Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 19(2): 842–866.
DOI: 10.3934/nhm.2024038
Received: 12 July 2024
Revised: 19 August 2024
Accepted: 22 August 2024
Published: 28 August 2024

Research article

Convergence analysis of finite element approximations for a nonlinear second order hyperbolic optimal control problems

Huanhuan Li, Meiling Ding, Xianbing Luo * and Shuwen Xiang

School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China.

* **Correspondence:** Email: xbluo1@gzu.edu.cn.

Abstract: This paper focused on approximating a second-order nonlinear hyperbolic optimal control problem. By introducing a new variable, the hyperbolic equation was converted into two parabolic equations. A second-order fully discrete scheme was obtained by combining the Crank-Nicolson formula with the finite element method. The error estimation for this scheme was derived utilizing the second-order sufficient optimality condition and auxiliary problems. To validate the effectiveness of the fully discrete scheme, a numerical example was presented.

Keywords: second order hyperbolic equation; optimal control; finite element method; a priori error estimates

1. Introduction

The optimal control problems (OCPs) of partial differential equations have been extensively studied in numerous fields of science and engineering applications, including fluid mechanics, earth science, petroleum engineering, telecommunication, etc., (see [1–3]). In the past few decades, scholars have conducted extensive research on OCPs (see [4–6]). From the perspective of control types, there is distributed control and boundary control (see [7–9]). With respect to the types of state equations, there are elliptic equations, parabolic equations, and second-order hyperbolic equations (SOHEs) [10]. Among the common numerical discretization schemes, there are finite element methods (FEM) [11], mixed element methods (MFVM) [12], and finite volume methods (FVM) [13] . In terms of handling optimization problems, there are two approaches: optimize-then-discretize and discretize-then-optimize [14].

The OCPs constrained by SOHEs is an active area of research, attracting significant attention from numerous scholars. Gugat et al. in [15] proposed a valid method based on Lavrentiev regularization and obtained a result similar to the penalty function method for second-order hyperbolic optimal control problems (SOHOCPs) with state constraint. Kröner in [16] used the space-time FEM to discretize SOHOCPs and derived a posteriori error estimates, which separately considers the influence of space, time, and control. In [17], Kröner et al. analyzed the convergence of three types of controls constrained by wave equations utilizing the semismooth Newton method, and numerically implemented the solution in conjunction with the space-time FEM. Lu et al. derived a priori error estimates using the mixed finite element method for a general SOHOCPs in [18]. Luo et al. in [19,20] studied linear SOHOCPs using the FVM and obtained priori error estimates for the Euler and Crank-Nicolson schemes. Lu et al. in [21] used the FVM to study nonlinear SOHOCPs and obtained optimal error estimates for a semi-discrete system. Li et al. in [22] used the FEM and variational discretization approach to investigate linear SOHOCPs by introducing an intermediate variable and obtained optimal priori error estimates.

Inspired by [21,22], we consider the following nonlinear SOHOCPs:

$$\min_{u\in U_{ad}} J(y,u) = \frac{1}{2}\int_0^T\int_\Omega |y-y_d|^2 dxdt + \frac{\alpha}{2}\int_0^T\int_\Omega |u|^2 dxdt, \tag{1.1}$$

such that

$$\begin{cases} y_{tt} - div(A\nabla y) + \phi(y) = f + Bu, & x\in\Omega, t\in(0,T), \\ y(x,t) = 0, & x\in\partial\Omega, t\in(0,T), \\ y(x,0) = y_0(x),\ y_t(x,0) = g(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset R^2$ is a bounded convex polygon domain with boundary $\partial\Omega$. α is a positive number. $T>0$ is a constant. $f, y_d\in L^2(0,T;L^2(\Omega))$ are given functions. $\phi(y)$ is a nonlinear function that satisfies $\phi(\cdot)\in C^2$. For any $R>0$ and $y\in H^1(\Omega)$, the function $\phi(\cdot)\in W^{2,\infty}(-R,R)$, $\phi'(\cdot)\in L^2(\Omega)$, and $\phi'(\cdot)\geq 0$. $A=(a_{ij}(x))_{2\times 2}\in(W^{1,\infty}(\bar{\Omega}))^{2\times 2}$ is symmetric and uniformly positive definite, i.e., for any $X\in R^2$, there exist two positive constants C_1, C_2 such that

$$0 < C_1X^TX \leq X^TA(x)X \leq C_2X^TX < +\infty, \forall x\in\Omega.$$

$B: L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is a bounded linear operator. The space U_{ad} is defined by

$$U_{ad} = \{u\in L^2(0,T;L^2(\Omega)) : u_a \leq u(x,t) \leq u_b,\ a.e.\ in\ \ \Omega\times(0,T]\}.$$

In this work, by introducing a new variable, we transform the hyperbolic equation into two parabolic equations. For the SOHOCPs (1.1) and (1.2), we obtain the continuous first-order necessary condition (FNC) and the second-order sufficient optimality condition (SSC). Using the discretize-then-optimize procedure, we derive the discrete optimality condition for the fully discrete scheme. Based on these, we obtain some optimal error estimates.

The paper is organized as follows. We give some notations in Section 2. In Section 3, we derive the first and second-order optimality conditions. The Crank-Nicolson finite element approximation and a priori error estimates are presented in Section 4. In Section 5, a numerical experiment is presented to confirm the validity of the proposed numerical scheme.

After this, C represents different positive constants in different places, each of which is independent of h and Δt.

2. Preliminary

Here, we first introduce some notations. $W^{m,p}(\Omega)$ is a standard Sobolev space with norm $\|\cdot\|_{W^{m,p}(\Omega)}$. $W^{1,2}(\Omega) = H^1(\Omega)$ with norm $\|\cdot\|_1$, and $H_0^1(\Omega) = \{\upsilon \in H^1(\Omega) : \upsilon|\partial\Omega = 0\}$. $L^k(\Omega)$ denotes a k-squared integrable function space in region Ω with norm $\|\cdot\|$. The norm of $L^k(\Omega)$ is denoted by $\|\cdot\|_{0,U}$. We denote by $L^k(0,T;W^{m,p}(\Omega))$ the Banach space of all L^k integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm $\|\upsilon\|_{L^k(W^{m,p})} = \|\upsilon\|_{L^k(0,T;W^{m,p}(\Omega))} = \left(\int_0^T \|\upsilon\|_{W^{m,p}(\Omega)}^k\right)^{\frac{1}{k}}$ for $k \in [1,\infty)$ and the standard modification for $k = \infty$.

For a positive integer N, define time step size Δt by $\Delta t = \frac{T}{N}$. For $n = 0, 1, ..., N-1$, $t^n = n\Delta t$, $I^{n+1} = [t^n, t^{n+1}]$. write

$$d_t\zeta^{n+1} = \frac{\zeta^{n+1} - \zeta^n}{\Delta t}, \quad \bar{d}_t\zeta^{n+1} = \frac{\zeta^n - \zeta^{n+1}}{\Delta t},$$

and $Q = \Omega \times (0,T]$. For any given sequence $\{\zeta^n\}_{n=0}^M$, $\zeta^n = \zeta(x,t^n)$ for the function $\zeta(x,t)$ defined in Q. For $1 \le q \le \infty$, a discrete time-dependent norm is given by

$$\|\upsilon\|_{l^q(W^{m,p})} := \|\upsilon\|_{l^q(0,T;W^{m,p})} = \left(\sum_{n=1}^N \Delta t \|\upsilon^n\|_{W^{m,p}}^q\right)^{\frac{1}{q}},$$

and the standard modification for $q = \infty$, where

$$l^q(W^{m,p}) := \{\upsilon : \|\upsilon\|_{l^q(0,T;W^{m,p}))} < \infty\}.$$

The inner product is noted by

$$(\upsilon_1, \upsilon_2) = \int_\Omega \upsilon_1\upsilon_2 dx, \quad \forall \upsilon_1, \upsilon_2 \in L^2(\Omega).$$

For convenience, we take A to be the identity matrix and write

$$a(\upsilon_1, \upsilon_2) = \int_\Omega A\nabla\upsilon_1 \cdot \nabla\upsilon_2 dx = (\nabla\upsilon_1, \nabla\upsilon_2), \quad \forall \upsilon_1, \upsilon_2 \in H_0^1(\Omega).$$

It is obvious that

$$a(\upsilon, \upsilon) \ge c_1\|\upsilon\|_1^2, \quad |a(\upsilon_1, \upsilon_2)| \le c_2\|\upsilon_1\|_1\|\upsilon_2\|_1, \quad \forall \upsilon, \upsilon_1, \upsilon_2 \in H_0^1(\Omega).$$

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω, h_K denotes the diameter of element K, and $h = \max_{K\in\mathcal{T}_h}\{h_K\}$. The space V_h associated with $\mathcal{T}_h$ is defined by

$$V_h := \{\nu_h | \nu_h \in C(\bar{\Omega}), \nu_h|_K \in P_1(K), \ \forall K \in \mathcal{T}_h\},$$

where $P_1(K)$ denotes the polynomials space with the degree being no more than one on $K \in \mathcal{T}_h$.

We consider the following piecewise constant finite element space

$$U_h := \{u_h \in L^2(0,T;\Omega) : u_h|_K \ is \ constant, \ \forall K \in \mathcal{T}_h\},$$

which is a finite dimensional subspace of U_{ad}.

For any $\mu \in U$, we define the orthogonal projection operator $\Pi_h : U \to U_h$ such that

$$(w_h, \mu - \Pi_h \mu) = 0, \quad \forall w_h \in U_h. \tag{2.1}$$

By the definition of Eq (2.1), we have

$$\Pi_h \mu|_K = \frac{1}{|K|} \int_K \mu, \quad \forall K \in \mathcal{T}_h, \tag{2.2}$$

where $|K|$ is the measure of K. For the operator Π_h defined in Eq (2.1), we have

$$|\mu - \Pi_h \mu|_{0,p,K} \leq Ch|\mu|_{1,p,K}, \tag{2.3}$$

for all $\mu \in W^{1,p}(\Omega)$ and $1 \leq p \leq \infty$ (see [23]).

For $t \in (0, T]$, define L^2 projection $\mathcal{R}_h \upsilon(t) \in V_h$ for $\upsilon(t) \in V$ by

$$(\upsilon(t) - \mathcal{R}_h \upsilon(t), \nu_h) = 0, \quad \forall \nu_h \in V_h. \tag{2.4}$$

As in [24], for $1 \leq r \leq 2$, the $\mathcal{R}_h \upsilon(t)$ satisfies

$$\|\upsilon - \mathcal{R}_h \upsilon(t)\|_{L^2(L^2)} + h\|\upsilon - \mathcal{R}_h \upsilon(t)\|_{L^2(H^1)} \leq Ch^r \|\upsilon\|_{L^2(H^r)}, \tag{2.5}$$

$$\|\frac{\partial(\upsilon - \mathcal{R}_h \upsilon(t))}{\partial t}\|_{L^2(L^2)} + h\|\frac{\partial(\upsilon - \mathcal{R}_h \upsilon(t))}{\partial t}\|_{L^2(H^1)} \leq Ch^r \|\upsilon\|_{H^1(H^r)}. \tag{2.6}$$

3. Optimality conditions

In this section, we first give the weak form of the state equation (1.2) as: We seek $y(\cdot, t) \in H_0^1(\Omega)$ such that

$$\begin{cases} (y_{tt}, v) + (\nabla y, \nabla v) + (\phi(y), v) = (f + Bu, v), \quad \forall v \in H_0^1(\Omega), t \in (0, T], \\ y(x, 0) = y_0(x), \quad \omega(x, 0) = g(x), \quad x \in \Omega. \end{cases} \tag{3.1}$$

Next, we introduce a new variable $\omega = y_t$, then the problem (1.1) and (1.2) can be written as

$$\min_{u \in U_{ad}} J(y, u) = \frac{1}{2} \int_0^T \int_\Omega |y - y_d|^2 dxdt + \frac{\alpha}{2} \int_0^T \int_\Omega |u|^2 dxdt, \tag{3.2}$$

subject to

$$\begin{cases} (\omega, v) = (y_t, v), \quad \forall v \in H_0^1(\Omega), t \in (0, T], \\ (\omega_t, v) + (\nabla y, \nabla v) + (\phi(y), v) = (f + Bu, v), \quad \forall v \in H_0^1(\Omega), t \in (0, T], \\ y(x, 0) = y_0(x), \quad \omega(x, 0) = g(x), \quad x \in \Omega. \end{cases} \tag{3.3}$$

The problem (3.2) and (3.3) is formulated in standard reduced functional form as

$$\begin{cases} \min \mathcal{J}(u) \\ u \in U_{ad}. \end{cases} \tag{3.4}$$

Since the problem (3.4) is non-convex, we cannot guarantee the global unique solutions of (3.4). Therefore, we consider the local optimal solutions (see [25–27]).

Definition 3.1. *(Local solution [28]). The control $\bar{u} \in U_{ad}$ is called a local solution of the problem* (3.4) *if for each fixed $t \in [0, T]$, there exists a constant $\iota > 0$, such that for all $u \in U_{ad}$ with $\|u - \bar{u}\| < \iota$, it satisfies*

$$\mathcal{J}(u) \geq \mathcal{J}(\bar{u}). \tag{3.5}$$

For the problem (3.4), the existence of the local solution can be guaranteed in [10]. Next, we can derive the following FNC for the local solution $\bar{u}$.

Theorem 3.1. *If $\bar{u}$ is a local optimal control for the problem* (3.4)*, then there exists a set of functions $(\omega(t), y(t), \bar{u}(t), q(t), p(t)) \in (H^1(L^2)\cap L^2(H^1))\times(H^2(L^2)\cap L^2(H^1))\times U_{ad}\times(H^1(L^2)\cap L^2(H^1))\times(H^2(L^2)\cap L^2(H^1))$ such that*

$$\begin{cases} (\omega, v_1) = (y_t, v_1), \quad \forall v_1 \in H_0^1(\Omega), t \in (0, T], \\ (\omega_t, v_2) + (\nabla y, \nabla v_2) + (\phi(y), v_2) = (f + B\bar{u}, v_2), \quad \forall v_2 \in H_0^1(\Omega), t \in (0, T], \\ y(x, 0) = y_0(x), \;\; \omega(x, 0) = g(x), \quad x \in \Omega, \end{cases} \tag{3.6}$$

$$\begin{cases} (q, v_1) = (p_t, v_1), \quad \forall v_1 \in H_0^1(\Omega), t \in (0, T], \\ (q_t, v_2) + (\nabla q, \nabla v_2) + (\phi'(y)p, v_2) = (y - y_d, v_2), \quad \forall v_2 \in H_0^1(\Omega), t \in (0, T], \\ p(x, T) = 0, \;\; q(x, T) = 0, \quad x \in \Omega, \end{cases} \tag{3.7}$$

$$\int_0^T (\alpha\bar{u} + B^* p, \; v - \bar{u})dt \geq 0, \quad \forall v \in U_{ad}, \tag{3.8}$$

where B^* is the adjoint operator of B.

Proof. Applying the variational rule, the optimal condition reads

$$\mathcal{J}'(\bar{u})(v - \bar{u}) = \int_0^T (y - y_d, \mathcal{D}_{\bar{u}}y(v - \bar{u}))dt + \alpha \int_0^T (\bar{u}, v - \bar{u})dt \geq 0, \tag{3.9}$$

where

$$\mathcal{D}_{\bar{u}}y(v - \bar{u}) = \lim_{t\to 0} \frac{y(\bar{u} + t(v - \bar{u})) - y(\bar{u})}{t}.$$

Next, differentiating the state equation (3.6) at $\bar{u}$ in the direction ϱ, we have

$$(\omega'\mathcal{D}_{\bar{u}}y(\varrho), v_1) = (\mathcal{D}_{\bar{u}}y(\varrho)_t, v_1), \tag{3.10}$$

$$(\omega'_t\mathcal{D}_{\bar{u}}y(\varrho), v_2) + (\nabla\mathcal{D}_{\bar{u}}y(\varrho)), \nabla v_2) + \phi'(y)\mathcal{D}_{\bar{u}}y(\varrho), v_2) = (B\varrho, v_2), \tag{3.11}$$

$$\mathcal{D}_{\bar{u}}y(\varrho)(t = 0) = 0, \;\; \omega'\mathcal{D}_{\bar{u}}y(\varrho)(t = 0) = 0. \tag{3.12}$$

Defining the co-state (p, q) satisfying Eq (3.7), and letting $v_1 = \omega'\mathcal{D}_{\bar{u}}y(\varrho)$, $v_2 = \mathcal{D}_{\bar{u}}y(\varrho)$ in Eq (3.7), we can obtain

$$(q, \omega'\mathcal{D}_{\bar{u}}y(\varrho)) = (p_t, \omega'\mathcal{D}_{\bar{u}}y(\varrho)), \tag{3.13}$$

$$(q_t, \mathcal{D}_{\bar{u}}y(\varrho)) + (\nabla p, \nabla\mathcal{D}_{\bar{u}}y(\varrho)) + (\phi'(y)p, \mathcal{D}_{\bar{u}}y(\varrho)) = (y - y_d, \mathcal{D}_{\bar{u}}y(\varrho)), \tag{3.14}$$

Meanwhile, letting $v_1 = q$ in Eq (3.10) and $v_2 = p$ in Eq (3.11), we get

$$(\omega'\mathcal{D}_{\bar{u}}y(\varrho), q) = (\mathcal{D}_{\bar{u}}y(\varrho)_t, q), \tag{3.15}$$

$$(\omega'_t\mathcal{D}_{\bar{u}}y(\varrho), p) + (\nabla\mathcal{D}_{\bar{u}}y(\varrho), \nabla p) + (\phi'(y)\mathcal{D}_{\bar{u}}y(\varrho), p) = (B\varrho, p). \tag{3.16}$$

Since $\omega'\mathcal{D}_{\bar{u}}y(\varrho)(t = 0) = 0, \mathcal{D}_{\bar{u}}y(\varrho)(t = 0) = 0,\ q(T) = 0,\ p(T) = 0$, integrating by parts and a direct calculation, we can yield

$$\int_0^T (\mathcal{D}_{\bar{u}}y(\varrho)_t, q)dt = \mathcal{D}_{\bar{u}}y(\varrho)q|_0^T - \int_0^T (\mathcal{D}_{\bar{u}}y(\varrho), q_t)dt = -\int_0^T (\mathcal{D}_{\bar{u}}y(\varrho), q_t)dt, \tag{3.17}$$

$$\int_0^T (\omega'_t\mathcal{D}_{\bar{u}}y(\varrho), p)dt = \omega'\mathcal{D}_{\bar{u}}y(\varrho)p|_0^T - \int_0^T (\omega'\mathcal{D}_{\bar{u}}y(\varrho), p_t)dt = -\int_0^T (\omega'\mathcal{D}_{\bar{u}}y(\varrho), p_t)dt. \tag{3.18}$$

Substituting Eqs (3.17) and (3.18) into Eqs (3.15) and (3.16), we derive

$$(\omega'\mathcal{D}_{\bar{u}}y(\varrho), q) = -(\mathcal{D}_{\bar{u}}y(\varrho), q_t), \tag{3.19}$$

$$-(\omega'\mathcal{D}_{\bar{u}}y(\varrho), p_t) + (\nabla\mathcal{D}_{\bar{u}}y(\varrho), \nabla p) + (\phi'(y)\mathcal{D}_{\bar{u}}y(\varrho), p) = (B\varrho, p), \tag{3.20}$$

Combining Eqs (3.19) and (3.20) with Eqs (3.13) and (3.14) yields

$$(\varrho, B^*p) = (B\varrho, p) = (y - y_d, \mathcal{D}_{\bar{u}}y(\varrho)). \tag{3.21}$$

Substituting Eq (3.21) into Eq (3.9) implies

$$\mathcal{J}'(\bar{u})(v - \bar{u}) = \int_0^T (\alpha\bar{u} + B^*p,\ v - \bar{u})dt \geq 0, \quad \forall v \in U_{ad}. \tag{3.22}$$

This completes the proof of Theorem 3.1. □

According to [26] and [29], we can know that for the non-convex problem (3.4), the FNC is not sufficient. So, we need to consider the SSC:

(SSC) There exist constants $\kappa > 0$ and $\lambda > 0$ such that

$$\mathcal{J}''(\bar{u})(v, v) \geq \kappa\|v\|^2_{L_2(L_2(\Omega))}, \tag{3.23}$$

for all $v \in L^2(0, T; L^2(\Omega))$ satisfying

$$v\begin{cases} = 0, & if\ \ |\alpha u + B^*p| \geq \lambda > 0, \\ \geq 0, & if\ \bar{u} = u_a, \\ \leq 0, & if\ \ \bar{u} = u_b, \end{cases} \tag{3.24}$$

where

$$\mathcal{J}''(\bar{u})(v, v) = \int_0^T (\mathcal{D}_{\bar{u}}y(v), \mathcal{D}_{\bar{u}}y(v))dt + \int_0^T (y(\bar{u}) - y_d, \mathcal{D}^2_{\bar{u}}y(v, v)dt$$

$$+ \alpha \int_0^T (v, v) dt, \tag{3.25}$$

and the notation $\mathcal{D}^2_{\bar{u}} y(v, v)$ is defined as follows: Let $\tilde{y} = \mathcal{D}_{\bar{u}} y(v)$. Then, its directional derivative in the direction v, denoted as $\tilde{y}'(v)$, is given by

$$\mathcal{D}^2_{\bar{u}} y(v, v) = \mathcal{D}_{\bar{u}} \tilde{y}(v) = \lim_{t \to 0} \frac{\tilde{y}(\bar{u} + tv) - \tilde{y}(\bar{u})}{t}.$$

More details about the notation $\mathcal{D}^2_{\bar{u}} y(v, v)$ can be found in [2].

Referring to [30] , we give the coercive property of the problem (3.4) in a neighborhood of the local solution by the following lemma.

Lemma 3.1. *Assume that $\bar{u}$ is a local solution of the problem* (3.4) *and satisfies the SSC* (3.23). *There exists a sufficient small constant $\iota > 0$ such that*

$$\mathcal{J}''(\bar{u})(v, v) \geq \frac{\kappa}{2} \|v\|^2_{L_2(L_2(\Omega))}, \tag{3.26}$$

for all $u \in U_{ad}$ with $\|u - \bar{u}\| < \iota$.

4. Crank-Nicolson scheme

In this section, we develop a fully discrete Crank-Nicolson scheme for the optimality system (3.2)–(3.3) as follows:

$$\mathcal{J}_\delta(y_h^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}}) = \Delta t \sum_{n=0}^{N-1} \Big[\frac{1}{2} \|y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}\|^2_{0,\Omega} + \frac{\alpha}{2} \|u_h^{n+\frac{1}{2}}\|^2_{0,\Omega} \Big], \tag{4.1}$$

$$(\omega_h^{n+\frac{1}{2}}, v) = \frac{1}{\Delta t}(y_h^{n+1} - y_h^n, v), \qquad \forall v \in V_h \tag{4.2}$$

$$\frac{1}{\Delta t}(\omega_h^{n+1} - \omega_h^n, v) + (\nabla y_h^{n+\frac{1}{2}}, \nabla v) + (\phi(y_h^{n+\frac{1}{2}}), v) = (f^{n+\frac{1}{2}} + B u_h^{n+\frac{1}{2}}, v), \qquad \forall v \in V_h \tag{4.3}$$

$$y_h^0 = \mathcal{R}_h y_0(x), \ \omega_h^0 = \mathcal{R}_h g(x), \qquad x \in \Omega, \tag{4.4}$$

where $\delta = \delta(h, \Delta t)$ denotes the fully discrete in space and time. We can reformulate the problem (4.1)–(4.4) as

$$\begin{cases} \min \mathcal{J}_\delta(u_h^{n+\frac{1}{2}}) \\ u_h^{n+\frac{1}{2}} \in U_h. \end{cases} \tag{4.5}$$

We also give the definition of the local solution of the discrete control problem (4.1)–(4.4).

Definition 4.1. *The control $\bar{u}_h^{n+\frac{1}{2}} \in U_h$ is called a fully discrete local solution of the problem* (4.1)–(4.4) *if for each fixed $t^{n+\frac{1}{2}}$, there exists a constant $\iota > 0$, such that for $\forall u_h^{n+\frac{1}{2}} \in U_h$ with $\|u_h^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}\| < \iota$, it satisfies*

$$\mathcal{J}_\delta(u_h^{n+\frac{1}{2}}) \geq \mathcal{J}_\delta(\bar{u}_h^{n+\frac{1}{2}}). \tag{4.6}$$

We present the first-order necessary optimality condition for problem (4.1)–(4.4) at the local solution $\bar{u}_h^{n+\frac{1}{2}}$ by the following theorem.

Theorem 4.1. *Assume that* $\bar{u}_h^{n+\frac{1}{2}}, n = 0, 1, ..., N-1$ *is a local solution of discrete control problem* (4.1)–(4.4)*, then there are state* $(\omega_h^{n+\frac{1}{2}}, y_h^{n+\frac{1}{2}}) \in V_h \times V_h$ *and co-state* $(q_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) \in V_h \times V_h$*,* $n = 0, 1, ..., N-1$*, such that the following optimality conditions hold:*

$$(\omega_h^{n+\frac{1}{2}}, v) = \frac{1}{\Delta t}(y_h^{n+1} - y_h^n, v), \qquad \forall v \in V_h, \tag{4.7}$$

$$\frac{1}{\Delta t}(\omega_h^{n+1} - \omega_h^n, v) + (\nabla y_h^{n+\frac{1}{2}}, \nabla v) + (\phi(y_h^{n+\frac{1}{2}}), v) = (f^{n+\frac{1}{2}} + B\bar{u}_h^{n+\frac{1}{2}}, v), \quad \forall v \in V_h, \tag{4.8}$$

$$y_h^0 = \mathcal{R}_h y_0(x),\ \omega_h^0 = \mathcal{R}_h g(x), \qquad x \in \Omega, \tag{4.9}$$

$$(q_h^{n+\frac{1}{2}}, v) = \frac{1}{\Delta t}(p_h^{n+1} - p_h^n, v), \qquad \forall v \in V_h, \tag{4.10}$$

$$\frac{1}{\Delta t}(q_h^{n+1} - q_h^n, v) + (\nabla p_h^{n+\frac{1}{2}}, \nabla v) + (\phi'(y_h^{n+\frac{1}{2}}) p_h^{n+\frac{1}{2}}, v) = (y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, v), \quad \forall v \in V_h, \tag{4.11}$$

$$p_h^N(x) = 0,\ q_h^N(x) = 0, \qquad x \in \Omega. \tag{4.12}$$

$$\mathcal{J}_\delta'(\bar{u}_h^{n+\frac{1}{2}})(v_h - \bar{u}_h^{n+\frac{1}{2}}) = \Delta t \sum_{n=0}^{N-1} (\alpha \bar{u}_h^{n+\frac{1}{2}} + B^* p_h^{n+\frac{1}{2}}, v_h - \bar{u}_h^{n+\frac{1}{2}}) \geq 0, \quad \forall v_h \in U_h. \tag{4.13}$$

Proof. Differentiate the Eq (4.5) at $\bar{u}_h^{n+\frac{1}{2}}$ in the direction $v_h - \bar{u}_h^{n+\frac{1}{2}}$, and the discrete optimal condition reads

$$\begin{aligned}\mathcal{J}_\delta'(\bar{u}_h^{n+\frac{1}{2}})(v_h - \bar{u}_h^{n+\frac{1}{2}}) &= \lim_{t\to 0} \frac{\mathcal{J}_\delta(\bar{u}_\delta + t(v_h - \bar{u}_\delta)) - \mathcal{J}_\delta(\bar{u}_\delta)}{t} \\ &= \Delta t \sum_{n=0}^{N-1} (y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, \mathcal{D}y_h^{n+\frac{1}{2}}(v_h - \bar{u}_h^{n+\frac{1}{2}})) \\ &\quad + \alpha \Delta t \sum_{n=0}^{N-1} (\bar{u}_h^{n+\frac{1}{2}}, v_h - \bar{u}_h^{n+\frac{1}{2}}) \geq 0. \end{aligned} \tag{4.14}$$

Similarly, differentiating the Eqs (4.7) and (4.8) at $\bar{u}_h^{n+\frac{1}{2}}$ in the direction v, we have

$$(\omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), v) = \frac{1}{\Delta t}(\mathcal{D}y_h^{n+1}(v) - \mathcal{D}y_h^n(v), v), \tag{4.15}$$

$$\begin{aligned}\frac{1}{\Delta t}(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v) - \omega'(y_h^n)\mathcal{D}y_h^n(v), v) + (\nabla \mathcal{D}y_h^{n+\frac{1}{2}}(v), \nabla v) \\ + (\phi'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), v) = (Bv, v),\end{aligned} \tag{4.16}$$

$$\mathcal{D}y_h^0(v) = 0,\ \ \omega'(y_h^0)\mathcal{D}y_h^0(v) = 0. \tag{4.17}$$

where

$$\mathcal{D}y_h^{n+\frac{1}{2}}(v) = \lim_{t\to 0} \frac{y_h^{n+\frac{1}{2}}(\bar{u}_h^{n+\frac{1}{2}} + tv) - y_h^{n+\frac{1}{2}}(\bar{u}_h^{n+\frac{1}{2}})}{t}.$$

Choosing the discrete co-state (p_δ, q_δ) satisfying Eqs (4.10)–(4.12), and selecting $v = \omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v)$ in Eq (4.10) and $v = \mathcal{D}y_h^{n+\frac{1}{2}}(v)$ in Eq (4.11), we get

$$(q_h^{n+\frac{1}{2}}, \omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v)) = \frac{1}{\Delta t}(p_h^{n+1} - p_h^n, \omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v)), \tag{4.18}$$

$$\frac{1}{\Delta t}(q_h^{n+1} - q_h^n, \mathcal{D}y_h^{n+\frac{1}{2}}(v)) + (\nabla p_h^{n+\frac{1}{2}}, \nabla\mathcal{D}y_h^{n+\frac{1}{2}}(v)) + (\phi'(y_h^{n+\frac{1}{2}})p_h^{n+\frac{1}{2}}, \mathcal{D}y_h^{n+1}(v)) = (y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, \mathcal{D}y_h^{n+\frac{1}{2}}(v)). \tag{4.19}$$

Taking $v = q_h^{n+\frac{1}{2}}$ in Eq (4.15) and $v = p_h^{n+\frac{1}{2}}$ in Eq (4.16), we have

$$(\omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), q_h^{n+\frac{1}{2}}) = \frac{1}{\Delta t}(\mathcal{D}y_h^{n+1}(v) - \mathcal{D}y_h^n(v), q_h^{n+\frac{1}{2}}), \tag{4.20}$$

$$\frac{1}{\Delta t}(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v) - \omega'(y_h^n)\mathcal{D}y_h^n(v), p_h^{n+\frac{1}{2}}) + (\nabla\mathcal{D}y_h^{n+\frac{1}{2}}(v), \nabla p_h^{n+\frac{1}{2}}) + (\phi'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), p_h^{n+\frac{1}{2}}) = (Bv, p_h^{n+\frac{1}{2}}), \tag{4.21}$$

Since $\mathcal{D}y_h^0(v) = 0$, $\omega'(y_h^0)\mathcal{D}y_h^0(v) = 0$, $p_h^N = 0$, $q_h^N = 0$, we know that

$$\begin{aligned}
&\sum_{n=0}^{N-1}(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v) - \omega'(y_h^n)\mathcal{D}y_h^n(v), p_h^{n+\frac{1}{2}}) \\
=&\sum_{n=0}^{N-1}\left(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v) - \omega'(y_h^n)\mathcal{D}y_h^n(v), \frac{p_h^n + p_h^{n+1}}{2}\right) \\
=&\frac{1}{2}\sum_{n=0}^{N-1}\Big[(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v), p_h^n + p_h^{n+1}) - (\omega'(y_h^n)\mathcal{D}y_h^n(v), p_h^n + p_h^{n+1})\Big] \\
=&\frac{1}{2}\sum_{n=0}^{N-1}\Big[(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v), p_h^n - p_h^{n+1}) + (\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v), p_h^{n+1}) \\
&\quad - (\omega'(y_h^n)\mathcal{D}y_h^n(v), p_h^{n+1} - p_h^n) - (\omega'(y_h^n)\mathcal{D}y_h^n(v), p_h^n)\Big] \\
=&\sum_{n=0}^{N-1}(\omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), p_h^n - p_h^{n+1}),
\end{aligned} \tag{4.22}$$

and

$$\sum_{n=0}^{N-1}(\mathcal{D}y_h^{n+1}(v) - \mathcal{D}y_h^n(v), q_h^{n+\frac{1}{2}}) = -\sum_{n=0}^{N-1}(q_h^{n+1} - q_h^n,\ \mathcal{D}y_h^{n+\frac{1}{2}}(v)). \tag{4.23}$$

Substituting Eqs (4.22) and (4.23) into Eqs (4.20) and (4.21), we obtain

$$(\omega'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), q_h^{n+\frac{1}{2}}) = -\frac{1}{\Delta t}(q_h^{n+1} - q_h^n,\ \mathcal{D}y_h^{n+\frac{1}{2}}(v)), \tag{4.24}$$

$$-\frac{1}{\Delta t}(\omega'(y_h^{n+1})\mathcal{D}y_h^{n+1}(v), p_h^{n+1} - p_h^n) + (\nabla\mathcal{D}y_h^{n+\frac{1}{2}}(v), \nabla p_h^{n+\frac{1}{2}})$$

$$+ (\phi'(y_h^{n+\frac{1}{2}})\mathcal{D}y_h^{n+\frac{1}{2}}(v), p_h^{n+\frac{1}{2}}) = (Bv, p_h^{n+\frac{1}{2}}). \tag{4.25}$$

Combining Eqs (4.24) and (4.25) with Eqs (4.18) and (4.19), and summing from 0 to $N-1$ leads to

$$\sum_{n=0}^{N-1} \Delta t(B^* p_h^{n+\frac{1}{2}}, v) = \sum_{n=0}^{N-1} \Delta t(Bv, p_h^{n+\frac{1}{2}}) = \sum_{n=0}^{N-1} \Delta t(y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, \mathcal{D}y_h^{n+\frac{1}{2}}(v)). \tag{4.26}$$

Therefore, substituting Eq (4.26) into Eq (4.14), we get

$$\mathcal{J}_\delta'(\bar{u}_h^{n+\frac{1}{2}})(v_h - \bar{u}_h^{n+\frac{1}{2}}) = \Delta t \sum_{n=0}^{N-1} (\alpha u_h^{n+\frac{1}{2}} + B^* p_h^{n+\frac{1}{2}}, v_h - u_h^{n+\frac{1}{2}}) \geq 0, \quad \forall v_h \in U_h. \tag{4.27}$$

This completes the proof of Theorem 4.1. □

Similarly, we also provide the discrete SSC for the local solution $\bar{u}_h^{n+\frac{1}{2}}$ as

$$\mathcal{J}_\delta''(\bar{u}_h^{n+\frac{1}{2}})(v_h, v_h) \geq \kappa \|v_h\|^2_{l_2(L_2(\Omega))}, \quad \forall v_h \in U_h. \tag{4.28}$$

where $\kappa > 0$, and $\bar{u}_h^{n+\frac{1}{2}}$ satisfies Eq (4.13). From Eq (4.1), we can get

$$\begin{aligned}\mathcal{J}_\delta''(\bar{u}_h^{n+\frac{1}{2}})(v_h, v_h) =& \Delta t \sum_{n=0}^{N-1} (\mathcal{D}y_h^{n+\frac{1}{2}}(v_h), \mathcal{D}y_h^{n+\frac{1}{2}}(v_h)) \\ &+ \Delta t \sum_{n=0}^{N-1} (y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, \mathcal{D}^2 y_h^{n+\frac{1}{2}}(v_h, v_h)) + \alpha \Delta t \sum_{n=0}^{N-1} (v_h, v_h).\end{aligned} \tag{4.29}$$

The following lemma shows the coercive property of the second derivative of the discrete objective function $\mathcal{J}_\delta$ in a neighborhood of a local solution $\bar{u}$.

Lemma 4.1. *Let $\bar{u}$ be a local solution of the problem* (3.4) *and the SSC* (3.23) *is valid. There exists sufficiently small constant $\iota > 0$, and h, for all $u \in U_{ad}$ with $\|u - \bar{u}\| < \iota$ and $v \in U_{ad}$,*

$$\mathcal{J}_\delta''(u)(v, v) \geq \frac{\kappa}{4} \|v\|^2_{l_2(L_2(\Omega))} \tag{4.30}$$

holds.

4.1. Auxiliary problems

So as to get a priori estimates, it is needed to introduce an auxiliary problem: find $(\omega_h^{n+\frac{1}{2}}(u), y_h^{n+\frac{1}{2}}(u))$ $\in V_h \times V_h$, $n = 0, 1, \ldots, N-1$, such that for all $v \in V_h$,

$$(\omega_h^{n+\frac{1}{2}}(u), v) = \frac{1}{\Delta t}(y_h^{n+1}(u) - y_h^n(u), v), \tag{4.31}$$

$$\frac{1}{\Delta t}(\omega_h^{n+1}(u) - \omega_h^n(u), v) + (\nabla y_h^{n+\frac{1}{2}}(u), \nabla v) + (\phi(y_h^{n+\frac{1}{2}}(u)), v) = (f^{n+\frac{1}{2}} + Bu^{n+\frac{1}{2}}, v), \tag{4.32}$$

$$y_h^0(u) = \mathcal{R}_h y_0(x), \ \omega_h^0(u) = \mathcal{R}_h g(x). \tag{4.33}$$

Another auxiliary problem: find $(q_h^{n+\frac{1}{2}}(u), p_h^{n+\frac{1}{2}}(u)) \in V_h \times V_h$, $n = 0, 1, \ldots, N-1$, such that for all $v \in V_h$,

$$(q_h^{n+\frac{1}{2}}(u), v) = \frac{1}{\Delta t}(p_h^{n+1}(u) - p_h^n(u), v), \tag{4.34}$$

$$\frac{1}{\Delta t}(q_h^{n+1}(u) - q_h^n(u), v) + (\nabla p_h^{n+\frac{1}{2}}(u), \nabla v) + (\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u), v)$$
$$= (y_h^{n+\frac{1}{2}}(u) - y_d^{n+\frac{1}{2}}, v), \tag{4.35}$$

$$q_h^N(u) = 0, \ p_h^N(u) = 0. \tag{4.36}$$

For convenience, we denote

$$\theta_\omega^n = \omega_h^n - \omega_h^n(u), \ \theta_y^n = y_h^n - y_h^n(u), n = 0, 1, \ldots, N,$$
$$\theta_q^n = q_h^n - q_h^n(u), \ \theta_p^n = p_h^n - p_h^n(u), n = N, \ldots, 1, 0.$$

It is clear that

$$\theta_\omega^0 = 0, \ \theta_y^0 = 0, \ \theta_q^N = 0, \ \theta_p^N = 0. \tag{4.37}$$

Next, we describe the error caused by the control discretization using the following lemma.

Lemma 4.2. *Let* $(\omega_\delta, y_\delta, q_\delta, p_\delta)$ *and* $(\omega_\delta(u), y_\delta(u), q_\delta(u), p_\delta(u))$ *be the solutions of Eqs* (4.7)–(4.12) *and Eqs* (4.31)–(4.36)*, respectively. Then,*

$$\|\omega_\delta - \omega_\delta(u)\|_{l^\infty(L^2)} + \|y_\delta - y_\delta(u)\|_{l^\infty(H^1)} \le C\|u_\delta - u\|_{l^2(L^2(\Omega))}, \tag{4.38}$$

$$\|q_\delta - q_\delta(u)\|_{l^\infty(L^2)} + \|p_\delta - p_\delta(u)\|_{l^\infty(H^1)} \le C\|u_\delta - u\|_{l^2(L^2(\Omega))}. \tag{4.39}$$

Proof. To begin, we develop the inequality for θ_ω and θ_y. Subtracting Eqs (4.7) and (4.8) from Eqs (4.31) and (4.32), we derive

$$(\theta_\omega^{n+\frac{1}{2}}, v) = \frac{1}{\Delta t}(\theta_y^{n+1} - \theta_y^n, v), \tag{4.40}$$

$$\frac{1}{\Delta t}(\theta_\omega^{n+1} - \theta_\omega^n, v) + (\nabla\theta_y^{n+\frac{1}{2}}, \nabla v) + (\phi(y_h^{n+\frac{1}{2}}) - \phi(y_h^{n+\frac{1}{2}}(u)), v) = (B(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}), v). \tag{4.41}$$

Choosing $v = \theta_y^{n+\frac{1}{2}}$, $v = \theta_\omega^{n+\frac{1}{2}}$ as the test function in Eqs (4.40) and (4.41), respectively, we have

$$\frac{1}{\Delta t}(\theta_y^{n+1} - \theta_y^n, \theta_y^{n+\frac{1}{2}}) = (\theta_\omega^{n+\frac{1}{2}}, \theta_y^{n+\frac{1}{2}}), \tag{4.42}$$

$$\frac{1}{\Delta t}(\theta_\omega^{n+1} - \theta_\omega^n, \theta_\omega^{n+\frac{1}{2}}) + (\nabla\theta_y^{n+\frac{1}{2}}, \nabla\theta_\omega^{n+\frac{1}{2}})$$
$$=(\phi(y_h^{n+\frac{1}{2}})(u) - \phi(y_h^{n+\frac{1}{2}}), \theta_\omega^{n+\frac{1}{2}}) + (B(u_h^{n+1} - u^{n+1}), \theta_\omega^{n+\frac{1}{2}}). \tag{4.43}$$

Substituting Eq (4.42) into Eq (4.43), we get

$$\frac{1}{2\Delta t}\Big[\|\theta_\omega^{n+1}\|^2 - \|\theta_\omega^n\|^2 + \|\nabla\theta_y^{n+1}\|^2 - \|\nabla\theta_y^n\|^2\Big]$$

$$=(\phi(y_h^{n+\frac{1}{2}})(u)-\phi(y_h^{n+\frac{1}{2}}),\theta_\omega^{n+\frac{1}{2}})+(B(u_h^{n+1}-u^{n+1}),\theta_\omega^{n+\frac{1}{2}}). \tag{4.44}$$

Summing up from $n=0$ up to $N-1$, it yields

$$\begin{aligned}&\|\theta_\omega^N\|^2+\|\nabla\theta_y^N\|^2\\ =&2\Delta t\sum_{n=0}^{N-1}(B(u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}),\theta_\omega^{n+\frac{1}{2}})+2\Delta t\sum_{n=0}^{N-1}(\phi(y_h^{n+\frac{1}{2}}(u))-\phi(y_h^{n+\frac{1}{2}}),\theta_\omega^{n+\frac{1}{2}}).\end{aligned} \tag{4.45}$$

For the first term, we derive

$$\begin{aligned}2\Delta t\sum_{n=0}^{N-1}(B(u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}),\theta_\omega^{n+\frac{1}{2}})\leq&C\Delta t\sum_{n=0}^{N-1}\left[\|u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\theta_\omega^{n+\frac{1}{2}}\|^2\right]\\ \leq&C\Delta t\sum_{n=0}^{N-1}\left[\|u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}\|_{0,\Omega}^2+(\|\theta_\omega^{n+1}\|^2+\|\theta_\omega^{n}\|^2)\right].\end{aligned} \tag{4.46}$$

For the second term, we get

$$\begin{aligned}2\Delta t\sum_{n=0}^{N-1}(\phi(y_h^{n+\frac{1}{2}}(u))-\phi(y_h^{n+\frac{1}{2}}),\theta_\omega^{n+\frac{1}{2}})\leq&C\Delta t\sum_{n=0}^{N-1}\left[\|\theta_y^{n+\frac{1}{2}}\|^2+\|\theta_\omega^{n+\frac{1}{2}}\|^2\right]\\ \leq&C\Delta t\sum_{n=0}^{N-1}\left[\|\theta_\omega^{n+1}\|^2+\|\theta_\omega^{n}\|^2+\|\theta_y^{n+1}\|^2+\|\theta_y^{n}\|^2\right].\end{aligned} \tag{4.47}$$

Now, combining Eqs (4.46) and (4.47) with Eq (4.45), we arrive at

$$\|\theta_\omega^N\|^2+\|\nabla\theta_y^N\|^2\leq C\Delta t\sum_{n=0}^{N-1}\|u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}\|^2+\Delta t\sum_{n=0}^{N-1}(\|\theta_\omega^{n+1}\|^2+\|\theta_y^{n+1}\|^2). \tag{4.48}$$

The discrete Gronwall's inequality leads to

$$\|\omega_\delta-\omega_\delta(u)\|_{l^\infty(L^2)}+\|y_\delta-y_\delta(u)\|_{l^\infty(H^1)}\leq C\|u_\delta-u\|_{l^2(L^2(\Omega))}. \tag{4.49}$$

Next, we develop the inequality for θ_q and θ_p. Subtracting Eqs (4.10) and (4.11) from Eqs (4.34) and (4.35), we get

$$(\theta_q^{n+\frac{1}{2}},v)=\frac{1}{\Delta t}(\theta_p^{n+1}-\theta_p^n,v), \tag{4.50}$$

$$\frac{1}{\Delta t}(\theta_q^{n+1}-\theta_q^n,v)+(\nabla\theta_p^{n+\frac{1}{2}},\nabla v)+(\phi'(y_h^{n+\frac{1}{2}})p_h^{n+\frac{1}{2}}-\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u),v)=(\theta_y^{n+\frac{1}{2}},v). \tag{4.51}$$

Choosing $v=-\theta_q^{n+\frac{1}{2}}$ as the test function in Eq (4.51), we have

$$\begin{aligned}&\frac{1}{2\Delta t}\|\theta_q^n\|^2-\frac{1}{2\Delta t}\|\theta_q^{n+1}\|^2-(\nabla\theta_p^{n+\frac{1}{2}},\nabla\theta_q^{n+\frac{1}{2}})\\ =&(\phi'(y_h^{n+\frac{1}{2}})p_h^{n+\frac{1}{2}}-\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u),\theta_q^{n+\frac{1}{2}})-(\theta_y^{n+\frac{1}{2}},\theta_q^{n+\frac{1}{2}}).\end{aligned} \tag{4.52}$$

Substituting Eq (4.50) into Eq (4.52), we know

$$\begin{aligned}&\frac{1}{2\Delta t}\Big[\|\theta_q^n\|^2-\|\theta_q^{n+1}\|^2+\|\nabla\theta_p^n\|^2-\|\nabla\theta_p^{n+1}\|^2\Big]\\&=(\phi'(y_h^{n+\frac12})p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}(u),\theta_q^{n+\frac12})-(\theta_y^{n+\frac12},\theta_q^{n+\frac12}).\end{aligned}\tag{4.53}$$

By calculation, the following formula holds:

$$\begin{aligned}&(\phi'(y_h^{n+\frac12})p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}(u),\theta_q^{n+\frac12})\\&=(\phi'(y_h^{n+\frac12})p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12},\ \theta_q^{n+\frac12})\\&\quad+(\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}(u),\ \theta_q^{n+\frac12}).\end{aligned}\tag{4.54}$$

Multiplying both sides of Eq (4.53) by $2\Delta t$, then summing it over n from M to $N-1$, it leads to

$$\begin{aligned}\|\theta_q^M\|^2+\|\nabla\theta_p^M\|^2=2\Delta t\sum_{n=M}^{N-1}\Big[&(\phi'(y_h^{n+\frac12})p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12},\ \theta_q^{n+\frac12})\\&+(\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}-\phi'(y_h^{n+\frac12}(u))p_h^{n+\frac12}(u),\ \theta_q^{n+\frac12})-(\theta_y^{n+\frac12},\theta_q^{n+\frac12})\Big]\\&\triangleq\sum_{i=1}^{3}\Pi_i.\end{aligned}\tag{4.55}$$

Here, the estimates of Π_1, Π_2, Π_3 are similar to those of Υ_1, Υ_2, Υ_3, and we have

$$\begin{aligned}\Pi_1&\le\Delta t\sum_{n=M}^{N-1}\|\phi'(y_h^{n+\frac12})-\phi'(y_h^{n+\frac12}(u))\|_{0,4}\|p_h^{n+\frac12}\|_{0,4}\|\|\theta_q^{n+\frac12}\|\\&\le C\Delta t\sum_{n=M}^{N-1}(\|\nabla\theta_y^{n+1}\|^2+\|\nabla\theta_y^n\|^2+\|\theta_q^n\|^2),\end{aligned}$$

$$\begin{aligned}\Pi_2&\le 2\Delta t\sum_{n=M}^{N-1}\|\phi'(y_h^{n+\frac12}(u))\|_{0,\infty}\|p_h^{n+\frac12}(u)-p_h^{n+\frac12}\|\ \|\theta_q^{n+\frac12}\|\\&\le C\Delta t\sum_{n=M}^{N-1}(\|\theta_p^n\|^2+\|\theta_q^n\|^2),\\\Pi_3&\le\Delta t\sum_{n=M}^{N-1}(\|\nabla\theta_y^n\|+\|\theta_q^n\|^2).\end{aligned}$$

Finally, inserting the above estimates of Π_1–Π_3 into Eq (4.55), we get

$$\|\theta_q^M\|^2+\|\nabla\theta_p^M\|^2\le C\Delta t\sum_{n=M}^{N-1}\|\nabla\theta_y^n\|+C\Delta t\sum_{n=M}^{N-1}(\|\theta_q^n\|^2+\|\theta_p^n\|^2).\tag{4.56}$$

Thus, combine the Poincaré inequality and the discrete Gronwall's inequality, such that

$$\|q_\delta-q_\delta(u)\|_{l^\infty(L^2)}+\|p_\delta-p_\delta(u)\|_{l^\infty(H^1)}\le C\|y_\delta-y_\delta(u)\|_{l^\infty(H^1)}.\tag{4.57}$$

Eqs (4.38) and (4.57) follows Eq (4.39), which also completes the proof of Lemma 4.2. □

Next, so as to estimate the error of the control discretization, we choose a local solution $\bar{u}$ of the continuous problem (3.6)–(3.8), and an associated approximate solution u_δ of the discrete problem (4.1)–(4.4). We introduce the following auxiliary problem:

$$\min_{u_h^{n+\frac{1}{2}} \in U_h^\iota} \mathcal{J}_\delta(u_h^{n+\frac{1}{2}}), \tag{4.58}$$

where $U_h^\iota(\bar{u}) = \{u_h^{n+\frac{1}{2}} \in U_h : \|u_\delta - \bar{u}\| < \iota\}$. In addition, since $\mathcal{J}_\delta''$ satisfies Lemma 4.1 for $\bar{u} \in U_h^\iota(\bar{u})$, $v \in U_{ad}$, the existence and uniqueness of the problem (4.58) are guaranteed; see [28].

From the definitions of the local solution (4.6) and U_h^ι, we can get

$$\mathcal{J}_\delta(\bar{u}_h^{n+\frac{1}{2}}) = \min_{u_h^{n+\frac{1}{2}} \in U_h^\iota(\bar{u})} \mathcal{J}_\delta(u_h^{n+\frac{1}{2}}), \qquad \text{for} \quad \|u_h^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}\| < \iota. \tag{4.59}$$

Utilizing (4.58) and (4.59), we can deduce that $\bar{u}_h^{n+\frac{1}{2}} \in U_h^\iota$ is the unique solution of the problem (4.58).

Lemma 4.3. *Let $\bar{u}$ be a local solution of the problem* (3.4) *and the SSC* (3.23) *is valid. Then, the discrete problem* (4.58) *has a unique solution $\bar{u}_h^{n+\frac{1}{2}}$, and the following estimate holds:*

$$\|\bar{u} - \bar{u}_\delta\|_{l^2(L^2(\Omega))} \leq C\|p - p_\delta(\bar{u})\|^2_{l^2(L^2)} + Ch^2(\|\bar{u}\|^2_{l^2(H^1(\Omega))} + \|p\|^2_{l^2(H^1)}). \tag{4.60}$$

for $\iota > 0, h > 0$ sufficiently small.

Proof. From Lemma 4.1, it is clear to infer that

$$\mathcal{J}_\delta''(u_\delta)(v, v) \geq \frac{\kappa}{4}\|v\|^2_{l^2(L^2(\Omega))}, \text{ for } \forall u_\delta \in U_h^\iota(\bar{u}) \text{ and } v \in U_{ad}.$$

By the FNC (4.13), we can get

$$\begin{aligned}
&\mathcal{J}_\delta'(\bar{u}^{n+\frac{1}{2}})(\bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}) \leq 0,\\
&\mathcal{J}_\delta'(\bar{u}_h^{n+\frac{1}{2}})(\bar{u}_h^{n+\frac{1}{2}} - \Pi_h\bar{u}^{n+\frac{1}{2}}) \leq 0.
\end{aligned}$$

for h sufficiently small. Utilizing $v^{n+\frac{1}{2}} = \theta\bar{u}^{n+\frac{1}{2}} + (1-\theta)\bar{u}_h^{n+\frac{1}{2}}$ with $\theta \in [0, 1]$, we get

$$\begin{aligned}
&\frac{\kappa}{4}\|\bar{u} - \bar{u}_\delta\|^2_{l^2(L^2(\Omega))}\\
\leq &\mathcal{J}_\delta''(v^{n+\frac{1}{2}})(\bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}, \bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}})\\
= &\mathcal{J}_\delta'(\bar{u}^{n+\frac{1}{2}})(\bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}) - \mathcal{J}_\delta'(\bar{u}_h^{n+\frac{1}{2}})(\bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}})\\
= &\Delta t\sum_{n=0}^{N-1}\Big[(\alpha\bar{u}^{n+\frac{1}{2}} + B^*p_h^{n+\frac{1}{2}}(\bar{u}), \bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}) - (\alpha\bar{u}_h^{n+\frac{1}{2}} + B^*p_h^{n+\frac{1}{2}}, \bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}})\Big]\\
\leq &\Delta t\sum_{n=0}^{N-1}\Big[(B^*(p_h^{n+\frac{1}{2}}(\bar{u}) - p^{n+\frac{1}{2}}), \bar{u}^{n+\frac{1}{2}} - \bar{u}_h^{n+\frac{1}{2}}) + (\alpha\bar{u}_h^{n+\frac{1}{2}}, \Pi_h\bar{u}^{n+\frac{1}{2}} - \bar{u}^{n+\frac{1}{2}})
\end{aligned}$$

$$+ (B^* p^{n+\frac{1}{2}}, \Pi_h \bar{u}^{n+\frac{1}{2}} - \bar{u}^{n+\frac{1}{2}}) + (B^*(p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}(\bar{u})), \bar{u}^{n+\frac{1}{2}} - \Pi_h \bar{u}^{n+\frac{1}{2}})$$
$$+ (B^*(p_h^{n+\frac{1}{2}}(\bar{u}) - p_h^{n+\frac{1}{2}}), \bar{u}^{n+\frac{1}{2}} - \Pi_h \bar{u}^{n+\frac{1}{2}})\Big]$$
$$\triangleq \sum_{i=1}^{5} \mathcal{S}_i. \tag{4.61}$$

To start, using the Cauchy-Schwarz inequality, we obtain

$$\mathcal{S}_1 \leq C\|p - p_\delta(\bar{u})\|^2_{l^2(L^2)} + \epsilon\|\bar{u} - \bar{u}_\delta\|^2_{l^2(L^2(\Omega))}.$$

For the $\mathcal{S}_2$ and $\mathcal{S}_3$, by the definition of the operator Π_h, we have

$$\mathcal{S}_2 = 0,$$

and

$$\mathcal{S}_3 = \sum_{n=0}^{N-1} \Delta t(B^*(p^{n+\frac{1}{2}} - \Pi_h p^{n+\frac{1}{2}}, \Pi_h \bar{u}^{n+\frac{1}{2}} - \bar{u}^{n+\frac{1}{2}})$$
$$\leq Ch^2(\|p\|^2_{l^2(0,T;H^l(\Omega))} + \|\bar{u}\|^2_{l^2(0,T;H^1(\Omega))}).$$

Then, for the $\mathcal{S}_4$ and $\mathcal{S}_5$, from Lemma 4.2 and the Cauchy inequality, it yields

$$\mathcal{S}_4 \leq C\|p - p_\delta(\bar{u})\|^2_{l^2(L^2)} + Ch^2\|\bar{u}\|^2_{l^2(0,T;H^l(\Omega))},$$

$$\mathcal{S}_5 \leq \epsilon\|\bar{u} - \bar{u}_\delta\|^2_{l^2(L^2(\Omega))} + Ch^2\|\bar{u}\|^2_{l^2(0,T;H^l(\Omega))}.$$

Substituting $\mathcal{S}_1$–$\mathcal{S}_5$ into Eq (4.61), we obtain

$$\|\bar{u} - \bar{u}_\delta\|^2_{l^2(0,T;L^2(\Omega))}$$
$$\leq C\|p - p_\delta(\bar{u})\|^2_{l^2(L^2)} + Ch^2(\|\bar{u}\|^2_{l^2(0,T;H^l(\Omega))} + \|p\|^2_{l^2(0,T;H^l(\Omega))}).$$

This completes the proof of Lemma 4.3. □

4.2. Error analysis

In this subsection, we will establish the error caused by the discretization of the Crank-Nicolson FEM scheme. To do this, we define the error function as

$$e^n_\omega = \omega(t_n) - \omega^n_h(u),\ e^n_y = y(t_n) - y^n_h(u),\ e^n_q = q(t_n) - q^n_h(u),\ e^n_p = p(t_n) - p^n_h(u),$$

and introduce the following truncation errors:

$$\mathcal{T}_\omega = \omega(t_{n+1}) - \omega(t_n) - \Delta t \omega_t(t_{n+\frac{1}{2}}),\ \ \eta_\omega = \frac{\omega(t_{n+1}) + \omega(t_n)}{2} - \omega(t_{n+\frac{1}{2}}),$$
$$\mathcal{T}_y = y(t_{n+1}) - y(t_n) - \Delta t y_t(t_{n+\frac{1}{2}}),\ \ \eta_y = \frac{y(t_{n+1}) + y(t_n)}{2} - y(t_{n+\frac{1}{2}}),$$
$$\mathcal{T}_q = q(t_{n+1}) - q(t_n) - \Delta t q_t(t_{n+\frac{1}{2}}),\ \ \eta_q = \frac{q(t_{n+1}) + q(t_n)}{2} - q(t_{n+\frac{1}{2}}),$$
$$\mathcal{T}_p = p(t_{n+1}) - p(t_n) - \Delta t p_t(t_{n+\frac{1}{2}}),\ \ \eta_p = \frac{p(t_{n+1}) + p(t_n)}{2} - p(t_{n+\frac{1}{2}}).$$

For these definitions, we first present the following estimates of the truncation error.

Lemma 4.4. *The following estimates hold:*

$$\|\mathcal{T}_\omega\|^2 \le C(\Delta t)^5 \int_{t_n}^{t_{n+1}} \|\omega_{ttt}\|^2 ds, \quad \|\eta_\omega\|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|\omega_{tt}\|^2 ds,$$
$$\|\mathcal{T}_y\|^2 \le C(\Delta t)^5 \int_{t_n}^{t_{n+1}} \|y_{ttt}\|^2 ds, \quad \|\eta_y\|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|y_{tt}\|^2 ds,$$
$$\|\mathcal{T}_q\|^2 \le C(\Delta t)^5 \int_{t_n}^{t_{n+1}} \|q_{ttt}\|^2 ds, \quad \|\eta_q\|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|q_{tt}\|^2 ds,$$
$$\|\mathcal{T}_p\|^2 \le C(\Delta t)^5 \int_{t_n}^{t_{n+1}} \|p_{ttt}\|^2 ds, \quad \|\eta_p\|^2 \le C(\Delta t)^3 \int_{t_n}^{t_{n+1}} \|p_{tt}\|^2 ds.$$

Lemma 4.5. *Let* (ω, y, u, q, p) *and* $(\omega_\delta(u), y_\delta(u), q_\delta(u), p_\delta(u))$ *be the solutions of Eqs* (3.6)–(3.7) *and Eqs* (4.31)–(4.36)*, respectively. Assume that* $\omega,\ q \in L^2(H^2) \cap H^1(H^2),\ \omega_{tt},\ q_{tt} \in L^2(H^1),\ y_{tt} \in L^2(L^2) \cap L^2(H^1) \cap L^2(H^2), p_{tt} \in L^2(L^2) \cap L^2(H^2),\ \omega_{ttt},\ q_{ttt} \in L^2(L^2),\ y_{ttt},\ p_{ttt} \in L^2(H^1),\ y_0(x),\ g(x) \in H^2(\Omega)$. *Then, we have*

$$\|\omega_\delta(u) - \omega\|_{l^\infty(L^2)} + \|y_\delta(u) - y\|_{l^\infty(H^1)} \le C((\Delta t)^2 + h), \tag{4.62}$$
$$\|q_\delta(u) - q\|_{l^\infty(L^2)} + \|p_\delta(u) - p\|_{l^\infty(H^1)} \le C((\Delta t)^2 + h). \tag{4.63}$$

Proof. From Eq (3.7), the exact solution (ω, y) satisfies

$$\frac{1}{2}(\omega(t_{n+1}) + \omega(t_n), v_h) = (\frac{y(t_{n+1}) - y(t_n)}{\Delta t}, v_h) - \frac{1}{\Delta t}(T_y, v_h) + (\eta_\omega, v_h), \tag{4.64}$$
$$(\frac{\omega(t_{n+1}) - \omega(t_n)}{\Delta t}, v_h) + (\nabla \frac{y(t_{n+1}) + y(t_n)}{2}, \nabla v_h) + (\phi(y(t_{n+\frac{1}{2}})), v_h) = (f(t_{n+\frac{1}{2}}) + Bu(t_{n+\frac{1}{2}}), v_h) + \frac{1}{\Delta t}(T_\omega, v_h) + (\nabla \eta_y, \nabla v_h). \tag{4.65}$$

From relations Eqs (4.64) and (4.65) and Eqs (4.31) and (4.32), it holds that

$$(e_\omega^{n+\frac{1}{2}}, v_h) = \frac{1}{\Delta t}(e_y^{n+1} - e_y^n, v_h) - \frac{1}{\Delta t}(T_y, v_h) + (\eta_\omega, v_h), \tag{4.66}$$
$$(e_\omega^{n+1} - e_\omega^n, v_h) + \Delta t(\nabla e_y^{n+\frac{1}{2}}, \nabla v_h) = \Delta t(\phi(y_h^{n+\frac{1}{2}}(u)) - \phi(y^{n+\frac{1}{2}}), v_h) + (T_\omega, v_h) + \Delta t(\nabla \eta_y, \nabla v_h). \tag{4.67}$$

Taking the discrete inner product of Eq (4.67) with $v_h = \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u) = \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega(t_{n+\frac{1}{2}}) + \omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u) = \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega(t_{n+\frac{1}{2}}) + e_\omega^{n+\frac{1}{2}}$ and rearranging the terms, we have

$$\begin{aligned}
&\frac{1}{2}\|e_\omega^{n+1}\|^2 - \frac{1}{2}\|e_\omega^n\|^2 + (\nabla e_y^{n+\frac{1}{2}}, \nabla e_\omega^{n+\frac{1}{2}})\Delta t \\
=&(e_\omega^{n+1} - e_\omega^n, \omega(t_{n+\frac{1}{2}}) - \mathcal{R}_h\omega(t_{n+\frac{1}{2}})) + (\nabla e_y^{n+\frac{1}{2}}, \nabla(\omega(t_{n+\frac{1}{2}}) - \mathcal{R}_h\omega(t_{n+\frac{1}{2}})))\Delta t \\
&+ (\phi(y_h^{n+\frac{1}{2}}(u)) - \phi(y^{n+\frac{1}{2}}), \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u))\Delta t \\
&+ (T_\omega, \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u)) + (\nabla\eta_y, \nabla(\mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u)))\Delta t.
\end{aligned} \tag{4.68}$$

Meanwhile, choosing $v_h = e_y^{n+\frac{1}{2}}$ in Eq (4.66), we obtain

$$\Delta t(e_\omega^{n+\frac{1}{2}}, e_y^{n+\frac{1}{2}}) = (e_y^{n+1} - e_y^n, e_y^{n+\frac{1}{2}}) - (T_y^n, e_y^{n+\frac{1}{2}}) + \Delta t(\eta_\omega, e_y^{n+\frac{1}{2}}). \tag{4.69}$$

Through Eqs (4.68) and (4.69), and summing from $n = 0$ up to N–1, we have that

$$\begin{aligned}
&\frac{1}{2}\Big[\|e_\omega^N\|^2 + \|\nabla e_y^N\|^2 - \|e_\omega^0\|^2 - \|\nabla e_y^0\|^2\Big] \\
=& \sum_{n=0}^{N-1}\Big[(e_\omega^{n+1} - e_\omega^n, \omega(t_{n+\frac{1}{2}}) - \mathcal{R}_h\omega(t_{n+\frac{1}{2}})) + (\nabla e_y^{n+\frac{1}{2}}, \nabla(\omega(t_{n+\frac{1}{2}}) - \mathcal{R}_h\omega(t_{n+\frac{1}{2}})))\Delta t \\
&+ (\phi(y_h^{n+\frac{1}{2}}(u)) - \phi(y(t_{n+\frac{1}{2}})), \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u))\Delta t + (T_\omega, \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u)) \\
&- \Delta t(\Delta\eta_y, \mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u))) + (\nabla T_y, \nabla e_y^{n+\frac{1}{2}}) - \Delta t(\nabla\eta_\omega, \nabla e_y^{n+\frac{1}{2}})\Big] \\
\triangleq& \sum_{i=1}^{7}\Theta_i.
\end{aligned} \tag{4.70}$$

By utilizing Young's and Hölder inequalities, we have

$$\begin{aligned}
\Theta_1 =& \Delta t\sum_{n=0}^{N-1}(d_t\omega^{n+1} - \mathcal{R}_h d_t\omega^{n+1}, \omega(t_{n+\frac{1}{2}}) - \mathcal{R}_h\omega(t_{n+\frac{1}{2}})) \\
\leq& Ch^4(\|\omega\|^2_{L^2(H^2)} + \|\omega\|^2_{H^1(H^2)}), \\
\Theta_2 \leq& C\sum_{n=0}^{N-1}\Delta t\|\nabla e_y^{n+1}\|^2 + Ch^2\|\omega\|^2_{L^2(H^2)}, \\
\Theta_3 \leq& C\sum_{n=0}^{N-1}\Delta t(\|e_y^{n+1}\|^2 + \|e_\omega^{n+1}\|^2) + Ch^4\|\omega\|^2_{L^2(H^2)}.
\end{aligned}$$

For Θ_4–Θ_7, by definition of the truncation error and Lemma 4.4, it can be obtained that

$$\begin{aligned}
\Theta_4 \leq& \frac{1}{\Delta t}\sum_{n=0}^{N-1}\|T_\omega\|^2 + \Delta t\sum_{n=0}^{N-1}\|\mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u))\|^2 \\
\leq& C(\Delta t)^4\|\omega_{ttt}\|^2_{L^2(L^2)} + Ch^4\|\omega\|^2_{L^2(H^2)} + C\sum_{n=0}^{N-1}\Delta t\|e_\omega^{n+1}\|^2, \\
\Theta_5 \leq& \sum_{n=0}^{N-1}\Delta t\|\Delta\eta_y\|\|\mathcal{R}_h\omega(t_{n+\frac{1}{2}}) - \omega_h^{n+\frac{1}{2}}(u))\| \\
\leq& C(\Delta t)^4\|y_{tt}\|^2_{L^2(H^2)} + C\sum_{n=0}^{N-1}\Delta t\|e_\omega^{n+1}\|^2 + Ch^4\|\omega\|^2_{L^2(H^2)}, \\
\Theta_6 \leq& C(\Delta t)^4\|y_{ttt}\|^2_{L^2(H^1)} + C\sum_{n=0}^{N-1}\Delta t\|\nabla e_y^{n+1}\|^2,
\end{aligned}$$

$$\Theta_7 \leq C(\Delta t)^4 \|\omega_{tt}\|^2_{L^2(H^1)} + C\sum_{n=0}^{N-1} \Delta t \|\nabla e_y^{n+1}\|^2.$$

Substituting Θ_1–Θ_7 into Eq (4.70), we get

$$\begin{aligned}\frac{1}{2}\|e_\omega^N\|^2 + \frac{1}{2}\|\nabla e_y^N\|^2 \leq & \frac{1}{2}\|e_\omega^0\|^2 + \frac{1}{2}\|\nabla e_y^0\|^2 + C\sum_{n=0}^{N-1} \Delta t(\|\nabla e_y^{n+1}\|^2 + \|e_\omega^{n+1}\|^2) \\ & + Ch^4(\|\omega\|^2_{L^2(H^2)} + \|\omega\|^2_{H^1(H^2)}) + Ch^2\|\omega\|^2_{L^2(H^2)} \\ & + C(\Delta t)^4(\|\omega_{ttt}\|^2_{L^2(L^2)} + \|y_{ttt}\|^2_{L^2(H^1)} + \|\omega_{tt}\|^2_{L^2(H^1)} + \|y_{tt}\|^2_{L^2(H^2)}). \end{aligned} \tag{4.71}$$

Note that

$$\begin{aligned}\frac{1}{2}\|e_\omega^0\|^2 + \frac{1}{2}\|\nabla e_y^0\|^2 &= \frac{1}{2}\|g(x) - \mathcal{R}_h g(x)\|^2 + \frac{1}{2}\|\nabla(y_0(x) - \mathcal{R}_h y_0(x)\|^2 \\ &\leq Ch^4\|g(x)\|^2_{H^2} + Ch^2\|y_0(x)\|^2_{H^2}\end{aligned}$$

Then, the discrete Gronwall's inequality implies that

$$\|\omega_\delta(u) - \omega\|_{l^\infty(L^2)} + \|y_\delta(u) - y\|_{l^\infty(H^1)} \leq C(h + (\Delta t)^2).$$

Furthermore, from Eq (3.7), we can find that the exact solution (p, q) satisfies

$$(\frac{q(t_{n+1}) + q(t_n)}{2}, v) = (\frac{p(t_{n+1}) - p(t_n)}{\Delta t}, v) - \frac{1}{\Delta t}(T_p, v) + (\eta_q, v), \tag{4.72}$$

$$\begin{aligned}&(\frac{q(t_{n+1}) - q(t_n)}{\Delta t}, v) + (\nabla\frac{p(t_{n+1}) + p(t_n)}{2}, \nabla v) + (\phi'(y(t_{n+\frac{1}{2}})p(t_{n+\frac{1}{2}}), v) \\ &\quad = (\frac{y(t_{n+1}) + y(t_n)}{2} - y_d(t_{n+\frac{1}{2}}), v) - (\eta_y, v) + \frac{1}{\Delta t}(T_q, v) - (\Delta\eta_p, v).\end{aligned} \tag{4.73}$$

From relations Eqs (4.72) and (4.73) and Eqs (4.34) and (4.35), we have

$$(e_q^{n+\frac{1}{2}}, v_h) = \frac{1}{\Delta t}(e_p^{n+1} - e_p^n, v_h) - \frac{1}{\Delta t}(T_p, v_h) + (\eta_q, v_h), \tag{4.74}$$

$$\begin{aligned}\frac{1}{\Delta t}(e_q^{n+1} - e_q^n, v_h) + (\nabla e_p^{n+\frac{1}{2}}, \nabla v_h) &= (\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u) - \phi'(y(t_{n+\frac{1}{2}})p(t_{n+\frac{1}{2}}), v_h) \\ &\quad + (e_y^{n+\frac{1}{2}}, v_h) + \frac{1}{\Delta t}(T_q, v_h) - (\Delta\eta_p, v_h) - (\eta_y, v_h).\end{aligned} \tag{4.75}$$

Choosing $v_h = -(\mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)) = q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}}) - e_q^{n+\frac{1}{2}}$ in Eq (4.75), we have

$$\begin{aligned}&\frac{1}{2}(\|e_q^n\|^2 - \|e_q^{n+1}\|^2) - \Delta t(\nabla e_p^{n+\frac{1}{2}}, \nabla e_q^{n+\frac{1}{2}}) \\ =&(e_q^n - e_q^{n+1}, q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}})) + \Delta t(\nabla e_p^{n+\frac{1}{2}}, \nabla(q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}}))) \\ &+ \Delta t(\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u) - \phi'(y(t_{n+\frac{1}{2}})p(t_{n+\frac{1}{2}}), q_h^{n+\frac{1}{2}}(u) - \mathcal{R}_h^{n+\frac{1}{2}}) \\ &- (T_q, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)) + \Delta t(\Delta\eta_p, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)))\end{aligned}$$

$$+\Delta t(\eta_y, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)) + \Delta t(e_y^{n+\frac{1}{2}}, q_h^{n+\frac{1}{2}}(u) - \mathcal{R}_h q(t_{n+\frac{1}{2}})). \tag{4.76}$$

Meanwhile, letting $v_h = -e_p^{n+\frac{1}{2}}$ in Eq (4.74), we get

$$-\Delta t(e_q^{n+\frac{1}{2}}, e_p^{n+\frac{1}{2}}) = -(e_p^{n+1} - e_p^n, e_p^{n+\frac{1}{2}}) + (T_p, e_p^{n+\frac{1}{2}}) - \Delta t(\eta_q, e_p^{n+\frac{1}{2}}). \tag{4.77}$$

Combining Eqs (4.76) and (4.77), then summing from $n = M$ to $N-1$, it can conclude that

$$\begin{aligned}
&\frac{1}{2}\|e_q^M\|^2 + \frac{1}{2}\|\nabla e_p^M\|^2 \\
=&\sum_{n=M}^{N-1}\Big[(e_q^n - e_q^{n+1}, q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}})) + \Delta t(\nabla e_p^{n+\frac{1}{2}}, \nabla(q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}}))) \\
&+ \Delta t(\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u) - \phi'(y(t_{n+\frac{1}{2}})p(t_{n+\frac{1}{2}}), q_h^{n+\frac{1}{2}}(u) - \mathcal{R}_h q(t_{n+\frac{1}{2}})) \\
&- (T_q, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)) + \Delta t(\Delta\eta_p, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u))) \\
&- (\nabla T_p, \nabla e_p^{n+\frac{1}{2}}) + \Delta t(\nabla\eta_q, \nabla e_p^{n+\frac{1}{2}}) + \Delta t(\eta_y, \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u)) \\
&+ \Delta t(e_y^{n+\frac{1}{2}}, q_h^{n+\frac{1}{2}}(u) - \mathcal{R}_h q(t_{n+\frac{1}{2}}))\Big] \\
\triangleq&\sum_{i=1}^{9}\mathcal{Y}_i.
\end{aligned} \tag{4.78}$$

By the properties of L^2 projection, we have

$$\begin{aligned}
\mathcal{Y}_1 =&\sum_{n=M}^{N-1}\Delta t(\bar{d}_t q^n - \mathcal{R}_h\bar{d}_t q^n, q(t_{n+\frac{1}{2}}) - \mathcal{R}_h q(t_{n+\frac{1}{2}})) \\
\le& Ch^4(\|q\|^2_{L^2(H^2)} + \|q\|^2_{H^1(H^2)}), \\
\mathcal{Y}_2 \le& C\sum_{n=M}^{N-1}\Delta t\|\nabla e_p^n\|^2 + Ch^2\|q\|^2_{L^2(H^2)}.
\end{aligned}$$

A standard algebraic manipulation implies that

$$\begin{aligned}
&\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u) - \phi'(y(t_{n+\frac{1}{2}}))p(t_{n+\frac{1}{2}}) \\
=&\phi'(y_h^{n+\frac{1}{2}}(u))p_h^{n+\frac{1}{2}}(u) - \phi'(y_h^{n+\frac{1}{2}}(u))\frac{p(t_{n+1}) + p(t_n)}{2} \\
&+ \phi'(y_h^{n+\frac{1}{2}}(u))(\frac{p(t_{n+1}) + p(t_n)}{2}) - \phi'(\frac{y(t_{n+1}) + y(t_n)}{2})\frac{p(t_{n+1}) + p(t_n)}{2} \\
&+ \phi'(\frac{y(t_{n+1}) + y(t_n)}{2})\frac{p(t_{n+1}) + p(t_n)}{2} - \phi'(\frac{y(t_{n+1}) + y(t_n)}{2})p(t_{n+\frac{1}{2}}) \\
&+ \phi'(\frac{y(t_{n+1}) + y(t_n)}{2})p(t_{n+\frac{1}{2}}) - \phi'(y(t_{n+\frac{1}{2}}))p(t_{n+\frac{1}{2}}).
\end{aligned}$$

For the bound of $\mathcal{Y}_3$, it holds

$$\mathcal{Y}_3 \le C\sum_{n=M}^{N-1}\Delta t(\|e_p^n\|^2 + \|e_q^n\|^2) + C(\Delta t)^4(\|y_{tt}\|^2_{L^2(H^1)} + \|p_{tt}\|^2_{L^2(L^2)})$$

$$+ Ch^4\|q\|^2_{L^2(H^2)} + C\|y - y_\delta(u)\|^2_{l^2(H^1)}.$$

Then, for $\mathcal{Y}_4$–$\mathcal{Y}_9$, noting that $\mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u) = \mathcal{R}_h q(t_{n+\frac{1}{2}}) - q(t_{n+\frac{1}{2}}) + e_q^{n+\frac{1}{2}}$ and applying Lemma 4.4, we have

$$\begin{aligned}\mathcal{Y}_4 \leq & \frac{1}{\Delta t}\sum_{n=M}^{N-1}\|T_q\|^2 + \Delta t\sum_{n=M}^{N-1}\|\mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u))\|^2 \\ \leq & C(\Delta t)^4\|q_{ttt}\|^2_{L^2(L^2)} + Ch^4\|q\|^2_{L^2(H^2)} + C\sum_{n=M}^{N-1}\Delta t\|e_q^n\|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{Y}_5 \leq & \sum_{n=M}^{N-1}\Delta t\|\Delta\eta_p\|\|\mathcal{R}_h q(t_{n+\frac{1}{2}}) - q_h^{n+\frac{1}{2}}(u))\| \\ \leq & C(\Delta t)^4\|p_{tt}\|^2_{L^2(H^2)} + C\sum_{n=M}^{N-1}\Delta t\|e_q^n\|^2 + Ch^4\|q\|^2_{L^2(H^2)}, \\ \mathcal{Y}_6 \leq & C(\Delta t)^4\|p_{ttt}\|^2_{L^2(H^1)} + C\sum_{n=M}^{N-1}\Delta t\|\nabla e_p^n\|^2, \\ \mathcal{Y}_7 \leq & C(\Delta t)^4\|q_{tt}\|^2_{L^2(H^1)} + C\sum_{n=M}^{N-1}\Delta t\|\nabla e_p^n\|^2, \\ \mathcal{Y}_8 \leq & C(\Delta t)^4\|y_{tt}\|^2_{L^2(L^2)} + C\sum_{n=M}^{N-1}\Delta t\|e_q^n\|^2 + Ch^4\|q\|^2_{L^2(H^2)}, \\ \mathcal{Y}_9 \leq & C\|y - y_\delta(u)\|^2_{l^2(H^1)} + C\sum_{n=M}^{N-1}\Delta t\|e_q^n\|^2 + Ch^4\|q\|^2_{L^2(H^2)}.\end{aligned}$$

Collecting the above bounds and using the discrete Gronwall's inequality, we deduce that

$$\begin{aligned}&\|q_\delta(u) - q\|^2_{l^\infty(L^2)} + \|p_\delta(u) - p\|^2_{l^\infty(H^1)} \\ \leq & C(\Delta t)^4(\|p_{tt}\|^2_{L^2(L^2)} + \|p_{tt}\|^2_{L^2(H^2)} + \|y_{tt}\|^2_{L^2(H^1)} + \|y_{tt}\|^2_{L^2(L^2)} + \|q_{tt}\|^2_{L^2(H^1)}) \\ &+C(\Delta t)^4(\|q_{ttt}\|^2_{L^2(L^2)} + \|p_{ttt}\|^2_{L^2(H^1)}) + Ch^4(\|q\|^2_{L^2(H^2)} + \|q\|^2_{H^1(H^2)}) \\ &+Ch^2\|q\|^2_{L^2(H^2)} + C\|y - y_\delta(u)\|^2_{l^2(H^1)}.\end{aligned} \tag{4.79}$$

The proof of Eq (4.63) can be completed by combining Eq (4.79) with Eq (4.62). □

Above all, the error of the Crank-Nicolson scheme (4.7)–(4.13) is given by the following theorem.

Theorem 4.2. *Let* $(\omega, y, \bar{u}, q, p)$ *and* $(\omega_\delta, y_\delta, \bar{u}_\delta, q_\delta, p_\delta)$ *be the local solutions of Eqs* (3.6)–(3.8) *and Eqs* (4.7)–(4.13)*, respectively. Moreover, we assume that all conditions in Lemmas 4.2–4.5 are valid. Then, we have*

$$\begin{aligned}&\|\omega - \omega_\delta\|_{l^\infty(L^2)} + \|y - y_\delta\|_{l^\infty(H^1)} \\ &+ \|q - q_\delta\|_{l^\infty(L^2)} + \|p - p_\delta\|_{l^\infty(H^1)} \\ &+ \|\bar{u} - \bar{u}_\delta\|_{l^2(L^2(\Omega))} \leq C(h + (\Delta t)^2).\end{aligned}$$

5. Numerical experiments

In this section, we provide a numerical example to verify the theoretical results, which will consider the following nonlinear SOHOCPs:

$$\min \frac{1}{2}\int_0^1 \int_\Omega (y-y_d)^2 dx + \int_\Omega u^2 dx,$$
$$s.t.\ \ y_{tt} - \Delta y + y^3 = f + u.$$

We adopt the same mesh triangular partition for the state and control. Furthermore, we choose

$$\begin{aligned}
&\Omega = (0,1)\times(0,1),\ T = 1,\ u_a = -1,\ u_b = 5,\\
&y_d = (e^t + 2 + 2\pi^2(t-T)^2)\sin\pi x_1 \sin\pi x_2 + 3e^{2t}(t-T)^2(\sin\pi x_1 \sin\pi x_2)^3,\\
&f = (1+2\pi^2)e^t \sin\pi x_1 \sin\pi x_2 + e^{3t}(\sin\pi x_1 \sin\pi x_2)^3\\
&\quad - \max\{u_a, \min\{u_b, (t-T)^2 \sin\pi x_1 \sin\pi x_2\}\}.
\end{aligned}$$

such that the exact (y, u, p) is

$$\begin{aligned}
&y = e^t \sin\pi x_1 \sin\pi x_2,\\
&u = \max\{u_a, \min\{u_b, (t-T)^2 \sin\pi x_1 \sin\pi x_2\}\},\\
&p = -(t-T)^2 \sin\pi x_1 \sin\pi x_2.
\end{aligned}$$

We show the convergence results of the Crank-Nicolson scheme in Table 1. The profile of the exact (y, p, u) is drawn in Figure 1. The simulated results for the second-order scheme is presented in Figure 2.

Table 1. Numerical results of the Crank-Nicolson scheme ($h = \Delta t^2$).

Δt	$\|\|y-y_\delta\|\|_{l^\infty(H^1(\Omega))}$	Rate	$\|\|p-p_\delta\|\|_{l^\infty(H^1(\Omega))}$	Rate	$\|\|u-u_\delta\|\|_{l^2(L^2(\Omega))}$	Rate
1/2	1.7731332397		0.6737539940		0.0349163048	
1/4	0.4450851439	1.9941	0.1680451731	2.0033	0.0056814014	2.6195
1/8	0.1111334138	2.0017	0.0414201514	2.0204	0.0013428612	2.0809
1/16	0.0277721376	2.0005	0.0103092780	2.0063	0.0003323956	2.0143
1/32	0.0069422975	2.0001	0.0025741990	2.0017	0.0000831732	1.9987

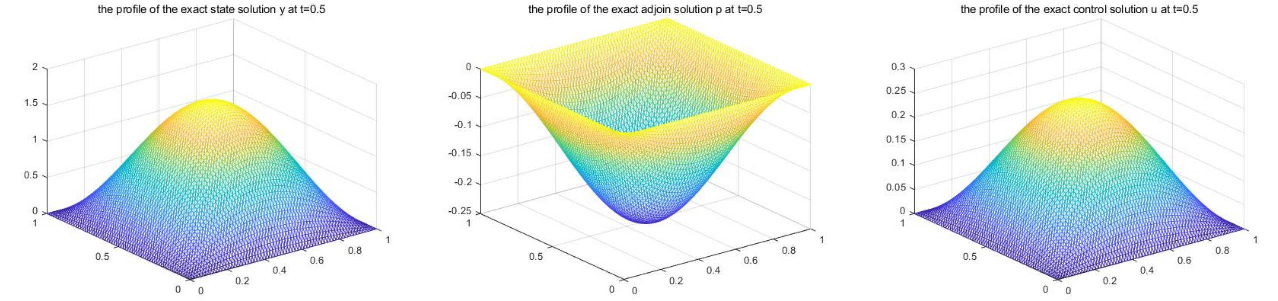

Figure 1. The exact (y, p, u) with $t = 0.5$.

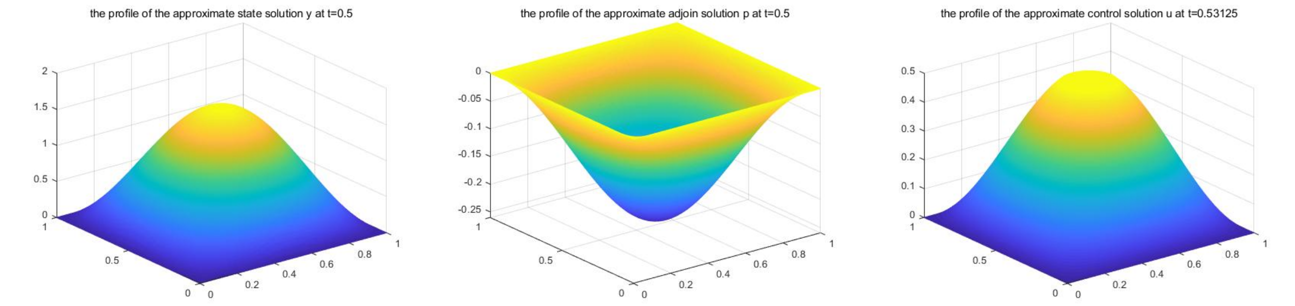

Figure 2. The approximate $(y_\delta, p_\delta, u_\delta)$ computed by the Crank-Nicolson scheme at $t = 0.5$.

From Table 1, it implies that the numerical results are consistent with the theoretical results. From Figure 1 and Figure 2, we can see the Crank-Nicolson scheme is efficient.

6. Conclusion

This paper presents a second-order fully discrete scheme for nonlinear SOHOCPs and, in conjunction with auxiliary problems, derives a priori error estimates. Furthermore, a numerical experiment is conducted to confirm the convergence order of the theoretical results.

In [31], Li et al. established a mixed-form discrete scheme for the nonlinear stochastic wave equations (SWEs) with multiplicative noise by defining a new variable. In [32], Sonawane et al. studied the existence of an optimal control problem for the bilinear SWEs. In the future, we plan to consider the method based on the definition of the new variable mentioned in this paper and in [31]. We will apply this method to the optimization system described in [32] and further conduct an error analysis of the resulting discrete scheme.

Author contributions

Huanhuan Li was responsible for the methodology and writing the original draft. Meiling Ding contributed the software. Xianbing Luo handled the review, editing, and supervision. Shuwen Xiang oversaw the supervision and validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Granted No. 11961008) and Guizhou University Doctoral Foundation (Granted NO. 15 (2022)).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. M. D. Gunzburger, *Perspective in Flow Control and Optimization*, SIAM, Philadelphia, 2002. https://doi.org/10.1137/1.9780898718720
2. F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, American Mathematical Society, Providence, 2010.
3. K. L. Teo, B. Li, C. Yu, V. Rehbock, *Applied and Computational Optimal Control: A Control Parametrization Approach*, Springer Cham, 2021.
4. Y. Chen, Z. Lu, *High Efficient and Accuracy Numerical Methods for Optimal Control Problems*, Science Press, Beijing, 2015.
5. W. Liu, N. Yan, *Adaptive Finite Element Methods for Optimal Control Governed by PDEs*, Science Press, Beijing, 2008.
6. M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, *Optimization with PDE Constraints*, Springer Science & Business Media, Dordrecht, 2009. https://doi.org/10.1007/978-1-4020-8839-1
7. F. Tröltzsch, *Optimal control of partial differential equations: theory, methods, and applications*, American Mathematical Society, Providence, 2010.
8. E. Casas, P. R. Jean, Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations, *SIAM J. Control Optim.*, **45** (2006), 1586–1611. https://doi.org/10.1137/050626600
9. T. Apel, J. Pfefferer, A. Rösch, Finite element error estimates for Neumann boundary control problems on graded meshes, *Comput. Optim. Appl.*, **52** (2012), 3–28. https://doi.org/10.1007/s10589-011-9427-x
10. J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.
11. W. Gong, M. Hinze, Z. Zhou, A priori error analysis for finite element approximation of parabolic optimal control problems with pointwise control, *SIAM J. Control Optim.*, **52** (2014), 97–119.https:://doi.org/10.1137/110840133
12. T. Hou, C. Liu, Y. Yang, Error estimates and superconvergence of a mixed finite element method for elliptic optimal control problems, *Comput. Math. Appl.*, **74** (2017), 714–726. https://doi.org/10.1016/j.camwa.2017.05.021
13. X. Luo, Y. Chen, Y. Huang, T. Hou, Some error estimates of finite volume element method for parabolic optimal control problems, *Optim. Control Appl. Methods*, **35** (2014), 145–165. https://doi.org/10.1002/oca.2059
14. C. Yang, T. Sun, BDF2 schemes for optimal parameter control problems governed by bilinear parabolic equations, *Optim. Control Appl. Methods*, **44** (2023), 2055–2081. https://doi.org/10.1002/oca.2964
15. M. Gugat, A. Keimer, G. Leugering, Optimal distributed control of the wave equation subject to state constraints, *Z. Angew. Math. Mech.*, **89** (2009), 420–444. https://doi.org/10.1002/zamm.200800196

16. A. Kröner, Adaptive finite element methods for optimal control of second order hyperbolic equations, *Comput. Methods Appl. Math.*, **11** (2011), 214–240. https://doi.org/10.2478/cmam-2011-0012

17. A. Kröner, K. Kunisch, B. Vexler, Semismooth Newton methods for optimal control of the wave equation with control constraints, *SIAM J. Control Optim.*, **49** (2011), 830–858. https://doi.org/10.1137/090766541

18. Z. Lu, X. Huang, A priori error estimates of mixed finite element methods for general linear hyperbolic convex optimal control problems, *Abstract and Appllied Analysis*, **2014** (2014), 547490. http://dx.doi.org/10.1155/2014/547490

19. X. Luo, Y. Chen, Y. Huang, A priori error estimates of finite volume element method for hyperbolic optimal control problems, *Sci. China Math.*, **56** (2013), 901–914. https://doi.org/10.1007/s11425-013-4573-5

20. X. Luo, A priori error estimates of Crank-Nicolson finite volume element method for a hyperbolic optimal control problem, *Numer. Methods Partial. Differ. Equ.*, **32** (2016), 1331–1356. https://doi.org/10.1002/num.22052

21. Z. Lu, L. Li, Y. Feng, L. Cao, W. Zhang, Error estimates of finite volume element method for nonlinear hyperbolic optimal control problems, *Ital. J. Pure Appl. Math.*, **41** (2019), 70–84.

22. X. Li, Y. Tang, A two-layer Crank-Nicolson linear finite element methods for second-order hyperbolic optimal control problems, *Results in Applied Mathematics*, **18** (2023), 100365. https://doi.org/10.1016/j.rinam.2023.100365

23. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Society for Industrial and Applied Mathematics, Philadelphia, 2002. https://doi.org/10.1137/1.9780898719208

24. M. F. Wheeler, A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations, *SIAM J. Numer. Anal.*, **10** (1973), 723–759. https://doi.org/10.1137/0710062

25. K. Chrysafinos, D. Plaka, Analysis and approximations of an optimal control problem for the Allen-Cahn equation, *Numer. Math.*, **55** (2023), 35–82. https://doi.org/10.1007/s00211-023-01374-8

26. E. Casas, K. Chrysafinos, A discontinuous Galerkin time-stepping scheme for the velocity tracking problem, *SIAM J. Numer. Anal.*, **50** (2012), 2281–2306. https://doi.org/10.1137/110829404

27. A. Kröner, B. Vexler, A priori error estimates for elliptic optimal control problems with a bilinear state equation, *J. Comput. Appl. Math.*, **230** (2009), 781–802. https://doi.org/10.1016/j.cam.2009.01.023

28. P. Shakya, R. K. Sinha, Finite element method for parabolic optimal control problems with a bilinear state equation, *J. Comput. Appl. Math.*, **367** (2020), 112431. https://doi.org/10.1016/j.cam.2019.112431

29. M. Winkler, Error estimates for the finite element approximation of bilinear boundary control problems, *Comput. Optim. Appl.*, **76** (2020), 155–199. https://doi.org/10.1007/s10589-020-00171-5

30. C. Yang, T. Sun, Second-order time discretization for reaction coefficient estimation of bilinear parabolic optimization problem with Neumann boundary conditions, *Comput. Math. Appl.*, **140** (2023), 211–224. https://doi.org/10.1016/j.camwa.2023.04.016

31. Y. Li, S. Wu, Y. Xing, Finite element approximations of a class of nonlinear stochastic wave equations with multiplicative noise, *J. Sci. Comput.*, **91** (2022), 53. https://doi.org/10.1007/s10915-022-01816-9

32. R. B. Sonawane, D. N. Kandekar, Bilinear optimal control for stochastic wave equations, *International Journal of Latest Engineering Research and Application*, **8** (2023), 17–22. https://doi.org/10.56581/IJLERA.8.7.17-22