-valued neural networks with delays: A switching control method without decomposition, *Neural Networks*, **148** (2022), 37–47. https://doi.org/10.1016/j.neunet.2021.12.012

27. T. Peng, J. Qiu, J. Lu, Z. Tu, J. Cao, Finite-time and fixed-time synchronization of quaternion-valued neural networks with/without mixed delays: An improved one-norm method, *IEEE Trans. Neural Networks Learn. Syst.*, **12** (2022), 7475–7487. https://doi.org/10.1109/TNNLS.2021.3085253

28. D. Ding, X. Yao, H. Zhang, Complex projection synchronization of fractional-order complex-valued memristive neural networks with multiple delays, *Neural Process. Lett.*, **51** (2020), 325–345. https://doi.org/10.1007/s11063-019-10093-x

29. Y. Zhang, S. Deng, Finite-time projective synchronization of fractional-order complex-valued memristor-based neural networks with delay, *Chaos, Solitons Fractals*, **128** (2019), 176-190. https://doi.org/10.1016/j.chaos.2019.07.043

30. Y. Cheng, Y. Shi, The exponential synchronization and asymptotic synchronization of quaternion-valued memristor-based Cohen-Grossberg neural networks with time-varying delays, *Neural Process. Lett.*, **55** (2023), 6637–6656. https://doi.org/10.1007/s11063-023-11152-0

31. Y. Cheng, Y. Shi, J. Guo, Exponential synchronization of quaternion-valued memristor-based Cohen-Grossberg neural networks with time-varying delays: Norm method, *Cognit. Neurodyn.*, **18** (2024), 1943–1953. https://doi.org/10.1007/s11571-023-10057-x

32. H. L. Li, L. Zhang, C. Hu, H. Jiang, J. Cao, Global Mittag-Leffler synchronization of fractional-order delayed quaternion-valued neural networks: Direct quaternion approach, *Appl. Math. Comput.*, **373** (2020), 125020. https://doi.org/10.1016/j.amc.2019.125020

33. Y. Kao, Y. Li, J. H. Park, X. Chen, Mittag-Leffler synchronization of delayed fractional memristor neural networks via adaptive control, *IEEE Trans. Neural Networks Learn. Syst.*, **32** (2020), 2279–2284. https://doi.org/10.1109/TNNLS.2020.2995718

34. J. Cheng, L. Xie, D. Zhang, H. Yan, Novel event-triggered protocol to sliding mode control for singular semi-Markov jump systems, *Automatica*, **151** (2023), 110906. https://doi.org/10.1016/j.automatica.2023.110906

35. J. Cheng, J. H. Park, Z. Wu, Observer-based asynchronous control of nonlinear systems with dynamic event-based try-once-discard protocol, *IEEE Trans. Cybern.*, **52** (2022), 12638–12648. https://doi.org/10.1109/TCYB.2021.3104806

36. Y. Gu, Y. Yu, H. Wang, Projective synchronization for fractional-order memristor-based neural networks with time delays, *Neural Comput. Appl.*, **31** (2019), 6039–6054. https://doi.org/10.1007/s00521-018-3391-7

37. G. Velmurugan, R. Rakkiyappan, Hybrid projective synchronization of fractional-order memristor-based neural networks with time delays, *Nonlinear Dyn.*, **83** (2016), 419–432. https://doi.org/10.1007/s11071-015-2337-1

38. H. Bao, J. Cao, Projective synchronization of fractional-order memristor-based neural networks, *Neural Networks*, **63** (2015), 1–9. https://doi.org/10.1016/j.neunet.2014.10.007

39. R. Li, X. Gao, J. Cao, K. Zhang, Exponential stabilization control of delayed quaternion-valued memristive neural networks: Vector ordering approach, *Circuits Syst. Signal Process.*, **39** (2020), 1353–1371. https://doi.org10.1007/s00034-019-01225-8