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Research article

Traffic network analysis via multidimensional split variational inequality problem with multiple output sets

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Abstract: In this study, we introduce and investigate a new class of split inverse problems, comprising a multidimensional parameter of evolution, which we call the multidimensional split variational inequality problem with multiple output sets. To demonstrate its applicability, we formulate the equilibrium flow of multidimensional traffic network models for an arbitrary number of locations. We define a multidimensional split Wardrop condition with multiple output sets and establish its equivalence with the formulated equilibrium flow of multidimensional traffic network models. We then establish the existence and uniqueness of equilibria for our proposed model. In addition, we propose a method for solving the introduced problem. We then validate our results using some numerical experiments.

Keywords: traffic network equilibrium problem; split inverse problem; multidimensional split variational inequality problem with multiple output sets; projected dynamical system; Wardrop condition

1. Introduction

The variational inequality problem was first introduced independently by Fichera [1] and Stampacchia [2] to model optimization problems arising from mechanics. The concept of multi-time has been employed in optimization theory, namely in the framework of multi-time optimal control problem. This problem is a particular case of the multidimensional variational problems. Several problems, in science and engineering, can be modelled in terms of optimization problems, which are governed by m-flow type partial differential equations (multi-time evolution systems) and cost functionals expressed as path-independent integrals or multiple integrals. Apart from optimization theory, the concept of multidimensional parameters of evolution has also been applied in space theory, where the space coordinates are represented by two-dimensional time parameters $t = (t_1, t_2)$, where t_1 and t_2 represent the intrinsic time and the observer time, respectively. For more details and recent studies in this direction, interested readers are referred to the studies in [3–5] and the references therein.

The study of variational inequality problems in finite dimensional spaces was initiated independently by Smith [6] and Dafermos [7]. They set up the traffic assignment problem in terms of a finite dimensional variational inequality problem (VIP). On the other hand, Lawphongpanich and Hearn [8], and Panicucci et al. [9] studied traffic assignment problems based on Wardrop user equilibrium principle via a variational inequality model.

Lions and Stampacchia [10], and Brezis [11] independently introduced the time-dependent (evolutionary) variational inequality problem, and developed an existence and uniqueness theory of the problem. Daniele et al. [12] formulated a dynamic traffic network equilibrium problem in terms of an evolutionary variational inequality problem. Ever since then, several other economics related problems like Nash equilibrium problem, spatial price equilibrium problems, internet problems, dynamic financial equilibrium problems and environmental network and ecology problems have been studied via time-dependent variational inequality problem (see [13–16]).

Censor et al. [17] introduced a new split inverse problem called the split variational inequality problem (SVIP). The authors proposed iterative methods for estimating the solution of the problem, they and analysed the convergence of the proposed iterative schemes. The SVIP has several areas of applications, including network problems, image reconstruction, cancer treatment planning and many more.

Very recently, Singh et al. [18] introduced another split inverse problem, which they called evolutionary split variational inequality problem. The authors demonstrated the applicability of this new problem through the formulation of the equilibrium flow of dynamic traffic network models, which comprised two given cities. Moreover, they established the existence and uniqueness of equilibria for the proposed model.

However, Singh in [19] noted that in an economic problem, other parameters in addition to time may also affect the values of the constraints and arguments associated with the problem. Similarly, in a traffic network problem the flow of traffic depends on several economic parameters other than the time parameter. For instance, traffic flow data are known to be strongly influenced by both space (location) and time. In addition, parameters related to road capacity, safety measures for averting road accidents and several other economic parameters could affect traffic flow. Based on this observation, Singh [19] introduced a new split inverse problem, called the multidimensional split variational inequality problem (MSVIP). This new problem includes a multidimensional parameter of evolution. As an application, the author formulated the equilibrium flow within two different traffic network models, e.g., traffic networks for two given cities.

More recently, Alakoya and Mewomo [20] studied a new class of split inverse problems, known as split variational inequality problem with multiple output sets. This class of split inverse problems is designed such that multiple variational inequality problems are solved simultaneously. The authors proposed an iterative method for estimating the solution of this problem, and they further presented some numerical experiments to demonstrate the feasibility of the proposed iterative method.

We note that the results of Singh et al. [18] and Singh [19] are only capable of dealing with two different traffic network models simultaneously. In other words, their results are not applicable when the goal is to study multiple (more than two) traffic network models simultaneously. Moreover, we also note that in formulating the split inverse problems introduced in [18, 19], the authors needed to define explicitly two inverse problems (one in each of the two spaces under consideration) such that the image of the solution of the first inverse problem under a bounded linear operator is the solution of the second inverse problem. This method of formulation made the proofs of the results in [18, 19] lengthy and not easily comprehensible. To overcome these shortcomings, in this study we introduce and study a new class of split inverse problems, which we call the multidimensional split variational inequality problem with multiple output sets. This newly introduced problem also includes a multidimensional parameter of evolution. Moreover, in formulating our problem we demonstrate that the inverse problems involved in the formulation need not to be explicitly defined. Instead, by introducing an index set our problem could be formulated succinctly and the proofs of the results presented more concisely. To demonstrate its applicability in the economic world, we formulate the equilibrium flow of multidimensional traffic network models for an arbitrary number of locations, e.g., traffic network models for different cities. Moreover, we define a multidimensional split Wardrop condition with multiple output sets (MSWC-MOS), and establish its equivalence with the formulated equilibrium flow of multidimensional traffic network models. Furthermore, we establish the existence and uniqueness of equilibria for our proposed model. We propose a method for solving the introduced problem, which will be useful in evaluating the equilibrium flow of multidimensional traffic network models for different cities simultaneously. Finally, we validate our results using some numerical experiments. To further illustrate the utilization of our newly introduced problem, we apply our results to study the network model of a city with heterogeneous networks. More precisely, we consider a city, which comprises connected automated vehicles (CAVs) and legacy (human-driven) vehicles, alongside electricity network, e.g. for charging the CAVs, and we formulate the equilibrium flow of this network model in terms of our newly introduced multidimensional split variational inequality problem with multiple output sets. We note that the results in [18, 19] cannot be applied to the numerical examples and application considered in our study.

2. Preliminaries and problem formulation

In this section, we formulate our multidimensional split variational inequality problem with multiple output sets. First, we introduce some important notations and mathematical concepts, which are needed for the problem formulation. In what follows, except otherwise stated, the abbreviation "a.e." means "almost everywhere" and $\mathbb{R}^m_+$ denotes the set of non-negative vectors in $\mathbb{R}^m$. We assume that our multidimensional traffic network model comprises a multi-parameter of evolution v, which is the multidimensional parameter of evolution, i.e., $v = (v^\alpha) \in \Omega_{v_0,v_1}$, where $\alpha = 1, 2, \ldots, m$. Geometrically, Ω_{v_0,v_1} is a hyper-parallelepiped in $\mathbb{R}^m_+$ with the opposite diagonal points $v_0 = (v_0^1, v_0^2, \ldots, v_0^m)$ and $v_1 = (v_1^1, v_1^2, \ldots, v_1^m)$, which by the product order on $\mathbb{R}^m_+$ is equivalent to the closed interval $v_0 \leq v \leq v_1$. Suppose that we have cities denoted by $C_i, i = 0, 1, \ldots, M$. The traffic network of each city C_i comprises the set of nodes N_i, representing railway stations, airports, crossings, etc., the set of directed links L_i between the nodes, the set of origin-destination pairs W_i and the set of routes V_i. Moreover, it is assumed that each route $r_i \in V_i$ connects exactly one origin-destination pair. We denote by $V_i(w_i)$ the set of all $r_i \in V_i$, which connects a given $w_i \in W_i$. Let $x_i(v) \in \mathbb{R}^{|V_i|}$ be the flow trajectory, and for each $r_i \in V_i$, let $x_{r_i}(v)$ represent the flow trajectory of the route r_i over the multidimensional parameter v. We take our functional setting for the flow trajectories to be the reflexive Banach space $L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, $p_i > 1$, with the dual space $L^{q_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 0, 1, \ldots, M$. We assume that every feasible flow satisfies the following multidimensional capacity constraints for each $i = 0, 1, \ldots, M$

$$\lambda_i(v) \leq x_i(v) \leq \mu_i(v), \quad \text{a.e. on } \Omega_{v_0,v_1},$$

and the multidimensional traffic conservation law/demand requirements

$$\Phi_i x_i(v) = \rho_i(v), \quad \text{a.e. on } \Omega_{v_0,v_1},$$

where $\lambda_i(v), \mu_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$ are given bounds such that $\lambda_i(v) \leq \mu_i(v)$ and $\rho_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|W_i|})$ is the given demand such that $\rho_i(v) \geq 0$, and $\Phi_i = (\phi_{r_i,w_i})$ is the pair-route incidence matrix, whose entries are equal to 1 if route r_i links the pair w_i and 0 otherwise. It is also assumed that

$$\Phi_i \lambda_i(v) \leq \rho_i(v) \leq \Phi_i \mu_i(v), \quad \text{a.e. on } \Omega_{v_0,v_1}.$$

This assumption implies the non-emptiness of the set of feasible flows

$$K_i = \{x_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|}) : \lambda_i(v) \leq x_i(v) \leq \mu_i(v) \text{ and } \Phi_i x_i(v) = \rho_i(v), \text{ a.e. on } \Omega_{v_0,v_1}, \ i = 0, 1, \ldots, M\}.$$

The canonical bilinear form on $L^{q_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|}) \times L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$ is defined as

$$\langle\langle f_i(v), x_i(v)\rangle\rangle_{C_i} = \int_{\Omega_{v_0,v_1}} \langle f_i(v), x_i(v)\rangle dv, \quad x_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$$

and

$$f_i(v) \in L^{q_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|}), \ \ i = 0, 1, \ldots, M,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $dv = dv^1 dv^2 \ldots dv^m$ denotes the volume element of Ω_{v_0,v_1}.

Remark 1. It is clear that for each $i = 0, 1, \ldots, M$, the feasible set K_i is closed, convex and bounded. From this, it follows that each K_i is weakly compact.

Moreover, for each $x_i(v) \in K_i$, $i = 0, 1, \ldots, M$, the cost trajectory is denoted by the mapping $A_i : K_i \to L^{q_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, and we let $T_i : L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_0|}) \to L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, $i = 0, 1, \ldots, M$ be bounded linear operators, where $T_0 = I^{L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_0|})}$ is the identity operator on $L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_0|})$.

Now, we formulate our multidimensional split variational inequality problem with multiple output sets (MSVIP-MOS) as follows:

find $x_0(v) \in K_0$ such that

$$\int_{\Omega_{v_0,v_1}} \langle A_0(x_0(v)), y_0(v) - x_0(v)\rangle dv \geq 0, \quad \forall y_0(v) \in K_0, \tag{2.1}$$

and such that

$$x_i(v) = T_i x_0(v) \in K_i \text{ solves } \int_{\Omega_{v_0,v_1}} \langle A_i(x_i(v)), y_i(v) - x_i(v)\rangle dv \geq 0, \quad \forall y_i(v) \in K_i, \quad (2.2)$$
$$i = 1, 2, \ldots, M.$$

Alternatively, the problem can be formulated in a more compact form as follows:
find $x_0(v) \in K_0$ such that

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0(v)), y_i(v) - T_i x_0(v)\rangle dv \geq 0, \ \forall y_i(v) \in K_i, T_i x_0(v) \in K_i, \quad i = 0, 1, \ldots, M. \quad (2.3)$$

We denote the solution set of the MSVIP-MOS by

$$\Gamma = \{x_0(v) \in C_0^* \quad \text{such that} \quad T_i x_0(v) \in C_i^*, \ i = 1, 2, \ldots, M\} = C_0^* \cap_{i=1}^M T_i^{-1}(C_i^*),$$

where $C_0^*, C_i^*, \ i = 1, 2, \ldots, M$ are the solution sets of VIPs (2.1) and (2.2), respectively.
We have the following specials cases of our formulated MSVIP-MOS:

1. if the multidimensional parameter of evolution $v = (t^\alpha), \alpha = 1, 2, \ldots, m$, then the MSVIP-MOS reduces to a multi-time split variational inequality problem with multiple output sets.
2. if $M = 1$, then our formulated MSVIP-MOS reduces to the *multidimensional split variational inequality problem* introduced by Singh [19].
3. if the multidimensional parameter of evolution $v = (v^\alpha) \in \Omega_{v_0,v_1}$, $\alpha = 1, \ldots, m$, is a single or linear dimensional parameter of evolution, that is, $m = 1$, then Ω_{v_0,v_1} is simply the closed real interval $[v_0, v_1]$ in $\mathbb{R}_+$ (set of non-negative real numbers). Moreover, for convenience we set $v_0 = 0$ and $v_1 = T$, where T denotes an arbitrary time. Thus, $\Omega_{v_0,v_1} = [0, T]$ (a fixed time interval). In this case, the MSVIP-MOS reduces to an evolutionary split variational inequality problem with multiple output sets. In addition, if $M = 1$, then the MSVIP-MOS reduces to the evolutionary split variational inequality problem studied by Singh et al. [18].
4. if all the functions are independent of the multidimensional parameter of evolution v, then the MSVIP-MOS reduces to the *split variational inequality problem with multiple output sets* studied by Alakoya and Mewomo [20]. In addition, if $M = 1$, then the MSVIP-MOS reduces to the split variational inequality problem introduced by Censor et al. [17].

In line with the definition of an equilibrium flow for a dynamic traffic network problem given by Danielle et al. [12], we put forward the following definition for a multidimensional traffic network model with multiple networks, in terms of the introduced MSVIP-MOS.

Definition 2.1. $x_0(v) \in K_0$ is an equilibrium flow if and only if $x_0(v) \in \Gamma$.

The equilibrium flow of a traffic network has been investigated by several authors in terms of the Wardrop condition. Danielle et al. [12] modelled the traffic network equilibrium problem as a classical variational inequality problem, thereby establishing an equivalent relationship between the Wardrop condition and the classical variational inequality problem. On the other hand, Raciti [21] examined the vector form of the Wardrop equilibrium condition. Motivated by these results, here we consider the following MSWC-MOS.

Definition 2.2. For an arbitrary $x_0(v) \in K_0$ and a.e. on Ω_{v_0,v_1}, the MSWC-MOS is defined as follows:

$$A_0^{u_0}(x_0(v)) < A_0^{s_0}(x_0(v)) \implies x_0^{u_0}(v) = \mu_0^{u_0}(v) \text{ or } x_0^{s_0}(v) = \lambda_0^{s_0}(v), \quad \forall w_0 \in W_0, \ \forall u_0, s_0 \in V_0(w_0)$$

and such that $x_i(v) = T_i x_0(v) \in K_i, \ i = 1, 2, \ldots, M,$ satisfies

$$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)) \implies x_i^{u_i}(v) = \mu_i^{u_i}(v) \text{ or } x_i^{s_i}(v) = \lambda_i^{s_i}(v), \quad \forall w_i \in W_i, \ \forall u_i, s_i \in V_i(w_i).$$

Alternatively, we can recast the definition as follows:

Definition 2.3. For an arbitrary $x_0(v) \in K_0$ and a.e. on Ω_{v_0,v_1}, the MSWC-MOS can be defined as

$$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)) \implies x_i^{u_i}(v) = \mu_i^{u_i}(v) \text{ or } x_i^{s_i}(v) = \lambda_i^{s_i}(v), \quad \forall w_i \in W_i, \ \forall u_i, s_i \in V_i(w_i), \qquad (2.4)$$

where $x_i(v) = T_i x_0(v) \in K_i, \ \ i = 0, 1, \ldots, M.$

3. User-oriented multidimensional traffic network equilibria with multiple networks

In this section, we present an equivalent form of the equilibria of our multidimensional traffic network model with multiple networks via the MSWC-MOS. We note that because of the form of the MSWC-MOS, it is more responsive to the user. Hence, we can conclude that it is a user-oriented equilibrium.

Now, we state and prove the following theorem, which is the main result of this section.

Theorem 3.1. *Let $x_0(v) \in K_0$ be an arbitrary flow. Then $x_0(v)$ is an equilibrium flow if and only if it satisfies the conditions of the MSWC-MOS.*

Proof. First, we suppose that $x_0(v) \in K_0$ satisfies the conditions of the MSWC-MOS. For a given origin-destination pair $w_i \in W_i, \ i = 0, 1, \ldots, M$, we define the following sets:

$$R_i = \{u_i \in V_i(w_i) : x_i^{u_i}(v) < \mu_i^{u_i}\}, \ \ i = 0, 1, \ldots, M,$$
$$S_i = \{s_i \in V_i(w_i) : x_i^{s_i}(v) > \lambda_i^{s_i}\}, \ \ i = 0, 1, \ldots, M.$$

By the MSWC-MOS, it follows that

$$A_i^{u_i}(T_i x_0(v)) \geq A_i^{s_i}(T_i x_0(v)), \quad \forall u_i \in R_i, \ \forall s_i \in S_i, \ i = 0, 1, \ldots, M, \ \text{ a.e. on } \Omega_{v_0,v_1}. \qquad (3.1)$$

It follows from Eq (3.1) that there exist real numbers $a_i \in \mathbb{R}, \ i = 0, 1, \ldots, M$, such that

$$\sup_{s_i \in S_i} A_i^{s_i}(T_i x_0(v)) \leq a_i \leq \inf_{u_i \in R_i} A_i^{u_i}(T_i x_0(v)), \quad \text{a.e. on } \Omega_{v_0,v_1}.$$

Suppose that $y_i(v) \in K_i, \ i = 0, 1 \ldots, M$, are arbitrary flows. Then, for a.e. on Ω_{v_0,v_1} we have

$$\forall r_i \in V_i(w_i), A_i^{r_i}(T_i x_0(v)) < a_i \implies r_i \notin R_i, \ i = 0, 1, \ldots, M.$$

Note that if $r_i \notin R_i$, then $(T_i x_0(v))^{r_i} = \mu^{r_i}(v)$ and $(y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \leq 0, \ i = 0, 1, \ldots, M$. Hence, it follows that $(A_i^{r_i}(T_i x_0(v)) - a_i)(y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \geq 0, \ i = 0, 1, \ldots, M$, a.e. on Ω_{v_0,v_1}. In a similar manner, for all $r_i \in V_i(w_i)$ such that $A_i^{r_i}(T_i x_0(v)) > a_i$ a.e. on Ω_{v_0,v_1}, we also have that $(A_i^{r_i}(T_i x_0(v)) -$

$a_i)(y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \geq 0,\ i = 0, 1, \ldots, M$, a.e. on Ω_{v_0,v_1}. Consequently, for each $i = 0, 1, \ldots, M$, we get

$$\begin{aligned}\langle A_i(T_i x_0(v)), y_i(v) - T_i x_0(v)\rangle &= \sum_{w_i \in W_i} \sum_{r_i \in V_i(w_i)} A_i^{r_i}(T_i x_0(v))(y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \\ &= \sum_{w_i \in W_i} \sum_{r_i \in V_i(w_i)} (A_i^{r_i}(T_i x_0(v)) - a_i)(y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \\ &\quad + a_i \sum_{w_i \in W_i} \sum_{r_i \in V_i(w_i)} (y_i^{r_i}(v) - (T_i x_0(v))^{r_i}) \\ &\geq 0, \quad \text{a.e. on } \Omega_{v_0,v_1}. \end{aligned} \tag{3.2}$$

Observe that in Eq (3.2), the value of the term $\sum_{w_i \in W_i} \sum_{r_i \in V_i(w_i)} (y_i^{r_i}(v) - (T_i x_0(v))^{r_i}),\ i = 0, 1, \ldots, M$, is zero by the traffic conservation law/demand requirements, i.e., $\sum_{r \in V(w)} x^r(v) = \rho_w(v)$ for all $x(v) \in K$ and $w \in W$ a.e. on Ω_{v_0,v_1}. Since each $y_i(v) \in K_i,\ i = 0, 1, \ldots, M$ is arbitrary, it follows from Eq (3.2) that

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0(v)), y_i(v) - T_i x_0(v)\rangle \geq 0, \quad \forall y_i(v) \in K_i,\ i = 0, 1, \ldots, M.$$

Consequently, $x_0(v)$ is an equilibrium flow.

Next, we prove the converse statement by contradiction, that is, we suppose that $x_0(v)$ is an equilibrium flow, but it does not satisfy the conditions of the MSWC-MOS. Then, it follows that there exists origin-destination pairs $w_0 \in W_0, w_i \in W_i$ and routes

$$u_0, s_0 \in V_0(w_0), u_i, s_i \in V_i(w_i),\ i = 1, 2, \ldots, M,$$

together with a set $\Psi \subset \Omega_{v_0,v_1}$ having a positive measure such that we have the following cases:

1. $$A_0^{u_0}(x_0(v)) < A_0^{s_0}(x_0(v)),\ \ x_0^{u_0}(v) < \mu_0^{u_0}(v), x_0^{s_0}(v) > \lambda_0^{s_0}(v),\ \ \text{a.e. on } \Psi,$$

   and such that
   $$x_i(v) = T_i x_0(v) \in K_i,\ i = 1, 2, \ldots, M,$$
   satisfies
   $$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)),\ \ x_i^{u_i}(v) < \mu_i^{u_i}(v), x_i^{s_i}(v) > \lambda_i^{s_i}(v),\ \ \text{a.e. on } \Psi.$$

2. $$A_0^{u_0}(x_0(v)) < A_0^{s_0}(x_0(v)),\ \ x_0^{u_0}(v) < \mu_0^{u_0}(v), x_0^{s_0}(v) > \lambda_0^{s_0}(v),\ \ \text{a.e. on } \Psi,$$

   and such that
   $$x_i(v) = T_i x_0(v) \in K_i,\ i = 1, 2, \ldots, M,$$
   satisfies
   $$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)) \implies x_i^{u_i}(v) = \mu_i^{u_i}(v)\ \text{ or }\ x_i^{s_i}(v) = \lambda_i^{s_i}(v),\ \ \text{a.e. on } \Psi.$$

3.
$$A_0^{u_0}(x_0(v)) < A_0^{s_0}(x_0(v)) \implies x_0^{u_0}(v) = \mu_0^{u_0}(v) \text{ or } x_0^{s_0}(v) = \lambda_0^{s_0}(v), \text{ a.e. on } \Psi,$$
and such that
$$x_i(v) = T_i x_0(v) \in K_i, \ i = 1, 2, \ldots, M,$$
satisfies
$$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)), \ x_i^{u_i}(v) < \mu_i^{u_i}(v), x_i^{s_i}(v) > \lambda_i^{s_i}(v), \text{ a.e. on } \Psi.$$

4.
$$A_0^{u_0}(x_0(v)) < A_0^{s_0}(x_0(v)) \implies x_0^{u_0}(v) = \mu_0^{u_0}(v) \text{ or } x_0^{s_0}(v) = \lambda_0^{s_0}(v), \text{ a.e. on } \Psi,$$
and such that
$$x_i(v) = T_i x_0(v) \notin K_i, \ i = 1, 2, \ldots, M,$$
satisfies
$$A_i^{u_i}(x_i(v)) < A_i^{s_i}(x_i(v)) \implies x_i^{u_i}(v) = \mu_i^{u_i}(v) \text{ or } x_i^{s_i}(v) = \lambda_i^{s_i}(v), \text{ a.e. on } \Psi.$$

5. Case 1. with $x_i(v) = T_i x_0(v) \notin K_i, \ i = 1, 2, \ldots, M$.
6. Case 2. with $x_i(v) = T_i x_0(v) \notin K_i, \ i = 1, 2, \ldots, M$.
7. Case 3. with $x_i(v) = T_i x_0(v) \notin K_i, \ i = 1, 2, \ldots, M$.

Starting with the Case 1., let

$$\delta_0(v) = \min\{\mu_0^{u_0}(v) - x_0^{u_0}(v), x_0^{s_0}(v) - \lambda_0^{s_0}(v)\} \text{ and } \delta_i(v) = \min\{\mu_i^{u_i}(v) - x_i^{u_i}(v), x_i^{s_i}(v) - \lambda_i^{s_i}(v)\}, \ i = 0, 1, \ldots, M,$$

where $v \in \Psi$.

Then, $\delta_0(v) > 0$ and $\delta_i(v) > 0$, $i = 0, 1, \ldots, M$, a.e. on Ψ. Next, we construct a flow trajectory $y_0(v) \in L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_0|})$ as follows:

$$y_0^{u_0}(v) = x_0^{u_0}(v) + \delta_0(v), y_0^{s_0}(v) = x_0^{s_0}(v) - \delta_0(v), y_0^{r_0}(v) = x_0^{r_0}(v),$$
$$\text{for } r_0 \neq u_0, s_0, \text{ a.e. on } \Psi, \text{and } y_0(v) = x_0(v) \text{ outside of } \Psi.$$

In the same manner, we can define a flow trajectory $y_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, $i = 1, 2, \ldots, M$ as

$$y_i^{u_i}(v) = x_i^{u_i}(v) + \delta_i(v), y_i^{s_i}(v) = x_i^{s_i}(v) - \delta_i(v), y_i^{r_i}(v) = x_i^{r_i}(v),$$
$$\text{for } r_i \neq u_i, s_i, \text{ a.e. on } \Psi, \text{and } y_i(v) = x_i(v) \text{ outside of } \Psi.$$

Hence, it is obvious that $y_0(v) \in K_0$ such that $y_0(v) = x_0(v)$ outside of Ψ and $y_i(v) \in K_i$ such that $y_i(v) = x_i(v)$, $i = 1, 2, \ldots, M$, outside of Ψ. Moreover, we have

$$\begin{aligned}\int_{\Omega_{v_0,v_1}} \langle A_0(x_0(v)), y_0(v) - x_0(v)\rangle dv &= \int_{\Psi} \langle A_0(x_0(v)), y_0(v) - x_0(v)\rangle dv \\ &= \int_{\Psi} \delta_0(v)(A_0^{u_0}(x_0(v)) - A_0^{s_0}(x_0(v)))dv \\ &< 0.\end{aligned}$$

By a similar argument, $x_i(v) = T_i x_0(v) \in K_i,\ i = 1, 2, \ldots, M$, satisfies

$$\int_{\Omega_{v_0,v_1}} \langle A_i(x_i(v)), y_i(v) - x_i(v)\rangle dv < 0, \quad i = 1, 2, \ldots, M.$$

It follows that $x_0(v)$ is not an equilibrium flow. Using a similar argument, we can easily show that $x_0(v)$ is not an equilibrium flow for Case 2 and Case 3. Furthermore, by the fact that $x_i(v) = T_i x_0(v) \in K_i, i = 1, 2, \ldots, M$, in Cases 4, 5, 6 and 7, it is clear that $x_0(v)$ is not an equilibrium flow. Consequently, we have a contradiction, and this completes the proof of the theorem.

4. Existence and uniqueness of equilibria of a multidimensional traffic network with multiple networks

Here, we establish the existence and uniqueness of equilibria of our multidimensional traffic network model with multiple networks, which is formulated as a MSVIP-MOS. To prove the existence and uniqueness theorem, we will employ the concept of graph theory of operators. First, we present the following definitions and lemma, which will be needed in establishing our results in this section (see [18, 19, 22]).

Definition 4.1. The graph of operator $T_i,\ i = 1, 2, \ldots, M$ is defined by

$$\text{Gr } T_i = \{(x_0(v), T_i x_0(v)) \in K_0 \times K_i : x_0(v) \in K_0\}.$$

We assume that $K_i \cap T_i K_0 \neq \emptyset$ for each $i = 1, 2, \ldots, M$, where $T_i K_0 = \{y_i(v) \in L^{p_i}(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|}) : \exists\ x_0(v) \in K_0 \text{ such that } y_i(v) = T_i x_0(v)\}$. It can easily be shown that Gr T_i is a convex set. Since T_i is a bounded linear operator for each $i = 1, 2, \ldots, M$, it follows that T_i is also continuous. Thus, by the closed graph theorem we have that Gr T_i is closed w.r.t. the product topology. Consequently, Gr T_i is a nonempty, closed and convex subset of $K_0 \times K_i,\ i = 1, 2, \ldots, M$. By Remark 1, we have that $K_0 \times K_i,\ i = 1, 2, \ldots, M$ is a weakly compact set. Thus, Gr $T_i,\ i = 1, 2, \ldots, M$ is a weakly compact set.

Definition 4.2. The cost operator A is said to be demi-continuous at the point $x(v) \in K_0$ if it is strongly-weakly sequentially continuous at this point, that is, if the sequence $\{A(x_n(v))\}$ weakly converges to $A(x(v))$ for each sequence $\{x_n(v)\} \subset K_0$ such that $x_n(v) \to x(v)$, where the symbol "$\to$" denotes strong convergence.

Definition 4.3. The cost operator A is said to be strictly monotone if

$$\langle\langle A(x) - A(y), x - y\rangle\rangle > 0, \quad \forall x, y \in K_0 \text{ and } x \neq y. \tag{4.1}$$

Definition 4.4. The convex hull of a finite subset $\{(x^1(v), Tx^1(v)), (x^2(v), Tx^2(v)), \ldots, (x^n(v), Tx^n(v))\}$ of Gr T is defined by

$$\begin{aligned}&\text{co}\{(x^1(v), Tx^1(v)), (x^2(v), Tx^2(v)), \ldots, (x^n(v), Tx^n(v))\}\\ &= \Big\{\sum_{j=1}^{n} \delta^j (x^j(v), Tx^j(v)) : \sum_{j=1}^{n} \delta^j = 1, \text{ for some } \delta^j \in [0, 1]\Big\}.\end{aligned}$$

Remark 2. Observe that

$$\begin{aligned}&\mathrm{co}\{(x^1(v),Tx^1(v)),(x^2(v),Tx^2(v)),\ldots,(x^n(v),Tx^n(v))\}\\&\subset(\mathrm{co}\{x^1(v),x^2(v),\ldots,x^n(v)\},\mathrm{co}\{Tx^1(v),Tx^2(v),\ldots,Tx^n(v)\}).\end{aligned}$$

Definition 4.5. ([19]) A set-valued mapping $Q:\mathrm{Gr}\ T\to 2^{K_0\times K_1}$ is said to be a KKM* mapping if, for any finite subset $(x^1(v),Tx^1(v)),(x^2(v),Tx^2(v)),\ldots,(x^n(v),Tx^n(v))$ of Gr T,

$$\mathrm{co}\{(x^1(v),Tx^1(v)),(x^2(v),Tx^2(v)),\ldots,(x^n(v),Tx^n(v))\}\subset\bigcup_{j=1}^{n}Q(x^j(v),Tx^j(v)).$$

Lemma 4.6. *([19] KKM-Fan theorem) Let $Q:Gr\ T\to 2^{K_0\times K_1}$ be a KKM mapping with closed set values. If $Q(x(v),Tx(v))$ is compact for at least one $(x(v),Tx(v))\in Gr\ T$, then*

$$\bigcap_{(x(v),Tx(v))\in Gr\ T}Q(x(v),Tx(v))\neq\emptyset.$$

We are now in a position to state and prove the existence theorem.

Theorem 4.7. *Suppose that for $i=1,2,\ldots,M$, the cost operators A_0,A_i are demi-continuous, and that there exist $B_0\times B_i\subseteq Gr\ T_i$ nonempty and compact, and $D_0\times D_i\subseteq Gr\ T_i$ compact such that for all $(x_0(v),T_ix_0(v))\in Gr\ T_i\backslash B_0\times B_i$ there exists $(y_0(v),T_iy_0(v))\in D_0\times D_i$ with $\int_{\Omega_{v_0,v_1}}\langle A_0(x_0(v)),y_0(v)-x_0(v)\rangle dv<0$ and $\int_{\Omega_{v_0,v_1}}\langle A_i(T_ix_0(v)),T_iy_0(v)-T_ix_0(v)\rangle dv<0$. Then, the MSVIP-MOS has a solution.*

Proof. First, we define the following set-valued mappings:

- for all $x_0^*(v)\in K_0$, we define the mapping $P_0:K_0\to 2^{K_0}$ by

$$P_0(x_0^*(v))=\{x_0(v)\in K_0:\int_{\Omega_{v_0,v_1}}\langle A_0(x_0^*(v)),x_0(v)-x_0^*(v)\rangle dv<0\},$$

- for all $y_i^*(v)\in K_i,\ i=1,2,\ldots,M$, we define $P_i:K_i\to 2^{K_i}$ by

$$P_i(y_i^*(v))=\{y_i(v)\in K_i:\int_{\Omega_{v_0,v_1}}\langle A_i(y_i^*(v)),y_i(v)-y_i^*(v)\rangle dv<0\},$$

- for all $(x_0(v),T_ix_0(v))\in\mathrm{Gr}\ T_i,\ i=1,2,\ldots,M$, we define the mappings $Q_i:\mathrm{Gr}\ T_i\to 2^{K_0\times K_i}$ by

$$Q_i(x_0(v),T_ix_0(v))=\{(x_0^*(v),T_ix_0^*(v))\in\mathrm{Gr}\ T_i:\int_{\Omega_{v_0,v_1}}\langle A_0(x_0^*(v)),x_0(v)-x_0^*(v)\rangle dv\geq 0$$

  and

$$\int_{\Omega_{v_0,v_1}}\langle A_i(T_ix_0^*(v)),T_ix_0(v)-T_ix_0^*(v)\rangle dv\geq 0\}.$$

*Knaster–Kuratowski–Mazurkiewicz lemma

Clearly, $(x_0(v), T_i x_0(v)) \in Q_i(x_0(v), T_i x_0(v))$, $i = 1, 2, \ldots, M$. Therefore, $Q_i(x_0(v), T_i x_0(v))$ is nonempty for each $i = 1, 2, \ldots, M$.

Next, we prove that for each $i = 1, 2, \ldots, M$, Q_i is a KKM mapping. We proceed by contradiction, i.e., by assuming that Q_i is not a KKM mapping for each $i = 1, 2, \ldots, M$. Then for each $i = 1, 2, \ldots, M$, there exists a finite subset $\{(x^1(v), T_i x^1(v)), (x^2(v), T_i x^2(v)), \ldots, (x^n(v), T_i x^n(v))\}$ of $\mathrm{Gr}\, T_i$ such that

$$\mathrm{co}\{(x^1(v), T_i x^1(v)), (x^2(v), T_i x^2(v)), \ldots, (x^n(v), T_i x^n(v))\} \not\subset \bigcup_{j=1}^{n} Q_i(x^j(v), T_i x^j(v)),\ i = 1, 2, \ldots, M. \quad (4.2)$$

By the definition of a convex hull, there exists the following, for each $i = 1, 2, \ldots, M$,

$$(\hat{y}_0(v), T_i \hat{y}_0(v)) \in \mathrm{co}\{(x^1(v), T_i x^1(v)), (x^2(v), T_i x^2(v)), \ldots, (x^n(v), T_i x^n(v))\}$$

such that

$$(\hat{y}_0(v), T_i \hat{y}_0(v)) = \sum_{j=1}^{n} \beta_i^j (x^j(v), T_i x^j(v)), \quad i = 1, 2, \ldots, M,$$

where $\beta_i^j \in [0, 1]$ and $\sum_{j=1}^{n} \beta_i^j = 1$ for each $i = 1, 2, \ldots, M$. The expression (4.2) implies that

$$(\hat{y}_0(v), T_i \hat{y}_0(v)) \notin \bigcup_{j=1}^{n} Q_i(x^j(v), T_i x^j(v)), \quad i = 1, 2, \ldots, M.$$

Consequently, for any $j = \{1, 2, \ldots, n\}$, we have the following cases:

1. $\int_{\Omega_{v_0,v_1}} \langle A_0(\hat{y}_0(v)), x^j(v) - \hat{y}_0(v)\rangle dv < 0$ and $\int_{\Omega_{v_0,v_1}} \langle A_i(T_i \hat{y}_0(v)), T_i x^j(v) - T_i \hat{y}_0(v)\rangle dv < 0$, $i = 1, 2, \ldots, M$.
2. $\int_{\Omega_{v_0,v_1}} \langle A_0(\hat{y}_0(v)), x^j(v) - \hat{y}_0(v)\rangle dv \geq 0$ and $\int_{\Omega_{v_0,v_1}} \langle A_i(T_i \hat{y}_0(v)), T_i x^j(v) - T_i \hat{y}_0(v)\rangle dv < 0$, $i = 1, 2, \ldots, M$.
3. $\int_{\Omega_{v_0,v_1}} \langle A_0(\hat{y}_0(v)), x^j(v) - \hat{y}_0(v)\rangle dv < 0$ and $\int_{\Omega_{v_0,v_1}} \langle A_i(T_i \hat{y}_0(v)), T_i x^j(v) - T_i \hat{y}_0(v)\rangle dv \geq 0$, $i = 1, 2, \ldots, M$.

Case 1 implies that

$$\{x^1(v), x^2(v), \ldots, x^n(v)\} \subset P_0(\hat{y}_0(v)) \text{ and } \{T_i x^1(v), T_i x^2(v), \ldots, T_i x^n(v)\} \subset P_i(T_i \hat{y}_0(v)), i = 1, 2, \ldots, M.$$

Moreover, it is clear that $P_0(x_0^*)$ and $P_i(T_i x_0^*)$ are convex, for each $x_0^* \in K_0$ and $T_i x_0^* \in K_i$, $i = 1, 2, \ldots, M$. Consequently, we have

$$\mathrm{co}\{x^1(v), x^2(v), \ldots, x^n(v)\} \subset P_0(\hat{y}_0(v))$$

and

$$\mathrm{co}\{T_i x^1(v), T_i x^2(v), \ldots, T_i x^n(v)\} \subset P_i(T_i \hat{y}_0(v)), \quad \text{for each } i = 1, 2, \ldots, M.$$

By the fact that

$$(\hat{y}_0(v), T_i \hat{y}_0(v)) \in \mathrm{co}\{(x^1(v), T_i x^1(v)), (x^2(v), T_i x^2(v)), \ldots, (x^n(v), T_i x^n(v))\}, i = 1, 2, \ldots, M$$

and by Remark 2, we have

$$(\hat{y}_0(v), T_i\hat{y}_0(v)) \in (\text{co}\{x^1(v), x^2(v), \ldots, x^n(v)\}, \text{co}\{T_ix^1(v), T_ix^2(v), \ldots, T_ix^n(v)\}),$$

which implies that $\hat{y}_0(v) \in P_0(\hat{y}_0(v))$ and $T_i\hat{y}_0(v) \in P_i(T_i\hat{y}_0(v))$, $i = 1, 2, \ldots, M$.
Thus, we have

$$\int_{\Omega_{v_0,v_1}} \langle A_0(\hat{y}_0(v)), \hat{y}_0(v) - \hat{y}_0(v)\rangle dv < 0 \text{ and } \int_{\Omega_{v_0,v_1}} \langle A_i(T_i\hat{y}_0(v)), T_ix^j(v) - T_i\hat{y}_0(v)\rangle dv < 0, \quad i = 1, 2, \ldots, M,$$

which are contradictions.
By a similar argument, we can easily show that the other cases also lead to contradictions. Hence, for each $i = 1, 2, \ldots, M$, Q_i is a KKM mapping.

Next, we claim that for each $i = 1, 2, \ldots, M$, Q_i is a closed set-valued mapping for each $(x_0(v), T_ix_0(v)) \in \text{Gr } T_i$ w.r.t. the weak topology of $K_0 \times K_i$, $i = 1, 2, \ldots, M$. Let $(x_0(v), T_ix_0(v)) \in \text{Gr } T_i$ be arbitrary and suppose that $\{(x_0^n(v), T_ix_0^n(v))\}_{n=0}^{\infty}$ is a sequence in $Q_i(x_0(v), T_ix_0(v))$, which converges strongly to $(y_0(v), T_iy_0(v))$, $i = 1, 2, \ldots, M$. Since for each $n \in \mathbb{N}$, $(x_0^n(v), T_ix_0^n(v)) \in Q_i(x_0(v), T_ix_0(v))$, $i = 1, 2, \ldots, M$, we have the following for each $n \in \mathbb{N}$

$$\int_{\Omega_{v_0,v_1}} \langle A_0(x_0^n(v)), x_0(v) - x_0^n(v)\rangle dv \geq 0 \text{ and } \int_{\Omega_{v_0,v_1}} \langle A_i(T_ix_0^n(v)), T_ix_0(v) - T_ix_0^n(v)\rangle dv \geq 0, \quad i = 1, 2, \ldots, M. \tag{4.3}$$

Since A_0, A_i, $i = 1, 2, \ldots, M$ are demi-continuous and T_0, T_i, $i = 1, 2, \ldots, M$ are continuous, by taking the limit as $n \to \infty$ in Eq (4.3), we obtain

$$\int_{\Omega_{v_0,v_1}} \langle A_0(y_0(v)), x_0(v) - y_0(v)\rangle dv \geq 0$$

and

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_iy_0(v)), T_ix_0(v) - T_iy_0(v)\rangle dv \geq 0, \quad i = 1, 2, \ldots, M,$$

which implies that

$$(y_0(v), T_iy_0(v)) \in Q_i(x_0(v), T_ix_0(v))$$

for each $i = 1, 2, \ldots, M$. Thus, $Q_i(x_0(v), T_ix_0(v))$ is closed (w.r.t. the strong topology) for each

$$(x_0(v), T_ix_0(v)) \in \text{Gr } T_i, \; i = 1, 2, \ldots, M.$$

By the hypothesis in Theorem 4.7, it follows that $Q_i(x_0(v), T_ix_0(v))$, $i = 1, 2, \ldots, M$ is compact (w.r.t. the strong topology) for each

$$(x_0(v), T_ix_0(v)) \in D_0 \times D_i \subseteq \text{Gr } T_i, \; i = 1, 2, \ldots, M.$$

Consequently, by the KKM-Fan theorem, we have

$$\bigcap_{(x_0(v), T_ix_0(v)) \in \text{Gr } T_i} Q_i(x_0(v), T_ix_0(v)) \neq \emptyset, \quad i = 1, 2, \ldots, M.$$

This implies that there exists

$$(x_0^*(v), T_i x_0^*(v)) \in \text{Gr } T_i, i = 1, 2, \ldots, M,$$

such that

$$(x_0^*(v), T_i x_0^*(v)) \in Q_i(x_0(v), T_i x_0(v))$$

for all

$$(x_0(v), T_i x_0(v)) \in \text{Gr } T_i,\ i = 1, 2, \ldots, M.$$

Now, we consider the subsets $F_0 \subset K_0,\ F_i \subset K_i$, $i = 1, 2, \ldots, M$, such that

$$(x_0^*(v), T_i x_0^*(v)) \in F_0 \times F_i \subseteq \text{Gr } T_i,\ i = 1, 2, \ldots, M.$$

Then, we can write that there exists

$$(x_0^*(v), T_i x_0^*(v)) \in F_0 \times F_i$$

such that

$$(x_0^*(v), T_i x_0^*(v)) \in Q_i(x_0^*(v), T_i x_0^*(v))$$

for all

$$(x_0(v), T_i x_0(v)) \in F_0 \times F_i,\ i = 1, 2, \ldots, M.$$

Consequently, we have that for all $(x_0(v), T_i x_0(v)) \in F_0 \times F_i$,

$$\int_{\Omega_{v_0,v_1}} \langle A_0(x_0^*(v)), x_0(v) - x_0^*(v)\rangle dv \geq 0 \text{ and } \int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0^*(v)), T_i x_0(v) - T_i x_0^*(v)\rangle dv \geq 0,\ \ i = 1, 2, \ldots, M. \quad (4.4)$$

Let

$$y_i^*(v) = T_i x_0^*(v), y_i(v) = T_i x_0(v), i = 1, 2, \ldots, M,$$

and observe that $x_0^*(v)$ and

$$y_i^*(v) = T_i x_0^*(v), i = 1, 2, \ldots, M$$

are fixed in Eq (4.4). Thus, Eq (4.4) can be rewritten as $x_0^*(v) \in F_0$, such that

$$\int_{\Omega_{v_0,v_1}} \langle A_0(x_0^*(v)), x_0(v) - x_0^*(v)\rangle dv \geq 0, \quad \forall x_0(v) \in F_0,$$

and such that

$$y_i^*(v) = T_i x_0^*(v) \in F_i \text{ solves } \int_{\Omega_{v_0,v_1}} \langle A_i(y_i^*(v)), y_i(v) - y_i^*(v)\rangle dv \geq 0, \quad \forall y_i(v) \in F_i, i = 1, 2, \ldots, M.$$

Hence, it follows that the MSVIP-MOS has a solution $x_0^*(v) \in F_0 \subset K_0$.

Next, we present the result on the uniqueness of the solution of the MSVIP-MOS in the following corollary.

Corollary 1. *If the cost operators $A_i,\ i = 0, 1, \ldots, M$ are strictly monotone on $K_i,\ i = 0, 1, \ldots, M$, then the MSVIP-MOS has a unique solution.*

Proof. Suppose to the contrary that the MSVIP-MOS does not have a unique solution. Let $x_0(v) \in K_0$ be a solution of the MSVIP-MOS. Then, we have

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0(v)), x_i(v) - T_i x_0(v)\rangle dv \geq 0, \quad \forall\ x_i(v) \in K_i, T_i x_0(v) \in K_i, \ \ i = 0, 1, \ldots, M. \tag{4.5}$$

Let $\hat{x}_0(v) \in K_0$ be another solution of the MSVIP-MOS such that $x_0(v) \neq \hat{x}_0(v)$. Then, it follows that

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i \hat{x}_0(v)), \hat{x}_i(v) - T_i \hat{x}_0(v)\rangle dv \geq 0, \quad \forall\ \hat{x}_i(v) \in K_i, T_i \hat{x}_0(v) \in K_i, \ \ i = 0, 1, \ldots, M. \tag{4.6}$$

We can rewrite Eq (4.5) as

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0(v)), T_i \hat{x}_0(v) - T_i x_0(v)\rangle dv \geq 0, \quad i = 0, 1, \ldots, M. \tag{4.7}$$

By the strict monotonicity of the $A_i,\ i = 0, 1, \ldots, M$, together with the fact that $x_0(v) \neq \hat{x}_0(v)$, we get

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0(v)) - A_i(T_i \hat{x}_0(v)), T_i x_0(v) - T_i \hat{x}_0(v)\rangle dv > 0, \quad i = 0, 1, \ldots, M. \tag{4.8}$$

By adding Eqs (4.7) and (4.8), we obtain

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i \hat{x}_0(v)), T_i x_0(v) - T_i \hat{x}_0(v)\rangle dv < 0, \quad i = 0, 1, \ldots, M,$$

which contradicts Eq (4.6). Therefore, it follows that $\hat{x}_0(v)$ is not a solution of the MSVIP-MOS. Consequently, the MSVIP-MOS has a unique solution.

5. Numerical experiments for the multidimensional traffic model with multiple networks

In this section, motivated by the work of Cojocaru et al. [23], we study our multidimensional traffic model with multiple networks by employing the theory of a projected dynamical system (PDS). Dupuis and Nagurney [24] were the first to introduce and study the PDS. Furthermore, they established the connections of PDS with the classical variational inequality problem. For more details about the various areas of applications of the PDS, we refer interested readers to [23, 25].

Inspired by the results from the aforementioned works, here, we introduce and formulate a multidimensional split projected dynamical system with multiple output sets (MSPDS-MOS) for $p_i = 2,\ \ i = 0, 1, \ldots, M$ as follows:

$$\text{Find } \ x_0(\cdot) \in K_0 \ \text{ such that } \ \frac{dx_0(\cdot, \tau)}{d\tau} = \Pi_{K_0}(x_0(\cdot, \tau), -A_0(x_0(\cdot, \tau))), \ \ x_0(\cdot, 0) = x_0^0(\cdot) \in K_0$$

and such that $x_i(\cdot) = T_i x_0(\cdot) \in K_i$ satisfies

$$\frac{dx_i(\cdot, \tau)}{d\tau} = \Pi_{K_i}(x_i(\cdot, \tau), -A_i(x_i(\cdot, \tau))), \ \ x_i(\cdot, 0) = x_i^0(\cdot) \in K_i, \ \ i = 1, 2, \ldots, M,$$

where $A_i : K_i \to L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, $i = 0, 1, \ldots, M$, are Lipschitz continuous vector fields and the operators $\Pi_{K_i} : K_i \times L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$, $i = 0, 1, \ldots, M$ are defined by

$$\Pi_{K_i}(x_i(\cdot), y_i(\cdot)) := \lim_{\delta \to 0^+} \frac{\text{proj}_{K_i}(x_i(\cdot) + \delta y_i(\cdot)) - x_i(\cdot)}{\delta}, \quad \forall\, x_i(\cdot) \in K_i, y_i(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|}),$$

where $\text{proj}_{K_i}(\cdot)$ are the nearest point projection of a given vector onto the sets given by K_i.

Alternatively, the MSPDS-MOS can be formulated as follows:

Find $x_0(\cdot) \in K_0$ such that

$$\frac{dT_i x_0(\cdot, \tau)}{d\tau} = \Pi_{K_i}(T_i x_0(\cdot, \tau), -A_i(T_i x_0(\cdot, \tau))), \;\; x_i(\cdot, 0) = x_i^0(\cdot) \in K_i, i = 0, 1, \ldots, M.$$

For clarity, here we have represented the elements of the space $L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V_i|})$ at fixed $v \in \Omega_{v_0,v_1}$ by $x(\cdot)$. Observe that for all $v \in \Omega_{v_0,v_1}$, a solution of the MSVIP-MOS represents a static state of the underlying system and the static states define one or more equilibrium curves when v varies over Ω_{v_0,v_1}. On the contrary, the time τ defines the dynamics of the system over the interval $[0, \infty)$ until it attains one of the equilibria on the curves. Clearly, the solutions to the MSPDS-MOS lie in the class of absolutely continuous functions with respect to τ, mapping $[0, \infty)$ to K_i, $i = 0, 1, \ldots, M$. Before we describe the procedure to solve the MSVIP-MOS, we present the following useful definitions motivated by [26,27].

Definition 5.1. A point $\hat{x}_0(\cdot) \in K_0$ is called a critical point for the MSPDS-MOS if

$$\Pi_{K_0}(\hat{x}_0(\cdot), -A_0(\hat{x}_0(\cdot))) = 0$$

and the point $\hat{y}_i(\cdot) = T_i \hat{x}_0(\cdot) \in K_i$ satisfies

$$\Pi_{K_i}(\hat{y}_i(\cdot), -A_i(\hat{y}_i(\cdot))) = 0, \;\; i = 1, 2, \ldots, M.$$

Alternatively, the critical point for the MSPDS-MOS can be defined as follows: $\hat{x}_0(\cdot) \in K_0$ is called a critical point for the MSPDS-MOS if

$$\Pi_{K_i}(T_i \hat{x}_0(\cdot), -A_i(T_i \hat{x}_0(\cdot))) = 0, \quad T_i \hat{x}_0(\cdot) \in K_i, \;\; i = 0, 1, \ldots, M.$$

Definition 5.2. The polar set K^o associated with K is defined by

$$K^o := \{x(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|}) : \langle\langle x(\cdot), y(\cdot)\rangle\rangle \leq 0, \;\; \forall\, y(\cdot) \in K\}.$$

Definition 5.3. The tangent cone to the set K at $x(\cdot) \in K$ is defined by

$$\hat{T}_K(x(\cdot)) = \text{cl}\Big(\bigcup_{\lambda > 0} \frac{K - x(\cdot)}{\lambda}\Big),$$

where cl denotes the closure operation.

Definition 5.4. The normal cone of K at $x(\cdot) \in K$ is defined by

$$N_K(x(\cdot)) := \{y(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|}) : \langle\langle y(\cdot), z(\cdot) - x(\cdot)\rangle\rangle \leq 0, \;\; \forall\, z(\cdot) \in K\}.$$

Alternatively, we can express this as $\hat{T}_K(x(\cdot)) = [N_K(x(\cdot))]^o$.

Definition 5.5. The projection of $x(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|})$ onto K is defined by

$$\text{proj}_K(x(\cdot)) := \arg\min_{y(\cdot)\in K} \|x(\cdot) - y(\cdot)\|.$$

Remark 3. The projection map $\text{proj}_K(\cdot)$ satisfies the following property for each $x(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|})$:

$$\langle\langle x(\cdot) - \text{proj}_K(x(\cdot)), y(\cdot) - \text{proj}_K(x(\cdot))\rangle\rangle \leq 0, \quad \forall\, y(\cdot) \in K.$$

We have the following results, which follow from Proposition 2.1 and 2.2 in [26].

Proposition 1. *For all $x(\cdot) \in K$ and $y(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|})$, $\Pi_K(x(\cdot), y(\cdot))$ exists and $\Pi_K(x(\cdot), y(\cdot)) = proj_{\hat{T}_K(x(\cdot))}(y(\cdot))$.*

Proposition 2. *For all $x(\cdot) \in K$, there exists $n(\cdot) \in N_K(x(\cdot))$ such that $\Pi_K(x(\cdot), y(\cdot)) = y(\cdot) - n(\cdot),\ \ \forall y(\cdot) \in L^2(\Omega_{v_0,v_1}, \mathbb{R}^{|V|})$.*

Now, we prove the following theorem, which establishes the relationship between solutions of MSVIP-MOS and the critical points of the MSPDS-MOS.

Theorem 5.6. *The point $x_0^*(\cdot) \in K_0$ is a solution of the MSVIP-MOS if and only if it is a critical point of the MSPDS-MOS.*

Proof. First, we suppose that $x_0^*(\cdot) \in K_0$ is a solution to the MSVIP-MOS, that is,

$$\int_{\Omega_{v_0,v_1}} \langle A_i(T_i x_0^*(\cdot)), y_i(\cdot) - T_i x_0^*(\cdot)\rangle dv \geq 0, \quad \forall\, y_i(\cdot) \in K_i,\ \ i = 0, 1, \ldots, M,$$

which implies that

$$\langle\langle A_i(T_i x_0^*(\cdot)), y_i(\cdot) - T_i x_0^*(\cdot)\rangle\rangle \geq 0, \quad \forall\, y_i(\cdot) \in K_i,\ \ i = 0, 1, \ldots, M.$$

From the last inequality, it follows that

$$-A_i(T_i x_0^*(\cdot)) \in N_{K_i}(T_i x_0^*(\cdot)), \quad i = 0, 1, \ldots, M.$$

By Proposition 2, we have

$$\Pi_{K_i}(T_i x_0^*(\cdot), -A_i(T_i x_0^*(\cdot))) = 0, \tag{5.1}$$

which implies that $x_0^*(\cdot)$ is a critical point of the MSPDS-MOS.

Conversely, suppose that $x_0^*(\cdot)$ is a critical point of the MSPDS-MOS. Then, Eq (5.1) holds. By Proposition 1, it follows that

$$\text{proj}_{\hat{T}_{K_i}(T_i x_0^*(\cdot))}(-A_i(T_i x_0^*(\cdot))) = 0, \quad i = 0, 1, \ldots, M.$$

Applying Remark 3, we obtain

$$\langle\langle -A_i(T_i x_0^*(\cdot)), z_i(\cdot)\rangle\rangle \leq 0, \quad \forall\, z_i(\cdot) \in \hat{T}_{K_i}(T_i x_0^*(\cdot)),\ \ i = 0, 1, \ldots, M,$$

which gives

$$-A_i(T_i x_0^*(\cdot)) \in N_{K_i}(T_i x_0^*(\cdot)), \quad i = 0, 1, \ldots, M.$$

From this, it follows that $x_0^*(\cdot)$ is a solution of the MSVIP-MOS.

At this point, we present the method for finding the solution of the MSVIP-MOS. In our numerical experiments, we consider the case in which $v = (t^{\alpha}), \alpha = 1, 2, \ldots, m$, that is, there are m-dimensional time parameters. We have established the existence and uniqueness of equilibria for the MSVIP-MOS in Section 4. Moreover, Theorem 5.6 guarantees that any point on a curve of equilibria in the set $\Omega_{v_{0},v_{1}}$ is a critical point of the MSPDS-MOS and vice versa. Taking into consideration all of these facts, now we discretize the set $\Omega_{v_{0},v_{1}}$ as follows: $\Omega_{v_{0},v_{1}} : (v_0^1, v_0^2, \ldots, v_0^m) = (t_0^1, t_0^2, \ldots, t_0^m) < (t_1^1, t_1^2, \ldots, t_1^m) < \ldots < (t_j^1, t_j^2, \ldots, t_j^m) < \ldots < (t_n^1, t_n^2, \ldots, t_n^m) = (v_1^1, v_1^2, \ldots, v_1^m)$. Consequently, for each $t_j = (t_j^1, t_j^2, \ldots, t_j^m),\ j = 0, 1, \ldots, n$, we obtain a sequence of the MSPDS-MOS on the distinct, finite-dimensional, closed and convex sets denoted by K_{t_j}. After evaluating all of the critical points of each MSPDS-MOS, we obtain a sequence of critical points and from this, we generate the curves of equilibria by interpolation.

5.1. Example 1

To demonstrate the implementation of this procedure, we consider the transportation network patterns of three cities C_0, C_1 and C_2 as shown in Figure 1 below.

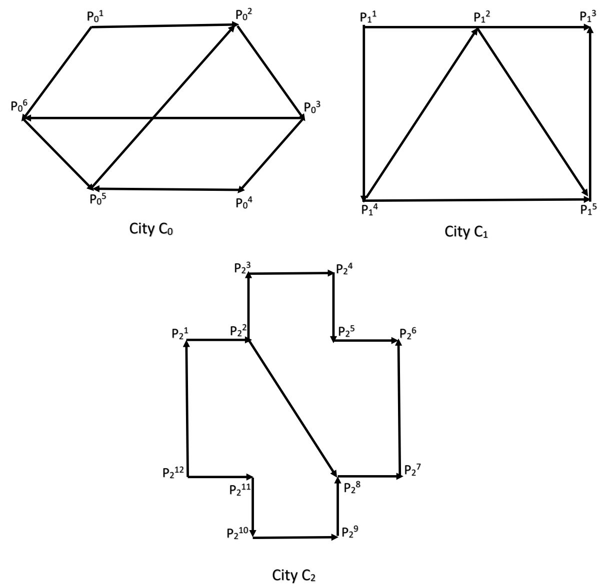

Figure 1. The transportation network patterns of the three cities C_0, C_1 and C_2.

We suppose that a bus company has stations at nodes P_0^1 and P_0^2 in City C_0, at nodes P_1^1 and P_1^4 in City C_1 and at nodes P_2^1 and P_2^{12} in City C_2. In City C_0, the buses from stations P_0^1 and P_0^2 have to deserve the locations P_0^3 and P_0^5, respectively. In City C_1, the buses from stations P_1^1 and P_1^4 have to deserve the locations P_1^2 and P_1^3, respectively. While in City C_2, the buses from stations P_2^1 and P_2^{12} have to deserve the locations P_2^6 and P_2^8, respectively.

Hence, the network of City C_0 comprises six nodes and eight links, and we assume that the origin-destination pairs are $w_0^1 = (P_0^1, P_0^3)$ and $w_0^2 = (P_0^2, P_0^5)$, which are respectively connected by the following routes:

$$w_0^1 : \begin{cases} r_0^1 = (P_0^1, P_0^2) \cup (P_0^2, P_0^3) \\ r_0^2 = (P_0^1, P_0^6) \cup (P_0^6, P_0^5) \cup (P_0^5, P_0^2) \cup (P_0^2, P_0^3), \end{cases}$$

$$w_0^2 : \begin{cases} r_0^3 = (P_0^2, P_0^3) \cup (P_0^3, P_0^4) \cup (P_0^4, P_0^5) \\ r_0^4 = (P_0^2, P_0^3) \cup (P_0^3, P_0^6) \cup (P_0^6, P_0^5). \end{cases}$$

Let $\Omega_{v_0,v_1} = \Omega_{0,3} = [0,3]^2$. The set of feasible flows, K_0, is given by

$$\begin{aligned} K_0 =& \{x(t) \in L^2(\Omega_{0,3}, \mathbb{R}^4) : \\ & (0,0,0,0) \leq (x_1(t), x_2(t), x_3(t), x_4(t)) \leq (t^1 + t^2 + 1, t^1 + t^2 + 2, 2t^1 + 2t^2 + 2, t^1 + t^2 + 3) \\ & \text{and } \ x_1(t) + x_2(t) = t^1 + t^2 + 2, \ \ x_3(t) + x_4(t) = 2t^1 + 2t^2 + 3, \ \ \text{a.e. in } \ \Omega_{0,3}\}, \end{aligned}$$

the cost function $A_0 : K_0 \to L^2(\Omega_{0,3}, \mathbb{R}^4)$ is defined by

$$A_0(x(t)) = (x_1(t), x_2(t), x_3(t), x_4(t))$$

and the bounded linear operator

$$T_0 : L^2(\Omega_{0,3}, \mathbb{R}^4) \to L^2(\Omega_{0,3}, \mathbb{R}^4)$$

is defined by $T_0x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$, where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$.

Moreover, the network of City C_1 is made up of five nodes and seven links, and we assume that the origin-destination pairs are $w_1^1 = (P_1^1, P_1^2)$ and $w_1^2 = (P_1^4, P_1^3)$, which are respectively connected by the following routes:

$$w_1^1 : \begin{cases} r_1^1 = (P_1^1, P_1^2) \\ r_1^2 = (P_1^1, P_1^4) \cup (P_1^4, P_1^2), \end{cases}$$

$$w_1^2 : \begin{cases} r_1^3 = (P_1^4, P_1^2) \cup (P_1^2, P_1^3) \\ r_1^4 = (P_1^4, P_1^5) \cup (P_1^5, P_1^3) \\ r_1^5 = (P_1^4, P_1^2) \cup (P_1^2, P_1^5) \cup (P_1^5, P_1^3). \end{cases}$$

The set of feasible flows, K_1, is given by

$$\begin{aligned} K_1 =& \{y(t) \in L^2(\Omega_{0,3}, \mathbb{R}^5) : \\ & (0,0,0,0,0) \leq (y_1(t), y_2(t), y_3(t), y_4(t), y_5(t)) \leq (t^1 + t^2 + 6, t^1 + t^2 + 6, 2t^1 + 2t^2 + 2, \\ & t^1 + t^2 + 4, 4t^1 + 4t^2 + 4) \ \text{ and } \ y_1(t) + y_2(t) = 3t^1 + 3t^2 + 5, \\ & y_3(t) + y_4(t) + y_5(t) = 2t^1 + 4t^2 + 6, \ \ \text{a.e. in } \ \Omega_{0,3}\}, \end{aligned}$$

the cost function $A_1 : K_1 \to L^2(\Omega_{0,3}, \mathbb{R}^5)$ is defined as

$$A_1(y(t)) = (y_1^2(t), y_2^2(t), y_3^2(t), y_4^2(t), y_5^2(t))$$

and the bounded linear operator

$$T_1 : L^2(\Omega_{0,3}, \mathbb{R}^4) \to L^2(\Omega_{0,3}, \mathbb{R}^5)$$

is defined by

$$T_1y(t) = (y_1(t) + y_4(t), y_2(t) + y_3(t), y_1(t) + y_2(t), 2y_1(t), 2y_2(t) + y_4(t) - y_3(t)),$$

where

$$y(t) = (y_1(t), y_2(t), y_3(t), y_4(t)).$$

Also, the network of City C_2 is composed of twelve nodes and thirteen links, and we assume that the origin-destination pairs are $w_2^1 = (P_2^1, P_2^6)$ and $w_2^2 = (P_2^{12}, P_2^8)$, which are respectively connected by the following routes:

$$w_2^1 : \begin{cases} r_2^1 = (P_2^1, P_2^2) \cup (P_2^2, P_2^3) \cup (P_2^3, P_2^4) \cup (P_2^4, P_2^5) \cup (P_2^5, P_2^6) \\ r_2^2 = (P_2^1, P_2^2) \cup (P_2^2, P_2^8) \cup (P_2^8, P_2^7) \cup (P_2^7, P_2^6), \end{cases}$$

$$w_2^2 : \begin{cases} r_2^3 = (P_2^{12}, P_2^{11}) \cup (P_2^{11}, P_2^{10}) \cup (P_2^{10}, P_2^9) \cup (P_2^9, P_2^8) \\ r_2^4 = (P_2^{12}, P_2^1) \cup (P_2^1, P_2^2) \cup (P_2^2, P_2^8). \end{cases}$$

The set of feasible flows, K_2, is given by

$$\begin{aligned} K_2 = \{&z(t) \in L^2(\Omega_{0,3}, \mathbb{R}^4) : \\ &(0,0,0,0) \le (z_1(t), z_2(t), z_3(t), z_4(t)) \le (2t^1 + 2t^2 + 3, t^1 + t^2 + 7, 3t^1 + 3t^2 + 4, 2t^1 + 2t^2 + 5) \\ &\text{and } \ z_1(t) + z_2(t) = 3t^1 + 3t^2 + 4, \ \ z_3(t) + z_4(t) = 2t^1 + 6t^2 + 7, \ \text{ a.e. in } \ \Omega_{0,3}\}, \end{aligned}$$

the cost function $A_2 : K_2 \to L^2(\Omega_{0,3}, \mathbb{R}^4)$ is defined by

$$A_2(z(t)) = (z_1(t) + z_1^2(t), z_2(t) + z_2^2(t), z_3(t) + z_3^2(t), z_4(t) + z_4^2(t))$$

and the bounded linear operator

$$T_2 : L^2(\Omega_{0,3}, \mathbb{R}^4) \to L^2(\Omega_{0,3}, \mathbb{R}^4)$$

is defined by

$$T_2 z(t) = (2z_3(t) - z_1(t), 2z_4(t) - z_2(t), 2z_1(t) + z_4(t), 2z_2(t) + z_3(t)),$$

where

$$z(t) = (z_1(t), z_2(t), z_3(t), z_4(t)).$$

It can easily be verified that all the hypotheses of Theorem 4.7 are satisfied and that the cost operators denoted by $A_i,\ i = 0, 1, 2$ are strictly monotone on the sets of feasible flows denoted by $K_i,\ i = 0, 1, 2$. Thus, the MSVIP-MOS has a unique solution. We select

$$t_j \in \{[\frac{k}{6}, \frac{k}{6}] : k \in \{0, 1, 2, \ldots, 18\}\}.$$

Then, we have a sequence of MSPDS-MOS defined on the feasible sets

$$\begin{aligned} K_{0,t_j} =\{&x(t_j) \in L^2(\Omega_{0,3}, \mathbb{R}^4) : \\ &(0,0,0,0) \le (x_1(t_j), x_2(t_j), x_3(t_j), x_4(t_j)) \le (t_j^1 + t_j^2 + 1, t_j^1 + t_j^2 + 2, 2t_j^1 + 2t_j^2 + 2, \\ &t_j^1 + t_j^2 + 3) \ \text{ and } \ x_1(t_j) + x_2(t_j) = t_j^1 + t_j^2 + 2, \ \ x_3(t_j) + x_4(t_j) = 2t_j^1 + 2t_j^2 + 3, \ \text{ a.e. in } \ \Omega_{0,3}\}, \end{aligned}$$

$$K_{1,t_j} = \{y(t_j) \in L^2(\Omega_{0,3}, \mathbb{R}^5) :$$

$(0,0,0,0,0) \leq (y_1(t_j), y_2(t_j), y_3(t_j), y_4(t_j), y_5(t_j)) \leq (t_j^1 + t_j^2 + 6, t_j^1 + t_j^2 + 6,$
$2t_j^1 + 2t_j^2 + 2, t_j^1 + t_j^2 + 4, 4t_j^1 + 4t_j^2 + 4)$ and $y_1(t_j) + y_2(t_j) = 3t_j^1 + 3t^2 + 5,\ \ y_3(t_j) + y_4(t_j) + y_5(t_j)$
$= 2t_j^1 + 4t_j^2 + 6,$ a.e. in $\Omega_{0,3}\},$

$K_{2,t_j} = \{z(t_j) \in L^2(\Omega_{0,3}, \mathbb{R}^4) :$
$(0,0,0,0) \leq (z_1(t_j), z_2(t_j), z_3(t_j), z_4(t_j)) \leq (2t_j^1 + 2t_j^2 + 3, t_j^1 + t_j^2 + 7, 3t_j^1 + 3t_j^2 + 4, 2t_j^1 + 2t_j^2 + 5)$
and $z_1(t_j) + z_2(t_j) = 3t_j^1 + 3t_j^2 + 4,\ \ z_3(t_j) + z_4(t_j) = 2t_j^1 + 6t_j^2 + 7,$ a.e. in $\Omega_{0,3}\}.$

For evaluating the unique equilibrium, we have the following system at t_j :

$$\text{find } x^*(t_j) \in K_{0,t_j} \text{ such that } -A_0(x^*(t_j)) \in N_{K_{0,t_j}}(x^*(t_j))$$
$$\text{and } T_i x^*(t_j) \in K_{i,t_j} \text{ solves } -A_i(T_i x^*(t_j)) \in N_{K_{i,t_j}}(T_i x^*(t_j)), \quad i = 1,2.$$

After some computations, we obtain the equilibrium points which are presented in Tables 1–3. Then, we interpolate the points in Tables 1–3 to get the curves of equilibria displayed in Figures 2–4.

Table 1 displays the equilibrium points at each instant for City C_0 while the traffic network pattern of City C_0 is presented in Figure 2. We observe from Table 1 that at the beginning of the equilibrium flow in City C_0, the flow on each of the routes connecting the origin-destination pair w_0^2 is about 1.5 times the flow on each of the routes connecting the origin destination pair w_0^1, and this factor increases gradually over the equilibrium flow time to about 1.9.

Table 1. Numerical results associated with the traffic network pattern of City C_0.

$t_i = \{t_i^1, t_i^2\}$	$x_1^*(t_i)$	$x_2^*(t_i)$	$x_3^*(t_i)$	$x_4^*(t_i)$
$\{0,0\}$	1.0000	1.0000	1.5000	1.5000
$\{\frac{1}{6},\frac{1}{6}\}$	1.1667	1.1667	1.8333	1.8333
$\{\frac{1}{3},\frac{1}{3}\}$	1.3333	1.3333	2.1667	2.1667
$\{\frac{1}{2},\frac{1}{2}\}$	1.5000	1.5000	2.5000	2.5000
$\{\frac{2}{3},\frac{2}{3}\}$	1.6667	1.6667	2.8333	2.8333
$\{\frac{5}{6},\frac{5}{6}\}$	1.8333	1.8333	3.1667	3.1667
$\{1,1\}$	2.0000	2.0000	3.5000	3.5000
$\{\frac{7}{6},\frac{7}{6}\}$	2.1667	2.1667	3.8333	3.8333
$\{\frac{4}{3},\frac{4}{3}\}$	2.3333	2.3333	4.1667	4.1667
$\{\frac{3}{2},\frac{3}{2}\}$	2.5000	2.5000	4.5000	4.5000
$\{\frac{5}{3},\frac{5}{3}\}$	2.6667	2.6667	4.8333	4.8333
$\{\frac{11}{6},\frac{11}{6}\}$	2.8333	2.8333	5.1667	5.1667
$\{2,2\}$	3.0000	3.0000	5.5000	5.5000
$\{\frac{13}{6},\frac{13}{6}\}$	3.1667	3.1667	5.8333	5.8333
$\{\frac{7}{3},\frac{7}{3}\}$	3.3333	3.3333	6.1667	6.1667
$\{\frac{5}{2},\frac{5}{2}\}$	3.5000	3.5000	6.5000	6.5000
$\{\frac{8}{3},\frac{8}{3}\}$	3.6667	3.6667	6.8333	6.8333
$\{\frac{17}{6},\frac{17}{6}\}$	3.8333	3.8333	7.1667	7.1667
$\{3,3\}$	4.0000	4.0000	7.5000	7.5000

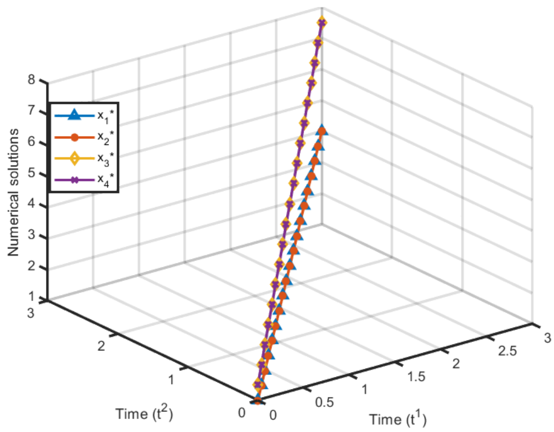

Figure 2. The traffic network pattern of City C_0.

Table 2 shows the equilibrium points at each instant for City C_1 and the traffic network pattern of the city is presented in Figure 3. We note from Table 2 that at the beginning of the equilibrium flow in City C_1, the flow on each of the routes connecting the origin-destination pair w_1^1 is about 1.3 times the flow on each of the routes connecting the origin destination pair w_1^2, and this factor increases gradually over the equilibrium flow time to about 1.4.

Table 2. Numerical results associated with the traffic network pattern of City C_1.

$t_i = \{t_i^1, t_i^2\}$	$y_1^*(t_i)$	$y_2^*(t_i)$	$y_3^*(t_i)$	$y_4^*(t_i)$	$y_5^*(t_i)$
$\{0, 0\}$	2.5000	2.5000	2.0000	2.0000	2.0000
$\{\frac{1}{6}, \frac{1}{6}\}$	3.0000	3.0000	2.3333	2.3333	2.3333
$\{\frac{1}{3}, \frac{1}{3}\}$	3.5000	3.5000	2.6667	2.6667	2.6667
$\{\frac{1}{2}, \frac{1}{2}\}$	4.0000	4.0000	3.0000	3.0000	3.0000
$\{\frac{2}{3}, \frac{2}{3}\}$	4.5000	4.5000	3.3333	3.3333	3.3333
$\{\frac{5}{6}, \frac{5}{6}\}$	5.0000	5.0000	3.6667	3.6667	3.6667
$\{1, 1\}$	5.5000	5.5000	4.0000	4.0000	4.0000
$\{\frac{7}{6}, \frac{7}{6}\}$	6.0000	6.0000	4.3333	4.3333	4.3333
$\{\frac{4}{3}, \frac{4}{3}\}$	6.5000	6.5000	4.6667	4.6667	4.6667
$\{\frac{3}{2}, \frac{3}{2}\}$	7.0000	7.0000	5.0000	5.0000	5.0000
$\{\frac{5}{3}, \frac{5}{3}\}$	7.5000	7.5000	5.3333	5.3333	5.3333
$\{\frac{11}{6}, \frac{11}{6}\}$	8.0000	8.0000	5.6667	5.6667	5.6667
$\{2, 2\}$	8.5000	8.5000	6.0000	6.0000	6.0000
$\{\frac{13}{6}, \frac{13}{6}\}$	9.0000	9.0000	6.3333	6.3333	6.3333
$\{\frac{7}{3}, \frac{7}{3}\}$	9.5000	9.5000	6.6667	6.6667	6.6667
$\{\frac{5}{2}, \frac{5}{2}\}$	10.0000	10.0000	7.0000	7.0000	7.0000
$\{\frac{8}{3}, \frac{8}{3}\}$	10.5000	10.5000	7.3333	7.3333	7.3333
$\{\frac{17}{6}, \frac{17}{6}\}$	11.0000	11.0000	7.6667	7.6667	7.6667
$\{3, 3\}$	11.5000	11.5000	8.0000	8.0000	8.0000

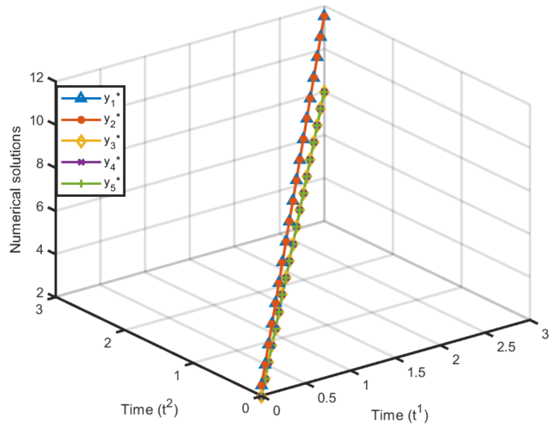

Figure 3. The traffic network pattern of City C_1.

Table 3 presents the equilibrium points at each instant for City C_2 while the traffic network pattern of the city is presented in Figure 4. It is observed from Table 3 that at the beginning of the equilibrium flow in City C_2, the flow on each of the routes connecting the origin-destination pair w_2^2 is about 1.8 times the flow on each of the routes connecting the origin destination pair w_2^1. Contrary to the observation in cities C_0 and C_1, this factor decreases gradually over the equilibrium flow time to about 1.4.

Table 3. Numerical results associated with the traffic network pattern of City C_2.

$t_i = \{t_i^1, t_i^2\}$	$z_1^*(t_i)$	$z_2^*(t_i)$	$z_3^*(t_i)$	$z_4^*(t_i)$
$\{0, 0\}$	2.0000	2.0000	3.5000	3.5000
$\{\frac{1}{6}, \frac{1}{6}\}$	2.5000	2.5000	4.1667	4.1667
$\{\frac{1}{3}, \frac{1}{3}\}$	3.0000	3.0000	4.8333	4.8333
$\{\frac{1}{2}, \frac{1}{2}\}$	3.5000	3.5000	5.5000	5.5000
$\{\frac{2}{3}, \frac{2}{3}\}$	4.0000	4.0000	6.1667	6.1667
$\{\frac{5}{6}, \frac{5}{6}\}$	4.5000	4.5000	6.8333	6.8333
$\{1, 1\}$	5.0000	5.0000	7.5000	7.5000
$\{\frac{7}{6}, \frac{7}{6}\}$	5.5000	5.5000	8.1667	8.1667
$\{\frac{4}{3}, \frac{4}{3}\}$	6.0000	6.0000	8.8333	8.8333
$\{\frac{3}{2}, \frac{3}{2}\}$	6.5000	6.5000	9.5000	9.5000
$\{\frac{5}{3}, \frac{5}{3}\}$	7.0000	7.0000	10.1667	10.1667
$\{\frac{11}{6}, \frac{11}{6}\}$	7.5000	7.5000	10.8333	10.8333
$\{2, 2\}$	8.0000	8.0000	11.5000	11.5000
$\{\frac{13}{6}, \frac{13}{6}\}$	8.5000	8.5000	12.1667	12.1667
$\{\frac{7}{3}, \frac{7}{3}\}$	9.0000	9.0000	12.8333	12.8333
$\{\frac{5}{2}, \frac{5}{2}\}$	9.5000	9.5000	13.5000	13.5000
$\{\frac{8}{3}, \frac{8}{3}\}$	10.0000	10.0000	14.1667	14.1667
$\{\frac{17}{6}, \frac{17}{6}\}$	10.5000	10.5000	14.8333	14.8333
$\{3, 3\}$	11.0000	11.0000	15.5000	15.5000

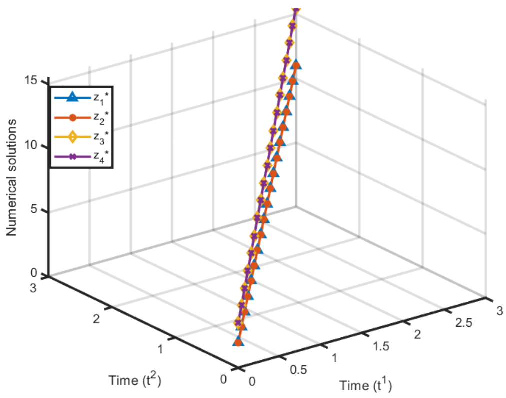

Figure 4. The traffic network pattern of City C_2.

We observe from the results that when the system is in equilibrium every route in each of the three cities is in use. Moreover, routes connecting the same origin-destination pair in each city have an equal amount of flow at each instant t within the equilibrium flow time.

5.2. Example 2: Extension to models with heterogeneous networks

In this section, we illustrate how our results can be applied to study models with heterogeneous networks. For that purpose, we consider a City C, which comprises a traffic network of human-driven vehicles (HDVs), traffic network of connected automated vehicles (CAVs) and an electricity network as shown in Figure 5 below.

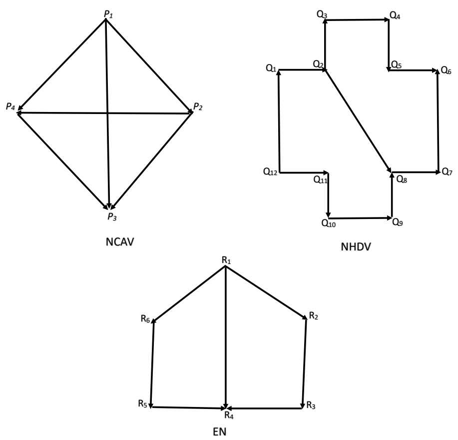

Figure 5. The network model of the three heterogeneous networks in City C.

We denote the traffic network of human-driven vehicles by NHDV, while we denote the traffic network of connected automated vehicles by NCAV and the electricity network by EN. Here, it is assumed that the EN is analogous to the traffic network. Suppose that within the network coverage of CAVs, we have commuters such that some of them need to be transported from location P^1 to location P^3 and others from location P^1 to location P^4, using CAVs. On the other hand, we assume that within the network coverage of HDVs, we have commuters who need to be transported by HDVs from locations Q^1 and Q^{12} to locations Q^6 and Q^8, respectively. Also, we suppose that within the EN, electricity needs to be transmitted from point R^1 to point R^4.

Therefore, the NCAV consists of four nodes and six links, and we assume that the origin destination pairs are $w_p^1 = (P^1, P^3)$ and $w_p^2 = (P^1, P^4)$, which are connected respectively by the following routes:

$$w_p^1 : \begin{cases} r_p^1 = (P^1, P^2) \cup (P^2, P^3) \\ r_p^2 = (P^1, P^4) \cup (P^4, P^3) \\ r_p^3 = (P^1, P^3), \end{cases}$$

$$w_p^2 : \begin{cases} r_p^4 = (P^1, P^4) \\ r_p^5 = (P^1, P^2) \cup (P^2, P^4). \end{cases}$$

The set of feasible flows K_0 is given by

$$K_0 = \{x_0(v) \in L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^5) : \lambda_0(v) \leq x_0(v) \leq \mu_0(v) \text{ and } \Phi_0 x_0(v) = \rho_0(v), \text{ a.e. on } \Omega_{v_0,v_1}\},$$

the cost function is given by $A_0 : K_0 \to L^{q_0}(\Omega_{v_0,v_1}, \mathbb{R}^5)$ and the bounded linear operator $T_0 : L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^5) \to L^{p_0}(\Omega_{v_0,v_1}, \mathbb{R}^5)$.

Similarly, the NHDV comprises twelve nodes and thirteen links, and we assume that the origin-destination pairs are $w_q^1 = (Q^1, Q^6)$ and $w_q^2 = (Q^{12}, Q^8)$, which are respectively connected by the following routes:

$$w_q^1 : \begin{cases} r_q^1 = (Q^1, Q^2) \cup (Q^2, Q^3) \cup (Q^3, Q^4) \cup (Q^4, Q^5) \cup (Q^5, Q^6) \\ r_q^2 = (Q^1, Q^2) \cup (Q^2, Q^8) \cup (Q^8, Q^7) \cup (Q^7, Q^6), \end{cases}$$

$$w_q^2 : \begin{cases} r_q^3 = (Q^{12}, Q^{11}) \cup (Q^{11}, Q^{10}) \cup (Q^{10}, Q^9) \cup (Q^9, Q^8) \\ r_q^4 = (Q^{12}, Q^1) \cup (Q^1, Q^2) \cup (Q^2, Q^8). \end{cases}$$

The set of feasible flows K_1 is given by

$$K_1 = \{x_1(v) \in L^{p_1}(\Omega_{v_0,v_1}, \mathbb{R}^4) : \lambda_1(v) \leq x_1(v) \leq \mu_1(v) \text{ and } \Phi_1 x_1(v) = \rho_1(v), \text{ a.e. on } \Omega_{v_0,v_1}\},$$

the cost function is given by $A_1 : K_1 \to L^{q_1}(\Omega_{v_0,v_1}, \mathbb{R}^4)$ and the bounded linear operator $T_1 : L^{p_1}(\Omega_{v_0,v_1}, \mathbb{R}^4) \to L^{p_1}(\Omega_{v_0,v_1}, \mathbb{R}^4)$.

On the other hand, the EN consists of six nodes and seven links, and we assume that the origin-destination pair is $w_r^1 = (R^1, R^4)$, which is connected by the following routes:

$$w_r^1 : \begin{cases} r_r^1 = (R^1, R^2) \cup (R^2, R^3) \cup (R^3, R^4) \\ r_r^2 = (R^1, R^6) \cup (R^6, R^5) \cup (R^5, R^4) \\ r_r^3 = (R^1, R^4). \end{cases}$$

The set of feasible flows K_2 is given by

$$K_2 = \{x_2(v) \in L^{p_2}(\Omega_{v_0,v_1}, \mathbb{R}^3) : \lambda_2(v) \leq x_2(v) \leq \mu_2(v) \text{ and } \Phi_2 x_2(v) = \rho_2(v), \text{ a.e. on } \Omega_{v_0,v_1}\},$$

the cost function is given by $A_2 : K_2 \to L^{q_2}(\Omega_{v_0,v_1}, \mathbb{R}^3)$ and the bounded linear operator

$$T_2 : L^{p_2}(\Omega_{v_0,v_1}, \mathbb{R}^3) \to L^{p_2}(\Omega_{v_0,v_1}, \mathbb{R}^3).$$

Then, it follows that $x_0(v) \in K_0$ is an equilibrium flow if and only if

$$\begin{aligned} &\int_{\Omega_{v_0,v_1}} \langle A_0(x_0(v)), y_0(v) - x_0(v)\rangle dv \geq 0, \quad \forall y_0(v) \in K_0,\\ &\text{and such that } \ x_i(v) = T_i x_0(v) \in K_i \ \text{ solves} \\ &\int_{\Omega_{v_0,v_1}} \langle A_i(x_i(v)), y_i(v) - x_i(v)\rangle dv \geq 0, \quad \forall y_i(v) \in K_i, \ \ i = 1, 2. \end{aligned} \tag{5.2}$$

Therefore, by employing the model (5.2), we can determine the equilibrium flows of the NCAV, NHDV and EN simultaneously.

Conclusion

We introduced and studied a new class of split inverse problem called the MSVIP-MOS. Our proposed model is finite-dimensional and essentially an assignment problem. It comprises a multidimensional parameter of evolution. To demonstrate the applicability of our proposed model in the economic world, we formulated the equilibrium flow of multidimensional traffic network models for an arbitrary number of locations. Moreover, we proposed a method for solving the introduced problem and validated our results with some numerical experiments. Finally, to further demonstrate the usefulness of our newly introduced model, we applied our results to study the network model of a city with heterogeneous networks that comprises CAVs and legacy (human-driven) vehicles, alongside the EN, e.g. for charging the CAVs, and we formulated the equilibrium flow of this network model in terms of the newly introduced MSVIP-MOS. However, we note that the problem investigated in this study belongs to the class of linear (split) inverse problems, and as such our results are not applicable to nonlinear traffic flow models. In our future study, we will be interested in extending our results to this class of models.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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