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Research article

Influence of environmental pollution and bacterial hyper-infectivity on dynamics of a waterborne pathogen model with free boundaries

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Abstract: In this paper, we mainly study the influence of environmental pollution and bacterial hyper-infectivity on the spreading of diseases by considering a waterborne pathogen model with free boundaries. At first, the global existence and uniqueness of the solution to this problem is proved. Then, we analyze its longtime behavior, which is determined by a spreading-vanishing dichotomy. Furthermore, we obtain the criteria for spreading and vanishing. Our results indicate that environmental pollution and bacterial hyper-infectivity can increase the chance of epidemic spreading.

Keywords: waterborne diseases; environmental pollution; bacterial hyper-infectivity; free boundary problem; spreading and vanishing

1. Introduction

Nowadays, infectious diseases pose a significant risk to human health, such as, waterborne diseases. To gain a comprehensive understanding of the transmission dynamics of such diseases, Eisenberg et al. [1] emphasized the necessity of considering various transmission pathways. A large number of models have been proposed to describe the transmission of waterborne diseases. For example, Codeco [2] used a compartmental ordinary differential equation (ODE) model to describe the human-water-human transmission mechanism, where the infectious population shed the pathogen into the water, and subsequently the susceptible population drink the contaminated water. However, this model overlooks human-human transmission. Later, Tien and Earn [3] added a compartment into the classical SIR model, and proposed the following model:

$$\begin{cases} S' = kN - \beta_1 S W - \beta_2 S I - kS, & t > 0, \\ I' = \beta_1 S W + \beta_2 S I - \gamma I - kI, & t > 0, \\ R' = \gamma I - kR, & t > 0, \\ W' = \alpha I - dW, & t > 0, \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ denote the densities of the susceptible, infectious, and recovered human population, respectively, and $W(t)$ represents the concentration of pathogen in the contaminated water. Assume that the birth and death rates are equal to k. A susceptible human can be infected through two primary pathways: human-water-human transmission and human-human contact, whose transmission rates are represented by the parameters β_1 and β_2, respectively. While γ is the recovery rate, and α is the pathogen shedding rate from infectious humans into the water. The removed rate of pathogen in the water is represented by d. The main results in [3] indicated that there exists a threshold parameter $\Re_0$ such that the disease will spread if $\Re_0 > 1$, and tend to extinction if $\Re_0 < 1$.

After the above work, numerous researchers have studied model (1.1) and related models. For example, reference [4] focused on the corresponding local diffusion version of (1.1) and provided insights into the global dynamics, while [5] considered the corresponding traveling waves. However, these works all showed that if the basic reproduction number is larger than 1, the disease will always spread regardless of the size of the initial infectious population. These results do not match well with the fact that the disease will not always spread for the small size of the initial infectious population. At the same time, the above works can not tell us the location of the spreading front. Motivated by the introduction of free boundary by Du and Lin in [6], Zhao [7] incorporated the free boundary into the partial differential equation (PDE) model discussed in [4], proposed a new model, and obtained the dynamics of the solution, which can be better to describe the spreading of diseases. Following the work of Du and Lin [6] on a logistic model, free boundary approaches similar to the problem considered in [7] have been studied by many researchers recently, for which the readers can refer to [8–13] and the references therein.

As society has developed, the issue of environmental pollution has attracted extensive attention. Findings from [14] demonstrated that environmental pollutants can suppress an individual's immune system and thereby increase the susceptibility of the human population to various infectious diseases. Thus, this will help epidemics spread rapidly. Thus, it is necessary to consider the effect of environmental pollutants when constructing mathematical models to describe disease transmission. Recently, Wang and Feng [15] proposed a PDE model to investigate the influence of environmental pollution on the spreading of waterborne diseases.

In addition, recent laboratory findings in [16] indicated that, for some diseases, the pathogen will be excreted by the infectious human via the gastrointestinal tract, and can remain viable, highly toxic, and infectious for several hours. Compared with the pathogen persisting in the environment for several months, these pathogens exhibit up to 700-fold infectivity. Therefore, bacterial hyper-infectivity should be considered during modeling. Wang and Wu [17] studied the different roles of two types of vibrios and the spatial heterogeneity of the environment on the transmission of cholera. Thus, it is significant to consider two types of vibrios distinguished by their infectivity.

Inspired by above works, we develop our model in [7] and study the influence of environmental pollution on dynamics of a waterborne pathogen model with bacterial hyper-infectivity and free boundaries. We categorize the human population into three classes: susceptible, infectious, and recovered, which are denoted by $S(t, x)$, $I(t, x)$, and $R(t, x)$, respectively. To explore the influence of environmental pollution, we further divide the susceptible human population into two subcategories: those unaffected by environmental pollutants, denoted as $U(t, x)$, and those affected, represented as $V(t, x)$. According to the infectivity of the pathogen, we also divide the pathogen into two classes denoted by $P(t, x)$ and $Q(t, x)$, where P exhibits hyper-infectivity. The susceptible human population U and V can be infected by two pathways: human-water-human and human-human contact. The direct transmission rate is represented by $\beta_3 I$. Recalling that pathogen stay highly toxic and infectious for a short time during disease transmission. Based on the number of pathogens P and Q, it follows from [18] that we can use the linear incidence rate $\beta_1 P$ and the saturated incidence rate $\beta_2 Q/(m + Q)$ (this can be derived from particles as in [19]) to describe the rate of indirect transmission. To describe the spreading of disease well, we suppose that the range of the initial infected area is the interval $[-h_0, h_0]$, and the infected area is increasing as the time goes on and is denoted by $[g(t), h(t)]$, where $g(t)$ and $h(t)$ represent the spreading fronts of the disease and satisfy the Stefan condition (the derivation of this free boundary condition can refer to [20]): $g'(t) = -\mu I_x(t, g(t))$ and $h'(t) = -\mu I_x(t, h(t))$ where μ is the spreading capacity. Before proposing our model, we put forward the following assumptions:

(i) the mobility of the pathogen is significantly lower than that of the human population and thus can be neglected; the dispersal rate of U, V, I, and R are represented by D_1, D_2, D_3, and D_4, respectively;

(ii) the recruitment rate of the human population is denoted by b, with a fraction p transitioning directly to the class V;

(iii) environmental pollutants cause some individuals to migrate from class U to class V, and we assume that the rate is the constant q as referenced in [21];

(iv) noting that the environmental pollutants increase the susceptibility of the human population to specific infectious diseases, we assume that the effects of pollution on β_i are equal and denoted by θ;

(v) assume that the death rate of U, V, I, and R are the same and denoted by k;

(vi) assume that the hyper-infective pathogen will not die before they transition into the lower-infective strains.

According to the above assumptions, we propose the following model:

$$\begin{cases} U_t = D_1 U_{xx} + (1-p)b - qU - \beta_1 UP - \frac{\beta_2 UQ}{m+Q} - \beta_3 UI - kU, & t > 0,\ x \in \mathbb{R}, \\ V_t = D_2 V_{xx} + pb + qU - \beta_1 \theta VP - \frac{\beta_2 \theta VQ}{m+Q} - \beta_3 \theta VI - kV, & t > 0,\ x \in \mathbb{R}, \\ I_t = D_3 I_{xx} + \beta_1 (U+\theta V)P + \frac{\beta_2 (U+\theta V)Q}{m+Q} + \beta_3 (U+\theta V)I - \gamma I - kI, & t > 0,\ x \in (g(t), h(t)), \\ R_t = D_4 R_{xx} + \gamma I - kR, & t > 0,\ x \in (g(t), h(t)), \\ P_t = \alpha I - \eta P, & t > 0,\ x \in (g(t), h(t)), \\ Q_t = \eta P - dQ, & t > 0,\ x \in (g(t), h(t)), \\ I(t,x) = R(t,x) = P(t,x) = Q(t,x) = 0, & t > 0,\ x \in \mathbb{R} \backslash (g(t), h(t)), \\ g'(t) = -\mu I_x(t, g(t)),\ h'(t) = -\mu I_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, & \\ U(0,x) = U_0(x),\ V(0,x) = V_0(x), & x \in \mathbb{R}, \\ I(0,x) = I_0(x),\ R(0,x) = R_0(x),\ P(0,x) = P_0(x),\ Q(0,x) = Q_0(x), & x \in [-h_0, h_0], \end{cases} \tag{1.2}$$

where γ is the recovery rate, α stands for the shedding rate of the pathogen from an infectious human into the water, η is the removal rates of the hyper-infective pathogen in contaminated water, d represents the removal rate of the lower-infective pathogen in contaminated water. Assume that the above parameters are all positive, $\theta > 1$, $0 < p, q < 1$, and $U_0(x)$, $V_0(x)$, $I_0(x)$, $R_0(x)$, $P_0(x)$, and $Q_0(x)$ satisfy

$$\begin{aligned}
&U_0(x), V_0(x) \in C^2(\mathbb{R}),\ I_0(x), R_0(x) \in C^2([-h_0, h_0]),\ P_0(x), Q_0(x) \in C^{1-}([-h_0, h_0]) \\
&I_0(x) = R_0(x) = P_0(x) = Q_0(x) = 0,\ x \in \mathbb{R}\backslash(-h_0, h_0), \\
&U_0(x) > 0, V_0(x) > 0, x \in \mathbb{R},\ I_0(x) > 0, R_0(x) > 0, P_0(x), Q_0(x) > 0, x \in (-h_0, h_0),
\end{aligned} \tag{1.3}$$

where C^{1-} is Lipschitz continuous functions space.

For convenience, we denote

$$\mathcal{R}_0 = \left(\frac{\beta_1\alpha}{\eta} + \frac{\beta_2\alpha}{dm} + \beta_3\right)\frac{(1-p)bk + \theta pb(k+q) + \theta q(1-p)b}{k(k+q)(\gamma+k)}, \tag{1.4}$$

and

$$\Lambda = \frac{(1-p)b}{k+q} + \frac{\theta pb}{k} + \frac{\theta q(1-p)b}{k(k+q)}. \tag{1.5}$$

Our main results are listed as follows.

Theorem 1.1. *For any given $h_0 > 0$ and U_0, V_0, I_0, R_0, P_0, Q_0 satisfying* (1.3)*, problem* (1.2) *admits a unique solution (U, V, I, R, P, Q, g, h) defined for all $t > 0$.*

Theorem 1.2. *Assume that the conditions in Theorem 1.1 hold. Let (U, V, I, R, P, Q, g, h) be the unique solution of* (1.2)*. Then, the following alternative holds:*

Either

(i) **Spreading**: $\lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = +\infty$ *(and necessarily $\mathcal{R}_0 > 1$),*

$$\lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|R(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} > 0,$$

and furthermore, if $\frac{\beta_3+\frac{\alpha}{\eta}\beta_1}{\gamma+k}\Lambda + \frac{k+\beta_2}{\Lambda k+b}\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1} < 1$, then

$$\lim_{t\to+\infty} (U(t,x), V(t,x), I(t,x), R(t,x), P(t,x), Q(t,x)) = (U^*, V^*, I^*, R^*, P^*, Q^*),$$

uniformly for x in any bounded set of $\mathbb{R}$, where $(U^*, V^*, I^*, R^*, P^*, Q^*)$ is given by (4.1)*;*

or

(ii) **Vanishing**: $\lim_{t\to\infty}[h(t) - g(t)] < \infty$*, and*

$$\begin{aligned}
&\lim_{t\to\infty} U(t,x) = \frac{(1-p)b}{k+q},\ \lim_{t\to\infty} V(t,x) = \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)} \text{ uniformly in } \mathbb{R}, \\
&\lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|R(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} = 0.
\end{aligned}$$

Theorem 1.3. *In Theorem 1.2, the dichotomy can be determined as follows: for fixed D_1, D_2, D_3, and D_4, we have:*

(i) *If $\mathcal{R}_0 \le 1$, then vanishing happens for any $(U_0, V_0, I_0, R_0, P_0, Q_0)$.*

(ii) *If $\mathcal{R}_0 > 1$, then there is a critical value $h^* > 0$ independent of $(U_0, V_0, I_0, R_0, P_0, Q_0)$ such that spreading happens when $h_0 \ge h^*$, and if $h_0 < h^*$ and $\|U_0\|_\infty \le \frac{(1-p)b}{k+q}$, $\|V_0\|_\infty \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)}$, then there exists $\mu^* \ge \mu_* > 0$ depending on $(U_0, V_0, I_0, R_0, P_0, Q_0)$ such that spreading happens for $\mu > \mu^*$, and vanishing happens for $\mu \le \mu_*$ and $\mu = \mu^*$.*

Remark 1.4. *In this paper, our primary focus is on the influence of the initial infected domain on the dynamics of* (1.2) *for a fixed dispersal rate. Additionally, for fixed* h_0*, we can follow the argument in [22] to investigate how the sign of the principal eigenvalue is affected by the dispersal rate. Then, we will know the impact of the dispersal rate on the dynamics of* (1.2)*.*

Remark 1.5. *It is crucial to highlight that we identify* $\mathcal{R}_0$ *as an important parameter in our analysis of the corresponding eigenvalue problem. The above results indicate that this parameter acts analogously as the basic reproduction number, and we call it* ***the risk index*** *rather than the basic reproduction number. Observing the expression of* $\mathcal{R}_0$*, we note that it decreases with respect to* η *and increases in* q *and* θ*. By understanding* η*,* q*, and* θ*, we conclude that greater environmental pollution correlates with elevated values of* q *and* θ*, while a reduced removal rate of the hyper-infective pathogen corresponds to a diminished value of* η*. Consequently, it follows from Theorem 1.3 that environmental pollution and bacterial hyper-infectivity can increase the chance of epidemic spreading.*

The rest of the paper is organized as follows. In Section 2, we first obtain the global existence and uniqueness of solution (Theorem 1.1). Then, the criteria for spreading and vanishing are established (Theorem 1.3) in Section 3. Finally, we give the longtime behavior of (U, V, I, R, P, Q) when spreading happens (Theorem 1.2) in Section 4.

2. The existence and uniqueness of solution

In this section, we mainly prove that problem (1.2) has a unique global solution. At first, we obtain the local existence and uniqueness of solution by the contraction mapping theorem.

Theorem 2.1. *For any given* $(U_0, V_0, I_0, R_0, P_0, Q_0)$ *and any* $\alpha \in (0, 1)$*, there is a* $T > 0$ *such that* (1.2) *admits a unique solution*

$$(U, V, I, R, P, Q, g, h) \in [\mathbb{C}_T^1]^2 \times [\mathbb{C}_T^2]^2 \times [C^{1,1-}(\overline{D}_{g,h}^T)]^2 \times [C^{1+\frac{\alpha}{2}}(0, T]]^2, \tag{2.1}$$

where

$$\begin{aligned}
&\mathbb{C}_T^1 = L^\infty(\Delta_T) \cap C_{loc}^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T),\ \mathbb{C}_T^2 = W_p^{1,2}(D_{g,h}^T) \cap C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_{g,h}^T),\\
&\Delta_T = \{(t, x) \in \mathbb{R}^2 :\ t \in [0, T],\ x \in \mathbb{R}\},\ D_{g,h}^T = \{(t, x) \in \mathbb{R}^2 :\ t \in (0, T],\ x \in [g(t), h(t)]\}.
\end{aligned}$$

Proof. The proof of this theorem can be done by following the steps of [23, Theorem 2.1] and [24, Theorem 1.1] with some modifications. In the following, we give the main steps for completeness.

Step 1: For any $T > 0$, let

$$\begin{aligned}
&A_1 = \max\left\{\frac{(1-p)b}{q+k}, \|U_0\|_\infty\right\},\ A_2 = \max\left\{\frac{pb + qA_1}{k}, \|V_0\|_\infty\right\},\\
&A_3 = \max\left\{\frac{b\alpha}{k\eta}, \|P_0\|_\infty\right\},\ A_4 = \max\left\{\frac{b\alpha}{kd}, \|Q_0\|_\infty\right\},
\end{aligned}$$

and

$$X_{U_0}^T := \{U \in C(\Delta_T) :\ U(0, x) = U_0(x),\ 0 \le U \le A_1\},$$

$$X_{V_0}^T := \{V \in C(\Delta_T): \ V(0,x) = V_0(x), \ 0 \le V \le A_2\},$$
$$X_{P_0}^T := \{P \in C(D_{g,h}^T): \ P(0,x) = P_0(x), \ 0 \le P \le A_3\},$$
$$X_{Q_0}^T := \{Q \in C(D_{g,h}^T): \ Q(0,x) = Q_0(x), \ 0 \le Q \le A_4\}.$$

For any given $(U, V, P, Q) \in X_{U_0}^T \times X_{V_0}^T \times X_{P_0}^T \times X_{Q_0}^T$, we consider

$$\begin{cases} I_t = D_3 I_{xx} + \beta_1(U+\theta V)P + \frac{\beta_2(U+\theta V)Q}{m+Q} + \beta_3(U+\theta V)I - \gamma I - kI, & t>0, \ x \in (g(t), h(t)), \\ I(t,x) = 0, & t>0, \ x \in \mathbb{R}\backslash(g(t), h(t)), \\ g'(t) = -\mu I_x(t, g(t)), \ h'(t) = -\mu I_x(t, h(t)), & t>0, \\ -g(0) = h(0) = h_0, \ I(0,x) = I_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{2.2}$$

Following the steps of [25, Theorem 1.1] with some modifications, we can find some $0 < T_1 \ll 1$ such that (2.2) admits a unique solution $(I, g, h) \in [W_p^{1,2}(D_{g,h}^{T_1}) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_{g,h}^{T_1})] \times [C^{1+\frac{\alpha}{2}}([0, T_1])]^2$ for any $\alpha \in (0, 1-3/p)$, and

$$-2h_0 \le g(t) < h(t) \le 2h_0, \ ||I||_{W_p^{1,2}(D_{g,h}^{T_1})} + ||g||_{C^{1+\frac{\alpha}{2}}([0,T_1])} + ||h||_{C^{1+\frac{\alpha}{2}}([0,T_1])} \le C_1.$$

Define

$$\widetilde{P}_0(x) = \begin{cases} P_0(x), & |x| \le h_0, \\ 0, & |x| > h_0 \end{cases} \quad \text{and} \quad \widetilde{Q}_0(x) = \begin{cases} Q_0(x), & |x| \le h_0, \\ 0, & |x| > h_0. \end{cases}$$

By $P_0, Q_0 \in C^{1-}([-h_0, h_0])$, we have $\widetilde{P}_0, \widetilde{Q}_0 \in C^{1-}([g(T_1), h(T_1)])$. For above $g(t)$ and $h(t)$, we define

$$t_x = \begin{cases} g^{-1}(x), & x \in [g(T_1), -h_0), \\ 0, & x \in [-h_0, h_0], \\ h^{-1}(x), & x \in (h_0, h(T_1)]. \end{cases}$$

For above $I(t, x)$ and any $x \in [g(T_1), h(T_1)]$, we consider

$$\begin{cases} \widetilde{P}_t = \alpha I(t,x) - \eta \widetilde{P}, & t_x < t \le T_1, \\ \widetilde{Q}_t = \eta \widetilde{P} - d\widetilde{Q}, & t_x < t \le T_1, \\ \widetilde{P}(t_x, x) = \widetilde{P}_0(x), \ \widetilde{Q}(t_x, x) = \widetilde{Q}_0(x), \end{cases} \tag{2.3}$$

and it follows from the standard theory of ODEs that there exists some $T_2 \in (0, T_1)$ such that $\widetilde{P}(t, x)$ and $\widetilde{Q}(t, x)$ are well defined on $[t_x, T_2]$ for any $x \in [g(T_2), h(T_2)]$, and then $\widetilde{P}(t, x)$ and $\widetilde{Q}(t, x)$ are also well defined on $\overline{D}_{g,h}^{T_2}$. Moreover, we can obtain that $\widetilde{P}$ and $\widetilde{Q}$ are Lipschitz continuous in x by similar arguments in step 2 of [24, Theorem 1.1], and then $\widetilde{P}, \widetilde{Q} \in C^{1,1-}(\overline{D}_{g,h}^{T_2})$.

For above $I(t, x)$, $\widetilde{P}(t, x)$, and $\widetilde{Q}(t, x)$, we consider

$$\begin{cases} U_t = D_1 U_{xx} + (1-p)b - qU - \beta_1 U\widetilde{P} - \frac{\beta_2 U\widetilde{Q}}{m+\widetilde{Q}} - \beta_3 UI - kU, & t>0, \ x \in \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.4}$$

By the standard theory in [26, 27], (2.4) has a unique solution $\widetilde{U} \in C_b(\Delta_T) \cap C_{loc}^{1+\frac{\alpha}{2}, 2+\alpha}(\Delta_T)$, where $C_b(\Delta_T)$ is the space of continuous and bounded functions in Δ_T.

For above $I(t,x)$, $\widetilde{P}(t,x)$, $\widetilde{Q}(t,x)$, and $\widetilde{U}(t,x)$, we consider

$$\begin{cases} V_t = D_2 V_{xx} + pb + q\widetilde{U} - \beta_1\theta V\widetilde{P} - \frac{\beta_2\theta V\widetilde{Q}}{m+\widetilde{Q}} - \beta_3\theta VI - kV, & t>0,\ x\in\mathbb{R}, \\ V(0,x) = V_0(x), & x\in\mathbb{R}. \end{cases} \tag{2.5}$$

By the standard theory in [26,27], (2.5) has a unique solution $\widetilde{V} \in C_b(\Delta_T) \cap C_{loc}^{1+\frac{\alpha}{2},2+\alpha}(\Delta_T)$.

Step 2: Denote $\Pi_T = [0,T]\times[-2h_0, 2h_0]$. By arguing as in the arguments in step 2 of [24, Theorem 1.1], we can find a constant M such that

$$|P(t,x) - P(t,y)| \le 2M|x-y|,\ |Q(t,x) - Q(t,y)| \le 2M|x-y| \text{ for } (t,x),(t,y)\in\Pi_T. \tag{2.6}$$

Define

$$\begin{aligned} Y_{P_0}^T &= \{P\in C(\Pi_T):\ P(0,x) = P_0(x),\ 0\le P\le A_3,\ |P(t,x)-P(t,y)|\le 2M|x-y|\}, \\ Y_{Q_0}^T &= \{Q\in C(\Pi_T):\ Q(0,x) = Q_0(x),\ 0\le Q\le A_4,\ |Q(t,x)-Q(t,y)|\le 2M|x-y|\}, \\ Y^T &= X_{U_0}^T \times X_{V_0}^T \times Y_{P_0}^T \times Y_{Q_0}^T. \end{aligned}$$

Obviously, Y^T is complete with the metric,

$$d((U_1,V_1,P_1,Q_1),(U_2,V_2,P_2,Q_2)) = \sup_{(t,x)\in\Delta_T}(|U_1-U_2| + |V_1-V_2|) + \max_{(t,x)\in\Pi_T}(|P_1-P_2| + |Q_1-Q_2|).$$

Define a map

$$\mathcal{F}(U,V,P,Q) = (\widetilde{U},\widetilde{V},\widetilde{P},\widetilde{Q}) \text{ for } (U,V,P,Q)\in Y^T.$$

In the following, we will prove $\mathcal{F}$ maps Y^T into itself and $\mathcal{F}$ is a contraction mapping on Y^T for all small T. Then we can obtain that $\mathcal{F}$ has a unique fixed point by the contraction mapping theorem.

By the comparison principle, we have $\widetilde{U}\le A_1$ and $\widetilde{V}\le A_2$ for $t>0$ and $x\in\mathbb{R}$, and $\widetilde{P}\le A_3$ and $\widetilde{Q}\le A_4$ for $t>0$ and $x\in[g(t),h(t)]$. Combined with (2.6), $\widetilde{U},\widetilde{V}\in C_b([0,T]\times\mathbb{R})\cap C_{loc}^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\mathbb{R})$ and $\widetilde{P},\widetilde{Q}\in C^{1,1-}(\overline{D}_{g,h}^{T_2})$, we have $(\widetilde{U},\widetilde{V},\widetilde{P},\widetilde{Q})\in Y^T$ for $T\le T_2$, namely, $\mathcal{F}$ maps $X_{S_0}^T$ into itself for $T\le T_2$.

For $(U_i,V_i,P_i,Q_i)\in X_{U_0}^T\times X_{V_0}^T\times Y_{P_0}^T\times Y_{Q_0}^T$ $(i=1,2)$, let (I_i,g_i,h_i) be the unique solution of (2.2) with $(U,V,P,Q) = (U_i,V_i,P_i,Q_i)$, let $(\widetilde{P}_i,\widetilde{Q}_i)$ be the unique solution of (2.3) with $I = I_i$, let $\widetilde{U}_i$ be the unique solution of (2.4) with $(I,\widetilde{P},\widetilde{Q}) = (I_i,\widetilde{P}_i,\widetilde{Q}_i)$, and let $\widetilde{V}_i$ be the unique solution of (2.5) with $(I,\widetilde{P},\widetilde{Q},\widetilde{U}) = (I_i,\widetilde{P}_i,\widetilde{Q}_i,\widetilde{U}_i)$. Denote $\Omega_T = D_{g_1,h_1}^T \cup D_{g_2,h_2}^T$ and

$$\begin{aligned} &\mathcal{U} = U_1 - U_2,\ \mathcal{V} = V_1 - V_2,\ \mathcal{P} = P_1 - P_2,\ \mathcal{Q} = Q_1 - Q_2, \\ &\widetilde{\mathcal{U}} = \widetilde{U}_1 - \widetilde{U}_2,\ \widetilde{\mathcal{V}} = \widetilde{V}_1 - \widetilde{V}_2,\ \widetilde{\mathcal{P}} = \widetilde{P}_1 - \widetilde{P}_2,\ \widetilde{\mathcal{Q}} = \widetilde{Q}_1 - \widetilde{Q}_2, \\ &\mathcal{I} = I_1 - I_2,\ \mathcal{G} = g_1 - g_2,\ \mathcal{H} = h_1 - h_2. \end{aligned}$$

Noting that $I_i(t,x) = \widetilde{P}_i(t,x) = \widetilde{Q}_i(t,x) = 0$ for $t>0$ and $x\in\mathbb{R}\backslash(g_i(t),h_i(t))$, we then have

$$\begin{cases} \widetilde{\mathcal{V}}_t = D_2\widetilde{\mathcal{V}}_{xx} + q\widetilde{\mathcal{U}} - \left(\beta_1\theta\widetilde{P}_1 + \beta_2\theta\frac{\widetilde{Q}_1}{m+\widetilde{Q}_1} + \beta_3\theta I_1 + k\right)\widetilde{\mathcal{V}} & \\ \qquad -\beta_1\theta\widetilde{V}_2\widetilde{\mathcal{P}} - \beta_2\theta\widetilde{V}_2\left(\frac{\widetilde{Q}_1}{m+\widetilde{Q}_1} - \frac{\widetilde{Q}_2}{m+\widetilde{Q}_2}\right) - \beta_3\theta\widetilde{V}_2\mathcal{I}, & t>0, x\in\mathbb{R}, \\ \widetilde{\mathcal{V}}(0,x) = 0, & x\in\mathbb{R}. \end{cases}$$

Similar to [28, (2.6)], one can apply the classical L^p estimate for parabolic equations to derive that

$$\|\widetilde{\mathcal{V}}\|_{L^\infty([0,T]\times\mathbb{R})} \le C_2\left(\|\widetilde{\mathcal{U}}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)} + \|\widetilde{\mathcal{Q}}\|_{C(\overline{\Omega}_T)} + \|\mathcal{I}\|_{C(\overline{\Omega}_T)}\right). \tag{2.7}$$

Noting that $I_i(t,x) = \widetilde{P}_i(t,x) = \widetilde{Q}_i(t,x) = 0$ for $t > 0$ and $x \in \mathbb{R}\backslash(g_i(t), h_i(t))$, then we have

$$\begin{cases} \widetilde{\mathcal{U}}_t = D_1\widetilde{\mathcal{U}}_{xx} - \left(q + \beta_1\widetilde{P}_1 + \beta_2\frac{\widetilde{Q}_1}{m+\widetilde{Q}_1} + \beta_3 I_1 + k\right)\widetilde{\mathcal{U}} & \\ \qquad -\beta_1\widetilde{U}_2\widetilde{\mathcal{P}} - \beta_2\widetilde{U}_2\left(\frac{\widetilde{Q}_1}{m+\widetilde{Q}_1} - \frac{\widetilde{Q}_2}{m+\widetilde{Q}_2}\right) - \beta_3\widetilde{U}_2\mathcal{I}, & t > 0, x \in \mathbb{R}, \\ \widetilde{\mathcal{U}}(0,x) = 0, & x \in \mathbb{R}. \end{cases}$$

It follows from the standard L^p theory and Sobolev's embedding theorem that we can obtain

$$\|\widetilde{\mathcal{U}}\|_{L^\infty([0,T]\times\mathbb{R})} \le C_3\left(\|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)} + \|\widetilde{\mathcal{Q}}\|_{C(\overline{\Omega}_T)} + \|\mathcal{I}\|_{C(\overline{\Omega}_T)}\right). \tag{2.8}$$

In the following, we estimate $\|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)}$ and $\|\widetilde{\mathcal{Q}}\|_{C(\overline{\Omega}_T)}$. Similar to the arguments in the proof of [24, (2.9)], we can have

$$\begin{aligned} \|\widetilde{\mathcal{Q}}\|_{C(\overline{\Omega}_T)} &\le C_4(\|\mathcal{G}\|_{C([0,T])} + \|\mathcal{H}\|_{C([0,T])}) + TC_5(\|\widetilde{\mathcal{Q}}\|_{C(\overline{\Omega}_T)} + \|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)}), \\ \|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)} &\le C_6(\|\mathcal{G}\|_{C([0,T])} + \|\mathcal{H}\|_{C([0,T])}) + TC_7(\|\widetilde{\mathcal{P}}\|_{C(\overline{\Omega}_T)} + \|\mathcal{I}\|_{C(\overline{\Omega}_T)}), \end{aligned} \tag{2.9}$$

where C_4 depends on $\eta A_3 + dA_4$ and σ, C_6 depends on $\alpha C_1 + \eta A_3$ and σ, C_5 depends on $\max\{\eta, d\}$, and C_7 depends on $\max\{\alpha, \eta\}$.

By following the steps in the proof of [24, (2.10)] with some modifications, we can have

$$\begin{aligned} &\|\mathcal{G}\|_{C([0,T])} + \|\mathcal{H}\|_{C([0,T])} \le T(\|\mathcal{G}\|_{C^1([0,T])} + \|\mathcal{H}\|_{C^1([0,T])}) \le C_8 T\|\mathcal{I}\|_{C(\overline{\Delta}_T)}, \\ &\|\mathcal{I}\|_{C(\overline{\Omega}_T)} \le C_9(\|\mathcal{U}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\mathcal{V}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\mathcal{P}\|_{C(\overline{\Delta}_T)} + \|\mathcal{Q}\|_{C(\overline{\Delta}_T)}). \end{aligned} \tag{2.10}$$

By (2.7)–(2.10), we have

$$\begin{aligned} &\|\widetilde{\mathcal{U}}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\widetilde{\mathcal{V}}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\widetilde{\mathcal{P}}\|_{C(\overline{\Delta}_T)} + \|\widetilde{\mathcal{Q}}\|_{C(\overline{\Delta}_T)} \\ \le\ &\frac{1}{3}(\|\mathcal{U}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\mathcal{V}\|_{L^\infty([0,T]\times\mathbb{R})} + \|\mathcal{P}\|_{C(\overline{\Delta}_T)} + \|\mathcal{Q}\|_{C(\overline{\Delta}_T)}), \end{aligned}$$

for $0 < T \ll 1$.

Therefore, $\mathcal{F}$ is a contraction mapping for small T, and then $\mathcal{F}$ has a unique fixed point denoted by (U, V, P, Q). For such (U, V, P, Q), we can obtain that (2.2) has a unique solution (I, g, h). For above (I, g, h), we can get a unique R satisfying the fourth equation of (1.2) and the corresponding initial condition in (1.3). By the fact that $U \le A_1$ and $V \le A_2$ for $t > 0$ and $x \in \mathbb{R}$, and $P \le A_3$ and $Q \le A_4$ for $t > 0$ and $x \in [g(t), h(t)]$, problem (1.2) has a unique local solution (U, V, I, R, P, Q, g, h). Moreover, we can obtain the regularity (2.1) by the above arguments. This completes the proof of the theorem.

In the following, we prove the global existence and uniqueness of solution by extending the local solution above.

Proof of Theorem 1.1: Applying the comparison principle, it is easy to obtain that $U \le A_1$ and $V \le A_2$ for $t > 0$ and $x \in \mathbb{R}$, and $P \le A_3$ and $Q \le A_4$ for $t > 0$ and $x \in [g(t), h(t)]$. In view of the equations satisfied by I and R, we can find two positive constants A_5 and A_6 such that $I(t,x) \le A_5$ and $R(t,x) \le A_6$ for $(t,x) \in \overline{D}^T_{g,h}$. By following the steps in the proof of [29, Lemma 2.1] with some modifications, we find an $A_7 > 0$ such that $0 < -g'(t), h'(t) \le A_7$ for $t \in [0,T]$. Using the above estimates, we can extend the local solution in Theorem 2.1 to the global solution by following the arguments in [25]. This completes the proof of the theorem.

3. Criteria for spreading and vanishing

By $g'(t) < 0$ and $h'(t) > 0$, we have that $g(t)$ is monotonically decreasing in t and $h(t)$ is monotonically increasing in t, which implies that there exist $g_\infty \in [-\infty, 0)$ and $h_\infty \in (0, \infty]$ such that $\lim_{t\to\infty} g(t) = g_\infty$ and $\lim_{t\to\infty} h(t) = h_\infty$. Since the spreading of disease depends on whether $h_\infty - g_\infty = \infty$ and $\lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} > 0$ or not, we give the following definition.

Definition 3.1. *The disease is* ***spreading*** *if*

$$h_\infty - g_\infty = \infty \text{ and } \lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} > 0;$$

the disease is ***vanishing*** *if*

$$h_\infty - g_\infty < \infty \text{ and } \lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

Before giving the criteria for spreading and vanishing, we first prove the following result, which shows that vanishing will happen if $\lim_{t\to\infty}[h(t) - g(t)] < \infty$.

Lemma 3.2. *If* $\lim_{t\to\infty}[h(t) - g(t)] < \infty$*, then*

$$\lim_{t\to\infty} U(t,x) = \frac{(1-p)b}{k+q},\ \lim_{t\to\infty} V(t,x) = \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)} \text{ uniformly in } \mathbb{R},$$
$$\lim_{t\to+\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|R(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

Proof. It follows from [30, Proposition 2] that

$$\lim_{t\to\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

By [31, Lemma 2.6], we have

$$\lim_{t\to\infty} \|R(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

Noting that $I(t,x) = 0$ for $t \geq 0$ and $x \in \mathbb{R}\backslash(g(t), h(t))$, then, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$I(t,x) \leq \varepsilon \text{ for } t > T \text{ and } x \in \mathbb{R}.$$

Then, P satisfies

$$\begin{cases} P_t \leq \alpha\varepsilon - \eta P, & t > T,\ x \in (g(t), h(t)), \\ P(t,x) = 0, & t > T,\ x = g(t) \text{ or } h(t), \\ P(T,x) > 0. \end{cases}$$

Applying the comparison principle, we get

$$\lim_{t\to\infty} \|P(t,\cdot)\|_{C([g(t),h(t)])} \leq \frac{\alpha\varepsilon}{\eta}.$$

By the arbitrariness of ε, we have

$$\lim_{t\to\infty} \|P(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

Similarly, we have

$$\lim_{t\to\infty} \|Q(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

It is easy to obtain that

$$U(t,x) \le \frac{(1-p)b}{k+q},\ V(t,x) \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)},\ t>0,\ x\in\mathbb{R}.$$

On the other hand, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$I(t,x) \le \varepsilon,\ P(t,x) \le \varepsilon,\ Q(t,x) \le \varepsilon \text{ for } t > T \text{ and } x \in \mathbb{R}.$$

Then,

$$\begin{cases} U_t \ge D_1 U_{xx} + (1-p)b - qU - \beta_1 U\varepsilon - \frac{\beta_2 U\varepsilon}{m+\varepsilon} - \beta_3 U\varepsilon - kU, & t > T,\ x\in\mathbb{R}, \\ U(T,x) > 0, & x \in \mathbb{R}. \end{cases}$$

Let $\underline{U}$ be the solution of

$$\begin{cases} \underline{U}_t = (1-p)b - q\underline{U} - \beta_1 \underline{U}\varepsilon - \frac{\beta_2 \underline{U}\varepsilon}{m+\varepsilon} - \beta_3 \underline{U}\varepsilon - k\underline{U}, & t > T, \\ \underline{U}(T,x) = 0. \end{cases}$$

It is well known that

$$\lim_{t\to\infty} \underline{U}(t) = \frac{(1-p)b}{k+q+\beta_1\varepsilon + \frac{\beta_2\varepsilon}{m+\varepsilon} + \beta_3\varepsilon}.$$

Applying the comparison principle, we have

$$U(t,x) \ge \underline{U}(t) \text{ for } t > T \text{ and } x \in \mathbb{R}.$$

Thus,

$$\liminf_{t\to\infty} U(t,x) \ge \frac{(1-p)b}{k+q+\beta_1\varepsilon + \frac{\beta_2\varepsilon}{m+\varepsilon} + \beta_3\varepsilon} \text{ uniformly in } \mathbb{R}.$$

By the arbitrariness of ε, we have

$$\liminf_{t\to\infty} U(t,x) \ge \frac{(1-p)b}{k+q} \text{ uniformly in } \mathbb{R}.$$

Hence,

$$\lim_{t\to\infty} U(t,x) = \frac{(1-p)b}{k+q} \text{ uniformly in } \mathbb{R}.$$

Repeating the same arguments as above, we can conclude that

$$\lim_{t\to\infty} V(t,x) = \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)} \text{ uniformly in } \mathbb{R}.$$

This completes the proof of the lemma.

In the following, we give the criteria for spreading and vanishing. The following arguments are divided into two cases according to the value of $\mathcal{R}_0$, which is given in (1.4).

3.1. The case of $\mathcal{R}_0 \le 1$

The next lemma shows that if $\mathcal{R}_0 \le 1$, then vanishing will happen no matter what the initial data are.

Lemma 3.3. *If $\mathcal{R}_0 \le 1$, then $\lim_{t\to\infty}[h(t) - g(t)] < \infty$.*

Proof. Noting that $\mathcal{R}_0 \le 1$ and

$$U(t,x) \le \frac{(1-p)b}{k+q},\ V(t,x) \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)} \text{ for } t > 0 \text{ and } x \in \mathbb{R},$$

we have

$$U + \theta V \le \frac{(1-p)b}{k+q} + \frac{\theta pb}{k} + \frac{\theta q(1-p)b}{k(k+q)} = \Lambda \text{ for } t > 0 \text{ and } x \in \mathbb{R},$$

and then

$$\begin{aligned}
&\frac{d}{dt}\int_{g(t)}^{h(t)}\left[I(t,x) + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)P(t,x) + \frac{\beta_2\Lambda}{dm}Q(t,x)\right]dx \\
=& \int_{g(t)}^{h(t)}\left[I_t(t,x) + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)P_t(t,x) + \frac{\beta_2\Lambda}{dm}Q_t(t,x)\right]dx \\
&+ h'(t)\left[I(t,h(t)) + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)P(t,h(t)) + \frac{\beta_2\Lambda}{dm}Q(t,h(t))\right] \\
&- g'(t)\left[I(t,g(t)) + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)P(t,g(t)) + \frac{\beta_2\Lambda}{dm}Q(t,g(t))\right] \\
=& \int_{g(t)}^{h(t)}\left[D_3 I_{xx} + \beta_1(U+\theta V)P + \frac{\beta_2(U+\theta V)Q}{m+Q} + \beta_3(U+\theta V)I - \gamma I - kI\right. \\
&\left. + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)(\alpha I - \eta P) + \frac{\beta_2\Lambda}{dm}(\eta P - dQ)\right]dx \\
\le& \int_{g(t)}^{h(t)}\left[D_3 I_{xx} + \beta_1\Lambda P + \frac{\beta_2\Lambda Q}{m} + \beta_3\Lambda I - \gamma I - kI\right. \\
&\left. + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)(\alpha I - \eta P) + \frac{\beta_2\Lambda}{dm}(\eta P - dQ)\right]dx \\
=& \int_{g(t)}^{h(t)}\left[D_3 I_{xx} + \beta_3\Lambda I - \gamma I - kI + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)\alpha I\right]dx \\
=& D_3[I_x(t,h(t)) - I_x(t,g(t))] + \int_{g(t)}^{h(t)}(\gamma + k)(\mathcal{R}_0 - 1)I dx \\
\le& -\frac{D_3}{\mu}[h'(t) - g'(t)].
\end{aligned}$$

Integrating from 0 to t yields

$$h(t) - g(t) \le 2h_0 + \frac{\mu}{D_3}\int_{-h_0}^{h_0}\left[I_0(x) + \left(\frac{\beta_1\Lambda}{\eta} + \frac{\beta_2\Lambda}{dm}\right)P_0(x) + \frac{\beta_2\Lambda}{dm}Q_0(x)\right]dx < \infty,\ t > 0.$$

Hence, $\lim_{t\to\infty}[h(t) - g(t)] < \infty$. This completes the proof of the lemma.

3.2. The case of $\mathcal{R}_0 > 1$

In this subsection, we always assume $\mathcal{R}_0 > 1$. Before giving the criteria for spreading and vanishing, we first study the corresponding eigenvalue problem.

It is well known that the eigenvalue problem

$$\begin{cases} D_3\phi'' + a_{11}\phi = \eta\phi, & x \in (-L, L), \\ \phi(x) = 0, & x = \pm L, \end{cases}$$

admits a principal eigenvalue denoted by η_0, and its corresponding eigenvector is $\widetilde{\phi}$.

Consider the following eigenvalue problem:

$$\begin{cases} D_3\phi'' + a_{11}\phi + a_{12}\varphi + a_{13}\psi = \lambda\phi, & x \in (-L, L), \\ a_{21}\phi + a_{22}\varphi = \lambda\varphi, & x \in (-L, L), \\ a_{32}\varphi + a_{33}\psi = \lambda\psi, & x \in (-L, L), \\ \phi(x) = \varphi(x) = \psi(x) = 0, & x = \pm L, \end{cases} \tag{3.1}$$

where $a_{12}, a_{13}, a_{21}, a_{32} > 0, a_{22}, a_{33} < 0$, and $a_{11} \in \mathbb{R}$ are constants. Then, we have the following lemma.

Lemma 3.4. *The following properties hold:*

(i) *Problem* (3.1) *has a principal simple eigenvalue λ_1 with a positive eigenfunction* (ϕ, φ, ψ)*;*

(ii) *λ_1 has the same sign as $\eta_0 - \frac{a_{12}a_{21}}{a_{22}} + \frac{a_{13}a_{32}a_{21}}{a_{33}a_{22}}$.*

Proof. (i) Define

$$\mathcal{L}_\lambda\phi = D_3\phi'' + \left(a_{11} + \frac{a_{12}a_{21}}{(\lambda - a_{22})} + \frac{a_{13}a_{32}a_{21}}{(\lambda - a_{33})(\lambda - a_{22})}\right)\phi,$$

with $\lambda > \max\{a_{22}, a_{33}\}$. Set

$$Q(\lambda) = \lambda^3 - (a_{22} + a_{33} + \eta_0)\lambda^2 + (a_{22}a_{33} + \eta_0 a_{22} + \eta_0 a_{33})\lambda - \eta_0 a_{22}a_{33} - a_{13}a_{32}a_{21}.$$

Let λ_0 be the largest root of $Q(\lambda) = 0$. Since $Q(a_{22}) = -a_{13}a_{32}a_{21} < 0$ and $Q(a_{33}) = -a_{13}a_{32}a_{21} < 0$, we have $\lambda_0 > \max\{a_{22}, a_{33}\}$. For such λ_0, it follows that

$$\begin{aligned} \mathcal{L}_{\lambda_0}\widetilde{\phi} &= D_3\widetilde{\phi}'' + \left[a_{11} + \frac{a_{12}a_{21}}{(\lambda_0 - a_{22})} + \frac{a_{13}a_{32}a_{21}}{(\lambda_0 - a_{33})(\lambda_0 - a_{22})}\right]\widetilde{\phi} \\ &= \left[\eta_0 + \frac{a_{12}a_{21}}{(\lambda_0 - a_{22})} + \frac{a_{13}a_{32}a_{21}}{(\lambda_0 - a_{33})(\lambda_0 - a_{22})}\right]\widetilde{\phi} \\ &\geq \left[\eta_0 + \frac{a_{13}a_{32}a_{21}}{(\lambda_0 - a_{33})(\lambda_0 - a_{22})}\right]\widetilde{\phi} \\ &= \lambda_0\widetilde{\phi}. \end{aligned}$$

Consequently, $e^{\lambda_0 t}\widetilde{\phi}(x)$ is a subsolution of $u_t = \mathcal{L}_{\lambda_0}u$. By [32, Theorem 2.3] and [32, Remark 2.1], problem (3.1) has an eigenvalue with geometric multiplicity one denoted by λ_1 and a nonnegative eigenpair $(\phi(x), \varphi(x), \psi(x))$. Using (3.1) and its associated parabolic system, we easily see that this eigenpair is positive.

(ii) It is clear that $\mathcal{L}_{\lambda_1}\phi = \lambda_1\phi$. Then,

$$\eta_0 = \lambda_1 - \frac{a_{12}a_{21}}{(\lambda_1 - a_{22})} - \frac{a_{13}a_{32}a_{21}}{(\lambda_1 - a_{33})(\lambda_1 - a_{22})},$$

namely,

$$\begin{aligned}&\eta_0 - \frac{a_{12}a_{21}}{a_{22}} + \frac{a_{13}a_{32}a_{21}}{a_{33}a_{22}}\\ =\ &\lambda_1 - \frac{a_{12}a_{21}}{(\lambda_1 - a_{22})} - \frac{a_{13}a_{32}a_{21}}{(\lambda_1 - a_{33})(\lambda_1 - a_{22})} + \frac{a_{12}a_{21}}{-a_{22}} + \frac{a_{13}a_{32}a_{21}}{a_{33}a_{22}} =: f(\lambda_1).\end{aligned}$$

Since $a_{22}, a_{33} < 0$, $f(\lambda_1)$ is monotone increasing in λ_1, and $f(0) = 0$, we can obtain that λ_1 has the same sign as $\eta_0 - \frac{a_{12}a_{21}}{a_{22}} + \frac{a_{13}a_{32}a_{21}}{a_{33}a_{22}}$. This completes the proof of the lemma.

In the following, we write $\lambda_1(L)$ instead of λ_1 to stress the dependence of λ_1 on L. Since

$$\eta_0 = a_{11} - \frac{D_3\pi^2}{4L^2},$$

we have the following corollary.

Corollary 3.5. *Define* $\Gamma = a_{11} - \frac{a_{12}a_{21}}{a_{22}} + \frac{a_{13}a_{32}a_{21}}{a_{33}a_{22}}$. *Then, we have:*
(i) *If* $\Gamma \le 0$, *then* $\lambda_1 < 0$ *for any* L;
(ii) *If* $\Gamma > 0$, *then there exists a unique* L^* *such that* $\lambda_1(L^*) = 0$, *and* $\lambda_1(L)(L - L^*) > 0$ *for* $L \neq L^*$.

Let us recall that Λ is given by (1.5). Let $(\lambda_1(L), \phi(x), \varphi(x), \psi(x))$ be the first eigenpair of (3.1) with $a_{11} = \beta_3\Lambda - \gamma - k$, $a_{12} = \beta_1\Lambda$, $a_{13} = \frac{\beta_2}{m}\Lambda$, $a_{21} = \alpha$, $a_{22} = -\eta$, $a_{32} = \eta$, and $a_{33} = -d$. Then, by $\mathcal{R}_0 > 1$, we have

$$\Gamma = \beta_3\Lambda - \gamma - k - \frac{\beta_1\Lambda\alpha}{-\eta} + \frac{\beta_2\Lambda\eta\alpha}{md\eta} = (\gamma + k)(\mathcal{R}_0 - 1) > 0.$$

By the above corollary, we can find a unique $h^* := L^*(\beta_3\Lambda - \gamma - k, \beta_1\Lambda, \frac{\beta_2}{m}\Lambda, \alpha, -\eta, \eta, -d) > 0$ such that $\lambda_1(h^*) = 0$ and $\lambda_1(L)(L - h^*) > 0$ for $L \neq h^*$.

The next result shows that if $h_0 \ge h^*$, then spreading will always happen no matter what the spreading capacity μ is.

Lemma 3.6. *If* $h_0 \ge h^*$, *then spreading happens.*

Proof. We only need to prove that if $\lim_{t\to\infty}[h(t) - g(t)] < \infty$, then $\lim_{t\to\infty}[h(t) - g(t)] \le 2h^*$. Assume on the contrary that $2h^* < \lim_{t\to\infty}[h(t) - g(t)] < \infty$. Then there exists $\varepsilon \in (0, \Lambda)$ such that

$$\lim_{t\to\infty}[h(t) - g(t)] > 2h^*_\varepsilon := 2L^*(\beta_3(\Lambda - \varepsilon) - \gamma - k, \beta_1(\Lambda - \varepsilon), \frac{\beta_2}{m}(\Lambda - \varepsilon), \alpha, -\eta, \eta, -d).$$

Then, we can obtain from Lemma 3.2 that, for the above ε, there exists $T > 0$ such that $h(T) - g(T) > 2h^*_\varepsilon$ and

$$U + \theta V \ge \frac{(1-p)b}{k+q} + \frac{\theta pb}{k} + \frac{\theta q(1-p)b}{k(k+q)} - \varepsilon = \Lambda - \varepsilon \text{ for } t \ge T \text{ and } x \in [g_\infty, h_\infty].$$

Therefore,

$$\begin{cases} I_t \geq D_3 I_{xx}+\beta_1(\Lambda-\varepsilon)P+\beta_2(\Lambda-\varepsilon)\frac{Q}{m+Q}+\beta_3(\Lambda-\varepsilon)I-\gamma I-kI, & t>T,\ x\in(g(T),h(T)), \\ P_t=\alpha I-\eta P, & t>T,\ x\in(g(T),h(T)), \\ Q_t=\eta P-dQ, & t>T,\ x\in(g(T),h(T)), \\ I(t,x)>0,\ P(t,x)>0,\ Q(t,x)>0, & t>T,\ x=g(T) \text{ or } h(T), \\ I(T,x)\geq 0,\ P(T,x)\geq 0,\ Q(T,x)\geq 0, & x\in[g(T),h(T)]. \end{cases} \tag{3.2}$$

Let $(\lambda_1(L), \phi(x), \varphi(x), \psi(x))$ be the eigenpair of (3.1) with $L = \frac{h(T)-g(T)}{2}$, $a_{11} = \beta_3(\Lambda-\varepsilon)-\gamma-k$, $a_{12} = \beta_1(\Lambda-\varepsilon)$, $a_{13} = \frac{\beta_2}{m}(\Lambda-\varepsilon)$, $a_{21} = \alpha$, $a_{22} = -\eta$, $a_{32} = \eta$, and $a_{33} = -d$. Then, $\lambda_1(L) > 0$. We define

$$\underline{I}(t,x) = \delta\phi\left(x - \frac{g(T)+h(T)}{2}\right),$$
$$\underline{P}(t,x) = \delta\varphi\left(x - \frac{g(T)+h(T)}{2}\right),$$
$$\underline{Q}(t,x) = \delta\psi\left(x - \frac{g(T)+h(T)}{2}\right),$$

for $t \geq T$ and $x \in [g(T), h(T)]$. By the direct computations, we have that, for $t > T$ and $x \in (g(T), h(T))$,

$$\begin{aligned} &\underline{I}_t - D_3\underline{I}_{xx} - \beta_1(\Lambda-\varepsilon)\underline{P} - \beta_2(\Lambda-\varepsilon)\frac{\underline{Q}}{m+\underline{Q}} - \beta_3(\Lambda-\varepsilon)\underline{I} + \gamma\underline{I} + k\underline{I} \\ =&\ -D_3\delta\phi'' - \beta_1(\Lambda-\varepsilon)\delta\varphi - \beta_2(\Lambda-\varepsilon)\frac{\delta\psi}{m+\delta\psi} - \beta_3(\Lambda-\varepsilon)\delta\phi + \gamma\delta\phi + k\delta\phi \\ =&\ \delta\left[\beta_2(\Lambda-\varepsilon)\left(\frac{\psi}{m} - \frac{\psi}{m+\delta\psi}\right) - \lambda_1\phi\right] \\ =&\ \delta\phi\left[\frac{a_{32}}{a_{33}-\lambda_1}\frac{a_{21}}{a_{22}-\lambda_1}\beta_2(\Lambda-\varepsilon)\left(\frac{1}{m} - \frac{1}{m+\delta\psi}\right) - \lambda_1\right] =: \Delta, \end{aligned}$$

$$\underline{P}_t - \alpha\underline{I} + \eta\underline{P} = -\alpha\delta\phi + \eta\delta\varphi = -\lambda_1\delta\varphi < 0,$$

and

$$\underline{Q}_t - \eta\underline{P} + d\underline{Q} = -\eta\delta\varphi + d\delta\psi = -\lambda_1\delta\psi < 0.$$

We can choose some $\delta > 0$ small enough such that $\Delta < 0$ and

$$I(0,x) \geq \underline{I}(0,x),\ P(0,x) \geq \underline{P}(0,x) \text{ and } Q(0,x) \geq \underline{Q}(0,x).$$

Recalling that $\underline{I}(t,x) = \underline{P}(t,x) = \underline{Q}(t,x) = 0$ for $x = g(T)$ or $h(T)$, we can apply the comparison principle to conclude that

$$I(t,x) \geq \underline{I}(t,x),\ P(t,x) \geq \underline{P}(t,x),\ Q(t,x) \geq \underline{Q}(t,x) \text{ for } t \geq T \text{ and } x \in [g(T), h(T)],$$

which implies that $\lim_{t\to\infty} \|I(t,\cdot)\|_{C([g(t),h(t)])} + \|P(t,\cdot)\|_{C([g(t),h(t)])} + \|Q(t,\cdot)\|_{C([g(t),h(t)])} > 0$. This contradicts Lemma 3.2. This completes the proof of the lemma.

In the following, we show that, under some conditions, if $h_0 < h^*$, then vanishing will happen for small μ.

Lemma 3.7. *Assume that* $\|U_0\|_\infty \le \frac{(1-p)b}{k+q}$ *and* $\|V_0\|_\infty \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)}$. *If* $h_0 < h^*$, *then there exists some* μ_0 *such that vanishing happens for* $\mu \le \mu_0$.

Proof. Thanks to $\|U_0\|_\infty \le \frac{(1-p)b}{k+q}$ and $\|V_0\|_\infty \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)}$, we can use the comparison principle to obtain

$$U(t,x) + \theta V(t,x) \le \Lambda \text{ for } t > 0 \text{ and } x \in \mathbb{R},$$

and then we have

$$\begin{cases} I_t \le D_3 I_{xx} + \beta_1 \Lambda P + \frac{\beta_2 \Lambda}{m} Q + \beta_3 \Lambda I - \gamma I - kI, & t > 0,\ x \in (g(t), h(t)), \\ P_t = \alpha I - \eta P, & t > 0,\ x \in (g(t), h(t)), \\ Q_t = \eta P - dQ, & t > 0,\ x \in (g(t), h(t)), \\ I(t,x) = P(t,x) = Q(t,x) = 0, & t > 0,\ x \le g(t) \text{ or } x \ge h(t), \\ g(0) = -h_0,\ g'(t) = -\mu I_x(t, g(t)), & t > 0, \\ h(0) = h_0,\ h'(t) = -\mu I_x(t, h(t)), & t > 0, \\ I(0,x) = I_0(x),\ P(0,x) = P_0(x),\ Q(0,x) = Q_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{3.3}$$

Let $(\lambda_1(L),\ \phi(x),\ \varphi(x),\ \psi(x))$ be the eigenpair of (3.1) with $L = h_0$, $a_{11} = \beta_3\Lambda - \gamma - k$, $a_{12} = \beta_1\Lambda$, $a_{13} = \frac{\beta_2}{m}\Lambda$, $a_{21} = \alpha$, $a_{22} = -\eta$, $a_{32} = \eta$, and $a_{33} = -d$, then $\lambda_1 < 0$. Set

$$\begin{aligned} &\sigma(t) = h_0(1 + \delta - \frac{\delta}{2} e^{-\delta t}),\ t \ge 0, \\ &\overline{I}(t,x) = Me^{-\delta t}\phi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \ge 0,\ x \in [-\sigma(t), \sigma(t)], \\ &\overline{P}(t,x) = Me^{-\delta t}\varphi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \ge 0,\ x \in [-\sigma(t), \sigma(t)], \\ &\overline{Q}(t,x) = Me^{-\delta t}\psi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \ge 0,\ x \in [-\sigma(t), \sigma(t)], \end{aligned}$$

where the positive parameters δ and M will be determined later. Direct computations yield that

$$\begin{aligned} &\overline{I}_t - D_3 \overline{I}_{xx} - \beta_1 \Lambda \overline{P} - \frac{\beta_2 \Lambda}{m} \overline{Q} - \beta_3 \Lambda \overline{I} + \gamma \overline{I} + k\overline{I} \\ &= Me^{-\delta t}\left(-\delta\phi - \frac{h_0 x \sigma'}{\sigma^2(t)}\phi' - D_3 \phi'' \frac{h_0^2}{\sigma^2} - \beta_1 \Lambda \varphi - \frac{\beta_2 \Lambda}{m}\psi - \beta_3 \Lambda \phi + \gamma\phi + k\phi\right) \\ &= Me^{-\delta t}\left[-\delta\phi + \left(\beta_1 \Lambda \varphi + \frac{\beta_2 \Lambda}{m}\psi + \beta_3 \Lambda \phi - \gamma\phi - k\phi\right)\left(\frac{h_0^2}{\sigma^2} - 1\right) - \lambda_1 \phi \frac{h_0^2}{\sigma^2}\right] - Me^{-\delta t}\frac{h_0 x \sigma'}{\sigma^2(t)}\phi' \\ &= Me^{-\delta t}\phi\left[-\delta + \left(\frac{a_{21}}{\lambda_1 - a_{22}}\beta_1 \Lambda + \frac{a_{32}}{\lambda_1 - a_{33}}\frac{a_{21}}{\lambda_1 - a_{22}}\frac{\beta_2 \Lambda}{m} + \beta_3 \Lambda - \gamma - k\right)\left(\frac{h_0^2}{\sigma^2} - 1\right) - \lambda_1 \frac{h_0^2}{\sigma^2}\right] \\ &\quad - Me^{-\delta t}\frac{h_0 x \sigma'}{\sigma^2(t)}\phi' =: \Delta_1, \end{aligned}$$

$$\begin{aligned}
&\overline{P}_t - \alpha\overline{I} + \eta\overline{P} \\
&= Me^{-\delta t}\left[-\delta\varphi - \frac{h_0 x\sigma'}{\sigma^2(t)}\varphi' - \alpha\phi + \eta\varphi\right] \\
&= Me^{-\delta t}(-\delta\varphi - \lambda_1\varphi) - Me^{-\delta t}\frac{h_0 x\sigma'}{\sigma^2(t)}\varphi' \\
&= Me^{-\delta t}\varphi(-\delta - \lambda_1) - Me^{-\delta t}\frac{h_0 x\sigma'}{\sigma^2(t)}\varphi' =: \Delta_2,
\end{aligned}$$

and

$$\overline{Q}_t - \eta\overline{P} + d\overline{Q} = Me^{-\delta t}\psi(-\delta - \lambda_1) - Me^{-\delta t}\frac{h_0 x\sigma'}{\sigma^2(t)}\psi' =: \Delta_3.$$

We choose sufficiently small $\delta > 0$ such that $\delta < -\lambda_1$ and

$$-\delta + \left(\frac{a_{21}}{\lambda_1 - a_{22}}\beta_1\Lambda + \frac{a_{32}}{\lambda_1 - a_{33}}\frac{a_{21}}{\lambda_1 - a_{22}}\frac{\beta_2\Lambda}{m} + \beta_3\Lambda - \gamma - k\right)\left(\frac{h_0^2}{\sigma^2} - 1\right) - \lambda_1\frac{h_0^2}{\sigma^2} > 0,$$

and then we can use the similar arguments as in [33, Lemma 3.5] to conclude that

$$\Delta_1 \geq 0,\ \Delta_2 \geq 0,\ \Delta_3 \geq 0.$$

We choose sufficiently large $M > 0$ such that, for $x \in [-h_0, h_0]$,

$$M\phi\left(\frac{h_0 x}{h_0(1+\delta/2)}\right) \geq I_0(x),\ M\varphi\left(\frac{h_0 x}{h_0(1+\delta/2)}\right) \geq P_0(x),\ M\psi\left(\frac{h_0 x}{h_0(1+\delta/2)}\right) \geq Q_0(x).$$

If $\mu \leq \frac{h_0\delta^2}{-2M\phi'(h_0)} =: \mu_0$, then

$$\sigma'(t) = h_0\frac{\delta^2}{2}e^{-\delta t} \geq -\mu Me^{-\delta t}\phi'(h_0) \geq -\mu Me^{-\delta t}\phi'(h_0)\frac{h_0}{\sigma(t)} = -\mu\overline{I}_x(t, \sigma(t)).$$

Similarly, $-\sigma'(t) \leq -\mu\overline{I}_x(t, -\sigma(t))$. By

$$\sigma(0) > h_0,\ \overline{I}(t, \pm\sigma(t)) = \overline{P}(t, \pm\sigma(t)) = \overline{Q}(t, \pm\sigma(t)) = 0 \text{ for } t > 0,$$

we can use the comparison principle to conclude that

$$-\sigma(t) \leq g(t),\ h(t) \leq \sigma(t) \text{ for } t \geq 0.$$

Then, we have that $\lim_{t\to\infty}[h(t) - g(t)] \leq 2\lim_{t\to\infty}\sigma(t) \leq 2h_0(1+\delta) < \infty$. Hence, vanishing will happen. This completes the proof of the lemma.

Finally, we show that if $h_0 < h^*$, then spreading will happen for large μ.

Lemma 3.8. *If $h_0 < h^*$, then there exists some μ^0 such that spreading happens for $\mu \geq \mu^0$.*

Proof. Consider the following problem:

$$\begin{cases} W_t = D_3 W_{xx} - (\gamma + k)W, & t > 0, x \in (r(t), s(t)), \\ W(t,x) = 0, & t > 0, x \le r(t) \text{ or } x \ge s(t), \\ r'(t) = -\mu W_x(t, r(t)),\ s'(t) = -\mu W_x(t, s(t)), & t > 0, \\ s(0) = -r(0) = h_0,\ W(0,x) = I_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{3.4}$$

By following the steps in the proof of [6] with some modifications, we can conclude that (3.4) admits a unique solution, denoted by (W, r, s). By the comparison principle, we have

$$I(t,x) \ge W(t,x),\ g(t) \le r(t),\ s(t) \le h(t) \text{ for } t > 0 \text{ and } x \in [r(t), s(t)].$$

Next, we show that there exists a $T > 0$ such that $r(T) - s(T) \ge 2h^*$. We first choose the smooth functions $\underline{r}(t)$, $\underline{s}(t)$, and $\underline{W}_0$ satisfying

$$\underline{s}(0) = -\underline{r}(0) = h_0,\ \underline{s}(T) - \underline{r}(T) = 2h^*,\ \underline{s}'(t) > 0,\ \underline{r}'(t) < 0 \text{ for } t > 0,$$

$$0 < \underline{W}_0(x) \le I_0(x) \text{ for } x \in [-h_0, h_0],\ \underline{W}_0(-h_0) = \underline{W}_0(h_0) = 0.$$

Consider the following problem:

$$\begin{cases} \underline{W}_t = D_3 \underline{W}_{xx} - (\gamma + k)\underline{W}, & t > 0, \underline{r}(t) \le x \le \underline{s}(t), \\ \underline{W}(t, \underline{r}(t)) = \underline{W}(t, \underline{s}(t)) = 0, & t > 0, \\ \underline{W}(0,x) = \underline{W}_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{3.5}$$

By the standard theory, this problem admits a unique positive solution $\underline{W}(t,x)$, $\underline{W}_x(t, \underline{s}(t)) < 0$ and $\underline{W}_x(t, \underline{r}(t)) > 0$. Then we can find a μ^0 such that, for $\mu \ge \mu^0$,

$$\underline{s}'(t) \le -\mu \underline{W}_x(t, \underline{s}(t)),\ \underline{r}'(t) \ge -\mu \underline{W}_x(t, \underline{r}(t)),\ t \in [0, T].$$

Thus, we have

$$W(t,x) \ge \underline{W}(t,x),\ r(t) \le \underline{r}(t),\ \underline{s}(t) \le s(t) \text{ for } 0 \le t \le T \text{ and } \underline{r}(t) \le x \le \underline{s}(t).$$

Therefore, $h(T) - g(T) \ge s(T) - r(T) \ge \underline{s}(T) - \underline{r}(T) = 2h^*$. By Lemma 3.6, we have $\lim_{t\to\infty}[h(t) - g(t)] = \infty$. This completes the proof of the lemma.

By similar arguments as in [29, Theorem 5.2], it follows from Lemmas 3.7 and 3.8 that we have the following lemma.

Lemma 3.9. *If $h_0 < h^*$ and $\|U_0\|_\infty \le \frac{(1-p)b}{k+q}$, $\|V_0\|_\infty \le \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)}$, then there exists $\mu^* \ge \mu_* > 0$ depending on $(U_0, V_0, I_0, R_0, P_0, Q_0)$ such that spreading happens for $\mu > \mu^*$, and vanishing happens for $\mu \le \mu_*$ and $\mu = \mu^*$.*

Theorem 1.3 can be obtained by Lemmas 3.3, 3.6, and 3.9.

4. The longtime behavior of (1.2) for spreading

In this section, we give the longtime behavior of the solution (U, V, I, R, P, Q) to (1.2) for spreading. At first, we give the following lemma, which implies that $[g(t), h(t)]$ will be $\mathbb{R}$ if $\lim_{t\to\infty}[h(t) - g(t)] = \infty$.

Lemma 4.1. *If* $\lim_{t\to\infty}[h(t) - g(t)] = \infty$, *then* $\lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = \infty$.

Proof. We can prove this lemma by following the steps in [33, Lemma 3.10] with some modifications. Here, we omit the details.

Without loss of generality, we assume on the contrary that $\lim_{t\to\infty} g(t) = -\infty$ and $\lim_{t\to\infty} h(t) < \infty$. Taking $L > 2h^* + 2$, we can find a $T_0 > 0$ such that $g(T_0) < -L$.

First, we use [23, Lemma 3.3] to conclude that

$$\lim_{t\to\infty} \|I(t,\cdot)\|_{C([-L,h(t)])} = 0.$$

Then, by a similar argument as in the proof of Lemma 3.2, we have

$$\lim_{t\to\infty} \max_{x\in[1-L,h(T_0)]} P(t, x) = 0, \ \lim_{t\to\infty} \max_{x\in[1-L,h(T_0)]} Q(t, x) = 0.$$

There exists some small ε_1 such that $L - 1 > 2h^*_\varepsilon$ for $\varepsilon \in (0, \varepsilon_1)$. We choose l_1 and l_2 such that $[l_1, l_2] \subset [1 - L, h(T_0)]$ and $l_2 - l_1 \geq 2h^*_\varepsilon$. Using the argument in step 3 of the proof in [33, Lemma 3.10], we can conclude that, for above L and small $\varepsilon \in (0, \varepsilon_1)$, we can find a $T_1 > 0$ such that

$$U(t, x) \geq \frac{(1-p)b}{k+q} - \frac{\varepsilon}{2}, \ V(t, x) \geq \frac{pb}{k} + \frac{q(1-p)b}{k(k+q)} - \frac{\varepsilon}{2\theta} \text{ for } t \geq T_1 \text{ and } x \in [l_1, l_2].$$

For $\varepsilon \in (0, \varepsilon_1)$ and $T > T_1$, (I, P, Q) satisfies

$$\begin{cases} I_t \geq D_3 I_{xx} + \beta_1(\Lambda-\varepsilon)P + \frac{\beta_2(\Lambda-\varepsilon)Q}{m+Q} + \beta_3(\Lambda-\varepsilon)I - \gamma I - kI, & t > T, \ x \in (l_1, l_2), \\ P_t = \alpha I - \eta P, & t > T, \ x \in (l_1, l_2), \\ Q_t = \eta P - dQ, & t > T, \ x \in (l_1, l_2), \\ I(t, x) > 0, \ P(t, x) > 0, \ Q(t, x) > 0, & t > T, \ x = l_1 \text{ or } l_2, \\ I(T, x) \geq 0, \ P(T, x) \geq 0, \ Q(T, x) \geq 0, & x \in [l_1, l_2]. \end{cases}$$

Finally, we can use similar arguments as in the proof of Lemma 3.6 to obtain

$$\liminf_{t\to\infty} I(t, x) > 0 \text{ for } x \in [l_1, l_2],$$

which is a contradiction. This completes the proof of the lemma.

In the following, we apply the iterative method to give the longtime behavior of the solution $(U,\ V,\ I,\ R,\ P,\ Q)$ to (1.2) for spreading under some additional condition.

Lemma 4.2. *Assume that* $\mathcal{R}_0 > 1$ *and* $\frac{\beta_3+\frac{\alpha}{\eta}\beta_1}{\gamma+k}\Lambda + \frac{k+\beta_2}{\Lambda k+b}\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1} < 1$. *If* $\lim_{t\to\infty}[h(t) - g(t)] = \infty$, *then*

$$\lim_{t\to\infty}(U, V, I, R, P, Q) = (U^*, V^*, I^*, R^*, P^*, Q^*),$$

uniformly for x in any bounded set of $\mathbb{R}$, where $(U^*, V^*, I^*, R^*, P^*, Q^*)$ is a unique positive constant root of

$$\begin{cases}(1-p)b - qU - \beta_1 UP - \frac{\beta_2 UQ}{m+Q} - \beta_3 UI - kU = 0,\\ pb + qU - \beta_1\theta VP - \frac{\beta_2\theta VQ}{m+Q} - \beta_3\theta VI - kV = 0,\\ \beta_1(U+\theta V)P + \frac{\beta_2(U+\theta V)Q}{m+Q} + \beta_3(U+\theta V)I - \gamma I - kI = 0,\\ \gamma I - kR = 0,\\ \alpha I - \eta P = 0,\\ \eta P - dQ = 0.\end{cases} \tag{4.1}$$

Proof. This lemma will be proved by the following iterative method:

Step 1: Clearly,

$$\lim_{t\to\infty} U(t,x) \le \frac{(1-p)b}{k+q} =: \overline{U}_1 \text{ uniformly in } \mathbb{R},$$

and then

$$\lim_{t\to\infty} V(t,x) \le \frac{pb + q\overline{U}_1}{k} =: \overline{V}_1 \text{ uniformly in } \mathbb{R}.$$

Then, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$U(t,x) \le \overline{U}_1 + \frac{\varepsilon}{2},\ V(t,x) \le \overline{V}_1 + \frac{\varepsilon}{2\theta} \text{ for } t \ge T \text{ and } x \in \mathbb{R}.$$

Thus, (I, P, Q) satisfies

$$\begin{cases} I_t \le D_3 I_{xx} + \beta_1(\overline{U}_1 + \theta\overline{V}_1 + \varepsilon)P + \frac{\beta_2(\overline{U}_1+\theta\overline{V}_1+\varepsilon)Q}{m+Q} & \\ \qquad + \beta_3(\overline{U}_1 + \theta\overline{V}_1 + \varepsilon)I - \gamma I - kI, & t > T,\ x \in (g(t), h(t)),\\ P_t = \alpha I - \eta P, & t > T,\ x \in (g(t), h(t)),\\ Q_t = \eta P - dQ, & t > T,\ x \in (g(t), h(t)),\\ I(t,x) = P(t,x) = Q(t,x) = 0, & t > T,\ x \le g(t) \text{ or } x \ge h(t),\\ I(T,x) \ge 0,\ P(T,x) \ge 0,\ Q(T,x) \ge 0, & x \in [g(T), h(T)].\end{cases}$$

Let $(\overline{I}, \overline{P}, \overline{Q})$ be the solution of

$$\begin{cases} \overline{I}'(t) = \beta_1(\overline{U}_1 + \theta\overline{V}_1 + \varepsilon)\overline{P} + \frac{\beta_2(\overline{U}_1+\theta\overline{V}_1+\varepsilon)\overline{Q}}{m+\overline{Q}} & \\ \qquad + \beta_3(\overline{U}_1 + \theta\overline{V}_1 + \varepsilon)\overline{I} - \gamma\overline{I} - k\overline{I}, & t > T,\\ \overline{P}'(t) = \alpha\overline{I} - \eta\overline{P}, & t > T,\\ \overline{Q}'(t) = \eta\overline{P} - d\overline{Q}, & t > T,\\ \overline{I}(T) \ge \|I(T,\cdot)\|_\infty,\ \overline{P}(T) \ge \|P(T,\cdot)\|_\infty,\ \overline{Q}(T) \ge \|Q(T,\cdot)\|_\infty. & \end{cases} \tag{4.2}$$

We can use the comparison principle to conclude that $I(t,x) \le \overline{I}(t)$, $P(t,x) \le \overline{P}(t)$, and $Q(t,x) \le \overline{Q}(t)$ for $t \ge T$ and $x \in \mathbb{R}$. In view of $\mathcal{R}_0 > 1$, we have that the basic reproduction number of (4.2) is larger than 1, and then $\lim_{t\to\infty}(\overline{I}(t), \overline{P}(t), \overline{Q}(t)) = (\overline{I}_1^\varepsilon, \overline{P}_1^\varepsilon, \overline{Q}_1^\varepsilon)$, where $(\overline{I}_1^\varepsilon, \overline{P}_1^\varepsilon, \overline{P}_1^\varepsilon)$ is the unique positive constant endemic equilibrium of (4.2). Thus,

$$\limsup_{t\to\infty} I(t,x) \le \overline{I}_1^\varepsilon,\ \limsup_{t\to\infty} P(t,x) \le \overline{P}_1^\varepsilon,\ \limsup_{t\to\infty} Q(t,x) \le \overline{Q}_1^\varepsilon \text{ uniformly in } \mathbb{R}.$$

By the arbitrariness of ε, we have

$$\limsup_{t\to\infty} I(t,x) \le \overline{I}_1, \limsup_{t\to\infty} P(t,x) \le \overline{P}_1,\ \limsup_{t\to\infty} Q(t,x) \le \overline{Q}_1 \text{ uniformly in } \mathbb{R},$$

where $(\overline{I}_1, \overline{P}_1, \overline{Q}_1)$ is the unique positive constant root of

$$\begin{cases} \beta_1(\overline{U}_1 + \theta\overline{V}_1)\overline{P} + \frac{\beta_2(\overline{U}_1+\theta\overline{V}_1)\overline{Q}}{m+\overline{Q}} + \beta_3(\overline{U}_1 + \theta\overline{V}_1)\overline{I} - \gamma\overline{I} - k\overline{I} = 0, \\ \alpha\overline{I} - \eta\overline{P} = 0, \\ \eta\overline{P} - d\overline{Q} = 0. \end{cases}$$

By direct calculations, we have

$$\overline{Q}_1 = \frac{\frac{\beta_2\alpha}{d}\Lambda}{\gamma + k - \beta_3\Lambda - \frac{\beta_1\alpha}{\eta}\Lambda} - m,\ \overline{I}_1 = \frac{d}{\alpha}\overline{Q}_1 \text{ and } \overline{P}_1 = \frac{d}{\eta}\overline{Q}_1,$$

where Λ is defined in (1.5). Moreover, $\overline{I}_1$, $\overline{P}_1$, and $\overline{Q}_1$ are positive by $\mathcal{R}_0 > 1$ and $\frac{\left(\frac{\beta_1\alpha}{\eta}+\beta_3\right)\Lambda}{\gamma+k} < 1$.

Step 2: For small $\varepsilon > 0$, there exists $T > 0$ such that

$$I(t,x) \le \overline{I}_1 + \varepsilon,\ P(t,x) \le \overline{P}_1 + \varepsilon,\ Q(t,x) \le \overline{Q}_1 + \varepsilon \text{ for } t \ge T \text{ and } x \in \mathbb{R}.$$

Thus, U satisfies

$$\begin{cases} U_t \ge D_1 U_{xx} + (1-p)b - qU - \beta_1 U(\overline{P}_1 + \varepsilon) - \frac{\beta_2 U(\overline{Q}_1+\varepsilon)}{m+\overline{Q}_1+\varepsilon} & \\ \quad -\beta_3 U(\overline{I}_1 + \varepsilon) - kU, & t > T,\ x \in \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

and then

$$\liminf_{t\to\infty} U(t,x) \ge \frac{(1-p)b}{q + \beta_1(\overline{P}_1 + \varepsilon) + \frac{\beta_2(\overline{Q}_1+\varepsilon)}{m+\overline{Q}_1+\varepsilon} + \beta_3(\overline{I}_1 + \varepsilon) + k} =: \underline{U}_1^{\varepsilon} \text{ uniformly in } \mathbb{R}.$$

By the arbitrariness of ε, we have

$$\liminf_{t\to\infty} U(t,x) \ge \underline{U}_1 \text{ uniformly in } \mathbb{R},$$

where $\underline{U}_1$ is the unique positive constant root of

$$(1-p)b - qU - \beta_1 U\overline{P}_1 - \frac{\beta_2 U\overline{Q}_1}{m + \overline{Q}_1} - \beta_3 U\overline{I}_1 - kU = 0.$$

By the direct calculation, we have

$$\underline{U}_1 = \frac{(1-p)b}{k + q + \frac{\gamma+k}{\Lambda}\overline{I}_1} > 0.$$

For small $\varepsilon > 0$, there exists $T > 0$ such that

$$U(t,x) \geq \underline{U}_1 - \varepsilon \text{ for } t \geq T \text{ and } x \in \mathbb{R}.$$

Thus, V satisfies

$$\begin{cases} V_t \geq D_2 V_{xx} + pb + q(\underline{U}_1 - \varepsilon) - \beta_1 \theta V(\overline{P}_1 + \varepsilon) - \frac{\beta_2 \theta V(\overline{Q}_1 + \varepsilon)}{m + \overline{Q}_1 + \varepsilon} & \\ \qquad -\beta_3 \theta V(\overline{I}_1 + \varepsilon) - kV, & t > T,\ x \in \mathbb{R}, \\ V(0,x) = V_0(x), & x \in \mathbb{R}, \end{cases}$$

and then

$$\liminf_{t\to\infty} V(t,x) \geq \frac{pb + q(\underline{U}_1 - \varepsilon)}{\beta_1 \theta(\overline{P}_1 + \varepsilon) + \frac{\beta_2 \theta(\overline{Q}_1 + \varepsilon)}{m + \overline{Q}_1 + \varepsilon} + \beta_3 \theta(\overline{I}_1 + \varepsilon) + k} =: \underline{V}_1^{\varepsilon} \text{ uniformly in } \mathbb{R}.$$

By the arbitrariness of ε, we have

$$\liminf_{t\to\infty} V(t,x) \geq \underline{V}_1 \text{ uniformly in } \mathbb{R},$$

where $\underline{V}_1$ is the unique positive constant root of

$$pb + q\underline{U}_1 - \beta_1 \theta V \overline{P}_1 - \frac{\beta_2 \theta V \overline{Q}_1}{m + \overline{Q}_1} - \beta_3 \theta V \overline{I}_1 - kV = 0.$$

By the direct calculation, we have

$$\underline{V}_1 = \frac{pb + q\underline{U}_1}{k + \frac{\theta(\gamma + k)}{\Lambda} \overline{I}_1} > 0.$$

For any $\varepsilon > 0$ and any given $L > L^*(\beta_3(\Lambda - \varepsilon) - \gamma - k, \beta_1(\Lambda - \varepsilon), \frac{\beta_2}{m}(\Lambda - \varepsilon), \alpha, -\eta, \eta, -d)$, it follows from $\lim_{t\to\infty}[h(t) - g(t)] = \infty$ that we can find a $T > 0$ such that

$$(g(t), h(t)) \supseteq [-L, L],\ U(t,x) \geq \underline{U}_1 - \frac{\varepsilon}{2},\ V(t,x) \geq \underline{V}_1 - \frac{\varepsilon}{2\theta} \text{ for } t \geq T \text{ and } x \in [-L, L].$$

Thus, (I, P, Q) satisfies

$$\begin{cases} I_t \geq D_3 I_{xx} + \beta_1(\underline{U}_1 + \theta \underline{V}_1 - \varepsilon)P + \frac{\beta_2(\underline{U}_1 + \theta \underline{V}_1 - \varepsilon)Q}{m + Q} & \\ \qquad + \beta_3(\underline{U}_1 + \theta \underline{V}_1 - \varepsilon)I - \gamma I - kI, & t > T,\ x \in (-L, L), \\ P_t = \alpha I - \eta P, & t > T,\ x \in (-L, L), \\ Q_t = \eta P - dQ, & t > T,\ x \in (-L, L), \\ I(t, \pm L) \geq 0,\ P(t, \pm L) \geq 0,\ Q(t, \pm L) \geq 0, & t > T, \\ I(T,x) \geq 0,\ P(T,x) \geq 0,\ Q(T,x) \geq 0, & x \in [-L, L]. \end{cases}$$

Let $(\lambda_1, \phi(x), \varphi(x), \psi(x))$ be the eigenpair of (3.1) with $a_{11} = \beta_3(\Lambda - \varepsilon) - \gamma - k$, $a_{12} = \beta_1(\Lambda - \varepsilon)$, $a_{13} = \frac{\beta_2}{m}(\Lambda - \varepsilon)$, $a_{21} = \alpha$, $a_{22} = -\eta$, $a_{32} = \eta$, and $a_{33} = -d$. Using the comparison principle, we can have that, for small enough δ,

$$(\underline{I}_0(x), \underline{P}_0(x), \underline{Q}_0(x)) = (\delta\phi(x), \delta\varphi(x), \delta\psi(x)) \text{ for } x \in [-L, L],$$

satisfies

$$I(t,x) \geq \underline{I}_0(x),\ P(t,x) \geq \underline{P}_0(x),\ Q(t,x) \geq \underline{Q}_0(x),\ t \geq T,\ x \in [-L, L].$$

Let $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ be the solution of the following auxiliary problem:

$$\begin{cases} \mathcal{U}_t = D_3\mathcal{U}_{xx} + \beta_1(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{V} + \frac{\beta_2(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{W}}{m + \mathcal{W}} & \\ \qquad + \beta_3(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{U} - \gamma\mathcal{U} - k\mathcal{U}, & t > T,\ x \in (-L, L), \\ \mathcal{V}_t = \alpha\mathcal{U} - \eta\mathcal{V}, & t > T,\ x \in (-L, L), \\ \mathcal{W}_t = \eta\mathcal{V} - d\mathcal{W}, & t > T,\ x \in (-L, L), \\ \mathcal{U}(t, \pm L) = \mathcal{V}(t, \pm L) = \mathcal{W}(t, \pm L) = 0, & t > T, \\ \mathcal{U}(T, x) = \underline{I}_0(x),\ \mathcal{V}(T, x) = \underline{P}_0(x),\ \mathcal{W}(T, x) = \underline{Q}_0(x), & x \in [-L, L]. \end{cases}$$

Applying the comparison principle, we derive

$$I(t,x) \geq \mathcal{U}(t,x),\ P(t,x) \geq \mathcal{V}(t,x),\ Q(t,x) \geq \mathcal{W}(t,x),\ t > T,\ x \in [-L, L].$$

By the choice of $(\underline{I}_0(x), \underline{P}_0(x), \underline{Q}_0(x))$, it follows from [34, Lemma 3.5] and [35, Theorem 4.5] that

$$\lim_{t\to\infty}(\mathcal{U}(t,x), \mathcal{V}(t,x), \mathcal{W}(t,x)) = (\mathcal{U}_L(x), \mathcal{V}_L(x), \mathcal{W}_L(x)) \text{ in } C^2([-L, L]),$$

where $(\mathcal{U}_L(x), \mathcal{V}_L(x), \mathcal{W}_L(x))$ is the solution of

$$\begin{cases} D_3\mathcal{U}_{xx} + \beta_1(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{V} + \frac{\beta_2(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{W}}{m + \mathcal{W}} & \\ \qquad + \beta_3(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)\mathcal{U} - \gamma\mathcal{U} - k\mathcal{U} = 0, & x \in (-L, L), \\ \alpha\mathcal{U} - \eta\mathcal{V} = 0, & x \in (-L, L), \\ \eta\mathcal{V} - d\mathcal{W} = 0, & x \in (-L, L), \\ \mathcal{U}(x) = \mathcal{V}(x) = \mathcal{W}(x) = 0, & x = -L \text{ or } L. \end{cases}$$

Moreover,

$$\lim_{L\to\infty}(\mathcal{U}_L(x), \mathcal{V}_L(x), \mathcal{W}_L(x)) = (\underline{I}_1^{\varepsilon}, \underline{P}_1^{\varepsilon}, \underline{Q}_1^{\varepsilon}) \text{ locally uniformly in } \mathbb{R},$$

where $(\underline{I}_1^{\varepsilon}, \underline{P}_1^{\varepsilon}, \underline{Q}_1^{\varepsilon})$ is the unique positive constant root of

$$\begin{cases} \beta_1(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)P + \frac{\beta_2(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)Q}{m + Q} + \beta_3(\underline{U}_1 + \theta\underline{V}_1 - \varepsilon)I - \gamma I - kI = 0, \\ \alpha I - \eta P = 0, \\ \eta P - dQ = 0. \end{cases}$$

By the arbitrariness of ε, we have

$$\liminf_{t\to\infty} I(t,x) \geq \underline{I}_1,\ \liminf_{t\to\infty} P(t,x) \geq \underline{P}_1,\ \liminf_{t\to\infty} Q(t,x) \geq \underline{Q}_1 \text{ locally uniformly in } \mathbb{R},$$

where $(\underline{I}_1, \underline{P}_1, \underline{Q}_1)$ is the unique positive constant root of

$$\begin{cases}\beta_1(\underline{U}_1+\theta\underline{V}_1)P+\frac{\beta_2(\underline{U}_1+\theta\underline{V}_1)Q}{m+Q}+\beta_3(\underline{U}_1+\theta\underline{V}_1)I-\gamma I-kI=0,\\ \alpha I-\eta P=0,\\ \eta P-dQ=0.\end{cases}$$

By direct calculations, we have

$$\underline{Q}_1=\frac{\beta_2(\underline{U}_1+\theta\underline{V}_1)}{\frac{d}{\alpha}[\gamma+k-\beta_3(\underline{U}_1+\theta\underline{V}_1)]-\frac{d}{\eta}\beta_1(\underline{U}_1+\theta\underline{V}_1)}-m,\ \underline{I}_1=\frac{d}{\alpha}\underline{Q}_1 \text{ and } \underline{P}_1=\frac{d}{\eta}\underline{Q}_1.$$

To make sure that $\underline{I}_1$, $\underline{P}_1$, and $\underline{Q}_1$ are positive, we should check that

$$\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1+\frac{\alpha}{dm}\beta_2}<\underline{U}_1+\theta\underline{V}_1<\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1}.$$

In the following, we check this result. According to

$$(1-p)b-q\underline{U}_1-\beta_1\underline{U}_1\overline{P}_1-\frac{\beta_2\underline{U}_1\overline{Q}_1}{m+\overline{Q}_1}-\beta_3\underline{U}_1\overline{I}_1-k\underline{U}_1=0,$$

and

$$pb+q\underline{U}_1-\beta_1\theta\underline{V}_1\overline{P}_1-\frac{\beta_2\theta\underline{V}_1\overline{Q}_1}{m+\overline{Q}_1}-\beta_3\theta\underline{V}_1\overline{I}_1-k\underline{V}_1=0,$$

we have

$$b-\frac{(\gamma+k)\overline{I}_1}{\Lambda}(\underline{U}_1+\theta\underline{V}_1)-k(\underline{U}_1+\underline{V}_1)=0,$$

and then it follows from $\theta>1$ that

$$\underline{U}_1+\theta\underline{V}_1\geq\frac{b}{k+\frac{\gamma+k}{\Lambda}\overline{I}_1}.$$

Then,

$$\begin{aligned}&\underline{U}_1+\theta\underline{V}_1-\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1+\frac{\alpha}{dm}\beta_2}\\ \geq&\frac{b}{k+\frac{\gamma+k}{\Lambda}\overline{I}_1}-\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1}\\ =&\frac{b}{k+\frac{\beta_2}{1-\frac{\beta_3+\frac{\alpha}{\eta}\beta_1}{\gamma+k}\Lambda}}-\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1}=\Delta.\end{aligned}$$

Let $\Pi=:\frac{\beta_3+\frac{\alpha}{\eta}\beta_1}{\gamma+k}\Lambda$. If $\Pi<1$, then $\Delta>0$ is equivalent to

$$\frac{b}{k+\frac{\beta_2}{1-\Pi}}-\frac{\Lambda}{\Pi}>0,$$

namely,

$$\frac{b\Pi}{b+\Lambda k}+\frac{\Lambda(k+\beta_2)}{(b+\Lambda k)\Pi}-1<0,$$

which must hold by $\frac{\beta_3+\frac{\alpha}{\eta}\beta_1}{\gamma+k}\Lambda+\frac{k+\beta_2}{\Lambda k+b}\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1}<1$, and then we have $\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1+\frac{\alpha}{dm}\beta_2}<\underline{U}_1+\theta\underline{V}_1$. On the other hand, $\underline{U}_1+\theta\underline{V}_1<\overline{U}_1+\theta\overline{V}_1<\Lambda<\frac{\gamma+k}{\beta_3+\frac{\alpha}{\eta}\beta_1}$.

Step 3: We can use the similar arguments as in Step 2 to obtain

$$\limsup_{t\to\infty}U(t,x)\le\overline{U}_2 \text{ locally uniformly in } \mathbb{R},$$

where $\overline{U}_2$ is the unique positive constant root of

$$(1-p)b-qU-\beta_1U\underline{P}_1-\frac{\beta_2U\underline{Q}_1}{m+\underline{Q}_1}-\beta_3U\underline{I}_1-kU=0.$$

Similarly, we can derive

$$\limsup_{t\to\infty}V(t,x)\le\overline{V}_2 \text{ locally uniformly in } \mathbb{R},$$

where $\overline{V}_2$ is the unique positive constant root of

$$pb+q\overline{U}_2-\beta_1\theta V\underline{P}_1-\frac{\beta_2\theta V\underline{Q}_1}{m+\underline{Q}_1}-\beta_3\theta V\underline{I}_1-kV=0.$$

Moreover, we have

$$\limsup_{t\to\infty}I(t,x)\le\overline{I}_2,\ \limsup_{t\to\infty}P(t,x)\le\overline{P}_2,\ \limsup_{t\to\infty}Q(t,x)\le\overline{Q}_2 \text{ locally uniformly in } \mathbb{R},$$

where $(\overline{I}_2,\overline{P}_2,\overline{Q}_2)$ is the unique positive constant root of

$$\begin{cases}\beta_1(\overline{U}_1+\theta\overline{V}_1)P+\frac{\beta_2(\overline{U}_1+\theta\overline{V}_1)Q}{m+Q}+\beta_3(\overline{U}_1+\theta\overline{V}_1)I-\gamma I-kI=0,\\ \alpha I-\eta P=0,\\ \eta P-dQ=0.\end{cases}$$

We can repeat the above steps to obtain ten monotone sequences $\{\underline{U}_i\}$, $\{\underline{V}_i\}$, $\{\underline{I}_i\}$, $\{\underline{P}_i\}$, $\{\underline{Q}_i\}$, $\{\overline{U}_i\}$, $\{\overline{V}_i\}$, $\{\overline{I}_i\}$, $\{\overline{P}_i\}$, and $\{\overline{Q}_i\}$ satisfying

$$\begin{aligned}&\underline{U}_i\le\liminf_{t\to\infty}U(t,x)\le\limsup_{t\to\infty}U(t,x)\le\overline{U}_i,\\&\underline{V}_i\le\liminf_{t\to\infty}V(t,x)\le\limsup_{t\to\infty}V(t,x)\le\overline{V}_i,\\&\underline{I}_i\le\liminf_{t\to\infty}I(t,x)\le\limsup_{t\to\infty}I(t,x)\le\overline{I}_i,\\&\underline{P}_i\le\liminf_{t\to\infty}P(t,x)\le\limsup_{t\to\infty}P(t,x)\le\overline{P}_i,\end{aligned}$$

$$\underline{Q}_i \le \liminf_{t\to\infty} Q(t,x) \le \limsup_{t\to\infty} Q(t,x) \le \overline{Q}_i,$$

locally uniformly in $\mathbb{R}$, $\overline{U}_1 = \frac{(1-p)b}{k+q}$, and $\overline{V}_1 = \frac{pb+q\overline{U}_1}{k}$,

$$\begin{cases}\beta_1(\overline{U}_i+\theta\overline{V}_i)\overline{P}_i + \frac{\beta_2(\overline{U}_i+\theta\overline{V}_i)\overline{Q}_i}{m+\overline{Q}_i} + \beta_3(\overline{U}_i+\theta\overline{V}_i)\overline{I}_i - \gamma\overline{I}_i - k\overline{I}_i = 0,\\ \alpha\overline{I}_i - \eta\overline{P}_i = 0,\\ \eta\overline{P}_i - d\overline{Q}_i = 0,\end{cases}$$

$$(1-p)b - q\underline{U}_i - \beta_1\underline{U}_i\overline{P}_i - \frac{\beta_2\underline{U}_i\overline{Q}_i}{m+\overline{Q}_i} - \beta_3\underline{U}_i\overline{I}_i - k\underline{U}_i = 0,$$

$$pb + q\underline{U}_i - \beta_1\theta\underline{V}_i\overline{P}_i - \frac{\beta_2\theta\underline{V}_i\overline{Q}_i}{m+\overline{Q}_i} - \beta_3\theta\underline{V}_i\overline{I}_i - k\underline{V}_i = 0,$$

$$\begin{cases}\beta_1(\underline{U}_i+\theta\underline{V}_i)\underline{P}_i + \frac{\beta_2(\underline{U}_i+\theta\underline{V}_i)\underline{Q}_i}{m+\underline{Q}_i} + \beta_3(\underline{U}_i+\theta\underline{V}_i)\underline{I}_i - \gamma\underline{I}_i - k\underline{I}_i = 0,\\ \alpha\underline{I}_i - \eta\underline{P}_i = 0,\\ \eta\underline{P}_i - d\underline{Q}_i = 0,\end{cases}$$

$$(1-p)b - q\overline{U}_{i+1} - \beta_1\overline{U}_{i+1}\underline{P}_i - \frac{\beta_2\overline{U}_{i+1}\underline{Q}_i}{m+\underline{Q}_i} - \beta_3\overline{U}_{i+1}\underline{I}_i - k\overline{U}_{i+1} = 0,$$

$$pb + q\overline{U}_{i+1} - \beta_1\theta\overline{V}_{i+1}\underline{P}_i - \frac{\beta_2\theta\overline{V}_{i+1}\underline{Q}_i}{m+\underline{Q}_i} - \beta_3\theta\overline{V}_{i+1}\underline{I}_i - k\overline{V}_{i+1} = 0,\ i = 1,2,\cdots.$$

From the above expressions, we have

$$\begin{aligned}&\underline{U}_1 \le \underline{U}_2 \le \cdots \le \underline{U}_i \le \cdots \le \overline{U}_i \le \cdots \le \overline{U}_2 \le \overline{U}_1,\\ &\underline{V}_1 \le \underline{V}_2 \le \cdots \le \underline{V}_i \le \cdots \le \overline{V}_i \le \cdots \le \overline{V}_2 \le \overline{V}_1,\\ &\underline{I}_1 \le \underline{I}_2 \le \cdots \le \underline{I}_i \le \cdots \le \overline{I}_i \le \cdots \le \overline{I}_2 \le \overline{I}_1,\\ &\underline{P}_1 \le \underline{P}_2 \le \cdots \le \underline{P}_i \le \cdots \le \overline{P}_i \le \cdots \le \overline{P}_2 \le \overline{P}_1,\\ &\underline{Q}_1 \le \underline{Q}_2 \le \cdots \le \underline{Q}_i \le \cdots \le \overline{Q}_i \le \cdots \le \overline{Q}_2 \le \overline{Q}_1.\end{aligned}$$

Thus,

$$\lim_{i\to\infty}(\underline{U}_i, \underline{V}_i, \underline{I}_i, \underline{P}_i, \underline{Q}_i) = (\underline{U}_\infty, \underline{V}_\infty, \underline{I}_\infty, \underline{P}_\infty, \underline{Q}_\infty),$$

and

$$\lim_{i\to\infty}(\overline{U}_i, \overline{V}_i, \overline{I}_i, \overline{P}_i, \overline{Q}_i) = (\overline{U}_\infty, \overline{V}_\infty, \overline{I}_\infty, \overline{P}_\infty, \overline{Q}_\infty),$$

are well defined, where $(\underline{U}_\infty, \underline{V}_\infty, \underline{I}_\infty, \underline{P}_\infty, \underline{Q}_\infty)$ and $(\overline{U}_\infty, \overline{V}_\infty, \overline{I}_\infty, \overline{P}_\infty, \overline{Q}_\infty)$ satisfy

$$\begin{cases}\beta_1(\overline{U}_\infty+\theta\overline{V}_\infty)\overline{P}_\infty + \frac{\beta_2(\overline{U}_\infty+\theta\overline{V}_\infty)\overline{Q}_\infty}{m+\overline{Q}_\infty} + \beta_3(\overline{U}_\infty+\theta\overline{V}_\infty)\overline{I}_\infty - \gamma\overline{I}_\infty - k\overline{I}_\infty = 0,\\ \alpha\overline{I}_\infty - \eta\overline{P}_\infty = 0,\\ \eta\overline{P}_\infty - d\overline{Q}_\infty = 0,\end{cases}$$

$$
\begin{aligned}
&(1-p)b - q\underline{U}_\infty - \beta_1 \underline{U}_\infty \overline{P}_\infty - \frac{\beta_2 \underline{U}_\infty \overline{Q}_\infty}{m+\overline{Q}_\infty} - \beta_3 \underline{U}_\infty \overline{I}_\infty - k\underline{U}_\infty = 0,\\
&pb + q\underline{U}_\infty - \beta_1\theta \underline{V}_\infty \overline{P}_\infty - \frac{\beta_2\theta \underline{V}_\infty \overline{Q}_\infty}{m+\overline{Q}_\infty} - \beta_3\theta \underline{V}_\infty \overline{I}_\infty - k\underline{V}_\infty = 0,\\
&\begin{cases}
\beta_1(\underline{U}_\infty + \theta\underline{V}_\infty)\underline{P}_\infty + \frac{\beta_2(\underline{U}_\infty+\theta\underline{V}_\infty)\underline{Q}_\infty}{m+\underline{Q}_\infty} + \beta_3(\underline{U}_\infty + \theta\underline{V}_\infty)\underline{I}_\infty - \gamma\underline{I}_\infty - k\underline{I}_\infty = 0,\\
\alpha\underline{I}_\infty - \eta\underline{P}_\infty = 0,\\
\eta\underline{P}_\infty - d\underline{Q}_\infty = 0,
\end{cases}\\
&(1-p)b - q\overline{U}_\infty - \beta_1 \overline{U}_\infty \underline{P}_\infty - \frac{\beta_2 \overline{U}_\infty \underline{Q}_\infty}{m+\underline{Q}_\infty} - \beta_3 \overline{U}_\infty \underline{I}_\infty - k\overline{U}_\infty = 0,\\
&pb + q\overline{U}_\infty - \beta_1\theta \overline{V}_\infty \underline{P}_\infty - \frac{\beta_2\theta \overline{V}_\infty \underline{Q}_\infty}{m+\underline{Q}_\infty} - \beta_3\theta \overline{V}_\infty \underline{I}_\infty - k\overline{V}_\infty = 0,\ i = 1, 2, \cdots.
\end{aligned}
$$

A series of calculations show that

$$
(\underline{U}_\infty, \underline{V}_\infty, \underline{I}_\infty, \underline{P}_\infty, \underline{Q}_\infty) = (\overline{U}_\infty, \overline{V}_\infty, \overline{I}_\infty, \overline{P}_\infty, \overline{Q}_\infty) = (U^*, V^*, I^*, P^*, Q^*),
$$

where $(U^*, V^*, I^*, P^*, Q^*)$ is a unique positive constant root of

$$
\begin{cases}
(1-p)b - qU - \beta_1 UP - \frac{\beta_2 UQ}{m+Q} - \beta_3 UI - kU = 0,\\
pb + qU - \beta_1\theta VP - \frac{\beta_2\theta VQ}{m+Q} - \beta_3\theta VI - kV = 0,\\
\beta_1(U+\theta V)P + \frac{\beta_2(U+\theta V)Q}{m+Q} + \beta_3(U+\theta V)I - \gamma I - kI = 0,\\
\alpha I - \eta P = 0,\\
\eta P - dQ = 0.
\end{cases}
$$

Finally, by [31, Lemma 2.6], we have that

$$
\lim_{t\to\infty} R = R^* \text{ locally uniformly in } \mathbb{R}.
$$

This completes the proof of the lemma.

Theorem 1.2 can be obtained by Lemmas 3.2 and 4.2.

5. Conclusions

In this paper, we investigate the influence of environmental pollution and bacterial hyper-infectivity on dynamics of a waterborne pathogen model with free boundaries. At first, we prove that the solution to this problem has a unique solution for all $t > 0$. Then, we show that the disease will either spread or vanish. Finally, we find a risk index $\mathcal{R}_0$ such that the disease will vanish if $\mathcal{R}_0 \le 1$, and whether the disease will spread or not depends on the initial data if $\mathcal{R}_0 > 1$, which is very different from that for the reaction diffusion equation without free boundaries. Specifically, under some assumptions, we can find some critical value h^* such that the disease will always spread as long as the initial infected domain is large than $2h^*$; otherwise, the disease will spread if the spreading capacity μ is large. These results will be helpful in taking measures to control the spreading of disease. For example, we can improve the environmental condition and decrease the density of the hyper-infective pathogen by sterilizing.

Although the results in this paper show that model (1.2) can describe the disease well, we only consider the most special situation, and there are many related problems deserving our further study. For example,

(i) we can study the heterogeneous environment to consider the different levels of environment stress in different parts of the spatial domain;

(ii) if we use the same function $\beta_1 P$ (or $\frac{\beta_2 Q}{m+Q}$) to describe the rate of indirect transmission due to contact with environments contaminated by hyper-infectivity and lower-infectivity state of the pathogen, it will be difficult to deal with as we can not calculate the specific expression of $\overline{Q}_1$ in Step 1 of Lemma 4.2;

(iii) it is interesting to study the case where the death rate of U, V, I, and R are different, but this problem is difficult as we can not deal with the term $U + V + I + R$;

(iv) if we do not ignore the diffusion of P and Q, then the corresponding eigenvalue problem will be complex and we will study this case in the future;

(v) if the effect of the pollution on β_i is not the same, this problem will be more complex and deserve our further study;

(vi) it is difficult to use MATLAB to carry out some numerical simulations to illustrate the spreading and vanishing of diseases since there are 19 parameters in (1.2), but taking some simulations is very meaningful and deserves our further study;

(vii) extending model (1.2) to two and three spatial dimensions is more realistic, so we will try to study the high-dimensional extension of (1.2) with radial symmetry in the future.

Author contributions

M. Zhao was responsible for writing the original draft. J. Liu handled the review and supervision. Y. Zhang worked on validating.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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